Journal of Combinatorial Theory, Series A 116 (2009) 314–333

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

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Factorisation of Littlewood–Richardson coefficients

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ARTICLE INFO

Article history:
Received 12 July 2007
Available online 9 August 2008

Keywords:
Littlewood–Richardson coefficients
Hive model
Horn inequalities
Puzzles
Factorisation
LR-polynomials

ABSTRACT

The hive model is used to show that the saturation of any essential Horn inequality leads to the factorisation of Littlewood–Richardson coefficients. The proof is based on the use of combinatorial objects known as puzzles. These are shown not only to account for the origin of Horn inequalities, but also to determine the constraints on hives that lead to factorisation. Defining a primitive Littlewood–Richardson coefficient to be one for which all essential Horn inequalities are strict, it is shown that every Littlewood–Richardson coefficient can be expressed as a product of primitive coefficients. Precisely the same result is shown to apply to the polynomials defined by stretched Littlewood–Richardson coefficients.

1. Introduction

Although Littlewood–Richardson coefficients, $c^{\nu}_{\lambda\mu}$, have been well studied, it is only recently that a conjecture [12] has been made regarding their possible factorisation. They are indexed by partitions λ, μ and ν. They count the number of Littlewood–Richardson tableaux [15] of skew shape ν/λ and weight μ. They are therefore non-negative integers. It is a non-trivial matter to determine whether or not $c^{\nu}_{\lambda\mu}$ is non-zero. However, it turns out [4,9,10,14] that this is the case if and only if $|\lambda| + |\mu| = |\nu|$ and certain partial sums of the parts of λ, μ and ν satisfy what are known as Horn inequalities [8]. Only a subset of these Horn inequalities is essential. It is proved here that a non-zero Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}$ can be expressed as a product of Littlewood–Richardson coefficients if λ, μ and ν are such that any essential Horn inequality is saturated.

Our approach is based on the use of a hive model [3,4,13] which allows Littlewood–Richardson coefficients to be evaluated through the enumeration of integer points of certain rational polytopes. Before defining hives, puzzles and labyrinths, which are the combinatorial constructs to be used in

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0097-3165/$ – see front matter © 2008 Published by Elsevier Inc.
doi:10.1016/j.jcta.2008.06.005

this context, it is worth recalling some definitions and properties of Littlewood–Richardson coefficients.

Let n be a fixed positive integer, let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ be a vector of indeterminates, and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition of weight $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ and of length $\ell(\lambda) \leqslant n$. Thus $\lambda_k \in \mathbb{Z}$ for $k = 1, 2, \dots, n$, with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{\ell(\lambda)} > 0$ and $\lambda_k = 0$ for $k > \ell(\lambda)$. The Schur functions $s_\lambda(\mathbf{x})$ specified by partitions λ with $\ell(\lambda) \leqslant n$ form a $\mathbb{Z}$-basis of the ring Λ_n of symmetric polynomials in $x_1, x_2, \dots, x_n$ [16].

Definition 1.1. Let λ, μ and ν be partitions of lengths $\ell(\lambda), \ell(\mu), \ell(\nu) \leqslant n$. Then the Littlewood–Richardson coefficient $c^\nu_{\lambda\mu}$ is the coefficient of $s_\nu(\mathbf{x})$ in the expansion of the Schur function product $s_\lambda(\mathbf{x})\, s_\mu(\mathbf{x})$, that is:

$$s_\lambda(\mathbf{x})\ s_\mu(\mathbf{x}) = \sum_\nu c^\nu_{\lambda\mu}\ s_\nu(\mathbf{x}). \tag{1.1}$$

To specify the necessary and sufficient conditions on λ, μ and ν for $c^\nu_{\lambda\mu}$ to be non-zero it is convenient to introduce the notion of partial sums of the parts of a partition and some other notational devices.

Let n be a fixed positive integer and $N = \{1, 2, \dots, n\}$. Then for any positive integer $r \leqslant n$ and any subset $I = \{i_1, i_2, \dots, i_r\}$ of N of cardinality $\#I = r$, the partial sum indexed by I of any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of length $\ell(\lambda) \leqslant n$ is defined to be

$$ps(\lambda)_I = \lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_r}. \tag{1.2}$$

Quite generally, given $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $I = \{i_1, i_2, \dots, i_r\}$ with $i_1 < i_2 < \cdots < i_r$ and $r \leqslant n$, then λ_I is the partition $(\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_r})$. In addition if we set $\tilde{I} = (i_r, \dots, i_2, i_1)$ and $\tilde{R} = (r, \dots, 2, 1)$ then $part(I) = \tilde{I} - \tilde{R}$ is a partition of length $\leqslant r$.

This notation allows us to define Horn triples as follows.

Definition 1.2. Let I, J and K be subsets of N, with $\#I = \#J = \#K = r$ for some r such that $0 < r < n$. Then the triple (I, J, K) is said to be a Horn triple if $c^{part(K)}_{part(I)part(J)} > 0$. Let N^n_r be the set of all such triples. Those triples (I, J, K) for which $c^{part(K)}_{part(I)part(J)} = 1$ are said to be essential. They constitute the subset R^n_r of N^n_r.

Building on a connection with the Horn conjecture [8] regarding eigenvalues of Hermitian matrices, the following theorem has been established by Klyachko [10], Belkale [1], Knutson and Tao [13] and Knutson, Tao and Woodward [14]. Comprehensive reviews of these developments have been provided by Fulton [9] and Zelevinsky [20].

Theorem 1.3. *Let λ, μ and ν be partitions of lengths $\ell(\lambda), \ell(\mu), \ell(\nu) \leqslant n$. Then $c^\nu_{\lambda\mu} > 0$ if and only if $|\nu| = |\lambda| + |\mu|$ and for all $r = 1, 2, \dots, n-1$,*

$$ps(\nu)_K \leqslant ps(\lambda)_I + ps(\mu)_J \tag{1.3}$$

for all triples $(I, J, K) \in N^n_r$. Moreover, not all of the Horn inequalities of type (1.3) *are essential, only those for which $(I, J, K) \in R^n_r$.*

Our main result is the following:

Theorem 1.4. *Let λ, μ and ν be partitions of lengths $\ell(\lambda), \ell(\mu), \ell(\nu) \leqslant n$ such that $c^\nu_{\lambda\mu} > 0$ and*

$$ps(\nu)_K = ps(\lambda)_I + ps(\mu)_J, \tag{1.4}$$

for some essential Horn triple $(I, J, K) \in R^n_r$ with $0 < r < n$. Then

$$c^\nu_{\lambda\mu} = c^{\nu_K}_{\lambda_I\mu_J}\, c^{\nu_{\overline{K}}}_{\lambda_{\overline{I}}\mu_{\overline{J}}} \tag{1.5}$$

where $\overline{I}, \overline{J}, \overline{K}$ are the complements of I, J, K, respectively, in $N = \{1, 2, \dots, n\}$.

2. The hive model

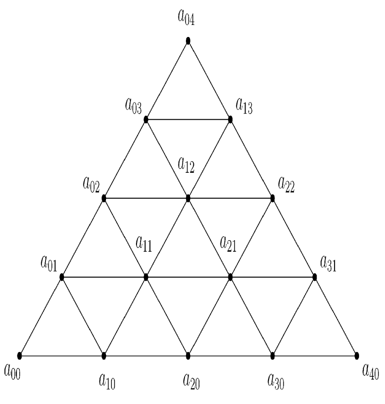
The expansion (1.1) may be effected by means of the Littlewood–Richardson rule [15]. This states that $c^{\nu}_{\lambda\mu}$ is the number of Littlewood–Richardson skew tableaux $T^{\nu/\lambda}$ of shape ν/λ and weight μ obtained by numbering the boxes of the skew Young diagram $F^{\nu/\lambda}$ with μ_i entries i for $i = 1, 2, \ldots, n$, in such a way that they (i) weakly increase across rows from left to right, (ii) strictly increase down columns, and (iii) satisfy the lattice permutation condition. This condition requires that in reading the entries of $T^{\nu/\lambda}$ from right to left across each row in turn from top to bottom, then at every stage the number of k's is greater than or equal to the number of $(k+1)$'s for all $k = 1, 2, \ldots, n-1$. For the sake of what is to follow, it is convenient to augment each Littlewood–Richardson skew tableau $T^{\nu/\lambda}$, with a tableau obtained by numbering all the boxes of the Young diagram F^{λ} with entries 0, as in [15, p. 122], thereby creating what we call an LR-tableau D of shape ν. The entries 0 contribute nothing to the weight of D, which remains that of the portion $T^{\nu/\lambda}$. By way of example, for $n = 3$, $\lambda = (3, 2, 0)$, $\mu = (2, 1, 0)$ and $\nu = (4, 3, 1)$, there exist just two LR-tableaux:

$$\begin{array}{|c|c|c|c|}\hline 0&0&0&1\\\hline 0&0&1\\\cline{1-3} 2\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 0&0&0&1\\\hline 0&0&2\\\cline{1-3} 1\\\cline{1-1}\end{array} \tag{2.1}$$

Hence $c^{431}_{32,21} = 2$.

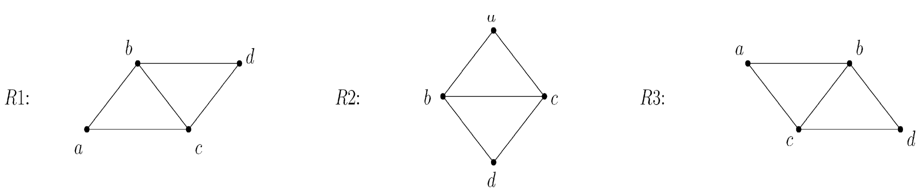
Although the Littlewood–Richardson rule provides a perfectly satisfactory combinatorial method of evaluating Littlewood–Richardson coefficients, it is more convenient here to make use of a *hive* model. This has its origin in the triangular arrays of Berenstein and Zelevinsky [3] that were used to specify individual contributions to Littlewood–Richardson coefficients. The model was then taken up by Knutson and Tao [13] in a manner described in an exposition by Buch [4].

An n-hive is an array of numbers a_{ij} with $0 \leqslant i, j, i+j \leqslant n$ labelling the vertices of an equilateral triangular graph satisfying certain *hive conditions*. For $n = 4$ their arrangement is as shown below:



Such an n-hive is said to be an integer hive if all of its entries are non-negative integers.

The hive conditions are a set of constraints on the vertex labels of each elementary rhombus consisting of a pair of neighbouring triangles. There are three distinct types of elementary rhombus, distinguished by their orientation:



In each case, with the vertex labelling as shown, the hive condition takes the form

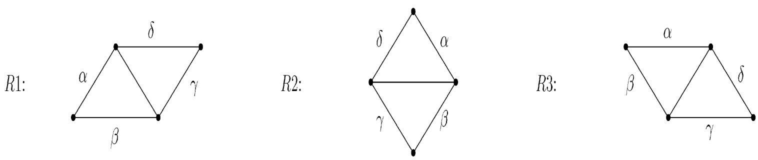
$$b + c \geqslant a + d. \tag{2.2}$$

In what follows we make use of edge labels more often than vertex labels. Each edge in the hive is labelled by means of the difference, $\epsilon = q - p$, between the labels, p and q, on the two vertices connected by this edge, with q always to the right of p. With this convention, in the case of the following two elementary triangles

we have $\sigma = q - p$, $\tau = r - q$ and $\rho = r - p$, so that automatically

$$\sigma + \tau = \rho. \tag{2.3}$$

With the following edge labelling of the three elementary rhombi



we have automatically in each case

$$\alpha + \delta = \beta + \gamma. \tag{2.4}$$

Then, in terms of edge labels, the hive conditions (2.2) take the form

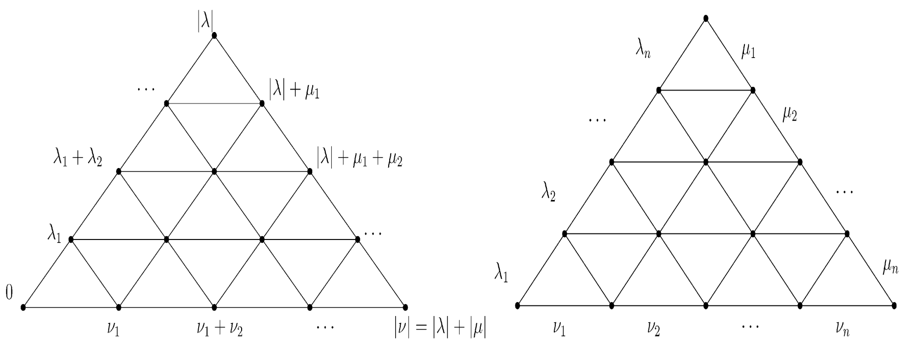
$$\alpha \geqslant \gamma \quad \text{and} \quad \beta \geqslant \delta, \tag{2.5}$$

where, of course, either one of the conditions $\alpha \geqslant \gamma$ or $\beta \geqslant \delta$ is sufficient to imply the other.

In order to enumerate contributions to Littlewood–Richardson coefficients, we require

Definition 2.1. An LR-hive is an integer n-hive, for some positive integer n, satisfying the hive conditions (2.2), or equivalently (2.5), for all its constituent rhombi of type R1, R2 and R3, with border labels determined by partitions λ, μ and ν, for which $\ell(\lambda), \ell(\mu), \ell(\nu) \leqslant n$ and $|\lambda| + |\mu| = |\nu|$, in such a way that $a_{00} = 0$, $a_{0,i} = ps(\lambda)_i$, $a_{j,n-j} = |\lambda| + ps(\mu)_j$ and $a_{k,0} = ps(\nu)_k$, for $i, j, k = 1, 2, \ldots, n$.

Schematically, in terms of either vertex or edge labels, we have



There exists a bijection between LR-tableaux D, of shape determined by ν/λ and of weight μ and LR-hives H, with border labels specified by λ, μ and ν. An illustration of this bijection is given below for a typical LR-tableau D in the case $n=3$, $\lambda=(3,2)$, $\mu=(2,1)$ and $\nu=(4,3,1)$.

$$D = \begin{array}{|c|c|c|c|}\hline 0&0&0&1\\ \hline 0&0&2\\ \cline{1-3} 1\\ \cline{1-1}\end{array} \iff H = \begin{array}{ccccccc} &&&5&&& \\ &&5&&7&& \\ &3&&6&&8& \\ 0&&4&&7&&8 \end{array} \tag{2.6}$$

For display purposes the hive edges have been omitted in the LR-hive H.

The bijection is such that for all (i,j) with $0 \leqslant i, j, i+j \leqslant n$ the entries of the LR-hive H are given by

$$a_{ij} = \#\text{ of entries} \leqslant i \text{ in first } (i+j) \text{ rows of } D, \tag{2.7}$$

with $a_{00}=0$ in the special case $i=j=0$. This forms the basis of a rather simple proof [12] of the following Proposition (see also [18] and the Appendix by Fulton in [4]):

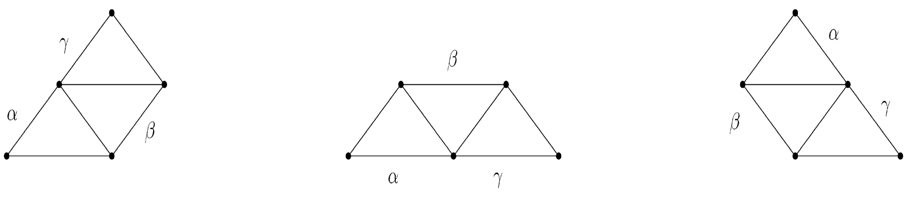
Proposition 2.2. *(See [4].) The Littlewood–Richardson coefficient $c_{\lambda\mu}^{\nu}$ is the number of LR-hives with border labels determined as above by λ, μ and ν.*

As an example of the application of this proposition, if $n=3$, $\lambda=(3,2,0)$, $\mu=(2,1,0)$ and $\nu=(4,3,1)$ then the corresponding LR-hives take the form

$$\begin{array}{ccccccc} &&&5&&& \\ &&5&&7&& \\ &3&&a&&8& \\ 0&&4&&7&&8 \end{array} \tag{2.8}$$

The LR-hive conditions for all the constituent rhombi then imply that $6 \leqslant a \leqslant 7$. Thus there are just two LR-hives with the given boundary labels, namely those with $a=7$ and $a=6$. It follows that $c_{32,21}^{431}=2$, in accordance with the result established earlier by enumeration of the LR-tableaux of (2.1).

It might be pointed out here, that when expressed in terms of edge labels, the hive conditions (2.5) for all constituent rhombi of types R1, R2 and R3 imply that in every LR-hive the edge labels along any line parallel to the north–west, north–east and southern boundaries of the hive are weakly decreasing in the north–east, south–east and easterly directions, respectively. This can be seen from the following 5-vertex sub-diagrams.



The edge conditions on the overlapping pairs of rhombi (R1, R2), (R1, R3) and (R2, R3) in the above diagrams give in each case $\alpha \geqslant \beta$ and $\beta \geqslant \gamma$, so that $\alpha \geqslant \gamma$ as claimed. This is of course consistent with the fact that edges of the three north–west, north–east and southern boundaries of each LR-hive are specified by partitions λ, μ and ν, respectively. Moreover, in each LR-hive the rhombus conditions (2.5), coupled with the fact that all the boundary edges are non-negative, implies that all edges of an LR-hive are non-negative.

3. Puzzles

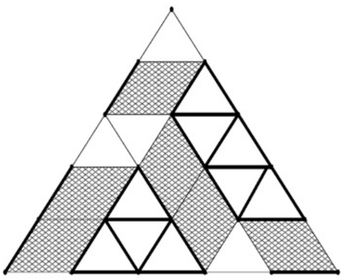
It was noted in the work of Berenstein and Zelevinsky [2] that some Kostka coefficients may factorise. Although rather easy to prove using semistandard tableaux, this factorisation property may also be established through the use of K-hives [11]. Analogous methods may be used to show that some Littlewood–Richardson coefficients may also factorise. Following the use of similar terminology in the case of Kostka coefficients, we propose the following

Definition 3.1. Let λ, μ and ν be partitions such that $|\nu| = |\lambda| + |\mu|$, $\ell(\lambda), \ell(\mu), \ell(\nu) \leqslant n$ and $c^{\nu}_{\lambda\mu} > 0$. Then the Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}$ is said to be *primitive* if $ps(\nu)_K < ps(\lambda)_I + ps(\mu)_J$ for all $(I, J, K) \in R^n_r$ and all $r = 1, 2, \ldots, n-1$. Conversely, $c^{\nu}_{\lambda\mu}$ is not primitive if there exists any $(I, J, K) \in R^n_r$ with $1 \leqslant r < n$ such that $ps(\nu)_K = ps(\lambda)_I + ps(\mu)_J$.

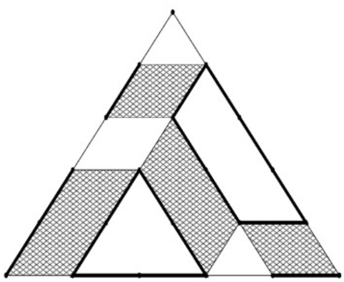
With this definition, we conjectured [12] that if $c^{\nu}_{\lambda\mu}$ is positive but not primitive, then $c^{\nu}_{\lambda\mu}$ factorises. This conjecture arose as a result of considering the consequences of saturating one or other of the essential Horn inequalities. Its origin can be exposed through a study of the properties of certain puzzles introduced by Knutson, Tao and Woodward [14].

A *puzzle* is a triangular diagram on a hive lattice built from three types of elementary piece: a thick-edged triangle, a thin-edged triangle and a shaded rhombus with its edges either thick or thin according as they are to the right or left, respectively of an acute angle of the rhombus, when viewed from its interior:

The puzzle is to put these together, oriented in any manner, so as to form a hive shape with all the edges matching. For example, one such puzzle takes the form shown below:



As pointed out by Danilov and Koshevoy [6], such a puzzle can be simplified, without loss of information, to give a *labyrinth*, or what we call a *hive plan*, by deleting all interior edges of the three types of region: corridors (also known [14] as rhombus regions) in the form of shaded parallelograms consisting of rhombi of just one type, either R1, or R2 or R3, and dark rooms (known as 0-regions) and light rooms (known as 1-regions) that are convex polygons consisting solely of just thick-edged triangles and just thin-edged triangles, respectively.



It is a remarkable fact [14] that for each positive integer $r < n$ and triple $(I, J, K) \in R^n_r$, there exists a unique puzzle, and correspondingly a unique labyrinth or hive plan of the above type. More precisely, we have the following:

Theorem 3.2. *(See [14].) The number of puzzles of side length n having thick edges on its north–west, north–east and southern boundaries specified by subsets I, J and K, respectively, of N, is given by* $c^{part(K)}_{part(I)part(J)}$.

This implies the following:

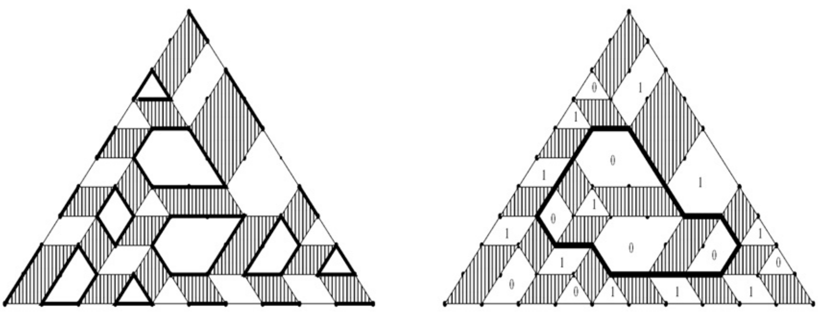
Corollary 3.3. *(See [14].) For each essential Horn triple $(I, J, K) \in R^n_r$ there exists just one puzzle of side length n having thick edges on its north–west, north–east and southern boundaries specified by subsets I, J and K, respectively, of N. Such a puzzle is said to be rigid.*

Still following Knutson, Tao and Woodward [14], a *gentle path* is a continuous path along the boundaries of the regions of the labyrinth corresponding to a puzzle, that is to say along the corridor walls, taken in such a direction that thick-edged regions, the dark rooms or 0-regions, are on the left and thin-edged regions, the light rooms or 1-regions, are on the right, with each angle of turn either 0 or $\pm\pi/3$. Such a path is a *gentle loop* if it forms a continuous closed path. With this terminology, we then have:

Theorem 3.4. *(See [14].) The labyrinth of a puzzle has no gentle loops if and only if the puzzle is rigid.*

In the example illustrated above, for which $n = 5$, $r = 3$, $I = \{1, 2, 4\}$, $J = \{2, 3, 4\}$, $K = \{2, 3, 5\}$, we have $part(I) = (1, 0, 0)$, $part(J) = (1, 1, 1)$ and $part(K) = (2, 1, 1)$. The fact that $(I, J, K) \in R^5_3$ then follows from the observation that $c^{211}_{1,111} = 1$. The puzzle is therefore rigid and its labyrinth possesses no gentle loops.

In contrast to this, in the case $n = 10$, $r = 5$, $I = \{1, 2, 4, 6, 8, \}$, $J = \{1, 3, 4, 7, 9\}$, $K = \{2, 4, 6, 8, 10\}$, we have $part(I) = (3, 2, 1, 0, 0)$, $part(J) = (4, 3, 1, 1, 0)$ and $part(K) = (5, 4, 3, 2, 1)$. In this case $(I, J, K) \notin R^5_3$ since $c^{54321}_{321,4311} = 6 > 1$. The puzzle is therefore not rigid and its labyrinth possesses gentle loops. This is illustrated below, with the corresponding labyrinth on the left and one example of a gentle loop shown on the right, with the loop to be taken anticlockwise.

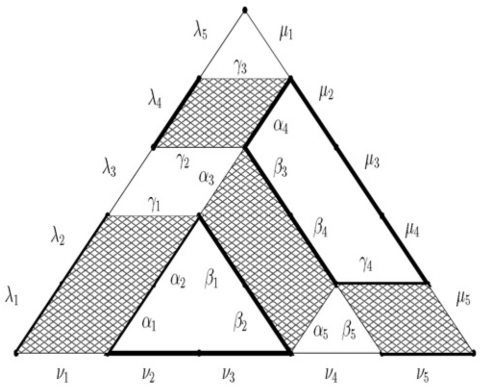


In order to display the loop in the second diagram, it has been convenient to drop the distinction between thick and thin edges, but to indicate thick-edged 0-regions and thin-edged 1-regions by the insertion of 0's and 1's, as appropriate.

4. Origin of the Horn inequalities

Each puzzle, whether rigid or not, gives rise to a Horn inequality of the form (1.3) that must be satisfied if a Littlewood–Richardson coefficient is to be non-zero. This comes about by using the labyrinth as a *hive plan*, that is by superimposing the labyrinth of the puzzle on the LR-hives with boundaries specified by λ, μ and ν and then exploiting the hive conditions.

To see this we return to our example with $n = 5$, $r = 3$, $I = \{1, 2, 4\}$, $J = \{2, 3, 4\}$ and $K = \{2, 3, 5\}$. To form the hive plan, the edges of the corresponding labyrinth are labelled with the parts of λ, μ and ν on the boundary, and with as yet unknown labels in the interior.



The successive application of the LR-hive conditions to each shaded sub-rhombus of the above hive plan gives in the case of thick-edge inequalities:

$$\begin{aligned}
\nu_2 + \nu_3 + \nu_5 &\leqslant (\nu_2 + \nu_3) + \gamma_4 = (\alpha_1 + \alpha_2 + \beta_1 + \beta_2) + \gamma_4 \\
&\leqslant \lambda_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_4 \leqslant \lambda_1 + \lambda_2 + \beta_1 + \beta_2 + \gamma_4 \\
&\leqslant \lambda_1 + \lambda_2 + \beta_3 + \beta_2 + \gamma_4 \leqslant \lambda_1 + \lambda_2 + (\beta_3 + \beta_4 + \gamma_4) \\
&= \lambda_1 + \lambda_2 + (\alpha_4 + \mu_2 + \mu_3 + \mu_4) \leqslant \lambda_1 + \lambda_2 + \lambda_4 + \mu_2 + \mu_3 + \mu_4,
\end{aligned} \tag{4.1}$$

where the intermediate equalities are just expressions of the fact that sums of any combination of positive edge lengths between any two fixed points are always the same, by virtue of the repeated use of (2.3). The result, as claimed quite generally, is a Horn inequality. In fact [14] all Horn inequalities, both essential and inessential, may be derived in this way from puzzles.

The same procedure applied to thin-edge inequalities gives

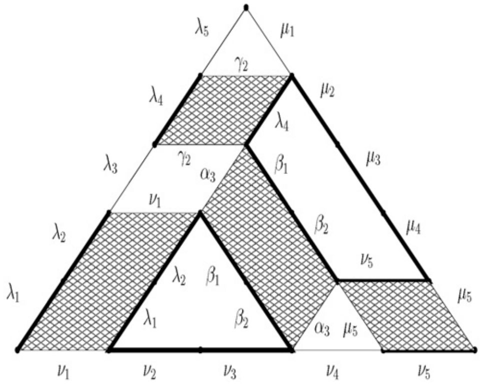
$$\begin{aligned}
\nu_1 + \nu_4 &\geqslant \gamma_1 + \nu_4 = \gamma_1 + (\alpha_5 + \beta_5) \\
&\geqslant \gamma_1 + \alpha_3 + \beta_5 \geqslant (\gamma_1 + \alpha_3) + \mu_5 = (\lambda_3 + \gamma_2) + \mu_5 \\
&\geqslant \lambda_3 + \gamma_3 + \mu_5 = \lambda_3 + (\lambda_5 + \mu_1) + \mu_5.
\end{aligned} \tag{4.2}$$

This is the complement of the Horn inequality (4.1) with respect to the identity $|\nu| = |\lambda| + |\mu|$.

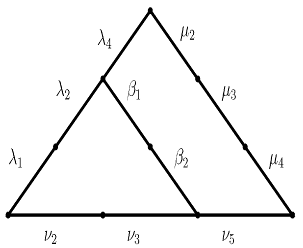
Clearly, if any Horn inequality is saturated, that is to say becomes an equality, then *all* of the individual inequalities arising in its derivation are also saturated, as well as the individual inequalities in its complement. In our example, this means that if the condition

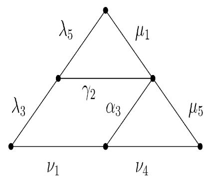
$$\nu_2 + \nu_3 + \nu_5 = \lambda_1 + \lambda_2 + \lambda_4 + \mu_2 + \mu_3 + \mu_4 \tag{4.3}$$

is satisfied, then we must have $\nu_5 = \gamma_4$, $\alpha_1 = \lambda_1$, $\alpha_2 = \lambda_2$, $\beta_1 = \beta_3$, $\beta_2 = \beta_4$ and $\alpha_4 = \lambda_4$, from (4.2), and $\nu_1 = \gamma_1$, $\alpha_5 = \alpha_3$, $\beta_5 = \mu_5$ and $\gamma_2 = \gamma_3$, from (4.3). The degrees of freedom are thereby greatly reduced, as shown in the following diagram:



In this particular example, for which the original puzzle is rigid, it is not difficult to see that the corridors, R_n, that is the shaded parallelograms, are completely redundant, and that the enumeration of all possible large LR-hives, H_n, is accomplished by enumerating pairs of small LR-hives, H_r and H_{n-r}, corresponding to the subdiagrams consisting of just the thick-edged 0-regions, and just the thin-edged 1-regions, respectively:





It follows that if $n = 5$ and $\nu_2 + \nu_3 + \nu_5 = \lambda_1 + \lambda_2 + \lambda_4 + \mu_2 + \mu_3 + \mu_4$ then

$$c^{(\nu_1\nu_2\nu_3\nu_4\nu_5)}_{(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)(\mu_1\mu_2\mu_3\mu_4\mu_5)} = c^{(\nu_2\nu_3\nu_5)}_{(\lambda_1\lambda_2\lambda_4)(\mu_2\mu_3\mu_4)}\ c^{(\nu_1\nu_4)}_{(\lambda_3\lambda_5)(\mu_1\mu_5)}. \tag{4.4}$$

5. Proof of factorisation

More generally, to complete the proof of the factorisation Theorem 1.4 it is necessary to show that if an essential Horn inequality is saturated, then the hive plan corresponding to the associated puzzle is such that:

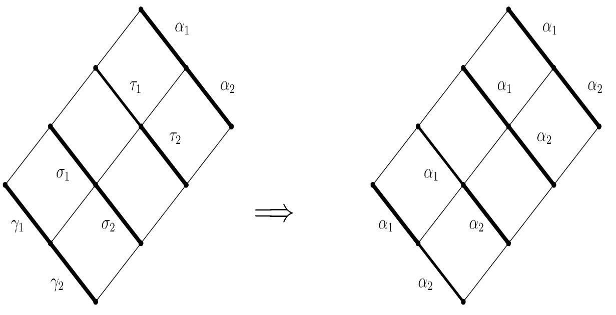
the corridors R_n of an LR-hive H_n are redundant;
the thick-edged 0-regions of an LR-hive H_n constitute an LR-hive H_r;
the thin-edged 1-regions of an LR-hive H_n constitute an LR-hive H_{n-r};
any LR-hives H_r and H_{n-r} that are first subdivided and then joined together by means of corridors R_n constitute an LR-hive H_n.

Throughout this section we assume that the boundary labels, λ, μ and ν, of the LR-hives under consideration are fixed and that they saturate the Horn inequality associated with a given puzzle. We consider the consequences of this for the edge labelling of the hives. Except where otherwise stated we do not have to assume that the Horn inequality is essential.

The redundancy of each corridor is rather easily established by showing that the hive conditions and the Horn equality are sufficient to ensure that all interior edge labels of a corridor are fixed in a trivial way by those on its boundary.

Lemma 5.1. *The saturation of the Horn inequality associated with a given puzzle, implies that the edge labels of an LR-hive do not vary either along or across each region of the hive corresponding to any corridor in the hive plan of the puzzle. This fixes all the interior edge labels of each corridor in terms of those on its boundary.*

Proof. This can be seen most simply by way of an example in which we may think of the corridor as running from south–west to north–east:



In the left-hand diagram, the hive conditions along the given corridor are $\gamma_1 \leqslant \sigma_1 \leqslant \tau_1 \leqslant \alpha_1$ and $\gamma_2 \leqslant \sigma_2 \leqslant \tau_2 \leqslant \alpha_2$. As we have seen the Horn inequality associated with the corresponding puzzle, implies that along any such corridor we have $\gamma_1 + \gamma_2 \leqslant \alpha_1 + \alpha_2$. If this is saturated to give $\gamma_1 + \gamma_2 = \alpha_1 + \alpha_2$, then we must have, $\gamma_1 = \alpha_1$ and $\gamma_2 = \alpha_2$ so that, as in the right-hand diagram, $\gamma_1 = \sigma_1 = \tau_1 = \alpha_1$ and $\gamma_2 = \sigma_2 = \tau_2 = \alpha_2$. Thus there is no variation in thick-edge labels in moving north–east along the corridor. The saturation of the complementary Horn inequality implies, in exactly the same way, that there is no variation in thin-edge labels in moving north–west across the corridor. Having fixed all thick and thin edge labels of the rhombi constituting the corridor, the labels of edges corresponding to the short diagonals of each elementary rhombus in the corridor are then fixed automatically by the triangle condition (2.3). Similar results apply to all corridors, whatever their orientation. □

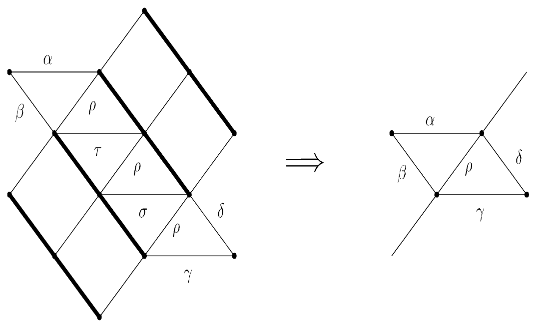
This observation allows us to show, as follows, that the hive conditions in the original hive H_n imply the validity of the hive conditions in both subhives H_r and H_{n-r} obtained by the deletion of the corridors R_n.

Lemma 5.2. *Let the hive plan corresponding to a puzzle be such that each LR-hive H_n consists of* 0*-regions and* 1*-regions separated by corridors. For the LR-hives under consideration, if the Horn inequality associated with this puzzle is saturated, then all the* 0*-regions taken together constitute an LR-hive H_r, and all the* 1*-regions taken together constitute an LR-hive H_{n-r}.*

Proof. To see that all the 0-regions of the original hive H_n constitute an equilateral triangle of side-length r it suffices to scale the lengths of all the thin-edges of the original puzzle by some parameter t, and to allow t to tend to 0. This deflation [14] causes all 1-regions and corridors to be deleted while retaining the overall equilateral triangular shape. It remains to show that the preserved thick edges together with their edge labels constitute an LR-hive H_r. A similar scaling procedure applied to the thick edges deletes the 0-regions and corridors, and maps the 1-regions of the original hive H_n and their edge labels to a candidate LR-hive H_{n-r}.

Since the hive conditions are automatically satisfied for each constituent rhombus that is contained wholly within one of the original 0-regions or one of the original 1-regions, it is only necessary to consider those constituent rhombi in H_r and H_{n-r} which are created by the deletion of some corridor.

This deletion is exemplified in the following diagram in which a corridor from some H_n is deleted to give a constituent rhombus of H_{n-r}:



First it should be noted that Lemma 5.1 has been exploited to ensure the equality of the three edge labels signified by ρ. Then the initial hive conditions $\gamma \leqslant \sigma$, $\sigma \leqslant \tau$ and $\tau \leqslant \alpha$ in H_n imply the final hive condition $\gamma \leqslant \alpha$ in H_{n-r}. Finally, the triangle constraints (2.3) in the initial hive H_n ensure that $\alpha - \beta = \rho = \gamma - \delta$, so that in the final hive H_{n-r} we have $\alpha + \delta = \beta + \gamma$, as required by (2.4).

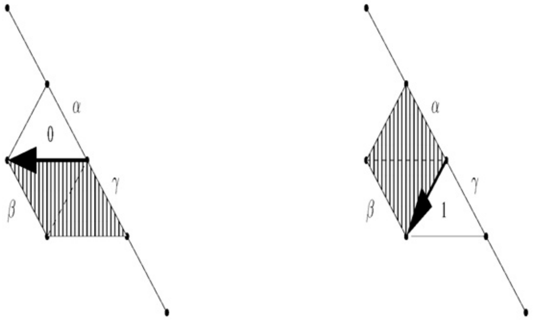
Similar results apply to the deletion of any corridor, whatever its orientation, and the creation of a rhombus formed from a pair of triangles one from each side, or one from each end of the corridor. Since this is the only way in which new rhombi can be created, this argument is sufficient to prove that each hive H_n maps to a pair of hives, H_r and H_{n-r}, by the edge scaling procedures applied to thick and thin edges, respectively, or equivalently by the deletion of corridors and the glueing together of all the 0-regions and then all the 1-regions. □

It only remains to examine whether or not cutting up a pair of hives H_r and H_{n-r} and glueing them together with the insertion of appropriate corridors, in a manner that is uniquely determined by some given puzzle, always yields a hive H_n. It is the fact that the puzzle is given that ensures that all the corridors to be inserted and all the edges to be cut and glued are pre-determined. Since the hive conditions are automatically satisfied for all rhombi contained wholly within either the corridors, or some initial 0-regions or some initial 1-regions, it is only necessary to consider the hive conditions (2.5) for those rhombi whose short diagonal forms part of a corridor wall, that is either a thick edge between a corridor and a 0-region or a thin edge between a corridor and a 1-region. To this end we consider gentle paths through the labyrinth or hive plan. These consist of a connected sequence of corridor walls with 0-regions on the left and 1-regions on the right. At each vertex the deviation is only through 0 or $\pm\pi/3$, not $\pm 2\pi/3$. An edge along a corridor wall is said to be *good* if the hive condition is satisfied for the rhombus of which it forms the short diagonal. If this hive condition is not satisfied then the edge is said to be *bad*. A gentle path is said to be a *good path* if all of its constituent edges are good.

We note

Lemma 5.3. *The first edge of each gentle path starting on the boundary of a hive plan is good.*

Proof. Since each gentle path follows a connected sequence of corridor walls, there are only two cases to consider, those starting with a 0-region on the right of the path and those starting with a 1-region on the left of the path. For paths starting from the north–east boundary these two possibilities are illustrated in the following two diagrams:

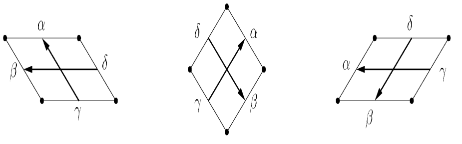


Each boundary has edges specified by a partition, so that in both diagrams $\alpha \geqslant \gamma$. The Horn equality applied to corridors gives in the left-hand diagram $\beta = \gamma$, so that $\alpha \geqslant \gamma = \beta$, and in the right-hand diagram $\beta = \alpha$, so that $\beta = \alpha \geqslant \gamma$. Thus in each case the required hive condition across the corridor wall is satisfied.

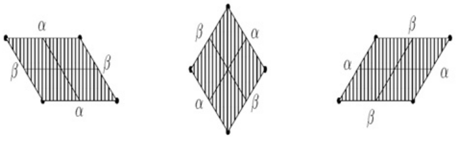
The same is true of all three boundaries, and we can conclude that the first edge of each gentle path starting from any boundary is good. □

Having taken at least one step from the boundary along a good gentle path, it may or may not be possible to extend the gentle path through the addition of further good edges. We will show that if the puzzle is rigid it is always possible to extend the gentle path with good edges until the boundary of the puzzle is again reached. To explore this it is convenient to introduce a diagrammatic notation which encapsulates the various hive conditions and their saturation.

First, wherever required, the hive conditions $\alpha \geqslant \gamma$ and $\beta \geqslant \delta$ of (2.4) are signified for each possible orientation of a rhombus by the use of arrows in the direction of weakly increasing edge labels:



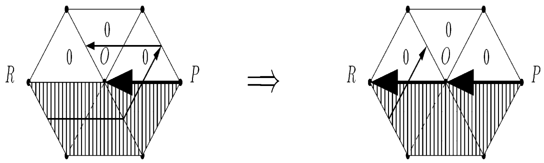
Second, we make use of Lemma 5.1, which implies the equality of pairs of opposite edge labels, $\alpha = \gamma$ and $\beta = \delta$ of any rhombus forming part of a corridor in a puzzle associated with a saturated Horn inequality. We signify this, wherever required, by means of straight lines from edge to edge across each shaded rhombus:



We now proceed to consider the possible extension of good gentle paths. If the extension of a gentle path is straight on, along a corridor wall then we have:

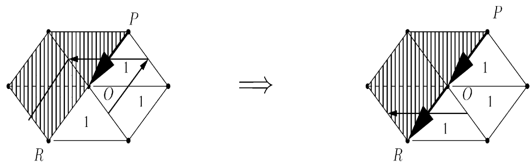
Lemma 5.4. *If any edge, PO, on a gentle path along a corridor wall is good, then, if it exists, so is the next edge OR along the same corridor wall. Equivalently, if any edge, OR, on a gentle path along a corridor wall is bad then, if it exists, so is the preceding edge PO along the same corridor wall.*

Proof. Up to reorientations, there are just two cases to consider. In the case of a gentle path along a corridor wall next to a 0-region we have:



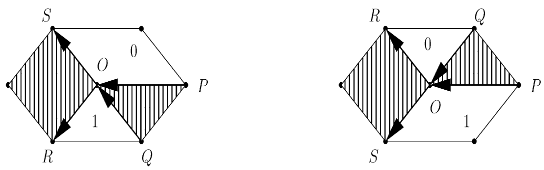
With the above diagrammatic convention it is clear that if the edge *PO* is good then so is the edge *OR*. It follows that if *OR* is bad then *PO* must also be bad.

In the case of a gentle path along a corridor wall next to a 1-region we have:



Once again it is clear that if the edge *PO* is good then so is the edge *OR*, and it follows that if *OR* is bad then *PO* must also be bad. □

It remains finally to consider the arrival of a gentle path at the common vertex of two different corridors. Up to reorientation, this can occur once again in just two different ways:

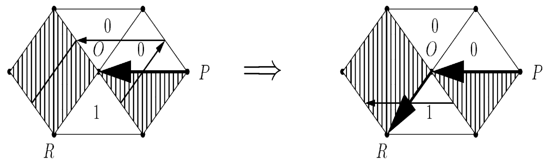


In each diagram the possible incoming paths edges are *PO* and *QO*, and the outgoing path edges are *OR* and *OS*. The possible gentle paths are *POR*, *POS* and *QOS*. Notice that *POR* turns through $\pm\pi/3$ and borders two distinct regions, both a 0-region and a 1 region, while *POS* also turns through $\pm\pi/3$ but borders a single region, either a 0-region or a 1-region, and *QOS* is straight and borders two distinct regions, both a 0-region and a 1 region.

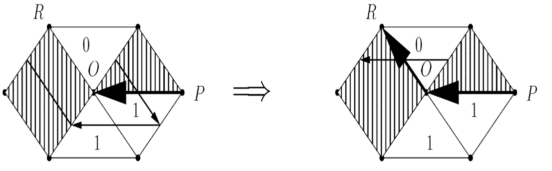
Lemma 5.5. *With the notation of the above pair of diagrams, we have*:

(i) *if the edge PO is good then the edge OR is also good*;
(ii) *if OR is bad then PO is also bad*;
(iii) *if both edges PO and QO are good then so are both OR and OS*;
(iv) *if OS is bad, then either PO or QO is bad.*

Proof. If *PO* is good then we have the following two possibilities for the path *POR*: a $\pi/3$ turn to the left:

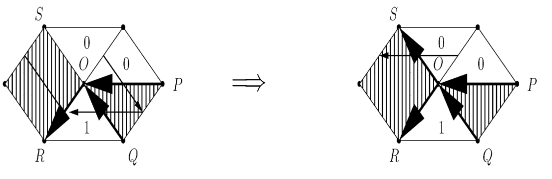


or a $\pi/3$ turn to the right:

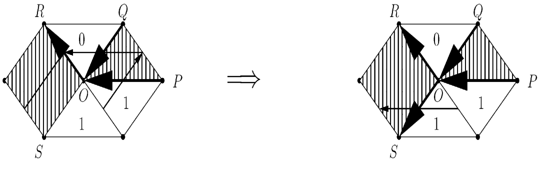


In each case, as indicated by the various cross-rhombus lines and arrows, if *PO* is good then *OR* must also be good. It follows that if *OR* is bad then *PO* must also be bad. Thus (i) and (ii) are both true.

Now, let us assume that both *PO* and *QO* are good. This gives rise to the following two possibilities, where in each case *OR* is necessarily good by virtue of (i):

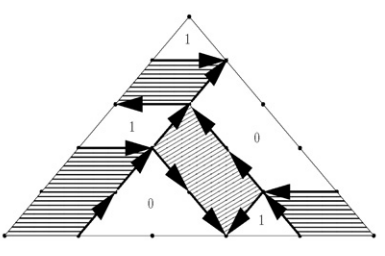


and



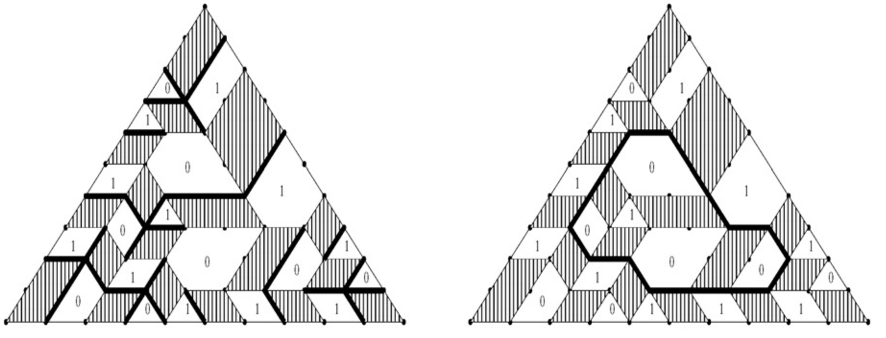
In both cases, as indicated by the cross-rhombus lines and arrows, if *PO* and *QO* are both good, then *OS* is also good. Taken together with (i) this implies the validity of (iii). It follows that if *OS* is bad, then at least one of *PO* and *QO* must also be bad, thereby proving (iv). □

In some instances, the union of good paths constructed in this way exhausts all interior edges of the hive plan. This happens in our earlier $n = 5$, $r = 3$, $I = \{1, 2, 4\}$, $J = \{2, 3, 4\}$, $K = \{2, 3, 5\}$ example. The good paths are generated as shown:



Since they cover all interior corridor walls, when combining the hives H_3 and H_2 with these corridors all the hive conditions of H_5 are automatically satisfied. Hence we have the factorisation (4.4) claimed earlier.

On the other hand, in the case of the example $n = 10$, $r = 5$, $I = (1, 2, 4, 6, 8)$, $J = (1, 3, 4, 7, 9)$, $K = (2, 4, 6, 8, 10)$ for which the corresponding puzzle is not rigid, the union of good paths is indicated by thick lines on the left-hand diagram shown below alongside a similar depiction of the gentle loop we had previously identified in this case:



Clearly, the union of good paths does not cover all interior corridor edges. They are obstructed from doing so by the existence of the gentle loop. It remains to show that such a gentle loop is the only possible obstruction to factorisation. This is done by means of the following:

Lemma 5.6. *Let two LR-hives H_r and H_{n-r} be subdivided into various* 0*-regions and* 1*-regions, respectively, that are then joined together in the shape of a potential LR-hive H_n by means of redundant corridors. Then if any edge of this potential LR-hive is bad there must exist a gentle loop in the corresponding puzzle.*

Proof. By construction, the only edges that may be bad are those that lie on some gentle path. Lemmas 5.4 and 5.5 imply that in each and every case if an edge is bad then it is immediately preceded on some gentle path by another bad edge. Iterating this, one can proceed backwards along a gentle path from any bad edge along a sequence of bad edges. This sequence cannot reach the boundary thanks to Lemma 5.3, which states that every gentle path emanating from the boundary starts with a good edge. It follows that the sequence of bad edges along the reverse gentle path cannot terminate. In a finite puzzle this means that the sequence of bad edges must eventually repeat itself, thereby forming a loop. Such a loop is, by construction, gentle. □

As a corollary to this we have:

Corollary 5.7. *Let a potential LR-hive H_n be formed by joining together by means of redundant corridors any two LR-hives H_r and H_{n-r}, subdivided in accordance with an appropriate puzzle into various* 0*-regions and* 1*-regions, respectively. Then H_n is an LR-hive if the puzzle is rigid.*

Proof. From Theorem 3.4, the labyrinth of a rigid puzzle contains no gentle loops. It then follows from Lemma 5.6 that in such a case all interior edges between corridors and either 0-regions or 1-regions are good. This means that the hive conditions are satisfied for all rhombi that are split by corridor walls. The remaining hive conditions are automatically satisfied for all rhombi contained wholly within a 0-region, or within a 1-region, or within a redundant corridor. It follows that each H_n, constructed in this way, is an LR-hive. □

This allows us to prove our main result, Theorem 1.4.

Proof of Theorem 1.4. Let an essential Horn inequality be saturated, as in the hypothesis (1.4). Then, by virtue of Lemma 5.2 each LR-hive H_n contributing to $c^{\nu}_{\lambda\mu}$ maps to a pair of LR-hives H_r and H_{n-r} contributing to $c^{\nu_K}_{\lambda_I\mu_J}\, c^{\nu_{\overline{K}}}_{\lambda_{\overline{I}}\mu_{\overline{J}}}$. However, this map is necessarily bijective as a result of Corollary 5.7, where it should be noted that the redundancy of the corridors implies the saturation of the essential Horn inequality that is associated with the corresponding rigid puzzle. □

As a final corollary we have:

Corollary 5.8. *Each Littlewood–Richardson coefficient may be expressed as a product of primitive Littlewood–Richardson coefficients.*

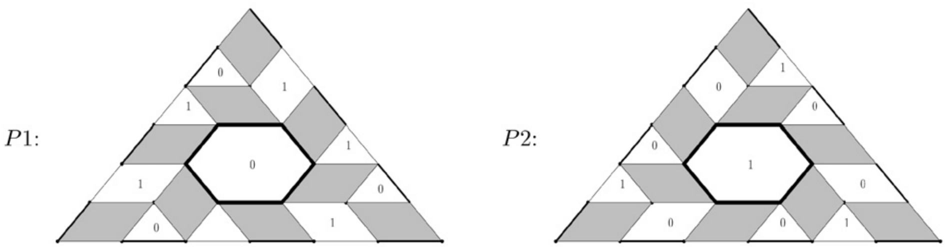
Proof. The condition (1.4) is precisely the condition that $c_{\lambda\mu}^{\nu}$ is not-primitive in the sense of Definition 3.1. It follows that the repeated application of Theorem 1.4 to all the factors that appear in the factorisation of a given Littlewood–Richardson coefficient, must eventually yield an expression for this coefficient as a product of Littlewood–Richardson coefficients that are all primitive. $\square$

It should be pointed out that Theorem 1.4 involves the hypothesis that the saturated Horn inequality be essential. The necessity of including this hypothesis may be seen by considering the simplest example of an inessential Horn inequality. This example has been discussed by both Fulton [9] and Knutsen, Tao and Woodward [14]. It is the case for which $n = 6$, $r = 3$, $I = J = \{1, 3, 5\}$ and $K = \{2, 4, 6\}$, so that the corresponding saturated inessential Horn inequality takes the form

$$\nu_2 + \nu_4 + \nu_6 = \lambda_1 + \lambda_3 + \lambda_5 + \mu_1 + \mu_3 + \mu_5. \tag{5.1}$$

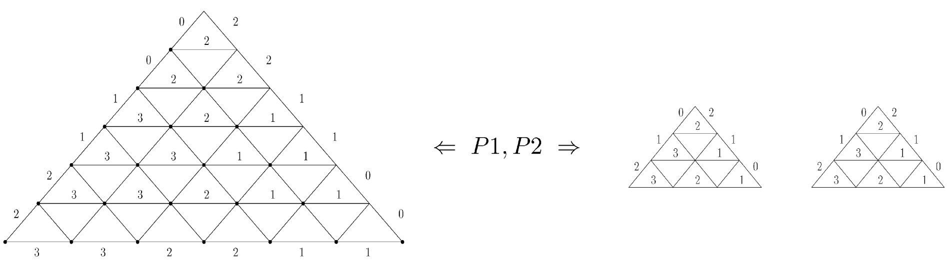
To exemplify this situation we take $\lambda = \mu = (221100)$ and $\nu = (332211)$, in which case $\lambda_I = \mu_J = \lambda_{\bar{I}} = \mu_{\bar{J}} = (210)$ and $\nu_K = \nu_{\bar{K}} = (321)$. Correspondingly we have $c_{\lambda\mu}^{\nu} = 3$, while $c_{\lambda_I\mu_J}^{\nu_K} = c_{\lambda_{\bar{I}}\mu_{\bar{J}}}^{\nu_{\bar{K}}} = 2$, so that the factorisation (1.5) certainly does not occur.

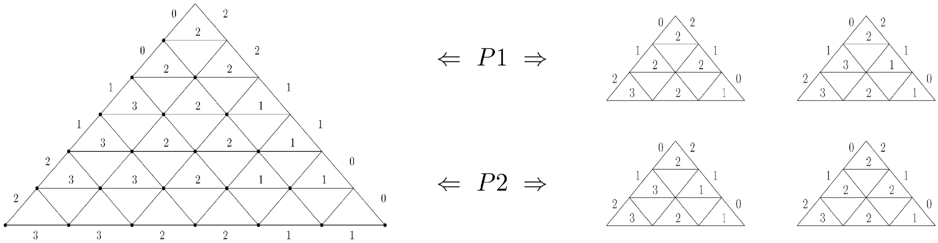
The explanation for this lies in the fact that in this case there exist two puzzles with thick edges on the boundary specified by the given I, J and K:

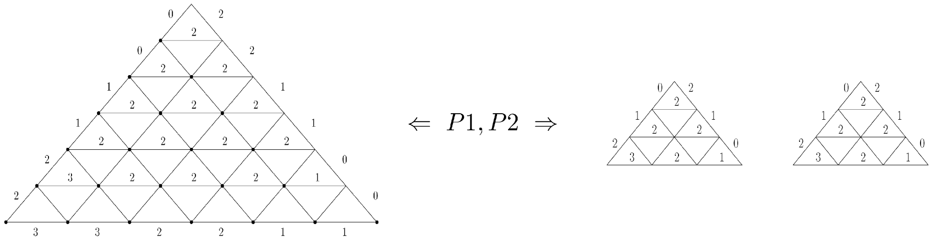


Neither puzzle is rigid. In each case there exists a single gentle loop, as shown, traversing the central hexagon anticlockwise in the case of $P1$ and clockwise in the case of $P2$.

In terms of hives and subhives, the three 6-hives corresponding to $c_{\lambda\mu}^{\nu} = 3$ map to pairs of subhives under the deletion of the corridors specified by the puzzles $P1$ and $P2$ as shown below:

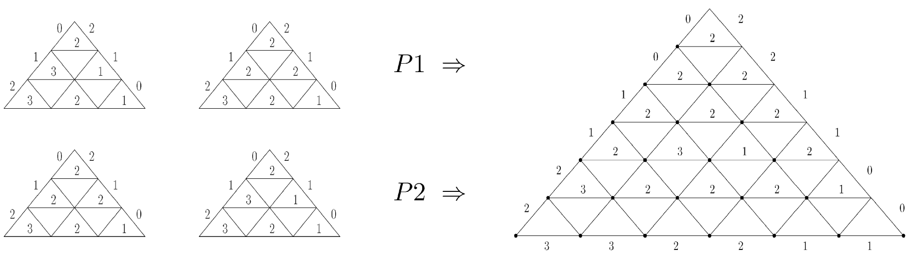




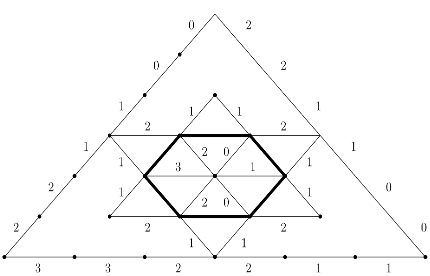


For clarity of display, only boundary edges and interior edges parallel to the base have been labelled. All other interior edge labels are fixed by the repeated application of the triangle condition (2.3).

In every case, as required by Lemma 5.2, the resulting subhives associated with the 0-regions and 1-regions of the puzzles take the form of one or other of the two 3-hives corresponding to $c^{\nu_K}_{\lambda_I\mu_J}=2$ and $c^{\nu_{\overline{K}}}_{\lambda_{\bar{I}}\mu_{\bar{J}}}=2$, respectively. However, the resulting pairs of 3-hives do not exhaust all possibilities. If the remaining possibilities are used to reconstruct a candidate 6-hive by means of the puzzles $P1$ and $P2$ one finds:



It can be seen immediately that the result is not a hive, since it contains a sequence of horizontal edges whose consecutive labels, (2312) do not form a partition. This is entirely consistent with our analysis, in particular Lemma 5.6, in that the puzzles are not rigid. They each contain a hexagonal gentle loop along which all six edges of our candidate hive are bad. The hive conditions (2.5) are violated for each of the twelve rhombi whose short diagonal coincides with an edge of the gentle loop, as can be seen from the following diagram in which all the relevant edges have been labelled:



This example is sufficient to show that Corollary 5.7 and Theorem 1.4 cannot be extended to cover cases in which the puzzle is not rigid and equivalently the Horn inequality is not essential.

6. Some special cases and applications

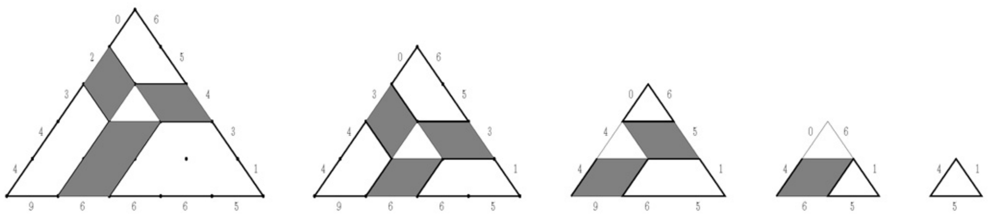
A special case of Theorem 1.4 has been established by Cho, Jung and Moon [5]. This is the case $r = n-1$, for which $n-r=1$. It follows that $\lambda_{\bar{I}}$, $\mu_{\bar{J}}$ and $\nu_{\overline{K}}$ are all one-part partitions with $\lambda_{\bar{I}} + \mu_{\bar{J}} = \nu_{\overline{K}}$, so that $c^{\nu_{\overline{K}}}_{\lambda_{\bar{I}}\mu_{\bar{J}}} = 1$. Hence under the hypotheses of Theorem 1.4, in the case $r = n-1$ we obtain the reduction formula:

$$c^{\nu}_{\lambda\mu} = c^{\nu_K}_{\lambda_I\mu_J}. \tag{6.1}$$

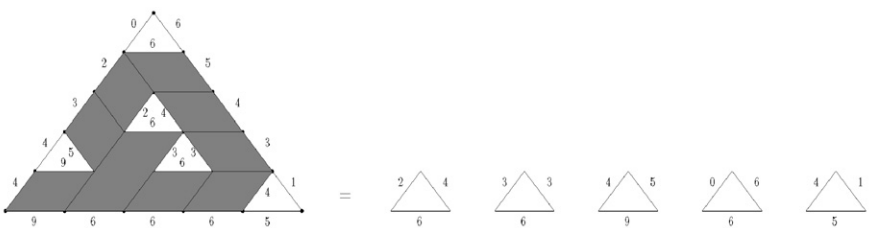
A similar result applies in the case $r=1$. This time λ_I, μ_J and ν_K are all one-part partitions with $\lambda_I + \mu_J = \nu_K$, so that $c^{\nu_K}_{\lambda_I\mu_J} = 1$, leading to the reduction formula:

$$c^{\nu}_{\lambda\mu} = c^{\nu_{\overline{K}}}_{\lambda_{\bar{I}}\mu_{\bar{J}}}. \tag{6.2}$$

The Example 4.9, offered in [5] for the evaluation of a Littlewood–Richardson coefficient through the repeated use of the reduction formula of type (6.1) along with a second reduction formula obtained by considering conjugates of tableaux, can be dealt with rather easily by the repeated use of Theorem 1.4. The resulting factorisation takes place in accordance with the following sequence of diagrams corresponding to the puzzles associated with saturating successive Horn inequalities with $n = 5, 4, 3, 2,$ and $r = 4, 3, 2, 1$, so that in each case $n - r = 1$.



If one marks all the redundant corridors on a single diagram, that is not itself a puzzle, then this yields the factorisation:



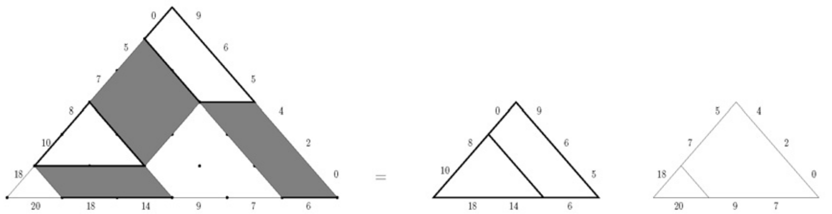
that is to say

$$c^{96665}_{44320,65431} = c^{6}_{2,4}\ c^{6}_{3,3}\ c^{9}_{4,5}\ c^{6}_{0,6}\ c^{5}_{4,1}. \tag{6.3}$$

Since $c^{a+b}_{a,b} = 1$ for all $a, b \geqslant 0$, it follows that $c^{96665}_{44320,65431} = 1$, as previously established in [5].

Cho, Jung and Moon also announced [5] an extension of the $r=1$ and $r=n-1$ reduction formulae to the case of $r=2$ and $r=n-2$. In each of these cases, reduction formulae of this type follow directly from our Theorem 1.4, since any non-vanishing Littlewood–Richardson coefficient involving three two-part partitions is necessarily equal to 1.

Moving away from the special case of reduction formulae, for $n = 6$, $r = 3$, $I = \{2, 3, 6\}$, $J = \{1, 2, 3\}$ and $K = \{2, 3, 6\}$ we have a single, rigid puzzle with no gentle loop, as illustrated in the hive plan illustrated below on the left:



In the case illustrated, $\lambda = (18, 10, 8, 7, 5, 0)$, $\mu = (9, 6, 5, 4, 2, 0)$ and $\nu = (20, 18, 14, 9, 7, 6)$. It is easy to check that the Horn inequality associated with the triple (I, J, K) is saturated, so that we necessarily have the factorisation illustrated on the right, from which we can infer that

$$c^{20\,18\,14\,9\,7\,6}_{18\,10\,8\,7\,5\,0,9\,6\,5\,4\,2\,0} = c^{18\,14\,6}_{10\,8\,0,9\,6\,5} \cdot c^{20\,9\,7}_{18\,7\,5,4\,2\,0} = 2 \cdot 3 = 6, \tag{6.4}$$

as may be checked by direct computation.

This work was motivated to a considerable extent by the authors' previous exploration of the properties of stretched Littlewood–Richardson coefficients [12]. If all the parts of the partitions λ, μ and ν are multiplied by a stretching parameter t, with t a positive integer, to give new partitions $t\lambda$, $t\mu$ and $t\nu$, the corresponding stretched Littlewood–Richardson coefficient is $c^{t\nu}_{t\lambda,t\mu}$. For each triple (λ, μ, ν) these stretched Littlewood–Richardson coefficients are known to be polynomial in the stretching parameter t [7,19]. We denote the corresponding LR-polynomial by

$$P^{\nu}_{\lambda,\mu}(t) = c^{t\nu}_{t\lambda,t\mu}. \tag{6.5}$$

Then, as a direct consequence of Theorem 1.4 we have

Corollary 6.1. *Let λ, μ and ν be partitions of lengths $\ell(\lambda), \ell(\mu), \ell(\nu) \leqslant n$ such that $c^{\nu}_{\lambda\mu} > 0$ and*

$$ps(\nu)_K = ps(\lambda)_I + ps(\mu)_J, \tag{6.6}$$

for some essential Horn triple $(I, J, K) \in R^n_r$ with $0 < r < n$. Then

$$P^{\nu}_{\lambda\mu}(t) = P^{\nu_K}_{\lambda_I\mu_J}(t)\, P^{\nu_{\overline{K}}}_{\lambda_{\overline{I}}\mu_{\overline{J}}}(t), \tag{6.7}$$

where $\overline{I}, \overline{J}, \overline{K}$ are the complements of I, J, K, respectively, in $N = \{1, 2, \ldots, n\}$.

Proof. Since $ps(t\lambda) = t\,ps(\lambda)$ for all partitions λ, all Horn inequalities, whether saturated or not, are unaffected by scaling all the parts of all the relevant partitions by a positive scaling parameter t. Then, precisely the same puzzle and hive plan as used in the proof of the factorisation $c^{\nu}_{\lambda\mu} = c^{\nu_K}_{\lambda_I\mu_J}\ c^{\nu_{\overline{K}}}_{\lambda_{\overline{I}}\mu_{\overline{J}}}$ serves to prove that $c^{t\nu}_{t\lambda,t\mu} = c^{t\nu_K}_{t\lambda_I,t\mu_J}\ c^{t\nu_{\overline{K}}}_{t\lambda_{\overline{I}},t\mu_{\overline{J}}}$ for all positive integers t. The polynomial nature of these coefficients then implies (6.7), as required. $\square$

It should be noted that the condition for primitivity given in Definition 3.1 is independent of any scaling by t. Accordingly, we say that the LR-polynomial $P^{\nu}_{\lambda,\mu}(t) = c^{t\nu}_{t\lambda,t\mu}$ is primitive if and only if $c^{\nu}_{\lambda\mu}$ is primitive. It is then a simple consequence of Corollary 5.8 that we have:

Corollary 6.2. *Each LR-polynomial may be expressed as a product of primitive LR-polynomials.*

As an illustration of this, it is not difficult to establish by carrying out a polynomial fit to the data on Littlewood–Richardson coefficients [17], that

$$P^{20\,18\,14\,9\,7\,6}_{18\,10\,8\,7\,5\,0,9\,6\,5\,4\,2\,0}(t) = 2t^2 + 3t + 1 = (t+1)(2t+1). \tag{6.8}$$

The factorisation of this LR-polynomial is no accident. The LR-polynomial is not-primitive, since as we have seen earlier the corresponding Littlewood–Richardson coefficient $c^{20\,18\,14\,9\,7\,6}_{18\,10\,8\,7\,5\,0,9\,6\,5\,4\,2\,0}$ is not-primitive. It factorises as in (6.4) as a consequence of the saturation of an essential Horn inequality. It follows from (6.7) that a similar factorisation must apply to the LR-polynomial itself, namely

$$P^{20\,18\,14\,9\,7\,6}_{18\,10\,8\,7\,5\,0,9\,6\,5\,4\,2\,0}(t) = P^{18\,14\,6}_{10\,8\,0,9\,6\,5}(t)\ P^{20\,9\,7}_{18\,7\,5,4\,2\,0}(t). \tag{6.9}$$

The validity of this can be verified by noting that

$$P^{18\,14\,6}_{10\,8\,0,9\,6\,5}(t) = t+1 \quad \text{and} \quad P^{20\,9\,7}_{18\,7\,5,4\,2\,0}(t) = 2t+1, \tag{6.10}$$

as required to recover (6.8). In this case no further factorisation is possible, since the two constituent LR-polynomials are primitive.

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