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Maximal Limit Spaces, Powerspaces, and Scott Domains

D. N. Hoover [1]

Odyssey Research Associates, Inc.
301 Dates Dr.
Ithaca, NY 14850-1326, USA
Internet: hoove@oracorp.com

Abstract

We explore an area that connects classical Hausdorff topology and the Scott domain theory and serves as a foundation for a denotational semantics of numerical programs.

Our key notion is that of a maximal limit space, a T_0 space $(X, \mathcal{T})$ in which every net that has a limit point has a unique limit point maximal in the specialization order induced by $\mathcal{T}$. Maximal limit spaces combine features of Hausdorff spaces and domains and form a bridge between those two categories. Every Hausdorff space is a maximal limit space, and maximal limit spaces are preserved under product, closed subspace, and function space constructions. A topological version of the lifting construction, familiar in domain theory, makes a maximal limit space into a compact maximal limit space. The upper powerspace construction makes a locally compact maximal limit space into a c.b.c. domain (continuous directed-complete partial order that is bounded-complete, i.e., any subset with an upper bound has a least upper bound) that is pointed (has a bottom element) if the original space was compact. C.b.c. domains are locally compact maximal limit spaces. The space of continuous functions from a locally compact topological space to a pointed c.b.c. domain is a pointed c.b.c. domain. The topology of pointwise convergence and the compact-open topology are identical on such function spaces.

The upper powerspace construction is functorial and behaves well in relation to function space formation.

1 Introduction

This paper introduces a category of topological spaces, the maximal limit spaces, which includes classical Hausdorff spaces and bounded-complete domains. Maximal limit spaces have Hausdorff-like properties, are closely related

[1] Thanks to Dana Hartman for her assistance in writing this paper.

to supersober spaces, and are closed under many familiar operations on topological spaces. Upper powerspaces of pointed, locally compact maximal limit spaces are pointed continuous, bounded-complete domains. The latter form a Cartesian closed category.

Our principal aim in developing the theory of maximal limit spaces and their powerspaces is to provide a domain theoretic basis for a denotational semantics of numerical programs adequate for describing the limit as roundoff error tends to zero [11,10]. A secondary aim is to suggest that the bridge between classical Hausdorff spaces and domains could be exploited in order to apply domain-theoretic ideas in mathematical analysis. Edalat [5] has demonstrated that powerdomains are also useful in connection with fractals and dynamical systems; [2] we anticipate further applications of powerdomains in set-valued analysis [3]. We suggest that in analytic applications it may also be possible to exploit the fact that the category of pointed continuous bounded complete domains is closed under function space formation.

Scott domains, or simply domains, and their relatives form a family of partial orders whose use in denotational semantics of computer programs is well known [18,28]. These semantics have a topological interpretation in terms of the Scott topology of these partial orders. In denotational semantics, computable functions are continuous and programming constructs induce continuous operations on computable functions. Scott domains have a connection with more general (including more classical) topological spaces via the Hofmann-Mislove Theorem (essentially 2.17, 2.19 and 2.21 in [9], stated as Theorem 4.2, below), which says that the space of compact upper (i.e., upward-closed with respect to the specialization order) subsets of a locally compact sober space is a domain and that a natural topology on this space is the same as its Scott topology. (This space of upper compact sets is called the *upper powerspace*.) Domains, however, are not adequate for all aspects of denotational semantics, because in considering denotations of functions and procedures, it is essential that the space of continuous functions between two semantic domains also be a domain. For this reason, *continuous, bounded-complete* (or *c.b.c.*) domains are more important than domains per se, because they have the property that the space of continuous functions between two pointed c.b.c. domains is a pointed c.b.c. domain (essentially Theorem 3.3 in [19]). [3] That is, the category of pointed c.b.c. domains is cartesian closed.

In this paper, we consider a particular class of T_0 spaces, the *maximal limit spaces*, a topological space $(X, \mathcal{T})$ in which every convergent sequence has a unique limit maximal in the specialization order of $\mathcal{T}$. Hausdorff spaces are characterized by the property that convergent sequences have unique limits. The defining property of maximal limit spaces is a natural weakening of this unique limit property that applies to T_0 topological spaces that are constructed using natural operations. In particular,

[2] We are grateful to one of the referees for pointing out the work of Edalat.

[3] We use the term c.b.c. domain instead of *bounded-complete* domain or BC-domain because in Gunter [8] BC-domains are algebraic, not just continuous.

- Hausdorff spaces are maximal limit spaces;
- maximal limit spaces are closed under lifting (adding a bottom element), products, and closed subspaces; and
- C.b.c. domains are maximal limit spaces.

Some other important properties of maximal limit spaces are the following.

- Maximal limit spaces are T_0.
- Maximal limit spaces are supersober, hence sober.
- Maximal limit spaces are bounded-complete (subsets with an upper bound in the specialization order have a least upper bound).
- Maximal limit spaces are characterized by a weakening of the Hausdorff separation property.
- The intersection of two compact upper subsets of a maximal limit space is compact upper.
- The upper powerspace of a locally maximal limit space is a c.b.c. domain.
- The space of continuous functions from a locally compact space to a maximal limit space is a maximal limit space (with the compact-open topology).

The second last result follows from the Hofmann-Mislove Theorem by the result about intersections of compact upper sets. Maximal limit spaces seem to be more useful in practise because their defining property is simpler and stronger and because they have better closure properties.

We also generalize the well-known result that c.b.c. domains are closed under forming spaces of continuous functions (essentially Theorem 3.3 in Scott [19]) by showing that in fact the space of continuous functions from any locally compact topological space to a pointed c.b.c. domain is a pointed c.b.c. domain. Another interesting result about function spaces is the in the space of continuous functions $D \to Y$, where D is a domain and Y is any topological space, the compact-open topology and the topology of pointwise convergence are the same.

The fact that locally compact Hausdorff spaces can be transformed by lifting and the upper powerspace construction into pointed c.b.c. domains, which are both quite well-behaved (in particular, locally compact) and closed under forming spaces of continuous functions, leads us to suggest that a useful and systematic approach to problems involving topologically difficult function spaces might be to start by lifting the basic spaces involved and forming their powerspaces. Then function spaces formed later will be better behaved. We present some basic results about the relationship between powerspace formation and function space formation that suggest that such an approach may be feasible. Although the Hausdorff property will be lost by the original lifting/powerspace construction, we think that the Hausdorff-like properties of maximal limit spaces and the better behavior of function spaces will provide good compensation.

An example of an application involving rather difficult function spaces (integrating path-dependent stochastic differential equations) is given in [12].

Related Work

The work most closely related to our use of powerspaces is that of Edalat [5,6], who has used upper powerspaces, especially the upper powerspace of the real numbers, in a number of areas of analysis, including dynamical systems, fractals, and integration theory.

Other work relating to topological powerspaces and domains is that of Nivat and his school [16], who use the fact that the space of closed subsets of a complete metric space (M, d) is both a complete metric space under the Hausdorff metric

$$d_H(C_1, C_2) = \max(\sup_{x \in C_1} \inf_{y \in C_2} d(x, y), \sup_{y \in C_2} \inf_{x \in C_1} d(x, y))$$

and a directed-complete partial order (DCPO) under the partial order

$$C_1 \sqsubseteq C_2 \iff C_1 \supseteq C_2$$

(the same as the specialization order in the upper powerspace). We feel that this approach has two drawbacks: the partial order and topology on the powerspace coexist but are not fully integrated; and topological limits can be used in semantic definitions only when convergence is unproblematic. For example, if $x_{2n} = 0$, $x_{2n+1} = 1$, then in the Hausdorff metric, $\{x_n\}$, $n \in \mathbb{N}$ does not converge, whereas in the upper topology it converges to $\{0, 1\}$. The latter seems more reasonable in the context of computer program semantics: it means that in the limit we do not know whether the answer will be 0 or 1, a nondeterministic result. In [11] we use maximal limit spaces to model convergence of a sequence of programs. There, the looser convergence criterion given by the upper topology is essential.

Certain results using Nivat's method are easily reproducible using upper powerspaces of suitable spaces; for example [11] contains results analogous to those of [16] on evaluation of infinitary real number expressions. Some results by de Bakker and his collaborators on semantics of communication streams (see [4] for a survey of their results) seem, however, to go deeper than this, since they use certain categories of metric spaces and nonexpansive mappings to solve domain equations. Perhaps the two approaches can be merged using *quasi-metric* spaces, first noted as important by Lawvere [15]. A quasi-metric is like a metric except that possibly $d(x, y) \neq d(y, x)$. A quasi-metric d on a space M induces a partial order by

$$x \sqsubseteq y \iff d(y, x) = 0$$

(same as the specialization order on the topology induced by d via $x_n \to y$ iff $d(x_n, y) \to 0$). The upper powerspace of a quasi-metric space is again a quasi-metric space under the Hausdorff quasi-metric,

$$d(C, D) = \sup_{x \in C} \inf_{y \in D} d(x, y).$$

Smyth [24] explores quasi-metric spaces in the context of computer program semantics. Quasi-uniform spaces, a generalization of quasi-metric spaces, are discussed in [23,26,27]. For our intended applications, however, plain topological spaces suffice.

Aberer [1] presents a quite different approach to denotational semantics involving real numbers, via combinatory differential fields.

We are grateful for advice from Achim Jung, which helped make us aware of work that might be related to this paper. Although we have not used the original results in it, the dissertation of Schalk [17] was useful in providing references on powerspaces.

2 Basic Definitions

A topology $\mathcal{T}$ on a set X induces a partial preorder $\sqsubseteq_{\mathcal{T}}$ on X, the *specialization (pre-) order* of $\mathcal{T}$, given by

$$x \sqsubseteq_{\mathcal{T}} y \iff \forall U \in \mathcal{T},\ x \in U \Rightarrow y \in U.$$

$\mathcal{T}$ is T_0 iff $\sqsubseteq_{\mathcal{T}}$ is a partial order.

In turn, a partial order $\sqsubseteq$ on a set D induces a Scott topology

$$\mathcal{S}_{\sqsubseteq} = \{U \subseteq D \mid U = U \text{ upper}, \big(S \text{ directed by } \sqsubseteq,\ S \cap U = \emptyset, \text{ and } \textstyle\bigsqcup S \text{ exists}\big) \Rightarrow \bigsqcup S \notin U\},$$

where U is *upper* if $x \in U$ and $x \sqsubseteq y$ implies $y \in U$. $\mathcal{S}_{\sqsubseteq}$ is always T_0 because if $y \not\sqsubseteq x$ then x and y are separated by the Scott-open set $\{z \mid z \not\sqsubseteq x\}$. In general, $\mathcal{S}_{\sqsubseteq_{\mathcal{T}}}$ and $\mathcal{T}$ may not be comparable, but it is always the case that $\sqsubseteq_{\mathcal{S}_{\sqsubseteq}} = \sqsubseteq$.

When $(X, \mathcal{T})$ is a topological space, $(N, \leq)$ is a directed set, and x_s, $s \in N$, is a net in X, we use limit notation $x_s \to x$ (or $x_s \xrightarrow{\mathcal{T}} x$ if we wish to identify or emphasize the topology) to indicate Moore-Smith convergence in $\mathcal{T}$. That is:

$$x_s \to x \iff \forall U \in \mathcal{T}\ (x \in U \to \exists s_0 \in N\ (\forall s \in N\, (s_0 \leq s \to x_s \in U))).$$

In a T_0 topological space, the limit of a net will not in general be unique, since $x_s \to x$ and $y \sqsubseteq_{\mathcal{T}} x$ implies $x_s \to y$.

The following definitions for the most part summarize one widespread form of terminology, the form used in [2]. In particular, here a DCPO or domain need not have a bottom element. This convention seems to be the most convenient convention for topological discussions, though the reverse is best for discussing denotational semantics, as in [8].

Definition 2.1

(i) A *DCPO* (directed-complete partial order) is a poset $(D, \sqsubseteq)$ such that every nonempty, $\sqsubseteq$-directed subset $S \subseteq D$ has a supremum $\bigsqcup S \in D$.

(ii) A T_0 topological space $(X, \mathcal{T})$ is *order consistent* ([25]) iff $\mathcal{T} \subseteq \mathcal{S}_{\sqsubseteq_{\mathcal{T}}}$ (iff for each directed $S \subseteq X$ that has a supremum, $\bigsqcup S$ is a $\mathcal{T}$-limit point of S).

(iii) A *DC* (directed-complete topological) space is an order consistent space $(D, \mathcal{T})$ such that $(D, \sqsubseteq_{\mathcal{T}})$ is a DCPO.

(iv) A topological space $(X, \mathcal{T})$ is a *maximal limit* space if for every convergent net x_s, $s \in N$, in X there exists a unique $x \in X$ such that

$$x_s \to x \quad \text{and} \quad x_s \to y \Rightarrow y \sqsubseteq_{\mathcal{T}} x.$$

We call x the *maximal limit* of x_s, $s \in S$, and write $x = \text{max-lim}_{s\in S}\, x_s$.

(v) An *H-space* is a locally compact maximal limit space. We call a poset $(D, \sqsubseteq)$ an H-space if $(D, \mathcal{S}_{\sqsubseteq})$ is an H-space.

(vi) For U a subset of a partially ordered set P, the *upper set* $\uparrow U$ of U is given by

$$\uparrow U = \{y \in P \mid \exists x \in U\, (x \sqsubseteq y)\}.$$

U is *upper* iff $U = \uparrow U$. We define the *lower set* $\downarrow U$ of U in the same way, with $\sqsupseteq$ replacing $\sqsubseteq$.

(vii) For x an element of a partial order $(P, \sqsubseteq)$, define

$$\mathord{\downarrow\!\!\downarrow} x = \{w \mid \uparrow w \text{ is a Scott neighborhood of } x\}.$$

(Such w are usually termed *way below* x, but we prefer to stick to a topological description. Note that $\uparrow w$ need not itself be open.)

(viii) A poset $(P, \sqsubseteq)$ is *continuous* iff for each $x \in P$, $\mathord{\downarrow\!\!\downarrow} x$ is a neighborhood basis for x in $\mathcal{S}_{\sqsubseteq}$. That is, if $x \in U \in \mathcal{S}_{\sqsubseteq}$ then there exists $w \in U \cap \mathord{\downarrow\!\!\downarrow} x$.

(ix) A *domain* is a continuous DCPO.

(x) A poset $(P, \sqsubseteq)$ is *bounded-complete* iff each nonempty $S \subseteq P$ that has an upper bound (lower bound) has a supremum (infimum). We say that a T_0 topological space is bounded-complete if it is complete in its specialization order.

(xi) A *c.b.c. domain* is a domain that is bounded-complete.

(xii) A poset $(P, \sqsubseteq)$ is pointed if it has a least element $\perp$.

It follows easily that in a domain $\mathord{\downarrow\!\!\downarrow} x$ is directed and $\bigsqcup \mathord{\downarrow\!\!\downarrow} x = x$.

Note that, by definition of $\sqsubseteq_{\mathcal{T}}$, $\mathcal{T}$-open sets are $\sqsubseteq_{\mathcal{T}}$-upper.

$\mathcal{S}_{\sqsubseteq_{\mathcal{T}}}$ may be finer than $\mathcal{T}$ in a DC-space $(X, \mathcal{T})$. For example, if $(X, \mathcal{T})$ is Hausdorff, then it is a DC-space (since all elements are $\sqsubseteq_{\mathcal{T}}$-incomparable), but $\mathcal{S}_{\sqsubseteq_{\mathcal{T}}}$ is the discrete topology.

Here are some easy results, which we state without proof.

Lemma 2.2 (i) *Any maximal limit space is T_0.*

(ii) *A Hausdorff space $(X, \mathcal{T})$ is a maximal limit space.*

(iii) *A pointed space is compact.*

(iv) *The lifting*

$$(X_\perp, \mathcal{T}_\perp) = (X \cup \{\perp\}, \mathcal{T} \cup \{X_\perp\})$$

of a DC-space $(X, \mathcal{T})$, where $\perp \notin X$, is a pointed DC space. A similar result holds for maximal limit spaces, H-spaces, DCPOs, domains and c.b.c. domains.

(v) *The Tykhonov product of a family of maximal limit spaces (respectively pointed H-spaces) is a maximal limit space (respectively pointed H-space).*

(vi) *The product of a family of pointed c.b.c. domains is a pointed c.b.c. domain.* □

For products of H-spaces or c.b.c. domains, pointedness is necessary in order to preserve local compactness or the continuity property.

We add one slightly more substantial result that shows when a topological space is a domain.

Lemma 2.3 *If a DC-space $(X, \mathcal{T})$ has a neighborhood basis of sets of the form $\uparrow x$ ($\uparrow$ with respect to $\sqsubseteq_{\mathcal{T}}$), then $(X, \sqsubseteq_{\mathcal{T}})$ is a domain and $\mathcal{T}$ is the Scott topology of $\sqsubseteq_{\mathcal{T}}$.*

Proof. Since $(X, \mathcal{T})$ is a DC-space, the Scott topology of $\sqsubseteq_{\mathcal{T}}$ refines $\mathcal{T}$. To show the converse inclusion, let S be a $\sqsubseteq_{\mathcal{T}}$-open subset of X and let $y \in S$. Let $\twoheaddownarrow y$ be the set of $x \in X$ such that $\uparrow x$ is a neighborhood of y. Since these sets $\uparrow x$ form a neighborhood basis for y, $\twoheaddownarrow y$ must be directed; hence $\bigsqcup \twoheaddownarrow y$ exists. Now clearly $y \sqsupseteq \bigsqcup \twoheaddownarrow y$ and no $\mathcal{T}$-open set contains y and not $\bigsqcup \twoheaddownarrow y$, so they must be equal. Since S is Scott-open, some $x \in S \cap \twoheaddownarrow y$. Since S is upper, $\uparrow x \subseteq S$. Thus S contains a $\mathcal{T}$-neighborhood of y. As y was an arbitrary element of S, S is $\mathcal{T}$-open. As S is an arbitrary $\sqsubseteq_{\mathcal{T}}$-Scott-open set, $\mathcal{S}_{\sqsubseteq_{\mathcal{T}}} \subseteq \mathcal{T}$. □

3 Maximal Limit Spaces and Supersober Spaces

In this section, we show that the maximal limit spaces are supersober (hence sober) spaces that have continuous bounded pairwise maxima. Equivalently, they are bounded-complete DC-spaces that have a continuous maximum function and satisfy a weak Hausdorff separation property. First we define the pertinent notions.

Definition 3.1 (i) A topological space $(X, \mathcal{T})$ *has continuous maxima* if the partial function $(x, y) \mapsto x \sqcup y$ is continuous (with the relative topology on its domain of definition).

(ii) A *c.b.c. space* is a bounded-complete DC-space with continuous maxima.

(iii) A *filter on a set* X is a set $\mathcal{F}$ of subsets of X such that:
- $\emptyset \notin \mathcal{F}$;
- $F, G \in \mathcal{F}$ implies $F \cap G \in \mathcal{F}$; and
- if $F \in \mathcal{F}$ and $F \subseteq G$ then $G \in \mathcal{F}$.

A *filter basis* on X is a set $\mathcal{F}$ of subsets of $\mathcal{F}$ that satisfies the first two conditions. An *ultrafilter* on X is a maximal filter on X.

(iv) A filter $\mathcal{F} \subset \mathcal{T}$ is *completely prime* (relative to $\mathcal{T}$) if $\mathcal{F} \neq \emptyset$, $\emptyset \notin \mathcal{F}$, and whenever $U_i \in \mathcal{T}$, $i \in I$, and $\bigcup_{i \in I} U_i \in \mathcal{F}$, there is some $i \in I$ such that $U_i \in \mathcal{F}$.

(v) A topological space $(X, \mathcal{T})$ is *sober* if every completely prime filter $\mathcal{F} \subset \mathcal{T}$ consists of all open neighborhoods of some point $x \in X$.

(vi) A filter basis $\mathcal{F}$ on a topological space $(X, \mathcal{T})$ *converges to* $x \in X$ iff $\mathcal{F}$ contains a neighborhood basis for x. We denote by $\lim \mathcal{F}$ the set of limits of $\mathcal{F}$.

(vii) A topological space is *supersober* if every convergent ultrafilter in it has a unique maximal limit.

(viii) A topological space is *weakly Hausdorff* if any two points $x, y \in X$ that do not have an upper bound have disjoint neighborhoods.

By Zorn's Lemma, every filter base can be extended to an ultrafilter.

Let $U \sqcup V = \{x \sqcup y \mid x \in U, y \in V, x \sqcup y \text{ exists}\}$.

Lemma 3.2 *A topological space $(X, \mathcal{T})$ has continuous maxima iff for all $x, y \in X$, if $x \bigsqcup y$ exists then it has a neighborhood basis of sets $U \cap V$, where $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$.*

Proof. ($\Rightarrow$) Suppose that $x, y \in X$ such that $x \sqcup y$ exists. Let W be a neighborhood of $x \sqcup y$. By continuity of $\sqcup$, there is $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that for $x' \in U$ and $y' \in V$, if $x' \sqcup y'$ exists, then $x' \sqcup y' \in W$. But if $x' = y' = z \in U \cap V$, then $z = x' \sqcup y' \in W$; hence $U \cap V \subseteq W$.

($\Leftarrow$) Similarly, for any two open sets U and V, $U \sqcup V = U \cap V$. Thus if $x \sqcup y$ exists and W is a neighborhood of $x \sqcup y$, let $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that $U \cap V \subseteq W$. Then $U \times V$ is a neighborhood of (x, y) such that $U \sqcup V \subseteq W$. Thus, $\sqcup$ is continuous. $\square$

Theorem 3.3 *The following are equivalent for a topological space $(X, \mathcal{T})$.*

(i) *$(X, \mathcal{T})$ is a maximal limit space.*

(ii) *$(X, \mathcal{T})$ is supersober, bounded complete and has continuous maxima.*

(iii) *$(X, \mathcal{T})$ is a weakly Hausdorff c.b.c. space.*

Proof. ($i \Rightarrow ii$) Let $(X, \mathcal{T})$ be a maximal limit space. Every filter (not just every ultrafilter) has a unique maximal limit. Therefore $(X, \mathcal{T})$ is supersober.

Suppose that $S \subseteq X$ is bounded above. Let x_s, $s \in N$, be a net such that $\{x_s \mid s \in N\}$ is the set of upper bounds of S, and such that for every upper bound y of S, $\{s \in N \mid x_s = y\}$ is cofinal in N. Then each $x \in S$ is a limit of x_s since $x_s \sqsupseteq x$ for all $s \in N$. Therefore max-lim$_s\, x_s$ is an upper bound of S. On the other hand, every upper bound y of S belongs to every neighborhood of max-lim$_s\, x_s$; hence $y \sqsupseteq$ max-lim$_s\, x_s$. Thus max-lim$_s\, x_s = \bigsqcup S$. Therefore $(X, \mathcal{T})$ is bounded-complete.

We show now $(X, \mathcal{T})$ has continuous maxima. Suppose that (x_s, y_s), $s \in N$ converges to (x, y), and suppose that $x_s \sqcup y_s$, $s \in N$, and $x \sqcup y$ all exist. Since $x_s \to x$ and $y_s \to y$,

$$\underset{s}{\text{max-lim}}\, x_s \sqsupseteq x, \; \underset{s}{\text{max-lim}}\, y_s \sqsupseteq y;$$

hence

$$\underset{s}{\text{max-lim}}\, (x_s \sqcup y_s) \sqsupseteq x \sqcup y.$$

Therefore $x_s \bigsqcup y_s \to x \bigsqcup y$. Thus $\sqcup$ is continuous.

($ii \Rightarrow iii$) First we show that a supersober space $(X, \mathcal{T})$ is weakly Hausdorff. If x and y do not have disjoint neighborhoods, then the set $\mathcal{F} = \{U \in \mathcal{T} \mid x \in U \text{ or } y \in U\}$ is a filter base. Extend it to an ultrafilter $\mathcal{U}$. Since $\mathcal{U}$ converges to both x and y, its maximal limit is an upper bound of x and y. Therefore $(X, \mathcal{T})$ is weakly Hausdorff.

By [7] Proposition IV.1.11, every supersober space is sober, and it is well-known that sober spaces are DC-spaces (e.g., [22]). Since $(X, \mathcal{T})$ is bounded-complete and has continuous maxima, it is a c.b.c. space.

$(iii \Rightarrow i)$ Let x_s, $s \in N$ be a convergent net in S. If $x_s \to x, y$ and $U \in \mathcal{N}_x$, $V \in \mathcal{N}_y$, then $x_s \in U \cap V$ for sufficiently large $s \in N$, so $U \cap V \neq \emptyset$. Therefore, by the weak Hausdorff property, x and y have an upper bound. Since $(X, \mathcal{T})$ is a c.b.c. space, $x \sqcup y$ exists and has a neighborhood basis of sets of the form $U \cap V$, $U \in N_x$, $V \in N_y$. Hence $x_s \to x \sqcup y$. Thus, the set L of limits of x_s, $s \in N$, is directed. Since L is closed and $(X, \mathcal{T})$ is a DC-space, $\bigsqcup L$ exists and belongs to L. Thus $\bigsqcup L$ is the unique maximal limit of x_s, $s \in N$. □

Corollary 3.4 *Every maximal limit space is sober.* □

The categories of supersober spaces, maximal limit spaces, and pointed H-spaces are all closed under Tychonov products. A closed subspace of a maximal limit space (or an H-space) is also a maximal limit space (H-space), but the same is not true of supersober spaces, since adding a top element to any space makes it supersober.

The following example shows that a supersober space, even if it is a domain with its Scott topology, need not be a maximal limit space.

Example 3.5 Consider a partial order $(P, \sqsubseteq)$ where $P = \{a, b, c, d\}$, $a, b \sqsubseteq c, d$ and no other order relations hold. P is a domain and is supersober in its Scott topology but is not bounded-complete, hence is not a c.b.c. space or a maximal limit space. It does not make any difference if we add a bottom element to make $(P, \sqsubseteq)$ pointed.

This counterexample is not surprising given that a domain need not be bounded-complete. Continuous bounded-complete domains, however, are maximal limit spaces.

Theorem 3.6 *Any c.b.c. domain with its Scott topology forms an H-space.*

Proof. Let $(D, \sqsubseteq)$ be a c.b.c. domain. Any set of the form $\uparrow x$ is compact, since any net in it converges to x. Since $\mathcal{S}_{\sqsubseteq}$ has a neighborhood basis of sets of this form, $(D, \mathcal{S}_{\sqsubseteq})$ is locally compact. To show that $(D, \mathcal{S}_{\sqsubseteq})$ has maximal limits, we use Theorem 3.3 and prove 3.3.iii. Suppose that $x, y \in D$ have no disjoint neighborhoods. Then each pair $u \in \downarrow x$ and $v \in \downarrow y$ has a upper bound, hence a supremum $u \sqcup v$. Since $\downarrow x$ and $\downarrow y$ are directed,

$$S = \{u \sqcup v \mid u \in \downarrow x,\ v \in \downarrow y\}$$

is directed. $\bigsqcup S \sqsupseteq x, y$ since it belongs to each neighborhood of x and y. In fact $\bigsqcup S = x \sqcup y$ since each upper bound of x and y is an upper bound of S. Finally, for each $u \in \downarrow x$ and $v \in \downarrow y$,

$$\uparrow u \cap \uparrow v = \uparrow (u \sqcup v)$$

is a neighborhood of $\bigsqcup S$. Since S is a limit point of $\bigsqcup S$, these sets must also form a neighborhood basis. It follows now by Theorem 3.3 that $(D, \mathcal{S}_{\sqsubseteq})$ is a maximal limit space. □

4 Compact Sets and Powerspaces

Definition 4.1 The *upper powerspace* [9,22] of a topological space $(X,\mathcal{T})$ is the space $(\mathcal{K}(X),\mathcal{K}(\mathcal{T}))$, where

$$\mathcal{K}(X)=\{K\subseteq X\mid K\text{compact upper}\},\quad \mathcal{K}(\mathcal{T})=\{N(U)\mid U\in\mathcal{T}\},$$

and

$$N(U)=\{K\in\mathcal{K}(X)\mid K\subseteq U\}.$$

Thus, a neighborhood of K in $\mathcal{K}(X)$ corresponds to what we would normally call a neighborhood of K.

The upper powerspace is the topological analog of the Smyth powerdomain. Smyth [22] showed that for ω-algebraic domains the upper powerspace is the same as the Smyth powerdomain [21]. Further results about powerspaces are given in Schalk [17]. Here we show that the upper powerspace of an H-space is a c.b.c. domain.

Theorem 4.2 (Hofmann and Mislove [9]) *If $(X,\mathcal{T})$ is a locally compact sober space, then $(\mathcal{K}(X),\supseteq)$ is a domain and its Scott topology is $\mathcal{K}(\mathcal{T})$. If X is also compact (in particular, if X is pointed), then $(\mathcal{K}(X),\mathcal{K}(\mathcal{T}))$ is pointed.*

Proof. By [9] Corollary 2.17, $(\mathcal{K}(X),\sqsupseteq)$ is a continuous partial order. By [9] Lemma 2.21, $\uparrow J$ is a Scott-neighborhood of K iff $K\subseteq$ *interior*(J). But that is equivalent to $N(J)$ being a neighborhood of K in $\mathcal{K}(\mathcal{T})$. Therefore $\mathcal{K}(\mathcal{T})$ is the Scott topology of $(\mathcal{K}(X),\supseteq)$. By [9] Proposition 2.19(i), the intersection of a $\supseteq$-directed family $S\subseteq\mathcal{K}(X)$ is compact and upper. By [9] 2.19(ii), such an intersection is nonempty and the limit of S. It follows that $(\mathcal{K}(X),\mathcal{K}(\mathcal{T}))$ is also a DC-space, hence a domain. If X is compact, then it is the bottom element of $\mathcal{K}(X)$, making the latter pointed. □

The unique maximal limit property of topological domains contributes the following.

Lemma 4.3 *In a maximal limit space, the intersection of two compact upper sets is a compact upper set.*

Proof. Let J and K be compact upper sets. Clearly $J\cap K$ is upper. Let x_s, $s\in N$ be a net in $J\cap K$. By compactness of J and K, x_s has a subnet that converges in J, which has a further subnet that converges in K. The maximal limit of this subnet must belong to both J and K since they are upper. Hence $J\cap K$ is compact. □

Theorem 4.4 *If $(X,\mathcal{T})$ is an H-space, then $(\mathcal{K}(X),\supseteq)$ is a c.b.c. domain and $\mathcal{K}(\mathcal{T})$ is $\mathcal{S}_{\supseteq}$. If X is also compact, then $\mathcal{K}(X)$ is pointed.*

Proof. Everything but the "BC" part follows by Theorem 3.4, and Theorem 4.2. Now if $J,K\supseteq L$, $J,K,L\in\mathcal{K}(X)$, then $J\cap K$ is nonempty, compact (by 4.3) and certainly upper; hence $J\cap K$ is the least upper bound of J and K in $(\mathcal{K}(X),\supseteq)$. But in a DCPO, existence of suprema of bounded pairs implies bounded completeness. □

By [14] Theorem 25, this theorem holds for locally compact supersober spaces.

We conclude this section by observing that $\mathcal{K}$ is in fact a functor. Given H-spaces X and Y and a continuous function $f : X \to Y$, define $\mathcal{K}(f)$ by

$$\mathcal{K}(f)(K) = \uparrow f[K] = \uparrow\{f(x) \mid x \in K\}. \tag{4.1}$$

Theorem 4.5 *$\mathcal{K}$ is a functor from H-spaces to c.b.c. domains.*

Proof. Let X, Y, and Z be H-spaces, and let $f : X \to Y$ and $g : Y \to Z$ be continuous mappings. We must show that $\mathcal{K}(f)$ maps $\mathcal{K}(X) \to \mathcal{K}(Y)$, is continuous, and that $\mathcal{K}(g \circ f) = \mathcal{K}(g) \circ \mathcal{K}(f)$. Now $\mathcal{K}(f)$ maps $\mathcal{K}(X)$ to $\mathcal{K}(Y)$ because the continuous image of a compact set is compact and the upper set of a compact set is compact. To show that $\mathcal{K}(f)$ is continuous, observe that for $V \subset Y$ open, $\mathcal{K}(f)^{-1}(N(V)) = N(f^{-1}(V))$. Because f is continuous, $f^{-1}(V)$ is open. Therefore $N(f^{-1}(V))$ is open, so $\mathcal{K}(f)$ is continuous. To show that $\mathcal{K}$ preserves composition of functions, observe that for

$$\mathcal{K}(g \circ f)(K) = \uparrow(g \circ f)[K] = \uparrow g[f[K]] = \uparrow g[\uparrow f[K]] = \mathcal{K}(g) \circ \mathcal{K}(f)(K).$$

The third equality follows because g, being continuous, is monotone. □

Since lifting $X \mapsto X_\perp$ is a functor from H-spaces to pointed H-spaces, which are compact, we have the following corollary.

Corollary 4.6 *$X \mapsto \mathcal{K}(X_\perp)$ is a functor from the category of H-spaces (which includes locally compact Hausdorff spaces) to the category of c.b.c. domains (both with continuous functions).* □

5 Spaces and Domains of Functions

If $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are topological spaces, let $C(X, Y)$ denote the space of continuous mappings from X to Y. Order $C(X, Y)$ by

$$f \sqsubseteq g \iff \forall x \in X \ (f(x) \sqsubseteq g(x)).$$

The *compact-open topology* on $C(X, Y)$ is the topology given by the subbasis of sets

$$N(K, U) = \{f \mid f(K) \subseteq U\},$$

where $K \subseteq X$ is compact, and $U \subseteq Y$ is open. Note that we need only consider K's that are $\sqsubseteq_{\mathcal{T}}$-upper, since any continuous function f is monotone and any open set U is $\sqsubseteq_{\mathcal{S}}$-upper. See Kelley [13] for a treatment of the compact-open topology in connection with Hausdorff topological spaces. The compact-open topology is important for the following reason.

Lemma 5.1 (i) *If X is locally compact and $C(X, Y)$ has the compact-open topology, then the evaluation mapping $e : X \times C(X, Y) \to Y$ is continuous.*

(ii) *If $\mathcal{D}$ is a topology on $C(X, Y)$ for which the evaluation mapping e continuous, then $\mathcal{D}$ refines the compact-open topology.* □

The proofs given in [13] for Hausdorff spaces apply without modification.

We need the following simple Lemma to help us show that c.b.c. domains are preserved under function space formation.

Lemma 5.2 *Let X be a topological space and let Y be a pointed DC-space. Let $U \subseteq X$ be open and let $f : U \to Y$ be continuous. Then the function $g : X \to Y$ given by*

$$g(x) = \begin{cases} f(x) & \text{if } x \in U, \\ \bot & \text{otherwise.} \end{cases}$$

also is continuous.

Proof. For $V \subseteq Y$ open,

$$g^{-1}(V) = \begin{cases} f^{-1}(V), & \text{if } \bot \notin V, \\ X, & \text{if } \bot \in U. \end{cases}$$

In either case, $g^{-1}(V)$ is open. Hence g is continuous. □

Theorem 5.3 (i) *The specialization order of the compact-open topology on $C(X,Y)$ is $\sqsubseteq$.*

(ii) *If Y is a DC-space, then $(C(X,Y), \sqsubseteq)$ is a DC-space. If Y is pointed, then $(C(X,Y), \sqsubseteq)$ is pointed.*

(iii) *If X is locally compact and Y is a maximal limit space, then $C(X,Y)$ with the compact-open topology is a maximal limit space.*

(iv) *If X is locally compact and D is a pointed c.b.c. domain, then the family $(C(X,D), \sqsubseteq)$ is a pointed c.b.c. domain and the compact-open topology is the Scott topology of $\sqsubseteq$.*

We remark that 5.3.iii does not mean that the category of H-spaces is closed under function space formation because the space $C(X,Y)$ need not be locally compact. Rather, it is a nontrivial analog of the trivial theorem that if Y is Hausdorff then $C(X,Y)$ is Hausdorff.

Proof.

Proof of 5.3.i: Let $\sqsubseteq_{CO}$ denote the specialization order of the compact-open topology on $C(X,Y)$.

Suppose that $f \sqsubseteq_{CO} g$. Let $x \in X$ and let U be any open set containing $f(x)$. Since $\{x\}$ is compact, $N(\{x\},U)$ is CO-open. Since $f \in N(\{x\},U)$, $g \in N(\{x\},U)$; that is $g(x) \in U$. As U was an arbitrary open neighborhood of $f(x)$, it follows that $f(x) \sqsubseteq_S g$. Thus $f \sqsubseteq g$.

On the other hand, suppose that $f \sqsubseteq g$ and that $f \in N(K,U)$; that is, for all $x \in K$, $f(x) \in U$. Since for all $x \in K$, $g(x) \sqsupseteq f(x)$, $g(x) \in U$. Therefore $g \in N(K,U)$. Since this is true of an arbitrary subbasic open set in the compact-open topology, $f \sqsubseteq_{CO} g$. Thus $\sqsubseteq$ and $\sqsubseteq_{CO}$ are identical.

Proof of 5.3.ii: If Y is pointed then the bottom element of $C(X,Y)$ is the function $\lambda x.\bot$. To prove that $C(X,Y)$ is a DC-space, we need to show that any directed family f_s, $s \in N$, of elements of $C(X,Y)$ has a supremum f such that $f_s \to f$ in the compact-open topology. Clearly, if the function

$$f(x) = \bigsqcup_s f_s(x)$$

is continuous then it is $\bigsqcup_s f_s$. Let U be an open subset of Y. Since U is open and upper,

$$f^{-1}(U) = \bigcup_s f_s^{-1}(U).$$

This set is open since each f_s is continuous; hence f is continuous. Suppose that for some compact $K \subseteq X$ and open $V \subseteq Y$, $f[K] \subseteq V$. Let $x \in K$. Since $f_s(x) \to f(x)$, there is s_x such that for $s \geq s_x$ (where $\leq$ is the order directing the net), $f_{s_x}(x) \in V$. By continuity of f_{s_x}, x has a neighborhood U_x such that $f_{s_x}[U_x] \subseteq V$. Since f_s is monotone in s, $f_s[U_x] \subseteq V$ for all $s \geq s_x$. Since K is compact, finitely many $U_{x_1}, \ldots, U_{x_n}$ cover K. Choose $s_0 \geq s_{x_1}, \ldots, s_{x_n}$. Then for $s \geq s_0$, $f_s[U_{x_1} \cup \ldots \cup U_{x_n}] \subseteq V$, hence $f_s[K] \subseteq V$. We have shown that if $f \in N(K, V)$ then there is s_0 such that for $s \geq s_0$, $f_s \in N(K, V)$. Since $N(K, V)$ is an arbitrary subbasic open set, this proves that $f_s \to f$ in the compact-open topology.

Proof of 5.3.iii: We use the characterization in Theorem 3.3. Since $C(X, Y)$ is a DC-space, it suffices to show that any pair of functions $f, g \in C(X, Y)$ that have no disjoint neighborhoods have a supremum $f \sqcup g \in C(X, Y)$ and that the function $f, g \mapsto (f \sqcup g)$ is continuous.

Suppose that $f, g \in C(X, Y)$ do not have disjoint neighborhoods. Consider $x \in X$. The map $e_x : h \mapsto h(x)$ is continuous because $e_x^{-1}(U) = N(\{x\}, U)$ is a subbasic open set. If U and V are respectively neighborhoods of $f(x)$ and $g(x)$, then $e_x^{-1}(U) = N(\{x\}, U)$ and $e_x^{-1}(V) = N(\{x\}, V)$ are neighborhoods of f and g respectively; hence there is $h \in N(\{x\}, U) \cap N(\{x\}, V) = N(\{x\}, U \cap V)$. We have $h(x) \in U \cap V$. Therefore $f(x)$ and $g(x)$ do not have disjoint neighborhoods. By 3.3, they have a least upper bound $f(x) \sqcup g(x)$. Since the map $(x, y) \mapsto x \sqcup y$ is continuous, it follows that the function

$$h(x) = f(x) \sqcup g(x)$$

is continuous; it must be $f \sqcup g$.

It remains to show that the mapping $(f, g) \mapsto f \sqcup g$ is continuous. We show that $f \sqcup g$ has a neighborhood basis of sets of the form $W_f \cap W_g$, where W_f and W_g are respectively neighborhoods of f and g. It suffices to show that if $f \sqcup g \in N(K, U)$ for some compact $K \subseteq X$ and open $U \subseteq Y$ then there are W_f and W_g as described such that $f \sqcup g \in W_f \cap W_g \subseteq N(K, U)$.

Consider $x \in K$. Since $f(x) \sqcup g(x) \in U$ and Y is a maximal limit space, $f(x)$ and $g(x)$ have respective neighborhoods $V_{f,x}$ and $V_{g,x}$ such that $f(x) \sqcup g(x) \in V_{f,x} \cap V_{g,x} \subseteq U$. By continuity of f and g and local compactness of X, x has a compact neighborhood K_x such that $f \in N(K_x, V_{f,x})$ and $g \in N(K_x, V_{g,x})$.

Find such K_x, $V_{f,x}$ and $V_{g,x}$ for each $x \in K$. By compactness of K, finitely many $K_{x_1}, \ldots, K_{x_n}$ cover K. Let

$$W_f = N(K_{x_1}, V_{f,x_1}) \cap \ldots \cap N(K_{x_n}, V_{f,x_n})$$

and

$$W_g = N(K_{x_1}, V_{g,x_1}) \cap \ldots \cap N(K_{x_n}, V_{g,x_n}).$$

W_f and W_g are as required.

It follows that $C(X,Y)$ is a maximal limit space.

Proof of 5.3.iv: It suffices to show that the compact-open topology has a neighborhood basis of sets of the form $\uparrow g$. The rest follows by Lemma 2.3, 5.3.i, and 5.3.iii.

We show that if $K \subseteq X$ is compact upper, $U \subseteq D$ is open and $f \in N(K,U)$, then there is $g \in N(K,U)$ such that $\uparrow g$ is a neighborhood of f in the compact-open topology. This is enough because if

$$N(K_1,U_1) \cap \ldots \cap N(K_n,U_n)$$

is a basic open neighborhood of f,

$$g_i \in N(K_i,U_i),\ i = 1,\ldots,n,$$

and $\uparrow g_i$ is a neighborhood of f, $i = 1,\ldots,n$, then

$$g_1 \sqcup \ldots \sqcup g_n \in N(K_1,U_1) \cap \ldots \cap N(K_n,U_n)$$

and

$$\uparrow(g_1 \sqcup \ldots \sqcup g_n) = \uparrow g_1 \cap \ldots \cap \uparrow g_n$$

is a neighborhood of f.

Suppose then that $f(K) \subseteq U$. Since $f(K)$ is compact and D is a continuous partial order, for some m there exist $y_1,\ldots,y_m \in U$ such that

$$W = \mathit{interior}(\uparrow y_1) \cup \ldots \cup \mathit{interior}(\uparrow y_m)$$

is an open neighborhood of $f(K)$. It follows that $f^{-1}(W)$ is an open neighborhood of K. By local compactness of D, K has a compact neighborhood $K' \subseteq f^{-1}(W)$. For $i = 1,\ldots,m$, Define

$$g_i(x) = \begin{cases} y_i, & \text{if } f(x) \in \mathit{interior}(\uparrow y_i), x \in \mathit{interior}(K'), \\ \bot, & \text{otherwise.} \end{cases}$$

By Lemma 5.2 applied to the constant function $x \mapsto y_i$ and the open set $\mathit{interior}(K') \cap f^{-1}(\mathit{interior}(\uparrow y_i))$, each g_i is continuous. Furthermore $g_1,\ldots,g_m$ have a common upper bound, namely f. Hence

$$g(x) = \bigsqcup_{i=1}^{m} g_i$$

exists and, by 5.3.iii, is continuous. We have

$$f \in N(K',W) \subseteq \uparrow g \subseteq N(K,U).$$

Hence $\uparrow g$ is a neighborhood of f contained in $N(K,U)$. □

The foregoing Lemma shows that the limit of a directed family of functions is just the pointwise supremum. The point of maximal limit spaces (including c.b.c. spaces and domains), however, is that all nets have a unique maximal limit. The following Theorem characterizes that limit in $C(X,Y)$.

Theorem 5.4 *Let X be locally compact, let Y be a c.b.c. domain, and let f_s, $s \in N$ be a net of functions in $C(X,Y)$ (not necessarily directed by $\sqsubseteq$). Then* $\text{max-lim}_s f_s$ *(in the compact-open topology) is the function*

$$f(x) = \operatorname*{max-lim}_{x' \to x, s} f_s(x').$$

Here, the notation $\text{max-lim}_{x'\to x,s} f_s(x')$ denotes

$$\operatorname*{max-lim}_{(x',U,s)} f_s(x'),$$

over the directed set $(M, \leq)$, where M is the set of triples (x', U, s) such that U is a neighborhood of x, $x' \in U$, and $s \in N$, and

$$(x', U, s) \leq (x'', V, t) \iff U \supseteq V \text{ and } s \leq t.$$

Proof. Since evaluation is continuous in the compact-open topology, $x' \to x$ and $f_s \to g$ implies that $f_s(x') \to g(x)$, hence $g(x) \sqsubseteq f(x)$. Thus, f will be the maximal limit of f_s, $s \in N$ if it is a limit at all.

First, we show that f is continuous. Fix $x \in X$. Since Y is a c.b.c. domain, there is $y \in Y$ such that $\uparrow y$ is a neighborhood of $f(x)$. By definition of f, there is a neighborhood V of x and $s \in N$ such that if $x' \in V$ and $t \geq s$ then $f_t(x') \sqsupseteq y$. If $K \subset V$ is compact neighborhood of x, then for $x'' \in K$,

$$f(x'') = \operatorname*{max-lim}_{x'\to x'', s\in N} f_s(x') \sqsupseteq y,$$

since y itself is a limit of $f_s(x')$ over $s \in N$ and $x' \to x''$. Since for each $x \in X$, $f(x)$ has a neighborhood basis of sets of the form $\uparrow y$, and for each such neighborhood x has a neighborhood K such that $f[K] \subseteq \uparrow y$, f is continuous.

Now we prove $f_s \to f$ in the compact-open topology. Let $N(K, U)$ be a subbasic open neighborhood of f, $K \subseteq X$ compact, $U \subseteq Y$ open. By definition of f, for each $x \in K$ there is $s_x \in N$ and a neighborhood V_x of x such that $s \leq t$ and $x' \in V_x$ implies $f_s(x') \in U$. Since K is compact, finitely many neighborhoods $V_{x_1}, \ldots, V_{x_n}$ cover K. Let $s \geq s_{x_1}, \ldots, s_{x_n}$. Then for $t \geq s$ and $x' \in K \subseteq V_{x_1} \cup \ldots \cup V_{x_n}$, $f_t(x') \in U$. Hence for $t \geq s$, $f_t \in N(K, U)$. Therefore $f_s \to f$ in the compact-open topology. □

When X is a domain, the compact-open topology on $C(X, Y)$ turns out to be very simple. First a lemma about the subbasis of the compact open topology.

Lemma 5.5 *Let X and Y be topological spaces. Let X be locally compact with neighborhood basis of compact subsets $\mathcal{K}$, and let $\mathcal{B}$ be a basis of open sets for the topology on Y. Then the compact-open topology has a subbasis of open sets of the form $N(K, B)$, $K \in \mathcal{K}$, $B \in \mathcal{B}$.*

Proof. We must show that for any $K \subseteq X$ compact, $U \subseteq Y$ open, and $f \in N(K, U)$, there are $n \in \mathbb{N}$, $K_1, \ldots, K_n \in \mathcal{K}$, and $B_1, \ldots, B_n \in \mathcal{B}$, such that

$$f \in N(K_1, B_1) \cap \ldots \cap N(K_n, B_n) \subseteq N(K, U).$$

Since $\mathcal{B}$ is a basis, $U = \bigcup_{i\in I} B_i$ for some $B_i \in \mathcal{B}$, $i \in I$. Since $f[K] \subseteq U$, for each $x \in K$, $f(x) \in B_i$ for some $i \in I$. For each $x \in K$, choose one such B_i and call it B_x. Since f is continuous, x has some compact neighborhood $K_x \in \mathcal{K}$ such that $f[K_x] \subseteq B_x$. Since K is compact, there exist n and $x_1, \ldots, x_n \in K$ such that $K_1 = K_{x_1}, \ldots, K_n = K_{x_n}$ cover K. If $B_1 = B_{x_1}, \ldots, B_n = B_{x_n}$, then $f \in N(K_j, B_j)$, $j = 1, \ldots, n$. On the other hand, $N(K_1, B_1) \cap \ldots \cap N(K_n, B_n) \subseteq N(K, U)$, since if $g[K_j] \subseteq B_j$, $j = 1, \ldots, n$, then $g[K] \subseteq g[\bigcup_{j=1}^n K_j] \subseteq \bigcup_{j=1}^n B_j \subseteq U$; hence $g \in N(K, U)$. □

Corollary 5.6 *If D is a domain, E is a topological space, and $\mathcal{B}$ is a basis of open sets for E, then the compact-open topology on $C(D,E)$ has a subbasis consisting of sets of the form $N(\uparrow x, B)$ where $x \in D$ and $B \in \mathcal{B}$.*

Proof. Apply the previous Lemma with $\mathcal{K} = \{\uparrow x \mid x \in D\}$. □

Theorem 5.7 *If D is a domain and Y is any topological space, then on $C(D,Y)$, the compact-open topology and the topology of pointwise convergence are identical.*

Proof. The compact-open topology contains the topology of pointwise convergence, so it suffices to show that if $f_s(x) \to f(x)$ for all $x \in D$, then $f_s \to f$ in the compact open topology. Suppose the former. By the preceding Corollary, it suffices to show that for each subbasic open set $N(\uparrow x, U)$ of the compact-open topology such that $f \in N(\uparrow x, U)$, $f_s[\uparrow x] \subseteq U$ for sufficiently large $s \in N$. But by pointwise convergence, there is s_0 such that for $s \geq s_0$, $f_s(x) \in U$. Hence $f_s(\uparrow x) \subseteq U$, since f is monotone and U is upper. □

6 Powerspaces and Function Spaces

In the introduction we suggested that problems involving function spaces might be approached by first converting the base spaces involved into c.b.c. domains using the upper powerspace construction, then forming function spaces. Although the usefulness of such an approach can be shown only by applying it to a nontrivial problem, which is beyond the scope of this paper, we will give some elementary results that show that this approach could conceivably work. In particular, we will show that for X and Y H-spaces, the embedding $\mathcal{K} : C(X,Y) \to C(\mathcal{K}(X), \mathcal{K}(Y))$ has reasonable properties.

In the following, we use $\uparrow K$ in the sense of $\mathcal{K}(X)$; that is, $\uparrow K = \{J \in \mathcal{K}(X) \mid J \subseteq K\}$.

Theorem 6.1 *The mapping $\mathcal{K} : C(X,Y) \to C(\mathcal{K}(X), \mathcal{K}(Y))$ is a homeomorphic embedding.*

Proof. $\mathcal{K}$ is trivially one to one. By Lemma 5.6, the compact-open topology on $C(\mathcal{K}(X), \mathcal{K}(Y))$ has a subbasis of open sets of the form $N(\uparrow K, N(U))$. But

$$\mathcal{K}^{-1}(N(\uparrow K, N(U))) = N(K, U).$$

Since the inverse image of each subbasic open set is open, $\mathcal{K}$ is continuous. The same applies to $\mathcal{K}^{-1}$. □

Another embedding of interest is that of $C(X, \mathcal{K}(Y))$ into $C(\mathcal{K}(X), \mathcal{K}(Y))$. In fact, the mapping

$$\Phi : C(X, \mathcal{K}(Y)) \to C(\mathcal{K}(X), \mathcal{K}(Y)) \text{ by } \Phi(f)(K) = \uparrow(\bigcup f[K]) = \uparrow(\bigcup\{f(x) \mid x \in K\})$$

and the mapping

$$\Phi_* : C(\mathcal{K}(X), \mathcal{K}(Y)) \to C(X, \mathcal{K}(Y)) \text{ by } \Phi_*(F)(x) = F(\uparrow x)$$

form a section-retraction pair in the sense of [20] (see also [2], Section 3.1.1). That is, Φ and Φ_* are continuous and $\Phi_* \circ \Phi$ is the identity function. It follows that Φ is a homeomorphic embedding.

Lemma 6.2 *Let $(X, \mathcal{T})$ be a topological space. The mapping $U : \mathcal{K} \mapsto \uparrow(\bigcup \mathcal{K})$ defines a continuous map $\mathcal{K}(\mathcal{K}(X)) \to \mathcal{K}(X)$.*

Proof. First we show that if $\mathcal{K} \in \mathcal{K}(\mathcal{K}(X))$ then $U(\mathcal{K}) \in \mathcal{K}(X)$.

It suffices to show that $\bigcup \mathcal{K}$ is compact, since the upper set of a compact set is compact. Let $\mathcal{U}$ be an open cover of $\bigcup \mathcal{K}$. For each $K \in \mathcal{K}$, $\mathcal{U}$ is an open cover of the compact set $K \subseteq X$; hence there is a finite $\mathcal{U}_K \subseteq \mathcal{U}$ such that $K \subseteq \bigcup \mathcal{U}_K$. Thus $K \in N(\bigcup \mathcal{U}_K)$; hence $\{\bigcup \mathcal{U}_K \mid K \in \mathcal{K}\}$ is an open cover of $\mathcal{K}$. Choose a finite subcover $\{\bigcup \mathcal{U}_{K_1}, \ldots, \bigcup \mathcal{U}_{K_n}\}$. Since each $\mathcal{U}_{K_i}$ is finite, $\mathcal{U}_{K_1} \cup \ldots \cup \mathcal{U}_{K_n}$ is a finite subcover of $\bigcup \mathcal{K}$. As $\mathcal{U}$ was arbitrary, it follows that $\bigcup \mathcal{K}$ is compact.

For any basic open set $N(U)$, $U \subseteq X$ open, $\uparrow(\bigcup \mathcal{K}) \in N(U)$ iff $\mathcal{K} \in N(N(U))$. Thus U is continuous. □

Theorem 6.3 *Let Φ and Φ_* be the mappings defined above. For each continuous map $f \in C(X, \mathcal{K}(Y))$, $\Phi(f)$ is in fact a continuous function $\mathcal{K}(X) \to \mathcal{K}(Y)$, and for each continuous map $F \in C(\mathcal{K}(X), \mathcal{K}(Y))$, $\Phi_*(F)$ is in fact a continuous function $X \to \mathcal{K}(Y)$. Furthermore, Φ and Φ_* are continuous and $\Phi_* \circ \Phi$ is the identity function.*

Proof. $\Phi = U \circ \mathcal{K}$ and Φ_* is just composition with the continuous mapping $x \mapsto \uparrow x$. That proves everything but the last statement. Suppose that $f \in C(X, \mathcal{K}(Y))$ and $x \in X$. Then

$$\bigcup f[\uparrow x] = f(x)$$

because f is monotone, and $\uparrow f(x) = f(x)$ since $f(x)$, being a member of $\mathcal{K}(Y)$, is already upper. Hence

$$\Phi_*(\Phi(f))(x) = \uparrow(\bigcup f[\uparrow x]) = f(x).$$

As x and f were arbitrary, $\Phi_* \circ \Phi$ is the identity function. □

Corollary 6.4 *The mapping $\Phi \colon C(X, \mathcal{K}(Y)) \to C(\mathcal{K}(X), \mathcal{K}(Y))$ is a homeomorphic embedding.* □

7 Summary

We have identified three key types of topological space that are closely related to domains: maximal limit spaces; H-spaces; and c.b.c. domains.

A maximal limit space is a topological space $(X, \mathcal{T})$ in which each net has a unique $\sqsubseteq_{\mathcal{T}}$-maximal limit. An H-space is a locally compact maximal limit space. A c.b.c. domain is a continuous DCPO in which each subset with an upper bound has a least upper bound.

The key results are as follows.

(i) Maximal limit spaces are closed under liftings and products. H-spaces are also closed under lifting and finite products. Pointed H-spaces (with a bottom element) are closed under arbitrary products.

(ii) The upper powerspace of an H-space is a c.b.c. domain.

(iii) The space of continuous functions $C(X,Y)$ from a locally compact space $(X,\mathcal{T})$ to a maximal limit space $(Y,\mathcal{S})$ is a maximal limit space in the compact-open topology.

(iv) The space of continuous functions $C(X,D)$ from a locally compact space $(X,\mathcal{T})$ to a pointed c.b.c. domain $(D,\sqsubseteq)$ is a c.b.c. domain with the Scott topology of the pointwise order equal to the compact-open topology.

(v) Under the foregoing conditions we characterize maximal limits in the function space $C(X,D)$.

(vi) In the space $C(Y,D)$ of continuous functions from a domain D to a topological space Y, the topology of pointwise convergence and the compact-open topology are identical.

(vii) We show that the embedding $C(X,Y) \to C(\mathcal{K}(X),\mathcal{K}(Y))$ is homeomorphic, and that the natural mappings $C(X,\mathcal{K}(Y)) \to C(\mathcal{K}(X),\mathcal{K}(Y))$ and $C(\mathcal{K}(X),\mathcal{K}(Y)) \to C(X,\mathcal{K}(Y))$ form a section-retraction pair.

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