Perturbation Theory for Commutative $m$-Tuples of Self-Adjoint Operators*

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Generalizing the Weyl-von Neumann theorem for normal operators, we show that a commutative $m$-tuple of self-adjoint operators in a separable Hilbert space may be changed into a diagonal one by adding compact perturbations of class $c_p$, for $p > m$. On the other hand it is shown that the absolutely continuous part, defined appropriately, of a commutative $m$-tuple of self-adjoint operators is stable under perturbations of class $c_p$, if $p < m$, $m > 3$, or if $p = 1$, $m = 2$ (the latter case $m = 2$ corresponding to the case of one normal operator). For the proof of these Kato-Rosenblum-type theorems a wave operator method for $m$-tuples is introduced.

INTRODUCTION

This paper treats problems arising in connection with the generalization of the Weyl-von Neumann theorem to normal operators. It was shown by Berg [2, 3], Sikonia [14], Weidmann [18], that a normal operator in a separable Hilbert space can be changed into a diagonal one by adding a compact perturbation of class $c_p$ ($p > 2$) (for "$c_p$", cf. Section 1(B)). The problem of showing that, in general, it is not possible to choose the perturbation in $c_2$ (i.e., the Hilbert-Schmidt operators) was the starting point of this work.

The observation that treating one normal operator is equivalent to treating two commuting self-adjoint operators was the motivation for the following Weyl-von Neumann-type theorem.

Given a commutative $m$-tuple $(T_1, ..., T_m)$ of self-adjoint operators and $p > m$, one can find compact operators $K_1, ..., K_m \in c_p$ such that the $m$-tuple $(T_1 + K_1, ..., T_m + K_m)$ is diagonal (Theorem 2.1).

For $m = 1$ the Weyl-von Neumann theorem cannot be essentially strengthened; this is shown by the Kato-Rosenblum theorem: Let $T$ be a self-adjoint operator, $K \in c_1$ (i.e., trace class) self-adjoint; then the wave operators

$$W_\pm(T + K, T) = \lim_{t \to \pm \infty} e^{it(T+K)} e^{-itT}$$

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exist on the absolutely continuous subspace $H_{ac}$, and the absolutely continuous parts of $T$ and $T + K$ are unitarily equivalent (Rosenblum [13], Kato [8]; see also Kato [9, X, Section 4.3]). In order to obtain related results for arbitrary $m \in \mathbb{N}$, we consider the unitary representations of $\mathbb{R}^m$ generated by commutative $m$-tuples of self-adjoint operators and introduce associated wave operators (Section 3), especially direction wave operators. These direction wave operators exist for almost all directions and implement the unitary equivalence of the absolutely continuous parts of $(T_1, ..., T_m)$ and $(T'_1, ..., T'_m)$ (both commutative $m$-tuples of self-adjoint operators, $m \geq 3$), under assumptions like $T'_j - T_j \in C_p$ $(j = 1, ..., m)$ for some $p < m$ (Section 4). The estimates yielding the existence of the wave operators for $m \geq 3$ do not apply for $m = 2$ (normal operators), however; so the problem stated initially remains open. Using the Kato–Rosenblum theorem we obtain some weaker results for normal operators (Section 5).

In the results sketched above the cases $m = 2$, $1 < p < 2$, and $m > 3$, $p = m$ remain undecided. In analogy to the Kato–Rosenblum theorem we conjecture that for all $m \in \mathbb{N}$ the absolutely continuous parts of the unperturbed and perturbed $m$-tuples are unitarily equivalent if the perturbations are in $C_m$. Also, in view of the results of Section 4, one might conjecture that this unitary equivalence should always be implemented by some direction wave operator; this latter conjecture is disproved by an example (Appendix).

1. Preliminaries

$\mathbb{N} = \{1, 2, ..., \}$, $\mathbb{R}$, $\mathbb{C}$ denote the natural numbers, the reals, the complex numbers, respectively; $c^*$ denotes the complex conjugate of $c \in \mathbb{C}$. For $m \in \mathbb{N}$ we denote by $\lambda^m$ the Lebesgue–Borel measure on $\mathbb{R}^m$.

All Hilbert spaces will be complex Hilbert spaces; the scalar product, denoted by $\langle \cdot | \cdot \rangle$, will be antilinear in the first argument, linear in the second. $\mathcal{B}(H)$ denotes the bounded linear operators $H \rightarrow H$.

1(A) Spectral Properties of Commutative $m$-Tuples of Self-Adjoint Operators

On a Hilbert space $H$ let $T = (T_1, ..., T_m)$ be a commutative $m$-tuple of self-adjoint operators, i.e., $T_1, ..., T_m$ are (not necessarily bounded) self-adjoint operators in $H$ commuting with one another (in the sense of commuting resolvents or spectral measures).

An element $x \in H$ will be called a joint eigenelement of $T$, if $x$ is an eigenelement of $T_j$ for all $j = 1, ..., m$. $T$ will be called diagonal if there is an orthonormal basis $B$ of $H$ consisting of joint eigenelements of $T$; $B$ is then called a joint eigenbasis.

$T$ can be represented unitarily by an $m$-tuple of multiplication operators in
perturbation theory for \( m \)-tuples

\( L_2(M, \mu) \) over some measure space. More precisely there exists a family \((\mu_\alpha; \alpha \in A)\) of Borel measures on \( \mathbb{R}^m \) and a unitary map

\[
J : \bigoplus (L_2(\mathbb{R}^m, \mu_\alpha); \alpha \in A) \to H,
\]

such that

\[
T_j = JM_{pr_j}J^* \quad (j = 1, \ldots, m)
\]

\((M_{pr_j} \) denotes multiplication by the \( j \)th coordinate in each component of \( \bigoplus (L_2(\mathbb{R}^m, \mu_\alpha); \alpha \in A) \), i.e., \( pr_j(\xi) = \xi_j \) \((\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m)\),

\[
D(M_{pr_j}) := \{ f = (f_\alpha); (f_\alpha), M_{pr_j}f := (pr_j f_\alpha) \in \bigoplus (L_2(\mathbb{R}^m, \mu_\alpha); \alpha \in A) \}.
\]

for this relation we also write

\[
T = JM_{pr}J^*,
\]

with the commutative \( m \)-tuple \( M_{pr} = (M_{pr_1}, \ldots, M_{pr_m}) \). In this spectral representation one may assume \( \text{supp} \mu_\alpha \) compact for each \( \alpha \in A \); and one can choose \( A \subset \mathbb{N} \) if \( H \) is separable. In terms of the spectral representation the functional calculus for \( T \) is given by

\[
h(T) = JM_{h}J^*,
\]

where \( h: \mathbb{R}^m \to \mathbb{C} \) is a Borel function and \( M_{h} \) is the operator of multiplication by \( h \) in \( \bigoplus (L_2(\mathbb{R}^m, \mu_\alpha); \alpha \in A) \). Especially \( E \), defined by

\[
E(\Omega) := \chi_\Omega(T) \quad (\Omega \subset \mathbb{R}^m \text{ Borel set}),
\]

is the spectral measure of \( T \).

If \( x \in H \), then \( \mu_x \), defined by \( \mu_x(\Omega) := \langle x, E(\Omega)x \rangle \) \((\Omega \subset \mathbb{R}^m \text{ Borel set})\), is a Borel measure on \( \mathbb{R}^m \). For the stability theorems (Section 4) we shall need the absolutely continuous subspace (cf. Kato [9, X, Section 1] for \( m = 1 \))

\[
H_{ac} := \{ x \in H; \mu_x \text{ is } \lambda^m\text{-absolutely continuous} \}
\]

of \( H \) with respect to \( T \). Like for \( m = 1 \), \( H_{ac} \) is a closed subspace of \( H \) which reduces \( T \) (i.e., it reduces \( T_j \) for all \( j = 1, \ldots, m \)). The part \( T_{ac} \) of \( T \) in \( H_{ac} \) is called the absolutely continuous part of \( T \). (It should be noted that the index "ac" belongs to the whole \( m \)-tuple and that in general \( T_{ac} \neq (T_{1,ac}, \ldots, T_{m,ac}) \); e.g., if \( m = 2 \), \( H = L_2(0, 1) \), \( T_1 = M_{id}, T_2 = 0 \), then the joint spectral measure is supported by \( \{ (\xi_1, 0); 0 \leq \xi_1 \leq 1 \} \), so \( H_{ac,T} = \{ 0 \} \), but \( H_{ac,T_1} = H \).) The spectral measure of \( T_{ac} \) will be denoted by \( E_{ac} \), the orthogonal projection onto \( H_{ac} \) by \( P_{ac} \). For the absolutely continuous part \( T_{ac} \) the spectral representation
can be given in a special form: There exists a family \((\Omega_\alpha; \alpha \in A)\) of Borel sets 
\(\Omega_\alpha \subseteq \mathbb{R}^m\) and a unitary map 
\[ J \colon \bigoplus (L_2(\Omega_\alpha); \alpha \in A) \to H_{ac}, \]
such that 
\[ T_{ac} = JM_{pr}J^*. \]
(We write \(L_2(\Omega) = L_2(\Omega, \lambda^m)\).) Again we can choose \(A \subseteq \mathbb{N}\) if \(H\) is separable.

If \((T_1, T_2)\) is a commutative pair of self-adjoint operators, then \(N = T_1 + iT_2\) 
is a normal operator. Conversely to each normal operator there corresponds a unique commutative pair of self-adjoint operators. The spectral measure of \(N\) on \(\mathbb{C}\) corresponds in the canonical way to the spectral measure of \((T_1, T_2)\) on \(\mathbb{R}^2\).

1(B) Classes \(c_p\) of Compact Operators

Let \(H\) be a Hilbert space. For a compact operator \(K \in \mathcal{B}(H)\) we denote by 
\(\mu_n(K)\) the eigenvalues of the nonnegative operator \((K^*K)^{1/2}\) and define for 
\[ 0 < p < \infty \]
\[ ||K||_p := \left(\sum \mu_n(K)^p\right)^{1/p}, \]
\[ ||K||_\infty := ||K|| = \max\{\mu_n(K)\}. \]
For \(0 < p \leq \infty\) the set 
\[ c_p := c_p(H) := \{K \in \mathcal{B}(H); K \text{ compact, } ||K||_p < \infty\} \]
is a linear space; \(c_1\) is the trace class, \(c_2\) the Hilbert–Schmidt class.

If \(0 < p < p' \leq \infty\) then \(||K||_p \geq ||K||_{p'}\), \(c_p \subseteq c_{p'}\).

If \(p \geq 1\), then \(\| \cdot \|_p\) defines a norm on \(c_p\), and \(c_p = (c_p, \| \cdot \|_p)\) is a Banach space (cf. [11]).

Let \(0 < p < \infty, H = \bigoplus (H_j; j \in \mathbb{N}), K_j \in c_p(H_j) (j \in \mathbb{N}).\) If \(K = \bigoplus (K_j; j \in \mathbb{N}), \ K^* = \bigoplus (K_j^*; j \in \mathbb{N}), \ K^*K = \bigoplus (K_j^*K_j; j \in \mathbb{N}), \ ||K||_p^p = \sum (||K_j||_p^p; j \in \mathbb{N}).\)

If \(H\) is a Hilbert space, \(A \subseteq \mathbb{N}\), and \((\varphi_j; j \in A), (\psi_j; j \in A)\) are orthogonal systems in \(H, K = \sum_{j \in A} \langle \psi_j | \cdot \varphi_j \rangle, K^* = \sum_{j \in A} \langle \varphi_j | \cdot \psi_j \rangle, K^*K = \sum ||\psi_j||_p^p \langle \psi_j | \cdot \varphi_j \rangle, ||K||_p^p = \sum (||\psi_j||_p \| \varphi_j\|)^p\) for \(0 < p < \infty\), and \(K\) is in \(c_p\) iff the last sum is finite.

2. Generalizations of the Weyl–von Neumann Theorem

2.1. Theorem. Let \(H\) be a separable Hilbert space, \(m \in \mathbb{N}, T = (T_1, \ldots, T_m)\) a commutative \(m\)-tuple of self-adjoint operators in \(H.\) Let \(p > m, \epsilon > 0.\)

Then there exist self-adjoint operators \(K_1, \ldots, K_m \in c_p, ||K_j||_p \leq \epsilon (j = 1, \ldots, m),\) such that the \(m\)-tuple \((D_j := T_j + K_j; j = 1, \ldots, m)\) is diagonal.
2.2. Remarks. (a) If we set $m = 1, p = 2$ in Theorem 2.1, we obtain the theorem of Weyl–von Neumann (Weyl [19] for $p = \infty$, von Neumann [12] for $p = 2$). Kuroda [10] proved for the case $m = 1$ an even sharper result. For proofs of these theorems see also Kato [9, X, 2.1, 2.3].

(b) If we set $m = 2$ in Theorem 2.1, we obtain the "Weyl–von Neumann theorem for normal operators." This statement, with $p = \infty$, was conjectured by Berberian [1] and Halmos [6], and proved by Berg [2], Sikonia [14], and Halmos [7]. It was shown by Berg [3] and Weidmann [18], that, essentially, the methods of Berg resp. Sikonia proved the theorem as well for $p > 2$.

(c) A "Weyl–von Neumann theorem for commutative sequences of self-adjoint operators" (i.e., "$m = \infty" in Theorem 2.1) has been proved by Brown et al. ([5], Corollary 5.4) and Thayer ([16], Corollary). Using the idea of Halmos' proof in [7], one can also obtain some of the estimates in Theorem 2.1 (cf. [17], Section 4).

Proof of Theorem 2.1. (We follow rather closely the lines of Weidmann's proof in [18].)

(i) Let $\Gamma \subset \mathbb{R}^m$ be a half open cube of side $l$, $\mu$ a finite Borel measure on $\Gamma$, $H = L_2(\Gamma, \mu)$, $T_j = M_{p_j}$ ($j = 1, \ldots, m$). We are going to construct self-adjoint operators $D_1, \ldots, D_m$, $(D_1, \ldots, D_m)$ diagonal, such that $K_j := D_j - T_j \in C_p$,

$$\|K_j\|_p \leq c(p)$$

$(j = 1, \ldots, m)$, where

$$c(p) := 2^m \sum_{i=0}^{\infty} 2^{(m/p)-1}i$$

$(\ll \infty$ because of $m/p < 1$).

First we choose inductively a joint eigenbasis for $(D_1, \ldots, D_m)$. In the 0th step we set

$$G_0 := \{\Gamma\}, \quad \psi_{\Gamma, 1} := \chi_{\Gamma}/\|\chi_{\Gamma}\|.$$ 

In the ith step we define

$$G_i := \left\{\text{set of half open cubes, obtained from the cubes in } G_{i-1} \text{ by cutting in half parallel to all coordinate hyperplanes.}\right\}$$

$G_i$ consists of $2^{im}$ cubes, the side of each cube being $l2^{-i}$. For $\Delta \in G_{i-1}$ we orthonormalize the set of functions $\{\chi_\Delta\} \cup \{\chi_{\Delta'} : \Delta' \in G_i, \Delta' \subset \Delta\}$ in $L_2(\Gamma, \mu)$ and obtain

$$\psi_{\Gamma, 1}, \psi_{\Gamma, 2}, \ldots, \psi_{\Gamma, k_\Gamma} \quad \text{if } i = 1 \text{ (then } \Delta = \Gamma),$$

$$\chi_\Delta/\|\chi_\Delta\|, \psi_{\Delta, 1}, \ldots, \psi_{\Delta, k_\Delta} \quad \text{if } i = 2, 3, \ldots ;$$

we have $k_\Delta \leq 2^m$ (in case $\|\chi_\Delta\| = 0$ we set $k_\Delta = 0$).
This construction yields an orthonormal sequence

\[(\varphi_{\Delta,k} ; k = 1,\ldots, k_{\Delta}, \Delta \in G)\]

where \(G := \bigcup_{i=0}^{\infty} G_i\).

This orthonormal sequence is an orthonormal basis, because \(\chi_\Delta \in \text{span}\{\varphi_{\Delta,k} ; k, \Delta\} \) for all \(\Delta \in G\) (in the \(L_2(\Gamma, \mu)\)-sense) and \(\{\chi_\Delta ; \Delta \in G\}\) is total in \(L_2(\Gamma, \mu)\).

For each \(\Delta \in G\) we choose \(\xi_\Delta \in \Delta, \xi_\Delta = (\xi_{\Delta,1}, \ldots, \xi_{\Delta,m})\). For \(j = 1,\ldots, m\) we define

\[D_j := \sum_{\Delta \in G} \sum_{k=1}^{k_{\Delta}} \xi_{\Delta,j} \varphi_{\Delta,k} | \cdot \varphi_{\Delta,k}^\perp.\]

Obviously \((D_1, \ldots, D_m)\) is a diagonal \(m\)-tuple of self-adjoint operators.

To estimate the \(c_\mu\)-norm of \(K_j = D_j - T_j\) we decompose

\[K_j = \sum_{i=0}^{\infty} K_{ji},\]

\[K_{ji} := \sum_{\Delta \in G_i} \sum_{k=1}^{k_{\Delta}} (\xi_{\Delta,j} - pr_j) \varphi_{\Delta,k} | \cdot \varphi_{\Delta,k} = \sum_{\Delta \in G_i} \sum_{k=1}^{2^m} \cdots\]

(replacing \(k_{\Delta}\) by \(2^m\) in the last summation we let the undefined terms be zero)

\[= \sum_{\Delta \in G_i} \sum_{k=1}^{2^m} (\xi_{\Delta,j} - pr_j) \varphi_{\Delta,k} | \cdot \varphi_{\Delta,k}^\perp.\]

For \(1 \leq k \leq 2^m\) \((\varphi_{\Delta,k} ; \Delta \in G_i)\) and \((\xi_{\Delta,j} - pr_j) \varphi_{\Delta,k} ; \Delta \in G_i)\) are orthogonal systems; for \(\xi = (\xi_1, \ldots, \xi_m) \in \Delta \in G_i\) we have \(|\xi_{\Delta,j} - \xi_j| \leq L^{-1}\). Using Section 1(B) we obtain

\[\left\| \sum_{\Delta \in G_i} (\xi_{\Delta,j} - pr_j) \varphi_{\Delta,k} | \cdot \varphi_{\Delta,k} \right\|_p^p = \sum_{\Delta \in G_i} \left(\| \varphi_{\Delta,k} \|^p (\| (\xi_{\Delta,j} - pr_j) \varphi_{\Delta,k} \|^p) \right) \leq \sum_{\Delta \in G_i} (L^{-1})^p = 2^{im}l^p 2^{-ip} = l^p 2^{i(m-p)},\]

\[\| K_j \|_p \leq \sum_{k=1}^{2^m} \left\| \sum_{\Delta \in G_i} \cdots \right\|_p \leq 2^{im} l 2^{i(m-p)},\]

\[\| K_j \|_p \leq \sum_{i=0}^{\infty} \| K_{ji} \|_p \leq 2^m \sum_{i=0}^{\infty} 2^{i(m-p)} = kl(p).\]

(ii) Now let \(\Gamma \subset \mathbb{R}^m\) be compact, \(\mu, T_1, \ldots, T_m\) as in (i). Let \(\Delta\) be a half open cube, \(\Gamma \subset \Delta,\) of side \(L\). We decompose \(\Delta\) into \(n^m\) half open cubes \(\Gamma_1, \ldots, \Gamma_{n^m}\) of side \(L/n\). Setting \(\mu_q := \mu |_{\Gamma_q} (q = 1,\ldots, n^m)\) we have

\[L_2(\Gamma, \mu) = \bigoplus (L_2(\Gamma_q, \mu_q); q = 1,\ldots, n^m),\]
and for \( j = 1, \ldots, m \) \( T_j \) is the orthogonal sum of the parts

\[
T_{j, q} := (T_j)_{L_q(\Gamma_q, \mu_q)}, \quad q = 1, \ldots, n^m,
\]

of \( T_j \) in \( L_q(\Gamma_q, \mu_q) \). For \( \Gamma_q, \mu_q, (T_1, q, \ldots, T_m, q) \) we determine \( D_{j, q} \) \((j = 1, \ldots, m)\) according to (i),

\[
K_{j, q} := D_{j, q} - T_{j, q} \in c_p(L_q(\Gamma_q, \mu_q)),
\]

\[
\| K_{j, q} \|_p \leq L n^{-1} c(p).
\]

With \( K_j := \bigoplus (K_{j, q} ; q = 1, \ldots, n^m) \) \((j = 1, \ldots, m)\) we obtain by Section 1(B)

\[
\| K_j \|_p^p = \sum_{q=1}^{n^m} \| K_{j, q} \|_p^p \leq n^m L^p n^{-p} c(p)^p = n^m - p L^p c(p)^p.
\]

\((D_j := T_j + K_j ; j = 1, \ldots, m)\) is diagonal by construction. Because of \( m < p \)

\[
\| K_j \|_p \quad (j = 1, \ldots, m)
\]

can be made arbitrarily small by an appropriate choice of \( n \in \mathbb{N} \).

(iii) Now let \( H, (T_1, \ldots, T_m) \) be as assumed in the theorem. Since the properties considered here are unitarily invariant, we may assume by Section 1(A) that

\[
H = \bigoplus (L_q(\mathbb{R}^m, \mu_q); \alpha \in A),
\]

\((T_1, \ldots, T_m) = (M_{pr_1}, \ldots, M_{pr_m}),\)

supp \( \mu_n \) is compact \((\alpha \in A)\) and \( A \subset \mathbb{N} \).

If we consider \( H_a := L_q(\mathbb{R}^m, \mu_a) \), in a canonical way, as a subspace of \( H \), then \((T_1, \ldots, T_m)\) is reduced by \( H_a \) and the part \((T_{1, \alpha}, \ldots, T_{m, \alpha}) := (T_1, \ldots, T_m)_{H_a}\) of \((T_1, \ldots, T_m)\) in \( H_a \) is of the type treated in (ii). So we find operators \( K_{1, \alpha}, \ldots, K_{m, \alpha} \in c_p(H_a) \) such that

\[
\| K_{j, \alpha} \|_p \leq 2^{-q/p} \quad (j = 1, \ldots, m).
\]

Then for \( K_j := \bigoplus (K_{j, \alpha}; \alpha \in A) \) we have the estimate (cf. Section 1(B))

\[
\| K_j \|_p^p = \sum_{\alpha \in A} \| K_{j, \alpha} \|_p^p \leq \epsilon^p \sum_{\alpha \in \mathbb{N}} 2^{-q} = \epsilon^p
\]

\((j = 1, \ldots, m)\), and \((D_j := T_j + K_j ; j = 1, \ldots, m)\) is diagonal by construction.

3. Wave Operators for Unitary Representations of \( \mathbb{R}^m \)

In this section we present an abstract method for deriving unitary equivalence of parts of commutative \( m \)-tuples of self-adjoint operators. This method will be applied in the subsequent sections.
Let $T$ be a commutative $m$-tuple of self-adjoint operators. For $t = (t_1, \ldots, t_m) \in \mathbb{R}^m$ the operator $e^{-itT} = e^{-it_1 T_1} \cdots e^{-it_m T_m}$ is defined by the functional calculus for $m$-tuples. $\mathbb{R}^m \ni t \mapsto e^{-itT}$ is a strongly continuous unitary representation of $\mathbb{R}^m$ (m-parameter group) in $H$, said to be generated by $T$.

3.1. Assumptions. Let $H$ be a Hilbert space, $m \in \mathbb{N}$. Let $T = (T_1, \ldots, T_m)$, $T' = (T'_1, \ldots, T'_m)$ be commutative $m$-tuples of self-adjoint operators in $H$. We denote $K_j := T'_j - T_j$ ($j = 1, \ldots, m$).

The notations belonging to $T$, e.g., $E$, $H_{ac}$, etc., will be used without index; the corresponding notations belonging to $T'$ will be denoted by $E'$, $H'_{ac}$, etc.

Let Assumptions 3.1 be given. Let the closed subspace $H_0$ of $H$ be a reducing subspace of $T$, $P_0$ the orthogonal projection onto $H_0$. Let there exist a net $(t^i)_{i \in J}$ in $\mathbb{R}^m$, $|t^i| \to \infty$, such that

$$W_0 := \lim_{i \to J} e^{it^i T'} e^{-it^i T} P_0$$

exists. Then obviously $W_0$ is a partial isometry with initial set $H_0$ and final set $R(W_0)$. For $t \in \mathbb{R}^m$ we consider the net $(t + t^i)_{i \in J}$ and find, that

$$W^t := \lim_{i \to J} e^{i(t+t^i) T'} e^{-i(t+t^i) T} P_0$$

$$= \lim_{i \to J} e^{it^i T'}(e^{it^i T'} e^{-it^i T} P_0) e^{-it T}$$

$$= e^{it T'} W_0 e^{-it T}$$

exists.

3.2. Theorem. Let Assumptions 3.1 be satisfied. Let $H_0$, $P_0$, $(t^i)_{i \in J}$, $W^t$ ($t \in \mathbb{R}^m$) be as just assumed; furthermore let $W := W_0 = W^t$ for all $t \in \mathbb{R}^m$.

Then $H'_0 := R(W)$ is a reducing subspace of $T'$; $T_{H_0}$ and $T'_{H'_0}$ (the parts of $T$ in $H_0$, $T'$ in $H'_0$, respectively) are unitarily equivalent,

$$T'_{H'_0} = W T_{H_0} W^*.$$

Proof. From the intertwining property $e^{-it T'} W = We^{-it T}$ ($t \in \mathbb{R}^m$) we obtain $e^{-it T'} H'_0 \subset H'_0$ ($t \in \mathbb{R}^m$), which implies that $H'_0$ is a reducing subspace of $T'$. The second statement is then a consequence of the intertwining property.

For $H_0 = H_{ac}$ the condition $"W^t = W_0"$ is satisfied under rather general conditions, as we shall see in Lemma 4.1.

Theorem 3.2 is the motivation for the following definition of wave operators.

3.3. Definition. Let Assumptions 3.1 be given. Let $(t^i)_{i \in J}$ be a net in $\mathbb{R}^m$, $|t^i| \to \infty$. 


We define the operator $W = W_0(T', T)$ in $H$ by

$$D(W) := \{ x \in H; \lim_{t \to 0} e^{i(t+t'\tau)}T' e^{-i(t+t'\tau)}T x \text{ exists for all } t \in \mathbb{R}^n \}$$

and does not depend on $t$,

$$Wx := \lim_{t \to 0} e^{i(t+t'\tau)}T' e^{-i(t+t'\tau)}T x \quad \text{for } x \in D(W).$$

$W = W_0 = W_0(T', T)$ is called the wave operator (along $(t')_{t \in I}$ associated with the pair $T, T'$). If $M$ is a subset of $H$ with $M \subseteq D(W)$, we shall say that $W$ exists on $M$.

For $\tau \in S_{m-1}$ (unit sphere in $\mathbb{R}^m$) we consider the net $(\delta_\tau)_{t \to \infty}$, and define $W_\tau := W_{(\delta_\tau)}$, the wave operator in direction $\tau$. Especially for $m = 1$ we obtain the usually defined wave operators $W_+ = W_0$, $W_- = W_{(1)}$ (see Kato [9, X, Section 3]).

3.4. Theorem. Under the assumptions of Definition 3.3 we have for $W := W_0(T', T)$, $W' := W_0(T, T')$:

(a) $D(W)$ is a closed subspace of $H$; $W: D(W) \to R(W)$ is a unitary map; the corresponding statements are valid for $D(W')$ and $W'$.

$$D(W) = R(W), \quad D(W') = R(W), \quad W' = W^{-1} = W^*.$$ ($W^*$ as adjoint of $W: D(W) \to R(W)$.)

(b) $D(W)$ is a reducing subspace of $T$; $R(W)$ is a reducing subspace of $T'$.

$$T_{R(W)} = WT_{D(W)}W^*;$$

$T_{D(W)}$ and $T_{D(W')}^*$ are unitarily equivalent.

Proof. (a) The first statements are obvious. The remaining statements follow from

$$\| x' - e^{i(t+t'\tau)}T' e^{-i(t+t'\tau)}T x \| = \| e^{i(t+t'\tau)}T' e^{-i(t+t'\tau)}T x' - x \|,$$

used for appropriate $x, x' \in H$.

(b) One shows easily $e^{-it' T} D(W) \subseteq D(W')$ ($t \in \mathbb{R}^n$), so the first statement follows. The remaining statements follow from Theorem 3.2. ∎

4. Unitary Equivalence of the Absolutely Continuous Parts of Operator $m$-Tuples, $m \geq 3$

4.1. Lemma. Let Assumptions 3.1 be given. For each bounded Borel set $\Omega \subseteq \mathbb{R}^m$ let $R(E_{a,b}(\Omega)) \subseteq D(T_j)$, and let $K_jE_{a,b}(\Omega)$ be compact ($j = 1, ..., m$). Let $(t')_{t \in I}$ be a net in $\mathbb{R}^n$, $| t' | \to \infty$. Then
(a) for all $x \in H_{ac}, t \in \mathbb{R}^m$

$$e^{it(t+s)t'}e^{-it(t+s)t}x - e^{itT}e^{-itT}x \to 0;$$

(b) if $x \in H_{ac}$ and $\lim e^{itT}e^{-itT}x$ exists, then $x \in D(W(t_0)(T', T));$ if $s\lim e^{itT}e^{-itT}P_{ac}$ exists, then $H_{ac} \subset D(W(t_0)(T', T)).$

Proof. (a) Because $\mathbb{R}^m$ is countable at infinity it is sufficient to consider sequences $(t^n)_{n \in \mathbb{N}}$ instead of nets $(t^i)_{i \in J}.$ By Section l(A) we may assume $H_{ac} = \bigoplus (L_\alpha(\Omega_\alpha); \alpha \in A), T_{ac} = M_{pr}.$ Let $\alpha \in A, f \in L_\alpha(\Omega_\alpha)$ (we consider $L_\alpha(\Omega_\alpha)$ canonically embedded in $\bigoplus (L_\alpha(\Omega_\alpha); \alpha \in A)$), let there exist a bounded Borel set $\Omega \subset \mathbb{R}^m$ such that $f = \chi_\Omega f.$ Then $f \in D(T_j)$ ($j = 1, \ldots, m),$ and $e^{-itT}f(\xi) = e^{-itT}f(\xi)$ implies $e^{-itT}f = \chi_\Omega e^{-itT}f = E_{ac}(\Omega) e^{-itT}f,$ therefore $e^{-itT}f \in D(T_j')$ ($j = 1, \ldots, m, t \in \mathbb{R}^m$) by assumption. This implies that for $t \in \mathbb{R}^m$ the function $\{0, 1\} \ni s \mapsto e^{it(t+s)t'}e^{-i(t+s)t}Tf$

is differentiable, with derivative $ie^{it(t+s)t'}(\sum t_jK_j)e^{-i(t+s)t}Tf,$ and the derivative is continuous because $K_jE_{ac}(\Omega)$ is continuous. From this we get the estimate

$$|| e^{it(t+s)t'}e^{-it(t+s)t}Tf - e^{itT}e^{-itT}f || \leq \int_0^1 || e^{it(t+s)t'}(\sum t_jK_j)e^{-it(t+s)t}Tf || ds$$

$$\leq \sum |t_j| \int_0^1 || K_jE_{ac}(\Omega) e^{-i(t+t)s}Tf || ds.$$

We are going to show $|| K_jE_{ac}(\Omega) e^{-i(t+s)t}Tf || \to 0$ ($n \to \infty$) for all $s \in [0, 1];$ from this we conclude $\int_0^1 || K_jE_{ac}(\Omega) e^{-i(t+s)t}Tf || ds \to 0$ ($n \to \infty$) by $|| K_jE_{ac}(\Omega) e^{-i(t+s)t}Tf || \leq || K_jE_{ac}(\Omega) || || f ||$ and the dominated convergence theorem. So by the above estimate the assertion of (a) will follow for the $f$ chosen above. Since the set of elements $f$ of this type is total in $H_{ac}$ and

$$|| e^{it(t+s)t'}e^{-i(t+s)t}Tf - e^{itT}e^{-itT}f || \leq 2,$$

this implies the assertion of (a) for all $f \in H_{ac}.$

If $g \in L_\alpha(\Omega_\alpha)$ then $g^*f \in L_\alpha(\mathbb{R}^m),$ and so by the Riemann–Lebesgue lemma $\langle g | e^{-itT}f \rangle = \int g(\xi)^* f(\xi) e^{-it\xi} d\xi \to 0$ ($l \to \infty$); so we see $e^{-i(t+s)t}Tf \to 0$ ($n \to \infty$), and the compactness of $K_jE_{ac}(\Omega)$ implies $K_jE_{ac}(\Omega) e^{-i(t+s)t}Tf \to 0$ ($n \to \infty$).

Assertion (b) is a direct consequence of (a) and the definition of $W(t_0)(T', T).$

4.2. Theorem. Let Assumptions 3.1 be given; let $m \geq 3.$ For each bounded Borel set $\Omega \subset \mathbb{R}^m$ let $R(E_{ac}(\Omega)) \subset D(T_j')$ ($j = 1, \ldots, m)$, and let there exist $0 < p < m$ such that $K_jE_{ac}(\Omega)P_{ac} \subset c_p$ ($j = 1, \ldots, m)$.
Then there exists a set \( N \subset S_{m-1} \) of measure zero (with respect to \((m - 1)\)-dimensional measure on \( S_{m-1} \)), such that

(a) \( \lim_{t \to \infty} e^{i\tau t}e^{-i\tau t}P_{ac} \) exists for all \( \tau \in S_{m-1}\setminus N \);

(b) \( H_{ac} \in D(W_{r}(T', T)) \) for all \( \tau \in S_{m-1}\setminus N \), i.e., for each \( \tau \in S_{m-1}\setminus N \) the wave operator in direction \( \tau \) (cf. Definition 3.3) exists on \( H_{ac} \).

4.3. Remark. The condition "\( K_{\varepsilon}E_{ac}(\Omega)P_{ac} \in c_{p} \)" is a "local" condition; "local" with respect to the set carrying the spectral measure. A similar, but sharper condition has been used by Birman ([4], Theorem 4.3) in the case \( m = 1 \). Birman obtains under a relatively mild additional assumption the completeness \( H'_{ac} \subset R(W_{r}(T', T)) \) of the wave operators. In our context we can achieve this only by adding to the assumptions of Theorem 4.2 the corresponding assumptions for \( E'_{ac} \); cf. Theorem 4.5 noticing \( D(W_{r}(T', T'')) = R(W_{r}(T', T')) \) from Theorem 3.4(a). One reason for this is that Birman's method is much more refined than ours. One could conjecture, however, that the assumptions of Theorem 4.2 together with some mild additional assumption imply the corresponding conditions for \( E'_{ac} \). Such an additional assumption is not known to the author.

4.4. ESTIMATE. Let \( m \in \mathbb{N} \), \( \Omega \subset \mathbb{R}^{m} \) a bounded Borel set, \( r > 0 \), \( q > m \frac{r}{r+1} \), and \( q \geq 1 \).

Then for each orthonormal system \((\varphi_{n})_{n \in \mathcal{A}} \subset L_{2}(\Omega)\)

\[
\sum_{n \in \mathcal{A}} \left( \int_{|t| > 1} |t|^{-r} |\varphi_{n}(t)|^{2} dt \right)^{q} \leq (2\pi)^{-m} \lambda^{m}(\Omega) \int_{|t| > 1} |t|^{-rq} dt < \infty.
\]

(\( f \) Fourier transform of \( f \in L_{2}(\Omega) \), \( f(t) = (2\pi)^{-m/2} \int_{\mathbb{R}} f(\xi) e^{-it\xi} d\xi \).)

Proof. By Hölder's inequality we have

\[
\int_{|t| > 1} |t|^{-r} |\varphi_{n}(t)|^{2} dt = \int_{|t| > 1} |t|^{-r} |\hat{\varphi}_{n}(t)|^{2/q} \hat{\varphi}_{n}(t)^{(2q-2)/q} dt \\
\leq \left( \int_{|t| > 1} |t|^{-rq} |\hat{\varphi}_{n}(t)|^{2} dt \right)^{1/q} \left( \int_{|t| > 1} |\hat{\varphi}_{n}(t)|^{2} dt \right)^{(q-1)/q} \\
\leq \left( \int_{|t| > 1} |t|^{-rq} |\hat{\varphi}_{n}(t)|^{2} dt \right)^{1/q}.
\]

Bessel's inequality implies \( \sum_{n \in \mathcal{A}} |\varphi_{n}(t)|^{2} = (2\pi)^{-m} \sum_{n \in \mathcal{A}} |e^{it\cdot} \varphi_{n} \rangle \langle \varphi_{n}|_{L^{2}(\Omega)} |^{2} \leq (2\pi)^{-m} \|e^{it\cdot}\|_{L^{2}(\Omega)}^{2} = (2\pi)^{-m} \lambda^{m}(\Omega) \). So we obtain

\[
\sum_{n \in \mathcal{A}} \left( \int_{|t| > 1} |t|^{-r} |\varphi_{n}(t)|^{2} dt \right)^{q} \leq \sum_{n \in \mathcal{A}} \int_{|t| > 1} |t|^{-rq} |\varphi_{n}(t)|^{2} dt \\
\leq (2\pi)^{-m} \lambda^{m}(\Omega) \int_{|t| > 1} |t|^{-rq} dt < \infty
\]

(\(< \infty \) because of \( rq > m \)).
Proof of Theorem 4.2(a). (i) By Section 1(A) we may assume $H_{ac} = \bigoplus (L_2(\Omega_\alpha); \alpha \in A)$, $T_{ac} = M_{pr}$.

(ii) There is a countable subset $A' \subset A$ such that $T'$ equals $T$ on $\bigoplus (L_2(\Omega_\alpha); \alpha \in A \setminus A')$. This easy consequence of the compactness of $K_f E_{ac}(\Omega)$ ($j = 1, \ldots, m$, $\Omega \subset \mathbb{R}^m$ bounded Borel set) will not be proved in detail. Since the existence of $W_n(T', T)$ ($\tau \in S_{m-1}$) on $\bigoplus (L_2(\Omega_\alpha); \alpha \in A \setminus A')$ is evident we may assume from the beginning $A = A' \subset \mathbb{N}$.

(iii) Let $\alpha \in A$, $\Omega \subset \Omega_\alpha$ a bounded Borel set, $f := \chi_\Omega \in L_2(\Omega_\alpha)$. We consider $L_2(\Omega_\alpha)$ canonically as a subspace of $H$; in the notation we shall not distinguish between $f$ and its canonical image.

We show that one can find a set $N_f \subset S_{m-1}$ of measure zero such that $\lim_{s \to \infty} e^{isT'} e^{-isTf}$ exists for all $\tau \in S_{m-1} \setminus N_f$.

For $\tau \in S_{m-1}$, $1 \leq s' < s < \infty$ we have

$$||e^{is'T'}e^{-is'Tf} - e^{is'T'}e^{-is'Tf}|| = \left|\left|\int_{s'}^{s} (d/ds) e^{is'T'}e^{-is'Tf} ds\right|\right|$$

$$\leq \sum_{j=1}^{m} \left|\tau_j\right| \int_{s'}^{s} ||K_f e^{-is'Tf}|| ds$$

(cf. the proof of Lemma 4.1(a); the idea is the estimate in "Cook's lemma," see Kato [9, X, 3.7]). So it suffices to find sets $N_{fj} \subset S_{m-1}$ ($j = 1, \ldots, m$) of measure zero, such that $\int_{1}^{\infty} ||K_f e^{-is'Tf}|| ds < \infty$ for $\tau \in S_{m-1} \setminus N_{fj}$; this will be done in (iv) below. Then the assertion of (iii) follows with $N_f := \bigcup_{j=1}^{m} N_{fj}$.

(iv) We take $j \in \{1, \ldots, m\}$ and show the existence of the set $N_{fj}$ mentioned in (iii); the index $j$ will be omitted in the sequel.

To show the existence of $N_f$ ($= N_{fj}$) we are going to show

$$\int_{S_{m-1}} \int_{1}^{\infty} ||K_f e^{-is'Tf}|| ds d\sigma(\tau) < \infty$$

(do surface measure on $S_{m-1}$). We rewrite

$$\int_{S_{m-1}} \int_{1}^{\infty} ||K_f e^{-is'Tf}|| ds d\sigma(\tau) = \int_{|\tau| > 1} t^{-\{m-1\}} ||K_f e^{-itTf}|| dt.$$

We calculate

$$||K_f e^{-itTf}|| = ||KE_{ac}(\Omega) P_\alpha e^{-itTf}||$$

$$= \left(\sum_{n \in \mathbb{N}} |c_n| ^2 <\varphi_n, e^{-itTf}> \psi_n\right)^{1/2},$$

with the following notations: $P_\alpha$ denotes the orthogonal projection onto $L_2(\Omega_\alpha) \subset H$. $E_{ac}(\Omega)$ and $P_\alpha$ can be inserted because of $e^{-itTf}(\xi) = e^{-it\chi_\Omega}(\xi)$. 
$KE_{ac}(\Omega)P_{ac} \in c_\phi$ implies $KE_{ac}(\Omega)P_a \in c_\phi$ ([11], Theorem 2.3), and there exist orthonormal systems $(\varphi_n)$, $(\psi_n)$ in $H$ and a sequence $(c_n) \subset \mathbb{C}$ such that $KE_{ac}(\Omega)P_a = \sum_{n \in \mathbb{N}} c_n \langle \varphi_n | \psi_n \rangle$ (cf. [9, V, Section 2.3]) and $\sum_{n \in \mathbb{N}} | c_n |^2 < \infty$ (cf. Section 1(B)). Because of $N(KE_{ac}(\Omega)P_a) \subset L_2(\Omega)^\perp$ (the subspace $L_2(\Omega)$ of $L_2(\Omega)$ canonically embedded in $H$) we may assume without restriction that $(\varphi_n)$ is an orthonormal system in $L_2(\Omega)$.

So we have to prove

$$\int_{|t| > 1} | t |^{-m-1} \left( \sum_{n \in \mathbb{N}} | c_n |^2 |\langle \varphi_n | e^{-itTf} \rangle|^2 \right)^{1/2} dt < \infty.$$  

We show this by applying Hölder’s inequality several times to integrals and sums, and using Estimate 4.4. Without restriction we may assume $2 < p < m$.

For $\epsilon > 0$, to be determined later more precisely, we define $r = r(\epsilon) := m - 2 - \epsilon$. Then by the Schwarz inequality

$$\int_{|t| > 1} | t |^{-m-1} \left( \sum_{n \in \mathbb{N}} | c_n |^2 |\langle \varphi_n | e^{-itTf} \rangle|^2 \right)^{1/2} dt \leq \left( \int_{|t| > 1} | t |^{-m-\epsilon} dt \right)^{1/2} \left( \int_{|t| > 1} | t |^{-r} \left( \sum_{n \in \mathbb{N}} \right) dt \right)^{1/2} \leq (\cdots)^{1/2} \left( \sum_{n \in \mathbb{N}} | c_n |^2 \int_{|t| > 1} | t |^{-r} |\langle \varphi_n | e^{-itTf} \rangle|^2 dt \right)^{1/2}.$$  

The first term is finite because of $m + \epsilon > m$. To show that the second term is finite, we note that $(| c_n |^2)_{n \in \mathbb{N}} \in \mathcal{L}_{p/2}$, with $1 < p/2 < m/2$; so it is sufficient to show

$$\left( \int_{|t| > 1} | t |^{-r} |\langle \varphi_n | e^{-itTf} \rangle|^2 dt \right)_{n \in \mathbb{N}} \in \mathcal{L}_q,$$

with $q = (p/2)/(p/2 - 1)$. To see that this follows from Estimate 4.4 for an appropriate $\epsilon$, we note first that

$$\langle \varphi_n | e^{-itTf} \rangle = \int_{\Omega_a} \varphi_n(\xi) \ast e^{-ittX_{\Omega}}(\xi) d\xi = (2\pi)^{m/2} \varphi_n^*(t),$$

and that $(\varphi_n^*)$ is an orthonormal system in $L_2(\Omega)$. Furthermore we have $r > 0$ provided that $\epsilon < 1$, and $q \geq 1$ since $q$ is the conjugate exponent of $p/2 > 1$.

To show that it is possible to achieve the remaining condition $q > m/r$, we remark that $1 < p/2 < m/2$ implies

$$q = \frac{p/2}{(p/2) - 1} > \frac{m/2}{(m/2) - 1} = \frac{m}{m - 2},$$

and that $r = r(\epsilon) \to m - 2$ ($\epsilon \to 0$) implies $m/r = m/r(\epsilon) \to m/(m - 2) < q$ ($\epsilon \to 0$).
(v) For each $\alpha \in A$ there exists a countable family $(\Omega_k)_{k \in \mathbb{N}}$ of bounded Borel sets $\Omega_k \subset \Omega_\alpha$, such that the set of functions $\{\chi_{\Omega_k} : k \in \mathbb{N}\}$ is total in $L_2(\Omega_\alpha)$. Since $A$ is countable (cf. part (ii) above), there is a countable total subset \( \{f_n : n \in \mathbb{N}\} \) of $\bigoplus (L_2(\Omega_\alpha)) ; \alpha \in A$ such that each $f_n \in \mathbb{N}$ is of the form described at the beginning of (iii). By (iii) and (iv) there are sets $N_n \subset S_{m-1}$ of measure zero, such that $\lim_{s \to \infty} e^{isT'} e^{-isT} f_n$ exists for all $\tau \in S_{m-1} \setminus N_n$, $n \in \mathbb{N}$. If we define the set $N := \bigcup_{n \in \mathbb{N}} N_n$ (of measure zero in $S_{m-1}$), then for $\tau \in S_{m-1} \setminus N$ the limit $\lim_{s \to \infty} e^{isT'} e^{-isT} f_n$ exists for all $f \in \text{span} \{f_n : n \in \mathbb{N}\}$, and so for all $f \in \bigoplus (L_2(\Omega_\alpha)) ; \alpha \in A = \text{span} \{f_n : n \in \mathbb{N}\}$ because $\|e^{isT'} e^{-isT}\| \leq 1$.

Assertion (b) follows from (a) and Lemma 4.1(b).

**4.5. Theorem.** Let Assumptions 3.1 be given; let $m \geq 3$. For each bounded Borel set $\Omega \subset \mathbb{R}^m$ let $R(E_{ac}(\Omega)) \subset D(T_j')$ and $R(E_{bc}(\Omega)) \subset D(T_j)$, and let there exist $0 < p < m$ such that $K_j E_{ac}(\Omega) P_{ac} \subset e_p$ and $K_j E_{bc}(\Omega) P_{bc} \subset e_p$ ($j = 1, \ldots, m$). Then

(a) there exists a set $N \subset S_{m-1}$ of measure zero, such that for all $\tau \in S_{m-1} \setminus N$

\[ H_{ac} \subset D(W_\tau(T', T)) \quad \text{and} \quad H_{ac} \subset D(W_\tau(T, T')) \]

(b) $T_{ac}$ and $T_{bc}$ are unitarily equivalent, a unitary equivalence being implemented by $W_\tau(T', T) : H_{ac} \to H_{bc}$ for each $\tau \in S_{m-1} \setminus N$ (with $N$ from (a)).

**Proof.** (a) By Theorem 4.2(b) there are sets $N_1 \subset N_2 \subset S_{m-1}$ of measure zero, such that $H_{ac} \subset D(W_\tau(T', T))$ for all $\tau \in S_{m-1} \setminus N_1$, $H_{bc} \subset D(W_\tau(T, T'))$ for all $\tau \in S_{m-1} \setminus N_2$. With $N := N_1 \cup N_2$ we obtain the assertion.

(b) Let $\tau \in S_{m-1} \setminus N$, with $N$ from (a); denote $W := W_\tau(T', T), W' := W_\tau(T, T')$. By Theorem 3.4(b) $W : D(W) \to D(W')$ implements a unitary equivalence between $T_{D(W)}$ and $T_{D(W')}$. This implies that $W(H_{ac})$ is a reducing subspace of $T'$, and $W(H_{bc}) \subset H_{ac'}$; likewise we get $W'(H_{bc'}) \subset H_{ac}$. From $W' = W^{-1}$ we conclude $W(H_{ac}) = H_{bc'}$.

**4.6. Corollary.** Let Assumptions 3.1 be given; let $m \geq 3$. Let $K_j (=T_j' - T_j)$ be densely defined and bounded, and let there exist $0 < p < m$ such that $K_j \subset e_p$ ($j = 1, \ldots, m$).

Then the conclusions of Theorem 4.5 are valid.

**Proof.** The assumptions on $K_j$ imply $D(T_j') = D(T_j)$ ($j = 1, \ldots, m$). So the assertion follows from Theorem 4.5.

**5. Results for Normal Operators ($m = 2$)**

Theorems 4.2 and 4.5 and Corollary 4.6 do not apply to normal operators (because of the assumption "$m \geq 3$"'). In this section we obtain a similar but
essentially weaker result for normal operators. Also we prove a related statement for not necessarily normal operators.

5.1. Theorem. Let Assumptions 3.1 be given; let \( m = 2 \). Let \( K_1, K_2 \) be densely defined and bounded, and let \( K_1 \in \mathcal{c}_1, K_2 \in \mathcal{c}_\infty \). Then

\begin{enumerate}[(a)]
    \item for \( \tau = (1, 0) \) and \( \tau = (-1, 0) \)
        \[ H_{ac} \subset D(W_\tau(T', T)), \quad H_{ac} \subset D(W_\tau(T, T')) \]
    \item \( T_{ac} \) and \( T_{ac}' \) are unitarily equivalent.
\end{enumerate}

**Proof.** (a) By the Kato–Rosenblum theorem \( K_1 \in \mathcal{c}_1 \) implies \( H_{ac, T_1} \subset D(W_\tau(T_1', T_1)) \). For the spectral measures \( E_\tau \) of \( T = (T_1, T_2) \), \( E_{\tau_1} \) of \( T_1 \) we have \( E_{\tau_1}(\Omega) = E_\tau(\Omega \times \mathbb{R}) \) \((\Omega \subset \mathbb{R} \text{ Borel set})\); with this fact \( H_{ac, T} \subset H_{ac, T_1} \) follows immediately from the definition of the absolutely continuous subspace. So we have \( H_{ac} \subset D(W_{\tau'}(T_1', T_1)) \), and Lemma 4.1(b) implies \( H_{ac} \subset D(W_{\tau}(T, T')) \). By the symmetry of the assumption we also get \( H_{ac}' \subset D(W_{\tau'}(T', T)) \).

Assertion (b) follows from (a) as in the proof of Theorem 4.5(b).

5.2. Corollary. Let \( \Omega \subset \mathbb{R}^2 \) be open, bounded, \( \Omega \neq \emptyset \), \( H := L_2(\Omega) \), \( N \in \mathcal{B}(H) \) defined by \( Nf(\xi) = (\xi_1 + it_2)f(\xi) \).

Then it is not possible to write \( N = D + K \) with diagonal \( D \) and \( K \in \mathcal{c}_1 \).

The statement of Corollary 5.2 is due to Berg ([3], Theorem 4). It just shows that there is no Weyl–von Neumann theorem for normal operators with \( K \in \mathcal{c}_1 \).

This reduced statement can also be obtained from the following generalization of Corollary 5.2, which does not require the results of the preceding sections.

5.3. Theorem. Let \( H \) be a Hilbert space, \( T \in \mathcal{B}(H) \) with \( H_{ac, \text{Re} T} \neq \{0\} \) \((\text{Re} T = \frac{1}{2}(T + T^*))\).

Then it is not possible to write \( T = D + K \) with diagonal \( D \) and \( K \in \mathcal{B}(H), \text{Re} K \in \mathcal{c}_1 \).

**Proof.** We assume there is \( K \in \mathcal{B}(H), \text{Re} K \in \mathcal{c}_1 \), such that \( T - K \) is diagonal. Then \( (T - K)^* \) is diagonal with the same eigenbasis as \( T - K \). So \( T - K + (T - K)^* = (T + T^*) - (K + K^*) \) is self-adjoint and diagonal, especially \( (T + T^*) - (K + K^*) \) is nontrivial by assumption. We have \( (T + T^*) - (K + K^*) \) is unitarily equivalent to \( (T + T^*)_{ac} \) which is nontrivial by assumption.

APPENDIX

We present an example of two commutative pairs of self-adjoint operators \( T = (T_1, T_2), T' = (T_1', T_2') \) such that for each bounded Borel set \( \Omega \subset \mathbb{R}^2 \) we
have $K_\tau E_{ac}(\tau)P_{ac} \in c_2$ ($j = 1, 2$) (i.e., the assumptions of Theorem 4.2 are “satisfied” with $p = m = 2$), and such that $D(W_\tau(T', T)) = \{0\}$ for all $\tau \in S_1$ although $H_{ac} \neq \{0\}$ (i.e., the conclusions of Theorem 4.2 do not hold).

It seems to the author that the nonexistence of the direction wave operators in this example might be a characteristic feature of the cases $p = m \in \mathbb{N}$ for $m \geq 2$. We note also that in the example there are sequences $(t^n)$ for which the wave operator $W(t^n)(T', T)$ exists on $H_{ac}$.

**Example.** (i) Let $H := L_2(\mathbb{R}^2)$. $U : \mathbb{R}^2 \to B(H)$, defined by $U(t)f(x) := f(x - t)$, is a strongly continuous unitary representation (translation representation); there exists a unique commutative pair $T = (T_1, T_2)$ of self-adjoint operators associated with $U$, i.e., $U(t) = e^{-itT}$. It is easily seen that $C_0^1(\mathbb{R}^2) \subset D(T_j)$ and $T_j\varphi = -i\partial_j\varphi$ ($\varphi \in C_0^1(\mathbb{R}^2)$) for $j = 1, 2$.

It is well known that a spectral representation of $T$ is given by the inverse of the Fourier-Plancherel transformation $F : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$, $Ff(\xi) = (2\pi)^{-1/2} \int e^{-ix\xi}f(x) \, dx$. $F$ is unitary by Plancherel's theorem (cf. [20, VI.21]); we use the notations $\hat{f} = Ff$, $\check{g} = F^*g$. From this one deduces $H_{ac} = H$, $E_{ac} = E$.

(ii) Let $G : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. We define the “perturbed” strongly continuous unitary representation $U' : \mathbb{R}^2 \to B(H)$ by

$$U'(t)f(x) := e^{iG(x - t)}e^{-itf(x)}f(x - t).$$

Let $T' = (T'_1, T'_2)$ be the pair of self-adjoint operators associated with $U'$. As above one can see that $C_0^1(\mathbb{R}^2) \subset D(T'_j)$ and $T'_j\varphi = -i\partial_j\varphi + (\partial_j G)\varphi$ ($\varphi \in C_0^1(\mathbb{R}^2)$) for $j = 1, 2$.

(iii) Assume now $\partial_j G \in L_2(\mathbb{R}^2) \cap L_{ac}(\mathbb{R}^2)$ ($j = 1, 2$). Then $K_j = \overline{T'_j - T_j} = (\partial_j G) \in B(H)$ ($j = 1, 2$) (by $\partial_j G$ we denote the operator of multiplication by the function $\partial_j G$), and we want to show $(\partial_j G) E(\Omega) \in c_2$ ($j = 1, 2$) for each bounded Borel set $\Omega \subset \mathbb{R}^2$. Now for $f \in L_2(\mathbb{R}^2)$

$$E(\Omega) f = \hat{F}^*(x)f = \widehat{\chi_\Omega Ff} = (2\pi)^{-1/2} \int \chi_\Omega(x - y) \widehat{Ff} = (2\pi)^{-1} \chi_{\Omega} \ast f$$

(cf. [20, VI.2, (28)]);

$$(\partial_j G) E(\Omega) f(x) = \partial_j G(x)(2\pi)^{-1} \int \chi_{\Omega}(x - y) f(y) \, dy = \int k(x, y) f(y) \, dy,$$

with $k(x, y) := (2\pi)^{-1} \partial_j G(x) \chi_{\Omega}(x - y)$. From

$$\int |k(x, y)|^2 \, dx \, dy = (2\pi)^{-1} \int |\partial_j G(x)|^2 \, |\chi_{\Omega}(x - y)|^2 \, dx \, dy$$

$$= (2\pi)^{-1} \|\partial_j G\|_2^2 \|\chi_{\Omega}\|_2^2 = (2\pi)^{-1} \|\partial_j G\|^2 \lambda^2(\Omega) < \infty$$
we see that $\partial_j G E(\Omega) = (\partial_j G) E(\Omega) \in c_2$ ($j = 1, 2$).

(iv) To complete the example we are going to construct a function $G$ with the properties assumed in (ii) and (iii), such that $D(W_\tau(T', T)) = \{0\}$ for all $\tau \in S_1$.

To do this let $a \in C_0^1(0, \infty; \mathbb{R})$, and for $\alpha > 0$ let $\varphi_\alpha(x) := \varphi(|x|^\alpha)$ ($x \in \mathbb{R}^2$). Then $\varphi_\alpha \in C_0^1(\mathbb{R}^2)$,

$$\dot{\varphi}_\alpha(x) = \varphi'(|x|^\alpha) \dot{x}, \quad (j = 1, 2).$$

$$\int (\partial_j \varphi_\alpha(x)^2 + \varphi_\alpha(x)^2) \, dx = \alpha^2 \int \varphi'(|x|^\alpha)^2 \, dx \geq 0$$

$$(\text{with } \alpha = \tau, \, dx = \frac{1}{\tau} \cos \theta \, dr \, d\theta)$$

$$= 2\pi \alpha^2 \int \varphi'(s)^2 \, ds$$

$$(\text{with } r = \frac{s}{\alpha}, \, dr = \frac{1}{\alpha} \cos \theta \, ds)$$

$$= 2\pi \alpha^2 \int_0^\infty \varphi'(s)^2 \, ds.$$

Now we choose $\alpha > 0$ such that $\text{supp } \varphi \subset (2, 3)$, $\varphi(5/2) = 1$; this implies

$$\text{supp } \varphi_\alpha \subset \{x \in \mathbb{R}^2; 2 < |x|^\alpha < 3\} = \{x \in \mathbb{R}^2; 2^{1/\alpha} < |x| < 3^{1/\alpha}\}.$$ 

Now we choose a sequence $(\alpha_n) \subset (0, \infty)$ such that $3^{1/\alpha_n} \leq 2^{1/\alpha_{n+1}}$ ($n \in \mathbb{N}$) (e.g., $\alpha_n = (\log 2/\log 3)^n$). Then we have $\sum\limits_{n \in \mathbb{N}} \alpha_n < \infty$, $\text{supp } \varphi_{\alpha_n} \cap \text{supp } \varphi_{\alpha_n} = \emptyset$ for $n \neq n'$. Finally we define $G(x) := \sum\limits_{n \in \mathbb{N}} \varphi_{\alpha_n}(x)$ ($x \in \mathbb{R}^2$); then $G \in C^1(\mathbb{R}^2)$,

$$\int (\partial_j G(x)^2 + \partial_j G(x)^2) \, dx = 2\pi \int \varphi'(s)^2 \, ds \sum\limits_{n \in \mathbb{N}} \alpha_n < \infty.$$ 

Now let $\tau \in S_1$. One verifies that for the sequence $(t^n)$, $t^n := 2^{1/\alpha_n}r$

$$e^{it^n T}e^{-it^n T f} = e^{iG(t^n+t^n)}e^{-iGf} \to e^{-iGf} \quad (n \to \infty)$$

for all $f \in L_2(\mathbb{R}^2)$ with $\text{supp } f$ compact, and so for all $f \in L_2(\mathbb{R}^2)$; whereas for the sequence $(t^n)$, $t^n := (5/2)^{1/\alpha_n}r$

$$e^{it^n T}e^{-it^n T f} = e^{iG(t^n+t^n)}e^{-iGf} \to e^{iGt^n}f \quad (n \to \infty).$$

This shows that $W_\tau(T', T)$ does not exist on any $f \neq 0$.

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