Multiplicity results for positive solutions of some semi-positone three-point boundary value problems

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Abstract

In this paper, some multiplicity results for positive solutions of some singular semi-positone three-point boundary value problem be obtained by using the fixed point index method. © 2003 Elsevier Inc. All rights reserved.

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1. Introduction

In this paper, we consider the following second order three-point boundary value problem:

\[
\begin{aligned}
\begin{cases}
y''(t) + f(t, y) = 0, \quad 0 < t < 1, \\
y(0) = 0, \quad \alpha y(\eta) = y(1),
\end{cases}
\end{aligned}
\]

(1.1)

where \(0 < \alpha, \eta, \alpha \eta < 1\), the nonlinear term \(f(t, y)\) satisfies

\[
\phi_0(t)h_0(y) - p(t) \leq f(t, y) \leq \phi(t)(g(y) + h(y))
\]

(1.2)

with \(\phi_0, \phi \in C((0, 1), (0, +\infty)), g \in ((0, +\infty), (0, +\infty)), h_0, h \in C(R^+, R^+), R^+ = [0, +\infty).\n
Nonlinear multi-point boundary value problems have been studied extensively in the literature by applying the Leray–Schauder continuation theorem, nonlinear alternatives of Leray–Schauder, coincidence degree theory, fixed point theorem in cones and so on. We refer the reader to [1–5, 10] for some recent results for nonlinear multi-point boundary
value problems. Recently, there have been some papers considered the existence of positive solutions of multi-point boundary value problems. For example, Ma [3] considered the following m-point boundary value problem:

\[
\begin{cases}
u''(t) + a(t)f(u) = 0, & t \in (0, 1), \\
u'(0) = \sum_{i=1}^{m-2} b_i u'(\xi_i), & u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i),
\end{cases}
\] (1.3)

where \( f \in C(\mathbb{R}^+, \mathbb{R}^+) \), \( \xi_i \in (0, 1) \) with \( 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1 \), \( a_i, b_i \in \mathbb{R}^+ \) with \( 0 < \sum_{i=1}^{m-2} a_i < 1 \), and \( 0 < \sum_{i=1}^{m-2} b_i < 1 \).

Set

\[
f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to +\infty} \frac{f(u)}{u}.
\]

Then \( f_0 = 0 \) and \( f_\infty = \infty \) correspond to the superlinear case, and \( f_0 = \infty \) and \( f_\infty = 0 \) correspond to the sublinear case. By applying the fixed point theorem in cones, Ma [3] showed that the m-point boundary value problem (1.3) has at least one positive solution if \( f \) is either superlinear or sublinear.

Obviously, the condition that \( f \) is nonnegative on \( \mathbb{R}^+ \) is necessary for the existence of positive solutions of the m-point boundary value problem (1.3). However, many nonlinearities in the multi-point boundary value problems which arise in applications are not nonnegative on \( \mathbb{R}^+ \). Different from [3], in this paper we will not assume that the nonlinear term \( f \) is nonnegative on \( \mathbb{R}^+ \). That is, we will study the so-called semi-positone problems.

Very recently, there are many papers considered the existence of positive solutions of two-point semi-positone boundary value problems (see [7, 8]). However, there are fewer papers considered the multiplicity of positive solutions of the semi-positone boundary value problems.

In this paper, by using the fixed point index method we will give some multiplicity results for positive solutions of semi-positone boundary value problem (1.1). In addition, in this paper we will assume that the nonlinear term \( f \) may have singularity at \( t = 0 \setminus 1 \) and \( y = 0 \).

In the sequel, we will always assume that (1.2) holds. By a positive solution of problem (1.1), we mean a function \( y \in C[0, 1] \cap C^2(0, 1) \) satisfying the boundary value problem (1.1) and \( y(t) > 0 \) for \( t \in (0, 1) \).

### 2. Several lemmas

Set

\[
c_1 = \frac{2[1 - \alpha \eta + |1 - \alpha|]}{\eta(1 - \eta)(1 - \alpha \eta) \min\{1 - \alpha \eta, \alpha - \alpha \eta\}} \int_0^1 (1 - s) p(s) \, ds,
\]

and

\[
q(t) = \begin{cases}
\frac{1 - \eta}{1 - \alpha \eta + |1 - \alpha|}, & t \in [0, \eta], \\
\frac{\eta((1 - \alpha \eta) - (1 - \alpha) \eta)}{1 - \alpha \eta + |1 - \alpha|}, & t \in [\eta, 1].
\end{cases}
\]
Let us list some conditions for convenience.

(\(H_1\)) \(g : (0, +\infty) \rightarrow (0, +\infty)\) is continuous and nonincreasing, \(h_0, h : R^+ \rightarrow R^+\) is continuous and nondecreasing.

(\(H_2\)) For any positive constant \(k_0\),
\[
\int_0^1 \left[ \phi(s)g(k_0s) + \phi(s) + p(s) \right] ds < +\infty.
\]

(\(H_3\)) There exists \(R_0 \geq 2c_1\) such that
\[
\frac{g(R_0)}{g(R_0) + h(R_0 + 1)} + \left( \int_0^{R_0} \frac{d\tau}{g(\tau/2)} - \frac{|1 - \alpha| R_0}{\alpha(1 - \eta)g(R_0/2)} \right) > \int_0^1 \left[ \phi(s) + p(s) \right] ds.
\]

(\(H_4\)) There exists \(u_1 > R_0\) such that
\[
\min\left\{ 1, \alpha \right\} \left( 1 - \eta \right) \eta \int_0^{u_1} \phi_0(s) ds > u_1.
\]

Let \(\|x\| = \max_{t \in [0,1]} |x(t)|\) for any \(x \in C[0, 1]\). It is well known that \(C[0,1]\) is a Banach space with the norm \(\|\cdot\|\). Let \(P = \{ x \mid x \in C[0, 1], x(t) \geq 0 \text{ for } t \in [0, 1] \}\) and \(Q = \{ x \mid x \in P, x(t) \geq \|x\| q(t) \text{ for } t \in [0, 1] \}\). Clearly, \(P\) and \(Q\) are cones of \(C[0, 1]\). For any \(y \in P\), let
\[
\left[ y(t) \right]^* = \max \left\{ y(t) - w(t), \frac{1}{2} R_0 q(t) \right\},
\]
where
\[
w(t) = \frac{t}{1 - \alpha \eta} \int_0^1 (1-s)p(s)ds - \int_0^t (t-s)p(s)ds - \frac{\alpha t}{1 - \alpha \eta} \int_0^\eta (\eta-s)p(s)ds, \quad t \in [0, 1].
\]

For any positive integer \(n\), let us define a mapping \(T_n : P \rightarrow C[0, 1]\) by
\[
(T_n y)(t) = \frac{t}{1 - \alpha \eta} \int_0^1 (1-s) \left[ f(x, \left[ y(s) \right]^* + n^{-1}) + p(s) \right] ds
\]
\begin{align*}
&- \int_0^t (t-s) \left[ f(s, [y(s)]^* + n^{-1}) + p(s) \right] ds \\
&- \frac{\alpha t}{1 - \alpha \eta} \int_0^\eta (\eta-s) \left[ f(s, [y(s)]^* + n^{-1}) + p(s) \right] ds,
\end{align*}
\quad t \in [0, 1]. \tag{2.1}

**Lemma 1** [9]. Let $X$ be a retract of the real Banach space $E$ and $X_1$ be a bounded convex retract of $X$. Let $U$ be a nonempty open set of $X$ and $U \subset X_1$. Suppose that $A : X_1 \mapsto X$ is completely continuous, $A(X_1) \subset X_1$ and $A$ has no fixed points on $X_1 \setminus U$. Then

\begin{equation*}
i(A, U, X) = 1.
\end{equation*}

**Lemma 2.** Suppose that $(H_1)$ and $(H_2)$ hold. Then $T_n : P \mapsto Q$ is a completely continuous operator for each positive integer $n$.

**Proof.** Let $n$ be a fixed positive integer and $y(t) = (T_n x)(t)$ for some $x \in P$. By direct computation, we get that

\begin{align*}
\left\{ \begin{array}{l}
y''(t) + [f(t, [x(t)]^* + n^{-1}) + p(t)] = 0, & t \in [0, 1], \\
y(0) = 0, & \alpha y(\eta) = y(1).
\end{array} \right.
\end{align*}

Thus, $y$ is a concave function on $[0, 1]$. It is easy to see that

\begin{align*}
y(\eta) &= \frac{\eta}{1 - \alpha \eta} \int_0^1 (1-s) \left[ f(s, [y(s)]^* + n^{-1}) + p(s) \right] ds \\
&\quad + \frac{1}{1 - \alpha \eta} \int_0^\eta (\eta-s) \left[ f(s, [y(s)]^* + n^{-1}) + p(s) \right] ds,
\end{align*}

and so, $y(1) = \alpha y(\eta) \geq 0$. Since the graph of the function $y$ passes through three points $(0, 0), (\eta, y(\eta))$ and $(1, \alpha y(\eta))$, then we get

\begin{align*}
y(t) \leq \left\{ \begin{array}{ll}
\frac{1-\alpha t}{1-\eta} y(\eta), & t \in [0, \eta], \\
\frac{t}{\eta} y(\eta), & t \in [\eta, 1],
\end{array} \right.
\end{align*}

\begin{align*}
y(t) \leq \left\{ \begin{array}{ll}
\frac{1-\alpha \eta + |1-\alpha|}{1-\eta} y(\eta), & t \in [0, \eta], \\
\frac{t}{\eta} y(\eta), & t \in [\eta, 1],
\end{array} \right.
\end{align*}

and so, $y(\eta) \geq c_0 \|y\|$, where

\begin{equation*}
c_0 = \frac{\eta (1-\eta)}{1 - \alpha \eta + |1-\alpha|}.
\end{equation*}
For any \( t \in [0, \eta] \), we have
\[
y(t) = y(t) \left( \eta \frac{t - \eta}{\eta} \cdot 0 \right) \geq \frac{t - \eta}{\eta} y(\eta) + \frac{\eta - t}{\eta} y(0) \geq \frac{c_0 t}{\eta} \| y \|.
\]
For any \( t \in [\eta, 1] \), we have
\[
y(t) = \left( \frac{1 - \eta - t}{1 - \eta} \cdot 1 \right) \left( 1 - \frac{1 - \eta - t}{1 - \eta} \cdot \eta \right) \geq \frac{t - \eta}{1 - \eta} y(1) + \frac{1 - t}{1 - \eta} y(\eta) \geq \frac{c_0 \alpha t}{1 - \eta} \| y \|.
\]
Hence, \( T_n : P \mapsto Q \).

Now we show that \( T_n \) is a completely continuous operator for every positive integer \( n \).
The continuity and the boundedness of \( T_n \) can be easily obtained. Let \( B \) be a bounded set of \( P \) such that \( \| x \| \leq L \) for all \( x \in B \) and some \( L > 0 \). It is easy to see that for any \( x \in B \),
\[
| (T_n x)(t_1) - (T_n x)(t_2) | \leq \frac{t_2 - t_1}{1 - \alpha \eta} \int_0^1 (1 - s) \left[ \phi(s) \left( g(n^{-1}) + h(L + \| w \| + R_0 \| q \| + 1) \right) + p(s) \right] ds
\]
\[
+ \int_{t_1}^{t_2} (1 - s) \left[ \phi(s) \left( g(n^{-1}) + h(L + \| w \| + R_0 \| q \| + 1) \right) + p(s) \right] ds
\]
\[
+ (t_2 - t_1) \int_0^1 \left[ \phi(s) \left( g(n^{-1}) + h(L + \| w \| + R_0 \| q \| + 1) \right) + p(s) \right] ds
\]
\[
+ \frac{\alpha (t_2 - t_1)}{1 - \alpha \eta} \int_0^\eta (\eta - s) \left[ \phi(s) \left( g(n^{-1}) + h(L + \| w \| + R_0 \| q \| + 1) \right) + p(s) \right] ds,
\] 0 \leq t_1 < t_2 \leq 1. \tag{2.2}
By (2.2) and (H_2), we easily see that \( T_n(B) \) is equicontinuous on \([0, 1]\). According to the Ascoli–Arzela theorem, we know that \( T_n B \) is a relatively compact set. Therefore, \( T_n \) is a completely continuous operator for every positive integer \( n \). The proof is completed.

**Lemma 3.** Suppose that (H_1)–(H_3) hold. Then for any positive integer \( n \),
\[
i(T_n, \Omega_0, Q) = 1,
\]
where \( \Omega_0 = \{ x \in Q \mid \| x \| < R_0 \} \).

**Proof.** It follows from Lemma 2 that \( T_n : Q \mapsto Q \) is a completely continuous operator for every positive integer \( n \). Now we will show that for every positive integer \( n \),
\[
z \neq \mu T_n z, \quad \mu \in [0, 1], \ z \in \partial \Omega_0.
\]  \tag{2.3}
In fact, if not, then there exist $\mu_0 \in [0, 1]$, $n_0 \in N$ and $z_0 \in \partial \Omega_0$ such that $z_0 = \mu_0 T_{n_0} z_0$.

Since $z_0 \in \mathcal{Q}$, we have

$$z_0(t) \geq \|z_0\|q(t) = R_0 q(t), \quad t \in [0, 1].$$

On the other hand, we have for any $t \in [0, \eta]$,

$$w(t) \leq \frac{t}{1 - \alpha \eta} \int_0^1 (1 - s) p(s) ds \leq c_1 \frac{1 - \eta}{1 - \alpha \eta + |1 - \alpha|} t = c_1 q(t),$$

and for any $t \in [\eta, 1]$,

$$w(t) \leq \frac{t}{1 - \alpha \eta} \int_0^1 (1 - s) p(s) ds \leq c_1 \frac{\eta \cdot \text{min}[1 - \alpha \eta, \alpha - \alpha \eta]}{1 - \alpha \eta + |1 - \alpha|}$$

$$\leq \begin{cases} c_1 \frac{\eta(1 - \alpha \eta)}{1 - \alpha \eta + |1 - \alpha|}, & \alpha \geq 1, \\ c_1 \frac{\eta(1 - \alpha \eta) - (1 - \alpha)}{1 - \alpha \eta + |1 - \alpha|}, & \alpha < 1, \end{cases}$$

$$\leq c_1 q(t).$$

Hence,

$$w(t) \leq c_1 q(t) \leq \frac{c_1}{R_0} z_0(t), \quad \forall t \in [0, 1].$$

Then we have

$$[z_0(t)]^\alpha = z_0(t) - w(t) \geq \left(1 - \frac{c_1}{R_0}\right) z_0(t) \geq \frac{1}{2} z_0(t) \geq \frac{1}{2} R_0 q(t), \quad t \in [0, 1]. \quad (2.4)$$

From $z_0 = \mu_0 T_{n_0} z_0$, by direct computation and (2.4) we have

$$\begin{cases} z'_0(t) + \mu_0 f(t, z_0(t) - w(t) + n_0^{-1} + p(t)) = 0, & t \in [0, 1], \\ z_0(0) = 0, & z_0(\eta) = z_0(1). \end{cases} \quad (2.5)$$

It follows from (2.5) that $z''_0(t) \leq 0$ for $t \in (0, 1)$. Thus $z_0(t)$ is a concave function on $[0, 1]$.

By (2.4), (2.5) and (H1), we have

$$-z''_0(t) \leq \phi(t) \left[ g(z_0(t) - w(t) + n_0^{-1}) + h(z_0(t) - w(t) + n_0^{-1}) \right] + p(t)$$

$$\leq \phi(t) g \left( \frac{1}{2} z_0(t) \right) \frac{g(R_0) + h(R_0 + 1)}{g(R_0)} + p(t), \quad t \in (0, 1). \quad (2.6)$$

Then we have the following two cases.

Case (a). There exists $t_0 \in (0, 1)$ such that $z'_0(t_0) = 0$. Then we easily see that

$$z'_0(t) \geq 0, \quad t \in (0, t_0), \quad z'_0(t) \leq 0, \quad t \in (t_0, 1), \quad \|z_0\| = z_0(t_0).$$

Since $g : (0, +\infty) \mapsto (0, +\infty)$ is nonincreasing and $z_0(t)$ is increasing on $(t, t_0)$, integration (2.6) from $t$ to $(0, t_0)$ to $t_0$ yields

$$z'_0(t) \leq g \left( \frac{1}{2} z_0(t) \right) \frac{g(R_0) + h(R_0 + 1)}{g(R_0)} \int_t^{t_0} \phi(s) ds + \int_t^{t_0} p(s) ds,$$
and so
\[ \frac{z'_0(t)}{g(z_0(t)/2)} \leq \frac{g(R_0) + h(R_0 + 1)}{g(R_0)} \int_0^1 [\phi(s) + p(s)] \, ds. \quad (2.7) \]

Then integration (2.7) from 0 to \( t_n \) yields
\[ \int_0^{t_n} \frac{dt}{g(t/2)} \leq \frac{g(R_0) + h(R_0 + 1)}{g(R_0)} \int_0^1 [\phi(s) + p(s)] \, ds. \quad (2.8) \]

Case (b). There exists no \( t_0 \in (0, 1) \) such that \( z_0(t_0) = \|z_0\| \). Clearly, in this case, \( \|z_0\| = z_0(1) \), and \( z'_0(t) > 0 \) for \( t \in (0, 1) \). Let \([t_n]\) be a number sequence such that \( \eta < t_n < 1 \) and \( t_n \to 1 \) as \( n \to +\infty \). From the concavity of \( z_0 \), we have for every positive integer \( n \),
\[ z'_0(t_n) \leq \frac{z_0(t_n) - z(\eta)}{t_n - \eta}. \quad (2.9) \]

Integrate (2.6) from \( t \in (0, t_n) \) to \( t_n \) to obtain
\[ z'_0(t) \leq z'_0(t_n) + g \left( \frac{1}{2} z_0(t) \right) \frac{g(R_0) + h(R_0 + 1)}{g(R_0)} \int_t^{t_n} \phi(s) \, ds + \int_t^{t_n} p(s) \, ds. \]

By (2.9), we have
\[ \frac{z'_0(t)}{g(z_0(t)/2)} \leq \frac{z_0(t_n) - z(\eta)}{(t_n - \eta) g(R_0/2)} + \frac{g(R_0) + h(R_0 + 1)}{g(R_0)} \int_0^1 [\phi(s) + p(s)] \, ds. \]

Then integrate from 0 to \( t_n \) to obtain
\[ \int_0^{t_n} \frac{dt}{g(t/2)} \leq \frac{z_0(t_n) - z(\eta)}{(t_n - \eta) g(R_0/2)} + \frac{g(R_0) + h(R_0 + 1)}{g(R_0)} \int_0^1 [\phi(s) + p(s)] \, ds. \]

Letting \( n \to +\infty \), we have
\[ \alpha \int_0^{\|z_0\|} \frac{dt}{g(t/2)} \leq \frac{|1 - \alpha| \cdot \|z_0\|}{(1 - \eta) g(R_0/2)} + \alpha \frac{g(R_0) + h(R_0 + 1)}{g(R_0)} \int_0^1 [\phi(s) + p(s)] \, ds. \quad (2.10) \]

Since \( z_0(t_0) = R_0 \), by (2.8) and (2.10), we have
\[ \frac{g(R_0)}{g(R_0) + h(R_0 + 1)} \left( \int_0^{R_0} \frac{dt}{g(t/2)} - \frac{|1 - \alpha| R_0}{\alpha (1 - \eta) g(R_0/2)} \right) \leq \int_0^1 [\phi(s) + p(s)] \, ds. \]
which is a contradiction to (H3). Therefore, (2.3) holds. By the properties of the fixed point
index, we know that the conclusion holds. The proof is completed. □

Remark 1. The nonlinear term \( f \) of the form (1.2) in the case \( \phi_0(t) = p(t) = 0 \) for all \( t \in [0, 1] \) has been studied by many authors (see [6,7]).

Remark 2. It is easy to see that (1.2), (H1) and (H2) naturally hold when \( f \) is nonnegative
and continuous on \([0, 1] \times \mathbb{R}^+\). In fact, if we set \( \phi_0(t) = p(t) = 0 \), \( \phi(t) = 2 \) for all \( t \in [0, 1] \), \( g(y) = 1 \) for all \( y \in \mathbb{R}^+ \), and
\[
h_0(y) = h(y) = \max_{(t,x) \in [0,1] \times [0,y]} f(t,s)
\]
for all \( y \in \mathbb{R}^+ \), then
\[
\phi_0(t)h_0(y) - p(t) \leq f(t,y) \leq \phi(t)\left(g(y) + h(y)\right)
\]
for \((t,y) \in (0,1) \times (0, +\infty)\).

3. Main results

**Theorem 1.** Suppose that (H1)–(H4) hold. Moreover,
\[
\lim_{y \to +\infty} \frac{h(y)}{y} = 0. \quad (3.1)
\]
Then the boundary value problem (1.1) has at least two positive solutions.

**Proof.** For each positive integer \( n \), let us define the operator \( T_n \) by (2.1). It follows from
Lemma 2 that \( T_n \) is a completely continuous operator. By (H4), there exists \( \tilde{u}_1 > u_1 \) such that
\[
\min\left\{1, \alpha\right\}(1 - \eta)\eta \frac{h_0\left(\frac{1}{2} \tilde{u}_1\right)}{2} \int_0^{\eta/2} \phi_0(s) \, ds > \tilde{u}_1. \quad (3.2)
\]
Let \( m \) be such that
\[
0 < m < \left(\int_0^1 \frac{(1-s)\phi(s)}{1 - \alpha \eta} \, ds\right)^{-1}.
\]
By (3.1), there exists \( \bar{R} > \tilde{u}_1 \) such that
\[
h(x) \leq mx, \quad \forall x \geq \bar{R}. \quad (3.3)
\]
Set
\[
R_1 = \max\left\{2\bar{R}, \frac{\int_0^1 \phi(s)\left(g(R_0q(s)/2) + m(\|w\| + R_0\|l\| + 1) + p(s)\right) \, ds}{1 - m \frac{1}{1 - \alpha \eta} \int_0^1 (1-s)\phi(s) \, ds}\right\}. \quad (3.4)
\]
Let $\Omega_0$ be defined as Lemma 3 and the sets $\Omega_1$, $\Omega_{01}$, $U_{01}$ be defined by

$$\Omega_1 = \{ x \in Q : \|x\| < R_1 \},$$

$$\Omega_{01} = \{ x \in Q : \|x\| < R_1 \text{ and } \inf_{t \in [\eta/2, \eta]} x(t) > u_1 \},$$

and

$$U_{01} = \{ x \in Q : \|x\| < R_1 \text{ and } \inf_{t \in [\eta/2, \eta]} x(t) > \tilde{u}_1 \},$$

respectively. It is easy to see that $\Omega_0$, $\Omega_1$, $\Omega_{01}$ and $U_{01}$ are bounded open convex sets of $Q$, and

$$\Omega_0 \subset \Omega_1, \quad \Omega_{01} \subset \Omega_1, \quad U_{01} \subset \Omega_1, \quad \Omega_0 \cap \Omega_{01} = \emptyset, \quad U_{01} \subset \Omega_{01}.$$ 

For any $y \in \bar{\Omega}_1$, by (3.3) and (3.4), we have

$$(T_n y)(t) \leq \frac{t}{1 - \alpha} \int_0^1 (1 - s) \left[ f(s, [y(s)]^n + n^{-1}) + p(s) \right] ds$$

$$\leq \frac{1}{1 - \alpha} \int_0^1 (1 - s) \left[ \phi(s) \left( g([y(s)]^n + h([y(s)]^n + 1)) + p(s) \right) ds \right.$$ \n
$$\leq \frac{1}{1 - \alpha} \int_0^1 (1 - s) \left[ \phi(s) \left( g \left( \frac{1}{2} R_0 q(s) \right) \right.$$

$$+ h(R_1 + \|w\| + R_0\|q\| + 1)) + p(s) \right] ds$$

$$\leq \frac{1}{1 - \alpha} \int_0^1 (1 - s) \left[ \phi(s) \left( g \left( \frac{1}{2} R_0 q(s) \right) \right.$$ \n
$$+ m(R_1 + \|w\| + R_0\|q\| + 1)) + p(s) \right] ds$$

$$< R_1, \quad \forall t \in [0, 1].$$ \hspace{1cm} (3.5)

This means that $\|T_n y\| < R_1$ for all $x \in \bar{\Omega}_1$. Therefore, $T_n \bar{\Omega}_1 \subset \Omega_1$. It follows from Lemma 1 that for every positive integer $n$,

$$i(T_n, \Omega_1, Q) = 1.$$ \hspace{1cm} (3.6)

For any $y \in \bar{\Omega}_{01}$, by (3.5), we get that $\|T_n y\| < R_1$. It is easy to see that for any $y \in \bar{\Omega}_{01}$,

$$[y(t)]^n = y(t) - w(t) \geq \frac{1}{2} y(t) \geq \frac{1}{2} u_1, \quad \forall t \in \left[ \frac{1}{2} \eta, \eta \right].$$

Then we have for any $y \in \bar{\Omega}_{01}$,
\[(T_n)(\eta) = \eta \frac{1}{1 - \alpha \eta} \int_0^{\eta} (1 - s) \left[ f(s, [y(s)]^n + n^{-1}) + p(s) \right] \, ds \]

\[+ \eta \frac{1}{1 - \alpha \eta} \int_{\eta/2}^{\eta} (1 - \eta s) \left[ f(s, [y(s)]^n + n^{-1}) + p(s) \right] \, ds \]

\[\geq \frac{1}{1 - \alpha \eta} \int_{\eta/2}^{\eta} (1 - \eta s) \phi_0(s) h_0([y(s)]^n) \, ds \]

\[\geq \frac{\min[1, \alpha](1 - \eta)}{2} h_0 \left( \frac{1}{1 - \alpha \eta} \right) \int_{\eta/2}^{\eta} \phi_0(s) \, ds > u_1 \quad \text{(3.7)} \]

and

\[(T_n)(\eta/2) = \frac{\eta}{2(1 - \alpha \eta)} \int_0^{\eta/2} (1 - s) \left[ f(s, [y(s)]^n + n^{-1}) + p(s) \right] \, ds \]

\[+ \frac{1}{2(1 - \alpha \eta)} \int_0^{\eta/2} \left[ \eta(1 - s) - \alpha \eta(n - s) \right] \left[ f(s, [y(s)]^n + n^{-1}) + p(s) \right] \, ds \]

\[\geq \frac{\eta}{2(1 - \alpha \eta)} \int_0^{\eta/2} \left[ (1 - \alpha \eta) - (1 - \alpha) s \right] \left[ f(s, [y(s)]^n + n^{-1}) + p(s) \right] \, ds \]

\[- \int_0^{\eta/2} \left( \frac{1}{2} \eta - s \right) \left[ f(s, [y(s)]^n + n^{-1}) + p(s) \right] \, ds \]

\[\geq \frac{\eta}{2(1 - \alpha \eta)} \int_0^{\eta/2} \left[ (1 - \alpha \eta) - (1 - \alpha) s \right] \left[ f(s, [y(s)]^n + n^{-1}) + p(s) \right] \, ds \]

\[- \int_0^{\eta/2} \left( \frac{1}{2} \eta - s \right) \left[ f(s, [y(s)]^n + n^{-1}) + p(s) \right] \, ds \]

\[= \frac{\eta}{2(1 - \alpha \eta)} \int_0^{\eta/2} \left[ (1 - \alpha \eta) - (1 - \alpha) s \right] \left[ f(s, [y(s)]^n + n^{-1}) + p(s) \right] \, ds \]

\[+ \frac{1}{2(1 - \alpha \eta)} \int_0^{\eta/2} \left[ \eta(1 - \alpha \eta) - \eta(1 - \alpha) s - (\eta - 2s)(1 - \alpha \eta) \right] \left[ f(s, [y(s)]^n + n^{-1}) + p(s) \right] \, ds \]

\[\times \left[ f(s, [y(s)]^n + n^{-1}) + p(s) \right] \, ds \]
\[
\eta \int_{\eta/2}^{\eta} \left[ (1 - \alpha \eta) - (1 - \alpha)s \right] f \left( s, \left[ y(s) \right]^* + n^{-1} \right) + p(s) \, ds
\]

\[
+ \frac{1}{2(1 - \alpha \eta)} \int_{0}^{\eta/2} \left[ 2 - \eta(1 + \alpha) \right] s \left[ f \left( s, \left[ y(s) \right]^* + n^{-1} \right) + p(s) \right] \, ds
\]

\[
\geq \frac{\eta}{2(1 - \alpha \eta)} \int_{\eta/2}^{\eta} \left[ (1 - \alpha \eta) - (1 - \alpha)s \right] \phi_0(s) h_0 \left( \left[ y(s) \right]^* \right) \, ds
\]

\[
\geq \begin{cases} 
\frac{\eta}{2(1 - \alpha \eta)} \int_{\eta/2}^{\eta} (1 - \alpha \eta) f_{\eta/2} \phi_0(s) h_0 \left( \left[ y(s) \right]^* \right) \, ds, & \alpha \geq 1, \\
\frac{\eta}{2(1 - \alpha \eta)} \int_{\eta/2}^{\eta} (1 - \eta) f_{\eta/2} \phi_0(s) h_0 \left( \left[ y(s) \right]^* \right) \, ds, & \alpha < 1,
\end{cases}
\]

\[
\geq \min \left\{ 1, \alpha \right\} (1 - \eta) \eta \int_{\eta/2}^{\eta} \phi_0(s) h_0 \left( \left[ y(s) \right]^* \right) \, ds > u_1. 
\]  

(3.8)

Since \((T_n y)(t)\) is a concave function on \([0,1]\), then by (3.7) and (3.8) we get that for any \(y \in \dot{\Omega}_0\),

\[
(T_n y)(t) \geq \min \left\{ (T_n y) \left( \frac{1}{2} \eta \right), (T_n y)(\eta) \right\} > u_1, \quad \forall t \in \left[ \frac{1}{2}, \eta \right].
\]

This implies that \(T_n(\dot{\Omega}_0) \subset \dot{\Omega}_0\). By Lemma 1, we have for every positive integer \(n\),

\[
i(T_n, \dot{\Omega}_0, Q) = 1. 
\]  

(3.9)

Similarly, by (3.2) we have for every positive integer \(n\),

\[
i(T_n, U_0, Q) = 1. 
\]  

(3.10)

It follows from (3.6), (3.9) and Lemma 3 that

\[
i(T_n, \Omega_1 \setminus \dot{\Omega}_0, Q) = i(T_n, \dot{\Omega}_0, Q) - i(T_n, \dot{\Omega}_0, Q) - i(T_n, \dot{\Omega}_0, Q) = -1
\]

for every positive integer \(n\). Therefore, \(T_n\) has at least one fixed point \(y_{n,1} \in \Omega_1 \setminus \dot{\Omega}_0\). Obviously, \(y_{n,1}\) satisfies

\[
\begin{cases} 
-y_{n,1}''(t) = f(t, \left[ y_{n,1}(t) \right]^* + n^{-1}) + p(t), & 0 \leq t \leq 1, \\
y_{n,1}(0) = 0, \quad \alpha y_{n,1}(1) = y_{n,1}(1),
\end{cases}
\]

Then we get that

\[
\left| y_{n,1}'(t) \right| \leq \int_{0}^{1} \left[ \phi(s) \left( g \left( \left[ y_{n,1}(s) \right]^* \right) + h \left( \left[ y_{n,1}(s) \right]^* + n^{-1} \right) \right) + p(s) \right] \, ds
\]

\[
\leq \int_{0}^{1} \phi(s) \left( g \left( \frac{1}{2} R_0 q(s) \right) + h \left( R_1 + \| w \| + R_0 \| q \| + 1 \right) + p(s) \right) \, ds,
\]

\(\forall t \in [0, 1]\).
By \((H_2)\), \(\{y_{n,1}\}\) is equicontinuous on \([0,1]\). For any \(y_{n,1} \in \Omega_1 \setminus (\overline{\Omega}_0 \cup \overline{\Omega}_1)\), there is at least one point \(t_n \in [\eta/2, \eta]\) such that \(y_{n,1}(t_n) \leq u_1\). According to the Arzela–Ascoli theorem and the boundedness of \(\{t_n\}\), there exist a subsequence \(\{y_{n,i}\}\) of the sequence \(\{y_{n,1}\}\), and a function \(y_1 \in \Omega_1 \setminus (\overline{\Omega}_0 \cup \overline{\Omega}_1)\) such that \(y_{n,i} \rightarrow y_1\) as \(i \rightarrow +\infty\), and there at least one point \(t_0 \in [\eta/2, \eta]\) such that \(y_1(t_0) \leq u_1\). Since \(\|y_1\| \geq R_0\), in a similar way as \((2.4)\), we see that

\[
\left[ y_1(t) \right]^b = y_1(t) - w(t) \geq \frac{1}{2} R_0 q(t), \quad t \in [0, 1].
\]

From \(y_{n,1} = T_n y_{n,1}\), by the Lebesgue dominated convergence theorem, we have

\[
y_1(t) = \frac{t}{1 - \alpha \eta} \int_0^1 (1 - s)f(s, y_1(s) - w(s)) + p(s) \, ds \\
- \int_0^t (t - s)f(s, y_1(s) - w(s)) + p(s) \, ds \\
- \frac{\alpha t}{1 - \alpha \eta} \int_0^\eta (\eta - s)f(s, y_1(s) - w(s)) + p(s) \, ds, \quad t \in [0, 1].
\]

Let \(x_1 = y_1 - w\). Then, we have

\[
x_1(t) = \frac{t}{1 - \alpha \eta} \int_0^1 (1 - s)f(s, x_1(s)) \, ds - \int_0^t (t - s)f(s, x_1(s)) \, ds \\
- \frac{\alpha t}{1 - \alpha \eta} \int_0^\eta (\eta - s)f(s, x_1(s)) \, ds, \quad t \in [0, 1].
\]

By direct computation, we can see that \(x_1\) is a positive solution of the boundary value problem \((1.1)\).

It follows from \((3.10)\) that \(T_n\) has at least one fixed point \(y_{n,2} \in \Omega_0\) for every positive integer \(n\). An essentially the same argument as above shows that there exist a subsequence \(\{y_{n,2}\}\) of the sequence \(\{y_{n,2}\}\), and a function \(y_2 \in \Omega_0\) such that \(y_{n,2} \rightarrow y_2\) as \(i \rightarrow +\infty\). Obviously,

\[
y_2(t) \geq \tilde{u}_1, \quad t \in \left[ \frac{1}{2} \eta, \eta \right].
\]

Let \(x_2 = y_2 - w\). Then \(x_2\) is a positive solution of the boundary value problem \((1.1)\). Since \(y_1(t_0) \leq u_1 < \tilde{u}_1 \leq y_2(t_0)\), then, \(x_1\) and \(x_2\) are two distinct positive solutions of the boundary value problem \((1.1)\). The proof is completed. \(\Box\)
Theorem 2. Suppose that \((H_1)-(H_4)\) hold. Moreover, there exists \(R_1 > u_1\) such that

\[
\frac{1}{1 - \alpha \eta} \int_0^1 \left[ \phi(s) \left( g \left( \frac{1}{2} R_0 q(s) \right) + h \left( R_1 + \|w\| + R_0 \|q\| + 1 \right) \right) + p(s) \right] ds < R_1
\]  

(3.11)

and

\[
\lim_{y \to +\infty} \frac{h_0(y)}{y} = + \infty.
\]  

(3.12)

Then the boundary value problem (1.1) has at least three positive solutions.

Proof. For any positive integer \(n\), let us define the operator \(T_n\) by (2.1). By (3.11), there exists a positive number \(\bar{R}_1 > R_1\) such that

\[
\frac{1}{1 - \alpha \eta} \int_0^1 \left[ \phi(s) \left( g \left( \frac{1}{2} R_0 q(s) \right) + h \left( \bar{R}_1 + \|w\| + R_0 \|q\| + 1 \right) \right) + p(s) \right] ds < \bar{R}_1.
\]  

(3.13)

Let us define the open sets \(\Omega_0, \Omega_1, \Omega_{01}\) and \(U_{01}\) as Theorem 2. Let \(U_1 = \{x \in Q: \|x\| < \bar{R}_1\}\). It is easy to see that for any \(y \in \Omega_1\),

\[
R_1 + \|w\| + R_0 \|q\| \geq \left[ y(t) \right]^n \geq \frac{1}{2} R_0 q(t), \quad \forall t \in [0, 1].
\]

Therefore, by (3.11) we have for any \(y \in \Omega_1\),

\[
| (T_n y)(t) | \leq \frac{1}{1 - \alpha \eta} \int_0^1 \left[ \phi(s) \left( g \left( \frac{1}{2} R_0 q(s) \right) 
+ h \left( R_1 + \|w\| + R_0 \|q\| + 1 \right) \right) + p(s) \right] ds < \bar{R}_1, \quad t \in [0, 1].
\]

This means that \(T_n(\Omega_1) \subset \Omega_1\) for every positive integer \(n\). Similarly, by (3.13), we can show that \(T_n(U_1) \subset U_1\) for any positive integer \(n\). By Lemma 1, we have

\[
i(T_n, U_1, Q) = 1.
\]  

(3.14)

In a similar way as Theorem 2, we can show that the boundary value problem (1.1) has at least two positive solutions \(x_1, x_2\) such that \(x_1 = y_1 - w, \ x_2 = y_2 - w,\) where \(y_1 \in \overline{\Omega_1 \setminus (\Omega_{01} \cup \Omega_0)}\) and \(y_2 \in \overline{U_{01}}\).

Now we will show the existence of the third positive solution of the boundary value problem (1.1). Set

\[
M > \left[ \frac{(1 - \eta)^2 \eta}{4(1 - \alpha \eta)(1 - \alpha \eta + |1 - \alpha|)} \int_{\eta/2}^n s \phi_0(s) ds \right]^{-1}.
\]  

(3.15)
By (3.12), there exists $\tilde{R} > \tilde{R}_1$ such that
\[
h_0(y) \geq M y, \quad y \geq \tilde{R}.
\] (3.16)

Set
\[
R_2 = \max \left\{ \tilde{R}, \frac{4(1 - \alpha \eta + |1 - \alpha|)}{\eta(1 - \eta)} \right\}.
\]

Let $\psi_0 \in Q \setminus \{ \theta \}$ and $\Omega_2 = \{ x \in Q : \| x \| < R_2 \}$. Now we will show that for any positive integer $n$,
\[
y \neq T_n y + \mu \psi_0, \quad y \in \partial \Omega_2, \quad \mu \geq 0.
\] (3.17)

In fact, if not, then there exist $y_0 \in \partial \Omega_2$, $n_0 \in N$ and $\mu_0 \geq 0$ such that
\[
y_0 = T_n y_0 + \mu_0 \psi_0.
\]

Obviously,
\[
\left[ y_0(t) \right]^* = y_0(t) - w(t) \geq \frac{1}{2} \| y_0 \| q(t)
\geq \frac{R_2(1 - \eta)}{4(1 - \alpha \eta + |1 - \alpha|)} \tilde{R}, \quad t \in \left[ \frac{1}{2} \eta, \eta \right].
\] (3.18)

It follows from (3.16) and (3.18) that
\[
R_2 = \| y_0 \| \geq \| T_{n_0} y_0 \| \geq (T_{n_0} y_0)(\eta)
\]
\[
= \frac{\eta}{1 - \alpha \eta} \int_0^1 (1 - s) \left[ f \left( s, \left[ y_0(s) \right]^* + n_0^{-1} \right) + p(s) \right] ds
\]
\[
+ \frac{1}{1 - \alpha \eta} \int_0^{\eta/2} (1 - \eta) s \left[ f \left( s, \left[ y_0(s) \right]^* + n_0^{-1} \right) + p(s) \right] ds
\]
\[
\geq \frac{(1 - \eta)}{1 - \alpha \eta} \int_\eta^{\eta/2} s \phi_0(s) h_0 \left[ \left[ y_0(s) \right]^* \right] ds
\]
\[
\geq \frac{M (1 - \eta)}{1 - \alpha \eta} \int_\eta^{\eta/2} s \phi_0(s) \left[ y_0(s) \right]^* ds
\]
\[
\geq \frac{M R_2(1 - \eta)^2 \eta}{4(1 - \alpha \eta)(1 - \alpha \eta + |1 - \alpha|)} \int_\eta^{\eta/2} s \phi_0(s) ds,
\]
and so
\[
M \leq \left[ \frac{(1 - \eta)^2 \eta}{4(1 - \alpha \eta)(1 - \alpha \eta + |1 - \alpha|)} \int_\eta^{\eta/2} s \phi_0(s) ds \right]^{-1}.
\]
which is a contradiction to (3.15). Thus (3.17) holds. By the properties of the fixed point
index, we know that
\[ i(T_n, \Omega_2, Q) = 0 \]  
for any positive integer \( n \). It follows from (3.14) and (3.19) that
\[ i(T_n, \Omega_2 \setminus \bar{U}_1, Q) = i(T_n, \Omega_2, Q) - i(T_n, U_1, Q) = -1 \]
for every positive integer \( n \). Therefore, \( T_n \) has at least one fixed point \( y_{n,3} \in \Omega_2 \setminus \bar{U}_1 \). In
the same way as Theorem 2, we can show that there exist a subsequence \( \{y_{n,i}\} \) of \( \{y_{n,3}\} \), and a function \( y_3 \in \Omega_2 \setminus \bar{U}_1 \) such that
\[ y_{n,i} \to y_3 \quad \text{as} \quad i \to + \infty. \]
Let \( x_3 = y_3 - w \). Then \( x_3 \) is the third positive solution of boundary value problem (1.1). The proof is completed. \( \square \)

**Corollary 1.** Suppose that \((H_1)-(H_3)\) hold. Moreover,
\[ \lim_{y \to +\infty} \frac{h_0(y)}{y} = +\infty. \]
Then the boundary value problem (1.1) has at least one positive solution.

**Corollary 2.** Suppose that \((H_1)-(H_4)\) hold. Moreover, there exist \( R_i, u_i \) \((i = 1, 2, \ldots, n)\) with \( R_0 < u_1 < R_1 < u_2 < R_2 < \cdots < u_n < R_n \) such that
\[ \frac{1}{1 - \alpha \eta} \int_0^1 \left[ \phi(s) \left( g \left( \frac{1}{2} R_0 q(s) \right) + h \left( R_i + \| w \| + R_0 \| q \| + 1 \right) \right) + p(s) \right] ds < R_i, \]
i = 1, 2, \ldots, n,  
\[ \frac{\min \{1, \alpha\} (1 - \eta) \eta}{2} h_0 \left( \frac{1}{2} u_i \right) \int_{\eta/2}^{\eta} \phi_0(s) ds > u_i, \quad i = 1, 2, \ldots, n, \]
and
\[ \lim_{y \to +\infty} \frac{h_0(y)}{y} = +\infty. \]
Then the boundary value problem (1.1) has at least \( 2n + 1 \) positive solutions.

**Example 1.** Consider the following three-point boundary value problem:
\[
\begin{cases}
y'' + \frac{3}{10^6} \left( \frac{1}{y^{1/5}} + h(y) \right) - \frac{1}{10^5} = 0, & t \in (0, 1), \\
y(0) = 0, & 4y(1_{10}) = y(1),
\end{cases}
\]  
(3.20)
where
\[ h(y) = \begin{cases}
y^2, & y \in [0, 9 \times 10^8], \\
2.7 \times 10^{13} y^{1/2}, & y \in [9 \times 10^8, +\infty). 
\end{cases} \]

**Conclusion.** The three-point boundary value problem (3.20) has at least two positive solutions.
Proof. Let $\alpha = 4$, $\eta = 1/16$, $g(y) = 1/y^{1/8}$ for $y \in (0, +\infty)$, $\phi(t) = \phi_0(t) = 3/10^3$ for $t \in (0, 1)$, $p(t) = 1/10^3$ for $t \in (0, 1)$, and $h_0(y) = h(y)$ for $y \in \mathbb{R}^+$. It is easy to see that (H1) and (H2) hold. Take $R_0 = 1$ and $u_1 = 4 \times 10^8$. By direct computation, we have that

$$R_0 > 2c_1 = 2 \times \frac{128}{1125} = \frac{256}{1125},$$

$$\frac{g(R_0)}{g(R_0) + h(R_0 + 1) + 1} \left( \int_0^{R_0} \frac{d\tau}{g(\tau/2)} - \frac{|1 - \alpha| R_0}{\alpha(1 - \eta) g(R_0/2)} \right)$$

$$= \frac{1}{6} \times \left( \frac{1}{2} \right)^{1/8} \left( \frac{8}{9} - \frac{4}{5} \right) > \frac{1}{6} \times \frac{9}{10} \times \frac{4}{45} > \frac{1}{6} \left( \frac{3}{10^3} + \frac{1}{10^3} \right) ds.$$

This means that (H3) holds. It is easy to check that (H4) holds. By Theorem 1, we see that the conclusion holds.

Remark 3. In this paper, some multiplicity results for positive solutions of semi-positone three-point boundary value problem be obtained. It is difficult to show the existence of multiple positive solutions of semi-positone boundary value problems. Obviously, we can use the ideas of this paper to establish multiplicity results for positive solutions of the more general m-point boundary value problems.

Remark 4. In a very recent paper, Liu [10] established some existence results for nontrivial solutions of some m-point boundary value problems. The condition that $f$ is nonnegative on $\mathbb{R}^+$ in [10] also is necessary for the existence of positive solutions of the m-point boundary value problem (see Theorem 3.3 in [10]). In this paper, we remove this condition and give some multiplicity results for positive solutions of semi-positone problems. Thus, the results of this paper are new. Also, the method of this paper is different from that in [10].

References