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Dirac particles tunneling from BTZ black hole

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Abstract

We calculated the Dirac particles' Hawking radiation from the outer horizon of BTZ black hole via tunneling formalism. Applying WKB approximation to the Dirac equation in $(2+1)$-dimensional BTZ spacetime background, we obtain the radiation spectrum for fermions and Hawking temperature of BTZ black hole. The results obtained by taking the fermion tunneling into account are consistent with the previous literatures.

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Hawking [1] discovered the thermal radiation of a collapsing black hole using the techniques of quantum field theory in curved spacetime. Since the Hawking radiation relates the theory of general relativity with quantum field theory and statistical thermodynamics, it is generally believed that a deeper understanding of Hawking radiation may shed some lights on seeking the underlying quantum gravity. Since then, several derivations of Hawking radiation have been proposed. The original method presented by Hawking is direct but complicated to be generalized to other spacetime backgrounds. In recent years, a semi-classical derivation of Hawking radiation as a tunneling process [2] has been developed and has already attracted a lot of attention. In this method, the imaginary part of the action is calculated using the null geodesic equation. Zhang and Zhao extended this method to Reissner–Nordström black hole [3] and Kerr–Newman black hole [4]. Angheben et al. [5] also proposed a derivation of Hawking radiation by calculating the particles' classical action from the Hamilton–Jacobi equation, which is an extension of the complex path analysis of Padmanabhan et al. [6]. Both of these approaches to tunneling used the fact that the tunneling probability for the classically forbidden trajectory from inside to outside the horizon is given by

$$\Gamma = \exp\left(-\frac{2}{\hbar}\,\mathrm{Im}\,I\right), \tag{1}$$

where I is the classical action of the trajectory. The crucial thing in tunneling formalism is to calculate the imaginary part of classical action. The difference between these two methods consists in how the classical action is calculated. For a detailed comparison of the Hamilton–Jacobi method and the Null-Geodesic method, one can see [7].

In this Letter, we extend the tunneling method presented in [8] to calculate the Dirac particles' Hawking radiation from $(2+1)$-dimensional BTZ black hole. Although many works [9–15] have been contributed to the Hawking radiation of BTZ black hole using the tunneling method, they all considered the scalar particles' radiation. Starting with the covariant Dirac equation in curved background, we calculate the radiation spectrum and Hawking temperature by using the WKB approximation.

The BTZ black hole solution is an exact solution to Einstein field equation in a $(2+1)$-dimensional theory of gravity with a negative cosmological constant $\Lambda = -1/l^2$:

$$S = \int dx^3 \sqrt{-g}\,(R + 2\Lambda). \tag{2}$$

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The BTZ black hole is described by the metric [16]

$$ds^2 = -N^2\,dt^2 + \frac{1}{N^2}dr^2 + r^2\big(N^\phi\,dt + d\phi\big)^2, \tag{3}$$

where

$$N^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \qquad N^\phi = -\frac{J}{2r^2}, \tag{4}$$

with M and J being the ADM mass and angular momentum of the BTZ black hole, respectively. This metric is stationary and axially symmetric, with two killing vectors $(\partial/\partial t)^\mu$ and $(\partial/\partial\phi)^\mu$. The line element (3) has two horizons which is determined by the equation

$$N^2 = \frac{1}{l^2 r^2}\big(r^2 - r_+^2\big)\big(r^2 - r_-^2\big) = 0, \tag{5}$$

where r_+ and r_- are defined by

$$r_\pm^2 = \frac{Ml^2}{2}\left[1 \pm \sqrt{1 - \frac{J^2}{M^2 l^2}}\right]. \tag{6}$$

We have assumed the non-extremal condition $Ml > J$, so that r_+ and r_- correspond to the outer event horizon and the inner event horizon respectively which is similar to the Reissner–Nordström spacetime. In the extremal case $Ml = J$, the two horizons coincide. In the present Letter, we mainly consider the non-extremal case. A brief comment regarding the extremal case appears at the end of this Letter.

Now we calculate the Dirac particles' Hawking radiation from the BTZ black hole. We consider the two component massive spinor field Ψ, with mass μ, obeys the covariant Dirac equation

$$i\hbar\gamma^a e_a^\mu \nabla_\mu \Psi - \mu\Psi = 0, \tag{7}$$

where ∇_μ is the spinor covariant derivative defined by $\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{[a}\gamma_{b]}$, and ω_μ^{ab} is the spin connection, which can be given in terms of the tetrad e_a^μ. The γ matrices in three spacetime dimensions are selected to be $\gamma^a = (i\sigma^2, \sigma^1, \sigma^3)$, where the matrices σ^k are the Pauli matrices. According to the line element (3), the tetrad field e_a^μ can be selected to be

$$\begin{aligned} e_0^\mu &= \left(\frac{1}{N}, 0, -\frac{N^\phi}{N}\right), \\ e_1^\mu &= (0, N, 0), \\ e_2^\mu &= \left(0, 0, \frac{1}{r}\right). \end{aligned} \tag{8}$$

We use the ansatz for the two component spinor Ψ as following

$$\Psi = \begin{pmatrix} A(t,r,\phi) \\ B(t,r,\phi) \end{pmatrix} \exp\left[\frac{i}{\hbar} I(t,r,\phi)\right]. \tag{9}$$

In order to apply WKB approximation, we can insert the ansatz for spinor field Ψ into the Dirac equation. Dividing by the exponential term and neglecting the terms with $\hbar$, one can arrive at the following two equations

$$\begin{cases} A(\mu + \frac{1}{r}\partial_\phi I) + B(N\partial_r I + \frac{1}{N}\partial_t I - \frac{N^\phi}{N}\partial_\phi I) = 0, \\ A(N\partial_r I - \frac{1}{N}\partial_t I + \frac{N^\phi}{N}\partial_\phi I) + B(\mu - \frac{1}{r}\partial_\phi I) = 0. \end{cases} \tag{10}$$

Note that although A and B are not constant, their derivatives and the components ω_μ are all of the factor $\hbar$, so can be neglected to the lowest order in WKB approximation. These two equations have a non-trivial solution for A and B if and only if the determinant of the coefficient matrix vanishes. Then we can get

$$N^2(\partial_r I)^2 - \frac{1}{N^2}\big(\partial_t I - N^\phi\partial_\phi I\big)^2 + \frac{1}{r^2}(\partial_\phi I)^2 - \mu^2 = 0. \tag{11}$$

Because there are two killing vectors $(\partial/\partial t)^\mu$ and $(\partial/\partial\phi)^\mu$ in the BTZ spacetime, we can separate the variables for $I(t,r,\phi)$ as following

$$I = -\omega t + j\phi + R(r) + K, \tag{12}$$

where ω and j are Dirac particle's energy and angular momentum respectively, and K is a complex constant. Insert it to Eq. (11) and solving for $R(r)$ yields

$$R_\pm(r) = \pm\int \frac{dr}{N^2}\sqrt{\big(\omega + jN^\phi\big)^2 + N^2\left(\mu^2 - \frac{j^2}{r^2}\right)}. \tag{13}$$

As discussed in the Hamilton–Jacobi method [17,18], one solution corresponds Dirac particles moving away from the outer event horizon and the other solution corresponds the particles moving toward the outer event horizon. The probabilities of crossing the outer horizon each way are respectively given by

$$\begin{aligned} P_{\text{out}} &= \exp\left[-\frac{2}{\hbar}\operatorname{Im} I\right] = \exp\left[-\frac{2}{\hbar}(\operatorname{Im} R_+ + \operatorname{Im} K)\right], \\ P_{\text{in}} &= \exp\left[-\frac{2}{\hbar}\operatorname{Im} I\right] = \exp\left[-\frac{2}{\hbar}(\operatorname{Im} R_- + \operatorname{Im} K)\right]. \end{aligned} \tag{14}$$

To ensure that the probability is normalized, we should note that the probability of any incoming classical particles crossing the outer horizon is unity [18]. So we get $\operatorname{Im} K = -\operatorname{Im} R_-$. Since $\operatorname{Im} R_+ = -\operatorname{Im} R_-$ this implies that the probability of a particle tunneling from inside to outside the outer horizon is given by

$$\Gamma = \exp\left[-\frac{4}{\hbar}\operatorname{Im} R_+\right]. \tag{15}$$

The imaginary part of R_+ can be calculated using Eq. (13). Integrating the pole at the horizon leads to the result (see [7,18] for a detailed similar process)

$$\operatorname{Im} R_+ = \frac{\pi}{2\kappa}(\omega - \omega_0), \tag{16}$$

where $\kappa = (r_+^2 - r_-^2)/(l^2 r_+)$ is the surface gravity of outer event horizon and $\omega_0 = j\Omega_+$ with $\Omega_+ = J/(2r_+^2)$ is the angular velocity of the outer event horizon. This leads to the tunneling probability

$$\Gamma = \exp\left[-\frac{2\pi}{\kappa}(\omega - \omega_0)\right], \tag{17}$$

which is consistent with the previous literatures (a recent discussion appeared in Ref. [15]). It should be noted that the higher terms about ω and j are neglected in our derivation and the expression (17) for tunneling probability implies the pure thermal radiation. The higher terms of the tunneling probability can arise from the energy and angular momentum conservation. In [15], the authors obtained the emission rate $\Gamma = e^{\Delta S_{BH}}$ when taking the back reaction into account and argued the result offers a possible mechanism to explain the information loss paradox.

From the emission probability (17), the fermionic spectrum of Hawking radiation of Dirac particles from the BTZ black hole can be deduced following the standard arguments [19,20]

$$N(\omega, j) = \frac{1}{e^{2\pi(\omega-\omega_0)/\kappa} + 1}. \tag{18}$$

From the tunneling probability and radiant spectrum, Hawking temperature of BTZ black hole can be determined as

$$T = \frac{\kappa}{2\pi} = \frac{1}{2\pi l^2 r_+}\left(r_+^2 - r_-^2\right). \tag{19}$$

At last, we present a brief comment regarding the extremal case $Ml = J$. In the extremal case, the two horizons coincide, i.e., $r_+ = r_-$. The factor $N^2(r)$ is of order two in terms of power series expansion near the horizon, namely $N^2(r) \sim (r - r_+)^2$, which is very different from the non-extremal case where $N^2(r) \sim (r - r_+)$. Then the integral (13) is divergent. This yields a diverging real component in the action while no imaginary part presented, which implies that the extremal black hole cannot emit particle in order to avoid the creation of naked singularity. Then we argue that Hawking temperature for extremal BTZ black hole is just zero. The result for the extremal black hole do not violate the cosmic censorship.

In summary, we have calculated the Dirac particles' Hawking radiation from BTZ black hole using the tunneling formalism. Starting with Dirac equation, we obtained the radiation spectrum and Hawking temperature of BTZ black hole by using the WKB approximation. The results coincide with the previous literatures. Whether the method presented in this Letter is also valid for other background spacetime is interesting to investigate in the future.

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