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Log-concavity and LC-positivity

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Abstract

A triangle $\{a(n,k)\}_{0\leqslant k\leqslant n}$ of nonnegative numbers is LC-positive if for each r, the sequence of polynomials $\sum_{k=r}^{n} a(n,k)q^k$ is q-log-concave. It is double LC-positive if both triangles $\{a(n,k)\}$ and $\{a(n,n-k)\}$ are LC-positive. We show that if $\{a(n,k)\}$ is LC-positive then the log-concavity of the sequence $\{x_k\}$ implies that of the sequence $\{z_n\}$ defined by $z_n=\sum_{k=0}^{n} a(n,k)x_k$, and if $\{a(n,k)\}$ is double LC-positive then the log-concavity of sequences $\{x_k\}$ and $\{y_k\}$ implies that of the sequence $\{z_n\}$ defined by $z_n=\sum_{k=0}^{n} a(n,k)x_k y_{n-k}$. Examples of double LC-positive triangles include the constant triangle and the Pascal triangle. We also give a generalization of a result of Liggett that is used to prove a conjecture of Pemantle on characteristics of negative dependence.

Keywords: Sequences; Linear transformations; Convolutions; Log-concavity; q-log-concavity; LC-positivity

1. Introduction

Let $x_0, x_1, x_2, \ldots$ be a sequence of nonnegative numbers and with no internal zeros. By the latter we mean that there are no three indices $i<j<k$ such that $x_i, x_k \neq 0$ and $x_j=0$. We say that the sequence is *log-concave* (LC) if $x_{i-1}x_{i+1}\leqslant x_i^2$ for all $i>0$. It is well known that the sequence $\{x_k\}$ is log-concave if and only if $x_{i-1}x_{j+1}\leqslant x_i x_j$ for all $j\geqslant i\geqslant 1$ (see [1, Proposition 2.5.1] for instance), or equivalently, all minors of order 2 of the infinite matrix $M=(x_{i-j})_{i,j\geqslant 0}$ are nonnegative (where $x_k=0$ if $k<0$). For this reason a log-concave

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sequence with no internal zeros is also called PF_2 (the notation actually has a precisely motivation, see [1,5]). Log-concave sequences arise often in combinatorics, algebra, geometry, analysis, probability and statistics. There have been many attempts to develop techniques for the log-concavity problems. We refer the reader to Stanley's survey article [12] and Brenti's supplement [2] for details.

Let $\{a(n,k)\}_{0\leqslant k\leqslant n}$ be a triangular array of nonnegative numbers. Define two linear transformations of sequences by

$$z_n=\sum_{k=0}^{n}a(n,k)x_k,\quad n=0,1,2,\ldots,\tag{1}$$

and

$$z_n=\sum_{k=0}^{n}a(n,k)x_k y_{n-k},\quad n=0,1,2,\ldots,\tag{2}$$

respectively. We say that the linear transformation (1) has *the PLC property* if it preserves the log-concavity of sequences, i.e., the log-concavity of $\{x_n\}$ implies that of $\{z_n\}$. We say that the linear transformation (2) has *the double PLC property* if the log-concavity of $\{x_n\}$ and $\{y_n\}$ implies that of $\{z_n\}$. The corresponding triangle $\{a(n,k)\}$ is also called *PLC* and *double PLC*, respectively. Clearly, the double PLC property implies the PLC property.

It is well known that the ordinary convolution

$$z_n=\sum_{k=0}^{n}x_k y_{n-k},\quad n=0,1,2,\ldots,$$

is double PLC, which can be obtained as a consequence of the fact that the product of TP_2 matrices is TP_2 (see Karlin [5, p. 394] for instance) or by a direct argument (see Menon [8] for instance). Using the same fact, Walkup can manage to prove that the binomial convolution

$$z_n=\sum_{k=0}^{n}\binom{n}{k}x_k y_{n-k},\quad n=0,1,2,\ldots,$$

is double PLC [13, Theorem 1]. A more general result is due to Liggett (see [7, Theorem 3] or Section 3 of this paper). However, there is no systematic study of linear transformations that are double PLC. The possible reason for this is that very few examples of such linear transformations are known. In the present paper we develop techniques to deal with the problems of finding these kind of linear transformations and apply these techniques to generate new log-concave sequences from existing ones.

When the triangle $\{a(n,k)\}$ is PLC, the linear transformation (1) has to send any log-concave sequence $\{x_k\}$ to a log-concave sequence $\{z_n\}$. So, by taking the special log-concave sequence $\{x_k\}$, we may obtain certain necessary conditions such that $\{a(n,k)\}$ is PLC from the log-concavity of the associated sequence $\{z_n\}$.

Remark 1.1. Let the triangle $\{a(n,k)\}$ be PLC. Then for $r\in\mathbb{N}$ and $p>0$:

(i) the column sequence $\{a(n,r)\}_{n\geqslant r}$ is log-concave;
(ii) the row-sum sequence $a(n)=\sum_{k=0}^{n}a(n,k)$ is log-concave; and
(iii) the sequence $\mathcal{A}_r(n;p)=\sum_{k=r}^{n}a(n,k)p^k$ is log-concave for $n\geqslant r$.

We can view $\mathcal{A}_r(n; p)$ as a polynomial in p. By (iii), the polynomial

$$\mathcal{A}_r^2(n; p) - \mathcal{A}_r(n-1; p)\mathcal{A}_r(n+1; p)$$

takes nonnegative values when $p > 0$, and so that its leading coefficient

$$a^2(n,n) - a(n-1,n-1)a(n+1,n+1)$$

has to be nonnegative. In other words, the diagonal sequence $\{a(n,n)\}_{n\geqslant 0}$ is log-concave.

In order to state our sufficient conditions for $\{a(n,k)\}$ to be PLC, we introduce some terminology and notation. Let q be an indeterminate and $\{f_n(q)\}_{n\geqslant 0}$ a sequence of polynomials in q. We say that the sequence $\{f_n(q)\}_{n\geqslant 0}$ is *q-log-concave* if for each $n \geqslant 1$, $f_n^2(q) - f_{n-1}(q)f_{n+1}(q)$ has nonnegative coefficients as a polynomial in q. The concept of q-log-concavity is first suggested by Stanley (see [11, p. 795]). We refer the reader to [3,4,6,10,11] for further information about q-log-concavity. Now for $0 \leqslant r \leqslant n$, define the polynomial

$$\mathcal{A}_r(n; q) = \sum_{k=r}^{n} a(n,k)q^k.$$

We say that the triangle $\{a(n,k)\}$ has *the LC-positive property* if for each $r \geqslant 0$, the sequence of polynomials $\{\mathcal{A}_r(n; q)\}_{n\geqslant r}$ is q-log-concave in n. (We remind the reader that the definition is different from Remark 1.1(iii).) Define the reciprocal triangle $\{a^*(n,k)\}$ of $\{a(n.k)\}$ by

$$a^*(n,k) = a(n, n-k), \quad 0 \leqslant k \leqslant n.$$

We say that the triangle $\{a(n,k)\}$ has *the double LC-positive property* if both $\{a(n,k)\}$ and $\{a^*(n,k)\}$ have the LC-positive property.

Example 1.2. Consider $a(n,k) \equiv 1$ for $0 \leqslant k \leqslant n$. Then $\mathcal{A}_r(n; q) = \sum_{k=r}^{n} q^k$ for $0 \leqslant r \leqslant n$. It immediately follows that

$$\mathcal{A}_r^2(n; q) - \mathcal{A}_r(n-1; q)\mathcal{A}_r(n+1; q) = q^{n+r},$$

and so that $\{\mathcal{A}_r(n; q)\}$ is q-log-concave in n. Thus the constant triangle $\{a(n,k)\}$ is LC-positive and therefore double LC-positive since $a^*(n,k) = a(n,k)$.

Example 1.3. Consider $a(n,k) = \binom{n}{k}$. Then $\mathcal{A}_r(n; q) = \sum_{k=r}^{n} \binom{n}{k} q^k$. We have

$$\mathcal{A}_r(n; q) = \sum_{k=r}^{n} \left[\binom{n-1}{k} + \binom{n-1}{k-1}\right] q^k = (q+1)\mathcal{A}_r(n-1; q) + \binom{n-1}{r-1} q^r.$$

It follows that

$$\begin{aligned}
&\mathcal{A}_r^2(n; q) - \mathcal{A}_r(n-1; q)\mathcal{A}_r(n+1; q) \\
&\quad = \mathcal{A}_r(n; q)\left[(q+1)\mathcal{A}_r(n-1; q) + \binom{n-1}{r-1} q^r\right] \\
&\qquad - \mathcal{A}_r(n-1; q)\left[(q+1)\mathcal{A}_r(n; q) + \binom{n}{r-1} q^r\right] \\
&\quad = \binom{n-1}{r-1} q^r \mathcal{A}_r(n; q) - \binom{n}{r-1} q^r \mathcal{A}_r(n-1; q)
\end{aligned}$$

$$=\sum_{k=r}^{n}\left[\binom{n-1}{r-1}\binom{n}{k}-\binom{n}{r-1}\binom{n-1}{k}\right]q^{k+r}$$
$$=\sum_{k=r}^{n}\left[\binom{n-1}{r-1}\binom{n-1}{k-1}-\binom{n-1}{r-2}\binom{n-1}{k}\right]q^{k+r},$$

which has nonnegative coefficients by the log-concavity of the binomial coefficients. Hence $\{\mathcal{A}_r(n;q)\}$ is q-log-concave in n. Thus the Pascal triangle $\{a(n,k)\}$ is LC-positive and therefore double LC-positive since $a^*(n,k)=a(n,k)$.

The object of this paper is twofold. First, we show that LC-positive triangles are PLC and that double LC-positive triangles are double PLC. Second, we present some examples of PLC and double PLC triangles by showing their LC-positivity. We also give a generalization of a result of Liggett that is used to prove a conjecture of Pemantle on characteristics of negative dependence.

2. Theorems

In this section we discuss the LC-positivity in detail and establish the relation between the (double) LC-positivity and (double) PLC property. The following simple result will be used repeatedly in our discussion.

Lemma 2.1. *Let $s\in\mathbb{P}$. Suppose that two sequences $a_0,\dots,a_s$ and $X_0,\dots,X_s$ of real numbers satisfy the following two conditions*:

(a) $\sum_{k=r}^{s} a_k \geqslant 0$ *for all* $0\leqslant r\leqslant s$;
(b) $0\leqslant X_0\leqslant X_1\leqslant\cdots\leqslant X_s$.

Then $\sum_{k=0}^{s} a_kX_k \geqslant X_0\sum_{k=0}^{s} a_k\geqslant 0$.

Proof. Applying the Abel's partial summation formula

$$\sum_{k=0}^{s} a_kX_k=(a_0+a_1+\cdots+a_s)X_0+(a_1+\cdots+a_s)(X_1-X_0)+\cdots+a_s(X_s-X_{s-1}),$$

the statement immediately follows. □

We first consider the relation between the LC-positivity and the PLC property. Let $\{a(n,k)\}_{0\leqslant k\leqslant n}$ be a triangle of nonnegative numbers and $\{x_k\}_{k\geqslant 0}$ be a log-concave sequence. It is convenient to extend the definition of x_k and $a(n,k)$ by setting $x_k=0$ for $k<0$ and $a(n,k)=0$ for $k<0$ or $k>n$. Let $\{z_n\}_{n\geqslant 0}$ be the sequence defined by (1) and denote $\Delta_n=z_n^2-z_{n-1}z_{n+1}$. Then we need that $\Delta_n\geqslant 0$ for each $n\geqslant 1$. Note that

$$\Delta_n=\left\{\sum_{k=0}^{n}a(n,k)x_k\right\}^2-\left\{\sum_{k=0}^{n-1}a(n-1,k)x_k\right\}\left\{\sum_{k=0}^{n+1}a(n+1,k)x_k\right\} \tag{3}$$

is a quadratic form in $n+2$ variables $x_0,x_1,\dots,x_{n+1}$. Such quadratic forms are generally not positive semidefinite. Hence the log-concavity of $\{x_k\}$ is indispensable for our purposes. To see this let us take $a(n,k)\equiv 1$ for $0\leqslant k\leqslant n$ as an example. In this case we have

$$\Delta_2=(x_0+x_1)^2-x_0(x_0+x_1+x_2)=x_1^2+x_0x_1-x_0x_2.$$

Clearly, Δ_2 may take negative values for nonnegative x_k's, but must be nonnegative when x_0, x_1, x_2 is log-concave.

To utilize the assumption for $\{x_k\}$, recall that $\{x_k\}$ is log-concave if and only if $x_{i-1}x_{j+1} \leqslant x_i x_j$ for $j \geqslant i \geqslant 1$. In other words, the $x_i x_j$'s with the same "weight" $i + j$ are comparable. Collect together those terms in Δ_n with the same weight t and denote their sum by S_t. For $0 \leqslant k \leqslant \lfloor t/2 \rfloor$, let $a_k(n,t)$ be the coefficient of the term $x_k x_{t-k}$ in Δ_n. Then $\Delta_n = \sum_{t=0}^{2n} S_t$ and $S_t = \sum_{k=0}^{\lfloor t/2 \rfloor} a_k(n,t) x_k x_{t-k}$. Thus it suffices that $S_t \geqslant 0$ for each $0 \leqslant t \leqslant 2n$. Note that $x_0 x_t \leqslant x_1 x_{t-1} \leqslant x_2 x_{t-2} \leqslant \cdots$. Hence by Lemma 2.1, it suffices that $\sum_{k=r}^{\lfloor t/2 \rfloor} a_k(n,t) \geqslant 0$ for each $0 \leqslant r \leqslant \lfloor t/2 \rfloor$. By (3),

$$a_k(n,t) = 2a(n,k)a(n,t-k) - a(n-1,k)a(n+1,t-k) - a(n+1,k)a(n-1,t-k)$$

for $k < t/2$, and

$$a_k(n,t) = a^2(n,k) - a(n-1,k)a(n+1,k)$$

for t even and $k = t/2$. Denote

$$A_r(n,t) = \sum_{k=r}^{\lfloor t/2 \rfloor} a_k(n,t). \tag{4}$$

Then it is not difficult to see that $A_r(n,t)$ is precisely the coefficient of q^t in the polynomial $\mathcal{A}_r^2(n;q) - \mathcal{A}_r(n-1;q)\mathcal{A}_r(n+1;q)$, i.e.,

$$\mathcal{A}_r^2(n;q) - \mathcal{A}_r(n-1;q)\mathcal{A}_r(n+1;q) = \sum_{t=2r}^{2n} A_r(n,t)q^t. \tag{5}$$

So the following lemma is immediate.

Lemma 2.2. *With the notation above, the triangle $\{a(n,k)\}_{0 \leqslant k \leqslant n}$ is LC-positive if and only if $A_r(n,t) \geqslant 0$ for all $2r \leqslant t \leqslant 2n$.*

We can now conclude the first main result of this paper from the discussion above.

Theorem 2.3. *The LC-positive triangles are PLC.*

We next relate the double LC-positivity with the double PLC property. We need the following.

Proposition 2.4. *Given a triangle $\{a(n,k)\}_{0 \leqslant k \leqslant n}$ of nonnegative numbers and two log-concave sequences $\{x_k\}_{k \geqslant 0}$ and $\{y_k\}_{k \geqslant 0}$, define three triangles $\{b(n,k)\}$, $\{c(n,k)\}$ and $\{d(n,k)\}$ by*

$$b(n,k) = a(n,k)x_k, \qquad c(n,k) = a(n,k)y_{n-k}, \qquad d(n,k) = a(n,k)x_k y_{n-k}.$$

For $2r \leqslant t \leqslant 2n$, define $B_r(n,t)$, $C_r(n,t)$ and $D_r(n,t)$ similar to $A_r(n,t)$ in (4):

(i) *If the triangle $\{a(n,k)\}$ is LC-positive, then the triangle $\{b(n,k)\}$ is LC-positive and $B_r(n,t) \geqslant A_r(n,t)x_r x_{t-r}$.*
(ii) *If the triangle $\{a(n,k)\}$ is double LC-positive, then the triangle $\{c(n,k)\}$ is LC-positive and $C_r(n,t) \geqslant A_r(n,t)y_{n-t+r}y_{n-r}$ for $t \leqslant n + r$.*

(iii) *If the triangle* $\{a(n,k)\}$ *is double LC-positive, then the triangle* $\{d(n,k)\}$ *is LC-positive and* $D_r(n,t) \geqslant A_r(n,t)x_r x_{t-r} y_{n-t+r} y_{n-r}$ *for* $t \leqslant n+r$.

Proof. Clearly, (iii) follows from (i) and (ii), so it suffices to prove (i) and (ii).

(i) Let $0 \leqslant t \leqslant 2n$. It is easy to see by definition that $b_k(n,t) = a_k(n,t)x_k x_{t-k}$ for $0 \leqslant k \leqslant \lfloor t/2 \rfloor$. Hence for $0 \leqslant r \leqslant \lfloor t/2 \rfloor$,

$$B_r(n,t) = \sum_{k=r}^{\lfloor t/2 \rfloor} b_k(n,t) = \sum_{k=r}^{\lfloor t/2 \rfloor} a_k(n,t)x_k x_{t-k}.$$

Now $\{a(n,k)\}$ is LC-positive and $x_0 x_t \leqslant x_1 x_{t-1} \leqslant x_2 x_{t-2} \leqslant \cdots$ by the log-concavity of $\{x_k\}$. From Lemma 2.1 it follows that

$$B_r(n,t) \geqslant x_r x_{t-r} \sum_{k=r}^{\lfloor t/2 \rfloor} a_k(n,t) = A_r(n,t)x_r x_{t-r} \geqslant 0.$$

So the triangle $\{b(n,k)\}$ is LC-positive.

(ii) Let $2r \leqslant t \leqslant 2n$. We need to prove $C_r(n,t) \geqslant 0$. For brevity, we do this only for the case t odd since the same technique is still valid for the case t even.

Let $t = 2s+1$. For $0 \leqslant k \leqslant s$, denote

$$\begin{aligned}
\alpha_k &= a(n,k)a(n,t-k), \\
\beta_k &= a(n-1,k)a(n+1,t-k), \\
\gamma_k &= a(n+1,k)a(n-1,t-k),
\end{aligned}$$

and $Y_k = y_{n-t+k} y_{n-k}$. Then

$$a_k(n,t) = 2\alpha_k - \beta_k - \gamma_k$$

and

$$c_k(n,t) = 2\alpha_k Y_k - \beta_k Y_{k+1} - \gamma_k Y_{k-1}$$

by definition. It follows that

$$\begin{aligned}
C_r(n,t) &= \sum_{k=r}^{s} (2\alpha_k Y_k - \beta_k Y_{k+1} - \gamma_k Y_{k-1}) \\
&= \sum_{k=r}^{s} (2\alpha_k - \beta_{k-1} - \gamma_{k+1}) Y_k + \beta_{r-1} Y_r - \gamma_r Y_{r-1},
\end{aligned}$$

where we use the fact that $Y_{s+1} = Y_s$ and $\gamma_{s+1} = \beta_s$. Note that $\{Y_k\}$ is nondecreasing by the log-concavity of $\{y_k\}$ and

$$\begin{aligned}
&2\alpha_k - \beta_{k-1} - \gamma_{k+1} \\
&\quad = 2a^*(n,n-k)a^*(n,n-t+k) - a^*(n-1,n-k)a^*(n+1,n-t+k) \\
&\qquad - a^*(n+1,n-k)a^*(n-1,n-t+k) = a^*_{n-t+k}(n,2n-t).
\end{aligned}$$

Hence by the LC-positivity of $\{a^*(n,k)\}$, we have

$$\begin{aligned} C_r(n,t) &= \sum_{j=n-t+r}^{\lfloor (2n-t)/2\rfloor} a_j^*(n,2n-t)Y_{j-n+t} + \beta_{r-1}Y_r - \gamma_r Y_{r-1} \\ &\geqslant Y_r \sum_{j=n-t+r}^{\lfloor (2n-t)/2\rfloor} a_j^*(n,2n-t) + \beta_{r-1}Y_r - \gamma_r Y_{r-1} \\ &= Y_r \sum_{k=r}^{s}(2\alpha_k - \beta_{k-1} - \gamma_{k+1}) + \beta_{r-1}Y_r - \gamma_r Y_{r-1} \\ &= Y_r \sum_{k=r}^{s}(2\alpha_k - \beta_k - \gamma_k) + \gamma_r(Y_r - Y_{r-1}) \\ &= A_r(n,t)Y_r + \gamma_r(Y_r - Y_{r-1}). \end{aligned} \tag{6}$$

Thus $C_r(n,t) \geqslant A_r(n,t)y_{n-t+r}y_{n-r} \geqslant 0$ since $Y_r \geqslant Y_{r-1}$, as desired. □

Now we present the second main result of this paper.

Theorem 2.5. *The double LC-positive triangles are double PLC.*

Proof. Let the triangle $\{a(n,k)\}$ be double LC-positive. Suppose that both $\{x_k\}$ and $\{y_k\}$ are log-concave. Then the triangle $\{a(n,k)x_k y_{n-k}\}$ is LC-positive by Proposition 2.4(iii) and is therefore PLC by Theorem 2.3. Thus the row-sum sequence

$$z_n = \sum_{k=0}^{n} a(n,k)x_k y_{n-k}, \quad n = 0,1,2,\ldots,$$

is log-concave. In other words, the triangle $\{a(n,k)\}$ is double PLC. □

We can give some more practicable conditions that imply the LC-positivity. We have seen that Lemma 2.1, especially Condition (a), plays a key role in the proof of the LC-positivity of Proposition 2.4. Clearly, Condition (a) is implied by the following two conditions:

(a1) $a_0, a_1, \ldots, a_s$ changes from nonpositive to nonnegative values;
(a2) $\sum_{k=0}^{s} a_k \geqslant 0$.

These two conditions are easier to check than Condition (a). For example, Condition (a1) can be obtained by showing that the sequence $\{a_k\}$ is nondecreasing and eventually nonnegative. In this case the analytic tools are often effective. On the other hand, Condition (a2) is just the simplest one of inequalities in Condition (a) and the methods of generating functions will be useful (see [15] for details). By Lemma 2.2, $\{a(n,k)\}$ is LC-positive if and only if the inequality $\sum_{k=r}^{\lfloor t/2\rfloor} a_k(n,t) \geqslant 0$ for all $2r \leqslant t \leqslant 2n$, so the following corollary is immediate.

Corollary 2.6. *Suppose that the following two conditions hold*:

(A) *There exists an index $m = m(n,t)$ such that $a_k(n,t) < 0$ for $k < m$ and $a_k(n,t) \geqslant 0$ for $k \geqslant m$;*
(B) *The sequence $\{\mathcal{A}_0(n;q)\}_{n\geqslant 0}$ is q-log-concave.*

Then the triangle $\{a(n,k)\}$ is LC-positive and therefore PLC.

Corollary 2.7. *Suppose that the triangle $\{a(n,k)\}$ satisfies Conditions* (A) *and* (B) *in Corollary* 2.6 *and $\{a^*(n,k)\}$ satisfies Condition* (A). *Then $\{a(n,k)\}$ is double LC-positive and therefore double PLC.*

Proof. Clearly, it suffices to show that $\{\mathcal{A}_0^*(n;q)\}$ is q-log-concave. We have

$$\mathcal{A}_0^*(n;q)=\sum_{k=0}^{n}a(n,n-k)q^k=\sum_{k=0}^{n}a(n,k)q^{n-k}=q^n\mathcal{A}_0\big(n;q^{-1}\big).$$

It follows that

$$\begin{aligned}&\mathcal{A}_0^{*2}(n;q)-\mathcal{A}_0^*(n-1;q)\mathcal{A}_0^*(n+1;q)\\&\quad=q^{2n}\big[\mathcal{A}_0^2\big(n;q^{-1}\big)-\mathcal{A}_0\big(n-1;q^{-1}\big)\mathcal{A}_0\big(n+1;q^{-1}\big)\big],\end{aligned}$$

which has nonnegative coefficients by the q-log-concavity of $\{\mathcal{A}_0(n;q)\}$, as desired. □

3. Applications

In this section we give some examples of PLC and double PLC triangles by showing their LC-positivity. In particular, we give a generalization of a result of Liggett that is used to prove a conjecture of Pemantle on characteristics of negative dependence.

Denote by $\mathfrak{S}$ the set of sequences $\{u_k\}_{k\in\mathbb{Z}}$ of nonnegative numbers. Given two nonnegative numbers λ and μ, define the linear operator $\mathcal{L}=\mathcal{L}[\lambda,\mu]$ on $\mathfrak{S}$ by

$$\mathcal{L}(u_k)=\lambda u_k+\mu u_{k-1},\quad k\in\mathbb{Z}.$$

For $n\geqslant 2$, define $\mathcal{L}^n=\mathcal{L}(\mathcal{L}^{n-1})$ by induction. It is convenient to view $\mathcal{L}^0$ as the identity operator. Let the sequence $\{u_k\}_{k\in\mathbb{Z}}$ be log-concave. Then the sequence $\{\mathcal{L}(u_k)\}_{k\in\mathbb{Z}}$ is also log-concave since

$$\begin{aligned}&\big[\mathcal{L}(u_k)\big]^2-\mathcal{L}(u_{k-1})\mathcal{L}(u_{k+1})\\&\quad=(\lambda u_k+\mu u_{k-1})^2-(\lambda u_{k-1}+\mu u_{k-2})(\lambda u_{k+1}+\mu u_k)\\&\quad=\lambda^2\big(u_k^2-u_{k-1}u_{k+1}\big)+\lambda\mu(u_{k-1}u_k-u_{k-2}u_{k+1})+\mu^2\big(u_{k-1}^2-u_{k-2}u_k\big).\end{aligned}$$

Thus we can conclude by induction that the sequence $\{\mathcal{L}^n(u_k)\}_{k\in\mathbb{Z}}$ is log-concave for each $n\geqslant 0$.

Theorem 3.1. *Given two nonnegative numbers λ,μ and a log-concave sequence $\{u_k\}$, define $a(n,k)=\mathcal{L}^n[\lambda,\mu](u_k)$ for $0\leqslant k\leqslant n$. Then the triangle $\{a(n,k)\}_{0\leqslant k\leqslant n}$ is double LC-positive and therefore double PLC.*

Proof. Denote $a_k=\mathcal{L}^{n-1}[\lambda,\mu](u_k)$ for $k\in\mathbb{Z}$. Then the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is log-concave and $\mathcal{A}_r(n-1;q)=\sum_{k=r}^{n-1}a_kq^k$. We have

$$\begin{aligned}\mathcal{A}_r(n;q)&=\sum_{k=r}^{n}(\lambda a_k+\mu a_{k-1})q^k=\lambda\sum_{k=r}^{n}a_kq^k+\mu\sum_{k=r}^{n}a_{k-1}q^k\\&=(\lambda+\mu q)\mathcal{A}_r(n-1;q)+\lambda a_nq^n+\mu a_{r-1}q^r,\end{aligned}$$

and similarly,

$$\mathcal{A}_r(n+1;q) = (\lambda+\mu q)\mathcal{A}_r(n;q) + \lambda(\lambda a_{n+1}+\mu a_n)q^{n+1} + \mu(\lambda a_{r-1}+\mu a_{r-2})q^r.$$

It follows that

$$\begin{aligned}
&\mathcal{A}_r^2(n;q) - \mathcal{A}_r(n-1;q)\mathcal{A}_r(n+1;q)\\
&\quad = \mathcal{A}_r(n;q)\big[(\lambda+\mu q)\mathcal{A}_r(n-1;q) + \lambda a_n q^n + \mu a_{r-1}q^r\big]\\
&\qquad - \mathcal{A}_r(n-1;q)\big[(\lambda+\mu q)\mathcal{A}_r(n;q) + \lambda(\lambda a_{n+1}+\mu a_n)q^{n+1} + \mu(\lambda a_{r-1}+\mu a_{r-2})q^r\big]\\
&\quad = \big(\lambda a_n q^n + \mu a_{r-1}q^r\big)\mathcal{A}_r(n;q)\\
&\qquad - \big[\lambda(\lambda a_{n+1}+\mu a_n)q^{n+1} + \mu(\lambda a_{r-1}+\mu a_{r-2})q^r\big]\mathcal{A}_r(n-1;q)\\
&\quad = \lambda\sum_{k=r}^{n}(\lambda a_k+\mu a_{k-1})a_n q^{n+k} + \mu\sum_{k=r}^{n} a_{r-1}(\lambda a_k+\mu a_{k-1})q^{k+r}\\
&\qquad - \lambda\sum_{k=r}^{n-1} a_k(\lambda a_{n+1}+\mu a_n)q^{n+k+1} - \mu\sum_{k=r}^{n-1}(\lambda a_{r-1}+\mu a_{r-2})a_k q^{k+r}\\
&\quad = \lambda^2\sum_{k=r+1}^{n}(a_k a_n - a_{k-1}a_{n+1})q^{n+k} + \mu^2\sum_{k=r}^{n}(a_{r-1}a_{k-1} - a_{r-2}a_k)q^{k+r}\\
&\qquad + \big(\lambda^2 a_r + 2\lambda\mu a_{r-1} + \mu^2 a_{r-2}\big)a_n q^{n+r},
\end{aligned} \tag{7}$$

which has nonnegative coefficients by the log-concavity of $\{a_k\}$. Hence the triangle $\{a(n,k)\}_{0\leqslant k\leqslant n}$ is LC-positive.

On the other hand, let $u_k^* = u_{-k}$ for $k \in \mathbb{Z}$. Then the sequence $\{u_k^*\}_{k\in\mathbb{Z}}$ is log-concave and $a^*(n,k) = \mathcal{L}^n[\mu,\lambda](u_k^*)$. Thus the triangle $\{a^*(n,k)\}_{0\leqslant k\leqslant n}$ is also LC-positive, and the triangle $\{a(n,k)\}_{0\leqslant k\leqslant n}$ is therefore double LC-positive. □

Remark 3.2. Let the triangle $\{a(n,k)\}$ be the same as Theorem 3.1. Then by (5) and (7), the inequality

$$A_r(n,t) \geqslant (a_{r-1}a_{t-r-1} - a_{r-2}a_{t-r})\mu^2$$

holds for $t \leqslant n+r$ (the equality holds when $t < n+r$). We will use this inequality repeatedly in the proof of Theorem 3.10.

Taking $\lambda = \mu = 1/2$ and $u_k \equiv 1$ in Theorem 3.1 leads to the following well-known result.

Corollary 3.3. *If the sequences $\{x_n\}$ and $\{y_n\}$ are log-concave, then so is their ordinary convolution $z_n = \sum_{k=0}^{n} x_k y_{n-k}$, $n = 0,1,2,\ldots.$*

Corollary 3.4. *Let a, b be two nonnegative integers and $a \geqslant b$. If the sequences $\{x_n\}$ and $\{y_n\}$ are log-concave, then so is the sequence*

$$z_n = \sum_{k=0}^{n}\binom{a+n}{b+k}x_k y_{n-k}, \quad n = 0,1,2,\ldots.$$

Proof. The statement follows by taking $\lambda = \mu = 1$ and $u_k = \binom{a}{b+k}$ in Theorem 3.1. (We remind the reader that $\binom{n}{k} = 0$ unless $0 \leqslant k \leqslant n$.) □

A special interesting case of Corollary 3.4 is the following.

Corollary 3.5. *If the sequences* $\{x_n\}$ *and* $\{y_n\}$ *are log-concave, then so is their binomial convolution* $z_n = \sum_{k=0}^{n} \binom{n}{k} x_k y_{n-k}$, $n = 0, 1, 2, \ldots.$

Remark 3.6. Corollaries 3.3–3.5 can also be followed directly from Theorem 2.5 by showing the double LC-positivity of the associated triangles. Actually, the double LC-positivity of the constant triangle and the Pascal triangle have been shown in Examples 1.2 and 1.3, respectively. In [14], we showed the LC-positivity of the triangle $a(n,k) = \binom{a+n}{b+k}$ for $0 \leqslant k \leqslant n$ by showing that Conditions (A) and (B) in Corollary 2.6 are satisfied. This result can also be followed by the same technique used in Example 1.3. Note that

$$a^*(n,k) = \binom{a+n}{b+(n-k)} = \binom{a+n}{(a-b)+k}.$$

Hence $a^*(n,k)$ is also LC-positive. Thus the triangle $\{a(n,k)\}$ is double LC-positive.

It is easy to extend Corollary 3.5 by induction to several log-concave sequences.

Corollary 3.7. *If* ℓ *sequences* $\{x_k^{(1)}\}, \{x_k^{(2)}\}, \ldots, \{x_k^{(\ell)}\}$ *are all log-concave, then so is the sequence*

$$X_n = \sum \binom{n}{k_1, k_2, \ldots, k_\ell} x_{k_1}^{(1)} x_{k_2}^{(2)} \cdots x_{k_\ell}^{(\ell)}, \quad n = 0, 1, 2, \ldots,$$

where the sum is over all nonnegative integers $k_1, \ldots, k_\ell$ *such that* $k_1 + k_2 + \cdots + k_\ell = n$.

The following theorem is in a sense "dual" to Theorem 3.1.

Theorem 3.8. *Let* α, β *be two nonnegative numbers and* $\{a(n,k)\}_{0 \leqslant k \leqslant n}$ *a triangle of nonnegative numbers. Suppose that each row of* $\{a(n,k)\}$ *is log-concave and satisfies the recurrence relation*

$$a(n,k) = \alpha a(n+1,k) + \beta a(n+1,k+1), \quad k = 0, 1, \ldots, n. \tag{8}$$

Then the triangle $\{a(n,k)\}$ *is double LC-positive and therefore double PLC.*

Proof. Denote $a(n+1,k) = v_k$ for $0 \leqslant k \leqslant n+1$. Then the sequence $\{v_k\}$ is log-concave and $\mathcal{A}_r(n+1;q) = \sum_{k=r}^{n+1} v_k q^k$. By the recurrence relation (8) we have

$$\mathcal{A}_r(n;q) = \sum_{k=r}^{n} (\alpha v_k + \beta v_{k+1}) q^k = \left(\alpha + \beta q^{-1}\right) \mathcal{A}_r(n+1;q) - \alpha v_{n+1} q^{n+1} - \beta v_r q^{r-1},$$

and similarly,

$$\mathcal{A}_r(n-1;q) = \left(\alpha + \beta q^{-1}\right) \mathcal{A}_r(n;q) - \alpha(\alpha v_n + \beta v_{n+1}) q^n - \beta(\alpha v_r + \beta v_{r+1}) q^{r-1}.$$

It follows that

$$
\begin{aligned}
&\mathcal{A}_r^2(n;q)-\mathcal{A}_r(n+1;q)\mathcal{A}_r(n-1;q)\\
&\quad=\mathcal{A}_r(n;q)\big[\big(\alpha+\beta q^{-1}\big)\mathcal{A}_r(n+1;q)-\alpha v_{n+1}q^{n+1}-\beta v_r q^{r-1}\big]\\
&\qquad-\mathcal{A}_r(n+1;q)\big[\big(\alpha+\beta q^{-1}\big)\mathcal{A}_r(n;q)-\alpha(\alpha v_n+\beta v_{n+1})q^n-\beta(\alpha v_r+\beta v_{r+1})q^{r-1}\big]\\
&\quad=\big[\alpha(\alpha v_n+\beta v_{n+1})q^n+\beta(\alpha v_r+\beta v_{r+1})q^{r-1}\big]\mathcal{A}_r(n+1;q)\\
&\qquad-\big(\alpha v_{n+1}q^{n+1}+\beta v_r q^{r-1}\big)\mathcal{A}_r(n;q)\\
&\quad=\alpha^2\sum_{k=r+1}^{n}(v_k v_n-v_{k-1}v_{n+1})q^{n+k}+\beta^2\sum_{k=r}^{n}(v_{r+1}v_{k+1}-v_r v_{k+2})q^{r+k}\\
&\qquad+v_r\big(\alpha^2 v_n+2\alpha\beta v_{n+1}+\beta^2 v_{n+2}\big)q^{n+r},
\end{aligned}
$$

which has nonnegative coefficients by the log-concavity of $\{v_k\}$. So the triangle $\{a(n,k)\}$ is LC-positive.

Clearly, the reciprocal triangle $\{a^*(n,k)\}$ possesses the same property as $\{a(n,k)\}$ does. Hence $\{a^*(n,k)\}$ is also LC-positive. Thus the triangle $\{a(n,k)\}$ is double LC-positive. □

In Theorem 3.8, taking $\alpha=\beta=1/2$ and $a(n,k)\equiv 1$ for $0\leqslant k\leqslant n$ leads to Corollary 3.3; and taking $\alpha=\beta=1$ and $a(n,k)=\binom{a-n}{b-k}$ for $0\leqslant k\leqslant n$ leads to the following.

Corollary 3.9. *Let $a,b\in\mathbb{N}$ and $a\geqslant b$. If the sequences $\{x_k\}$ and $\{y_k\}$ are log-concave, then so is the sequence*

$$z_n=\sum_{k=0}^{n}\binom{a-n}{b-k}x_k y_{n-k},\quad n=0,1,2,\ldots.$$

In what follows we generalize a result of Liggett. Let $\{x_k\}_{k\geqslant 0}$ be a sequence of nonnegative numbers and with no internal zeros. Following Pemantle [9] and Liggett [7], the sequence is *ultra-log-concave of order m* (**ULC(*m*)**) if $x_k=0$ for $k>m$ and the sequence $\{x_k/\binom{m}{k}\}_{k=0}^{m}$ is log-concave. The sequence $\{x_k\}_{k\geqslant 0}$ is *ULC*(∞) if the sequence $\{k!x_k\}_{k\geqslant 0}$ is log-concave. It is clear from definitions that ULC(m) implies ULC(ℓ) for $0\leqslant m\leqslant\ell\leqslant\infty$. The concept of ultra-log-concavity is closely related to negatively dependent Bernoulli sequences (see [9] for details). Pemantle speculates that ultra-log-concavity is characteristic of negative dependence in the exchangeable case. This leads to a conjecture that the ordinary convolution of a ULC(m) sequence and a ULC(ℓ) sequence is ULC($m+\ell$) where m and ℓ may be infinity [9, Conjecture 7]. It is not difficult to see that the conjecture actually consists of two parts:

(i) The Pascal triangle $\{\binom{n}{k}\}$ is double PLC;
(ii) The triangle $\{\binom{n}{k}\binom{a-n}{b-k}\}$ is double PLC.

Liggett verified the conjecture by establishing the following stronger result.

Liggett Theorem. [7] *Given three log-concave sequences $\{v_k\}$, $\{x_k\}$ and $\{y_k\}$, let*

$$z_{n-1}=\sum_{k=0}^{n-1}\binom{n-1}{k}(v_k+2v_{k+1}+v_{k+2})x_k y_{n-1-k},$$

$$z_n = \sum_{k=0}^{n} \binom{n}{k} (v_k + v_{k+1}) x_k y_{n-k},$$

$$z_{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} v_k x_k y_{n+1-k}.$$

Then $z_{n-1} z_{n+1} \leqslant z_n^2$.

Liggett's proof for his theorem, essentially using the double LC-positivity of the Pascal triangle, is not simple. To see his idea more clearly, we show the following more general result.

Theorem 3.10. *Given four nonnegative numbers* $\alpha, \beta, \lambda, \mu$ *and four log-concave sequences* $\{u_k\}_{k \in \mathbb{Z}}$, $\{v_k\}_{k \geqslant 0}$, $\{x_k\}_{k \geqslant 0}$ *and* $\{y_k\}_{k \geqslant 0}$, *let* $a(n,k) = \mathcal{L}^n[\lambda, \mu](u_k)$ *and*

$$z_{n-1} = \sum_{k=0}^{n-1} a(n-1,k) \bigl(\alpha^2 v_k + 2\alpha\beta v_{k+1} + \beta^2 v_{k+2}\bigr) x_k y_{n-1-k},$$

$$z_n = \sum_{k=0}^{n} a(n,k)(\alpha v_k + \beta v_{k+1}) x_k y_{n-k},$$

$$z_{n+1} = \sum_{k=0}^{n+1} a(n+1,k) v_k x_k y_{n+1-k}.$$

Then $z_{n-1} z_{n+1} \leqslant z_n^2$.

Proof. Clearly, $z_n^2 - z_{n-1} z_{n+1}$ can be viewed as a quadratic form in $n+2$ variables $v_0, v_1, \ldots, v_{n+1}$. Let

$$z_n^2 - z_{n-1} z_{n+1} = \sum_{t=0}^{2(n+1)} \sum_{k=0}^{\lfloor t/2 \rfloor} e_k(n,t) v_k v_{t-k}.$$

Then we need to show that $\sum_{k=r}^{\lfloor t/2 \rfloor} e_k(n,t) \geqslant 0$ for $2r \leqslant t \leqslant 2(n+1)$. For brevity, we do this only for the case t odd. Let $t = 2s+1$.

Define $d(n,k) = a(n,k) x_k y_{n-k}$ for $0 \leqslant k \leqslant n$. For convenience, set $x_k = y_k = 0$ for $k < 0$ and $d(n,k) = 0$ for $k < 0$ or $k > n$. The triangle $\{a(n,k)\}$ is double LC-positive by Theorem 3.1, and so is the triangle $\{d(n,k)\}$ by Proposition 2.4. Rewrite

$$z_{n-1} = \sum_{k=0}^{n+1} \bigl[\alpha^2 d(n-1,k) + 2\alpha\beta d(n-1,k-1) + \beta^2 d(n-1,k-2)\bigr] v_k,$$

$$z_n = \sum_{k=0}^{n+1} \bigl[\alpha d(n,k) + \beta d(n,k-1)\bigr] v_k,$$

$$z_{n+1} = \sum_{k=0}^{n+1} d(n+1,k) v_k.$$

Then

$$\begin{aligned}
e_k(n,t) &= 2\big[\alpha d(n,k)+\beta d(n,k-1)\big]\big[\alpha d(n,t-k)+\beta d(n,t-k-1)\big] \\
&\quad - \big[\alpha^2 d(n-1,k)+2\alpha\beta d(n-1,k-1)+\beta^2 d(n-1,k-2)\big]d(n+1,t-k) \\
&\quad - d(n+1,k)\big[\alpha^2 d(n-1,t-k)+2\alpha\beta d(n-1,t-k-1) \\
&\quad + \beta^2 d(n-1,t-k-2)\big] \\
&= \alpha^2 P_k + 2\alpha\beta Q_k + \beta^2 R_k,
\end{aligned}$$

where

$$\begin{aligned}
P_k &= 2d(n,k)d(n,t-k) - d(n-1,k)d(n+1,t-k) - d(n+1,k)d(n-1,t-k), \\
Q_k &= d(n,k)d(n,t-k-1) + d(n,k-1)d(n,t-k) - d(n-1,k-1)d(n+1,t-k) \\
&\quad - d(n+1,k)d(n-1,t-k-1), \\
R_k &= 2d(n,k-1)d(n,t-k-1) - d(n-1,k-2)d(n+1,t-k) \\
&\quad - d(n+1,k)d(n-1,t-k-2).
\end{aligned}$$

Thus it suffices to show the inequality

$$\alpha^2 \sum_{k=r}^{s} P_k + 2\alpha\beta \sum_{k=r}^{s} Q_k + \beta^2 \sum_{k=r}^{s} R_k \geqslant 0. \tag{9}$$

Note that $P_k = d_k(n,t)$ and $R_k = d^*_{n-t+k+1}(n, 2n-t+2)$. Hence both

$$\sum_{k=r}^{s} P_k = D_r(n,t) \tag{10}$$

and

$$\sum_{k=r}^{s} R_k = D^*_{n-t+r+1}(n, 2n-t+2) \tag{11}$$

are nonnegative by the double LC-positivity of the triangle $\{d(n,k)\}$. Also,

$$\begin{aligned}
\sum_{k=r}^{s} Q_k &= \sum_{k=r}^{s}\big[d(n,k)d(n,t-k-1) + d(n,k-1)d(n,t-k) \\
&\quad - d(n-1,k-1)d(n+1,t-k) - d(n+1,k)d(n-1,t-k-1)\big] \\
&= \big[d^2(n,s) - d(n-1,s)d(n+1,s)\big] + \sum_{k=r-1}^{s-1}\big[2d(n,k)d(n,t-1-k) \\
&\quad - d(n-1,k)d(n+1,t-1-k) - d(n+1,k)d(n-1,t-1-k)\big] \\
&\quad + \big[d(n+1,r-1)d(n-1,t-r) - d(n,r-1)d(n,t-r)\big] \\
&= D_{r-1}(n,t-1) + \big[d(n+1,r-1)d(n-1,t-r) - d(n,r-1)d(n,t-r)\big].
\end{aligned} \tag{12}$$

Assume that $r=0$ or $t>n+r$. Then $\sum_{k=r}^{s} Q_k = D_{r-1}(n,t-1) \geqslant 0$. Thus inequality (9) is trivial. So let $r \geqslant 1$ and $t \leqslant n+r$.

If we can show that there exists a nonnegative number $E = E(n,t,r)$ such that

$$\begin{cases} \sum_{k=r}^{s} P_k \geqslant \mu^2 E x_r x_{t-r} y_{n-t+r} y_{n-r}, \\ \sum_{k=r}^{s} Q_k \geqslant -\lambda\mu E x_{r-1} x_{t-r} y_{n-t+r} y_{n-r+1}, \\ \sum_{k=r}^{s} R_k \geqslant \lambda^2 E x_{r-1} x_{t-r-1} y_{n-t+r+1} y_{n-r+1}, \end{cases} \tag{13}$$

then the arithmetic–geometric mean inequality and the log-concavity of $\{x_k\}$ and $\{y_k\}$ will give

$$\alpha^2 \sum_{k=r}^{s} P_k + \beta^2 \sum_{k=r}^{s} R_k \geqslant -2\alpha\beta \sum_{k=r}^{s} Q_k,$$

the required inequality. So, to prove (9), it suffices to prove (13).

We use Proposition 2.4 to estimate the lower bounds for $\sum_{k=r}^{s} P_k$, $\sum_{k=r}^{s} Q_k$ and $\sum_{k=r}^{s} R_k$. From (10) and Proposition 2.4(iii) it is immediate that

$$\sum_{k=r}^{s} P_k \geqslant A_r(n,t) x_r x_{t-r} y_{n-t+r} y_{n-r}. \tag{14}$$

Similarly, note that $d^*(n,k) = a^*(n,k) y_k x_{n-k}$, it follows from (11) and Proposition 2.4(iii) that

$$\sum_{k=r}^{s} R_k \geqslant A^*_{n-t+r+1}(n, 2n-t+2) y_{n-t+r+1} y_{n-r+1} x_{t-r-1} x_{r-1}. \tag{15}$$

To get an analogous lower bound for $\sum_{k=r}^{s} Q_k$ from (12), let $c(n,k) = a(n,k) y_{n-k}$. Then $d(n,k) = c(n,k) x_k$ and so

$$D_{r-1}(n,t-1) \geqslant C_{r-1}(n,t-1) x_{r-1} x_{t-r}$$

by Proposition 2.4(i). However,

$$\begin{aligned} C_{r-1}(n,t-1) \geqslant{}& A_{r-1}(n,t-1) y_{n-t+r} y_{n-r+1} \\ &+ a(n+1,r-1) a(n-1,t-r) (y_{n-t+r} y_{n-r+1} - y_{n-t+r-1} y_{n-r+2}) \end{aligned}$$

by inequality (6). Hence we have by (12)

$$\begin{aligned} \sum_{k=r}^{s} Q_k \geqslant{}& \big[A_{r-1}(n,t-1) y_{n-t+r} y_{n-r+1} \\ &+ a(n+1,r-1) a(n-1,t-r) (y_{n-t+r} y_{n-r+1} - y_{n-t+r-1} y_{n-r+2})\big] x_{r-1} x_{t-r} \\ &+ \big[a(n+1,r-1) x_{r-1} y_{n-r+2} a(n-1,t-r) x_{t-r} y_{n-t+r-1} \\ &- a(n,r-1) x_{r-1} y_{n-r+1} a(n,t-r) x_{t-r} y_{n-t+r}\big] \\ ={}& Q x_{r-1} x_{t-r} y_{n-t+r} y_{n-r+1}, \end{aligned} \tag{16}$$

where

$$Q = A_{r-1}(n,t-1) + a(n+1,r-1) a(n-1,t-r) - a(n,r-1) a(n,t-r). \tag{17}$$

It remains to show that three coefficients $A_r(n,t)$, $A^*_{n-t+r+1}(n, 2n-t+2)$ and Q in inequalities (14)–(16) have the lower bounds of the forms in (13). We do this by Remark 3.2.

Denote $a_k = \mathcal{L}^{n-1}[\lambda, \mu](u_k)$. It follows from Remark 3.2 that

$$A_r(n,t) \geqslant (a_{r-1}a_{t-r-1} - a_{r-2}a_{t-r})\mu^2$$

and that

$$\begin{aligned} Q &\geqslant (a_{r-2}a_{t-r-1} - a_{r-3}a_{t-r})\mu^2 + \big(\lambda^2 a_{r-1} + 2\lambda\mu a_{r-2} + \mu^2 a_{r-3}\big)a_{t-r} \\ &\quad - (\lambda a_{r-1} + \mu a_{r-2})(\lambda a_{t-r} + \mu a_{t-r-1}) \\ &= -(a_{r-1}a_{t-r-1} - a_{r-2}a_{t-r})\lambda\mu \end{aligned}$$

by (17). Also, note that $a^*(n,k) = \mathcal{L}^n[\mu, \lambda](u_{-k})$. Again by Remark 3.2,

$$\begin{aligned} A^*_{n-t+r+1}(n, 2n-t+2) &\geqslant \big[a^*(n-1, n-t+r)a^*(n-1, n-r) \\ &\quad - a^*(n-1, n-t+r-1)a^*(n-1, n-r+1)\big]\lambda^2 \\ &= (a_{r-1}a_{t-r-1} - a_{r-2}a_{t-r})\lambda^2. \end{aligned}$$

Finally, recall that the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is log-concave, so for $r \leqslant \lfloor t/2 \rfloor$,

$$E = a_{r-1}a_{t-r-1} - a_{r-2}a_{t-r} \geqslant 0,$$

as required. This completes our proof. □

Remark 3.11. Let $\{a(n,k)\}$ and $\{a'(n,k)\}$ be the double LC-positive triangles appearing in Theorems 3.1 and 3.8, respectively. Although the triangle $\{a(n,k)a'(n,k)\}$ is not double LC-positive in general, it is double PLC by Theorem 3.10.

4. Concluding remarks

In this paper we provide some sufficient conditions for linear and bilinear transformations preserving the log-concavity. As shown in Remark 1.1(iii), the LC-positivity is "almost" necessary for the PLC property. It is a challenge to give a necessary and sufficient condition for the PLC property. On the other hand, we believe that the techniques developed in the present paper can be used to deal with various combinatorial inequalities. For example, it is possible that the log-convexity problems can be treated with the same approach.

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