Laplacian coefficients of trees with a given bipartition

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**A R T I C L E I N F O**

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**A B S T R A C T**

Let $G$ be a graph of order $n$ and $\mu(G, \lambda) = \sum_{k=0}^{n} (-1)^k c_k \lambda^{n-k}$ the Laplacian characteristic polynomial of $G$. Zhou and Gutman [19] proved that among all trees of order $n$, the $k$th coefficient $c_k$ is largest when the tree is a path and is smallest for a star. In this paper, for two given positive integers $p$ and $q$ ($p \leq q$), we characterize the trees with a given bipartition $(p, q)$ which have the minimal and second minimal Laplacian coefficients.

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1. Introduction

Let $G$ be a graph with $n$ vertices and $A(G)$ the adjacency matrix of $G$. The characteristic polynomial of $G$ is defined as $\phi(G, \lambda) = \det(\lambda I_n - A(G))$, where $I_n$ is the unit matrix of order $n$. It is well known [3] that if $T$ is a tree with $n$ vertices, then

$$\phi(T) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k m_k(T) \lambda^{n-2k},$$

(1.1)

where $m_k(T)$ equals the number of matchings with $k$ edges of $T$, and $\lfloor \frac{n}{2} \rfloor$ denotes the largest integer no more than $\frac{n}{2}$.

Gutman [5] introduced a quasi-ordering relation “$\geq$” on the set of all forests (acyclic graphs) with $n$ vertices: if $T_1$ and $T_2$ are two trees with $n$ vertices and with characteristic polynomials in the form...
The Laplacian-like energy of graph $G$, then

\[ T_1 \geq T_2 \iff m_k(T_1) \geq m_k(T_2) \tag{1.2} \]

for all $k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$. If $T_1 \geq T_2$ and there exists a $j$ such that $m_j(T_1) > m_j(T_2)$, then we write $T_1 > T_2$.

The Laplacian polynomial $\mu(G, \lambda)$ of $G$ is the characteristic polynomial of its Laplacian matrix $L(G) = D(G) - A(G)$, that is,

\[ \mu(G, \lambda) = \det(\lambda I_n - L(G)) = \sum_{k=0}^{n} (-1)^k c_k(G)\lambda^{n-k}. \tag{1.3} \]

The Laplacian matrix $L(G)$ has non-negative eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \mu_n = 0$ [4]. It is easy to see that $c_0(G) = 1$, $c_1(G) = 2|E(G)|$, $c_n(G) = 0$, $c_{n-1}(G) = n\tau(G)$, where $\tau(G)$ denotes the number of spanning trees of $G$. If $G$ is a tree, coefficient $c_{n-2}(G)$ is equal to its Wiener index $W(G)$, which is the sum of distances between all pairs of vertices, that is,

\[ c_{n-2}(G) = W(G) = \sum_{u,v\in V(G)} d(u, v). \]

The Laplacian-like energy of graph $G$ is defined as follows [11]:

\[ \text{LEL}(G) = \sum_{k=1}^{n-1} \sqrt{\mu_k}. \]

Zhou and Gutman [19] proved that for any tree $T$ of order $n$ and for any $k$, $c_k(K_{1,n-1}) \leq c_k(T) \leq c_k(P_n)$ ($0 \leq k \leq n$), where $K_{1,n-1}$ and $P_n$ are the star and the path of order $n$, respectively. Mohar [12] presented a different proof of this result and proposed the problem “how to order trees by the Laplacian coefficients”. Ilić [6] proved that among $n$-vertex trees with fixed diameter $d$, the caterpillar $C_{n,d}$ (see Fig. 1) has minimal Laplacian coefficients. Ilić and Ilić [8] characterized the trees with $k$ leaves which simultaneously minimize all Laplacian coefficients. They proved that graph $S(n, k)$ has minimal Laplacian coefficients among all trees with $n$ vertices and $k$ leaves, where $S(n, k)$ is a tree of order $n$ with just one center vertex $v$ and each of the $k$ branches of $T$ at $v$ is a path of length $\lceil \frac{(n-1)}{k} \rceil$ or $\lfloor \frac{(n-1)}{k} \rfloor$. They also proved that $S(n, n - 1 - p)$ has minimal Laplacian coefficients among vertices trees with $p$ vertices of degree two. Stevanović and Ilić [14] showed that among all connected unicyclic graphs of order $n$, the $k$th coefficient $c_k$ is largest when the graph is a cycle $C_n$ and smallest when the graph is the star $K_{1,n-1}$ with an additional edge between two of its pendant vertices. Zhang et al. [18] investigated a partial ordering of trees with diameters 3 and 4 by the Laplacian coefficients. Some related work on Laplacian coefficients can be found for example in [2,7,9,13].

The subdivision graph $S(G)$ of a graph $G$ is a graph obtained by inserting a new vertex (called the subdivision vertex) on each edge of $G$. If $G$ has $n$ vertices and $m$ edges, then $S(G)$ has $n + m$ vertices and $2m$ edges.

![Fig. 1. Trees $C_{n,d}$ and $D(p, q)$.](image)
Let $G$ be a connected bipartite graph with $n$ vertices. Hence its vertex set can be partitioned into two subsets $V_1$ and $V_2$, such that each edge joins a vertex in $V_1$ with a vertex in $V_2$. Suppose that $V_1$ has $p$ vertices and $V_2$ has $q$ vertices, where $p + q = n$. Then we say that $G$ has a $(p, q)$-bipartition $(p \leq q)$. Denote by $\Psi_n$ the class of trees with $n$ vertices, each of which has a $(p, q)$-bipartition $(p + q = n)$. Consider a star $K_{1,p}$ with $p + 1$ vertices and attach $q − 1$ pendent edges to a non-central vertex of the star $K_{1,p}$. The resulting tree with $p + q$ vertices has a $(p, q)$-bipartition. Denote the resulting tree by $D(p, q)$ (see Fig. 1). Obviously, $D(p, q) \in \Psi_n$. Call $D(p, q)$ to be a double star. If $q \geq p \geq 3$, suppose that $B(p, q)$ is the tree obtained from $D(p − 1, q)$ by attaching a pendent edge to one of the vertices of degree one which join the vertex of degree $q$ in $D(p − 1, q)$ (see Fig. 2). If $q \geq p = 2$, we assume that $B(2, q)$ is the tree obtained from the path $P_4$ by attaching $q − 2$ pendent edges to an end vertex of $P_4$ (see Fig. 2).

In this paper, we prove the following results.

**Theorem 1.1.** Let $p$ and $q$ be two positive integers and $p + q = n$. Let $T$ be a tree with $n$ vertices and with Laplacian polynomial in the form (1.3) and $T \in \Psi_n$. Then

1. $c_k(T) \geq c_k(D(p, q))$ for every $k \leq n$, where all equalities hold if and only if $T \cong D(p, q)$.
2. $LEL(T) \geq LEL(D(p, q))$, where equality holds if and only if $T \cong D(p, q)$.
3. $W(T) \geq W(D(p, q))$.

**Theorem 1.2.** Let $p$ and $q$ be two positive integers such that $q \geq p \geq 2$, and let $T$ be a tree with a $(p, q)$-bipartition such that $T \not\cong D(p, q)$. Then

1. $c_k(T) \geq c_k(B(p, q))$ for every $k \leq n$, where all equalities hold if and only if $T \cong B(p, q)$.
2. $LEL(T) \geq LEL(B(p, q))$, where equality holds if and only if $T \cong B(p, q)$.
3. $W(T) \geq W(B(p, q))$.

2. Proofs

In order to prove the main results, we need to introduce some lemmas as follows.

**Lemma 2.1** [19]. Let $G$ be a bipartite graph with $n$ vertices and $m$ edges and let $S(G)$ be its subdivision graph. Then $\phi(S(G), \lambda) = \lambda^{m−n} \mu(G, \lambda^2)$. Hence $m_k(S(G)) = c_k(G)$.

Given a graph $G$ and an edge $uv$, we denote by $G − uv$ (resp. $G − u$) the graph obtained from $G$ by deleting the edge $uv$ (resp. the vertex $u$ and edges adjacent to it).

**Lemma 2.2** [3]. Let $T$ be a tree with $n$ vertices and $e = uv$ an edge of $T$. Then

$$\phi(T) = \phi(T−uv) − \phi(T−u−v).$$
Proof. We prove this result by induction on $T$ and Lemma 2.5.

Hence, by Lemma 2.3,

$$T = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} b_i x^{n-2i}, \quad \phi(T_2) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} b'_i x^{n-2i},$$

respectively. Then $T_1 \succeq T_2$ if and only if $b_0 - b'_0 = 0$ and $(-1)^i (b_i - b'_i) \geq 0$ for $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$; and $T_1 \succ T_2$ if and only if $T_1 \succeq T_2$ and there exists a $j \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ such that $(-1)^j (b_j - b'_j) > 0$.

Lemma 2.6. Let $T$ and $T'$ be two trees with $n$ vertices. Suppose that $uv$ (resp. $u'v'$) is a pendant edge of $T$ (resp. $T'$), $u$ (resp. $u'$) is a pendant vertex of $T$ (resp. $T'$), $d(v) = 2$, $d(v') = 2$, and $w$ (resp. $w'$) is another neighbor of $v$ (resp. $v'$). Let $T_1 = T - vw$, $T_2 = T - v - w$, and $T_1' = T' - v'w'$, $T_2' = T' - v' - w'$. If $T_1 \succeq T_1'$ and $T_2 \succeq T_2'$, then $T \succeq T'$, with equality if and only if $T_1 \simeq T_1'$, $T_2 \simeq T_2'$.

Proof. By Lemma 2.2,

$$\phi(T) = \phi(T - vw) - \phi(T - v - w) = (x^2 - 1) \phi(T_1) - \phi(T_2),$$

$$\phi(T') = \phi(T' - v'w') - \phi(T' - v' - w') = (x^2 - 1) \phi(T_1') - \phi(T_2').$$

Hence

$$\phi(T) - \phi(T') = (x^2 - 1) \left( \phi(T_1) - \phi(T_1') \right) - \left( \phi(T_2) - \phi(T_2') \right).$$

Suppose that

$$(x^2 - 1) \left( \phi(T_1) - \phi(T_1') \right) = \sum_{i \geq 0} a_i x^{n-2i} \quad \text{and} \quad \phi(T_2) - \phi(T_2') = \sum_{i \geq 0} b_i x^{n-2i}.$$

Then if $T_1 \succeq T_1'$ and $T_2 \succeq T_2'$, $a_0 = b_0 = 0$ and $(-1)^i a_i \geq 0$ and $(-1)^i b_i \geq 0$ for $i \geq 1$. Hence $(-1)^i (a_i - b_{i-1}) \geq 0$ for $i \geq 1$. Note that

$$\phi(T) - \phi(T') = \sum_{i \geq 1} (a_i - b_{i-1}) x^{n-2i}.$$

Hence, by Lemma 2.3, $T \succeq T'$.

If $T_1 \succ T_1'$ and $T_2 \succeq T_2'$, then there is at least one $k$ such that $(-1)^k a_k > 0$. Hence $(-1)^k (a_k - b_{k-1}) > 0$. By Lemma 2.3, $T \succ T'$. Similarly, if $T_1 \succeq T_1'$ and $T_2 \succ T_2'$, then $T \succeq T'$. The lemma thus follows. □

Lemma 2.5 [1]. Let $G$ be a tree. Then $G$ has a perfect matching if and only if $o(G - w) = 1$ for all $w \in V(G)$, where $o(G - w)$ denotes the number of odd components of $G - w$.

Lemma 2.6. Let $T$ be a tree with $n$ ($n \geq 2$) vertices and $u$ is a vertex of $T$. Let $S(T)$ be the subdivision graph of $T$. Construct a new graph $T'$ from $S(T)$ by adding a new vertex $v$ and a new edge joining two vertices $u$ and $v$. Then $T'$ has a perfect matching.

Proof. We prove this result by induction on $n$.

If $n = 2$, $T = P_2$ and $T' = P_4$. Then $T'$ has a perfect matching and the lemma holds. If $n = 3$, $T = P_3$ and $T' = P_6$ or $T' = T_6$ (see Fig. 3). Obviously, both $P_6$ and $T_6$ have a perfect matching.

We assume inductively that the theorem holds if the number of vertices of a tree is less than $n$. Let $d(u) = k$ ($k \geq 1$), and $w_1, w_2, \ldots, w_k$ be the neighbors of $u$ in $T$, and $w'_1, w'_2, \ldots, w'_k$ be the subdivision vertices on the edges $uw_1, uw_2, \ldots, uw_k$ (see Fig. 3). By induction, all $S(T_1) + w_1w'_1,$
$S(T_2) + w_2w'_2, \ldots, S(T_k) + w_kw'_k$ have a perfect matching. Suppose that $M_i$ is the perfect matching of $S(T_i) + w_iw'_i$. Then $M = M_1 \cup M_2 \ldots \cup M_k \cup \{uv\}$ is a perfect matching of $T'$. Hence the lemma holds. □

Let $F_n$ be a tree with $n$ vertices obtained by adding a pendent edge to each vertex of the star $K_{1, \frac{n}{2} - 1}$, where $n$ is even and $K_{1, \frac{n}{2} - 1}$ is the star with $\frac{n}{2}$ vertices. Let $B_n$ be a tree obtained from $F_{n-2}$ by attaching a $P_2$ to the 2-degree vertex of a pendent edge, and $L_n$ is obtained from $F_{n-4}$ by attaching two $P_2$ to the 2-degree vertex of a pendent edge, $M_n$ is obtained from $F_{n-2}$ by attaching a $P_2$ to a 1-degree vertex to form a path of length 6. Trees $F_n, B_n, L_n$ and $M_n$ are illustrated in Fig. 4.

**Lemma 2.7** [15]. Let $T$ be a tree with $n$ vertices which has a perfect matching, and $T \not\leq F_n, B_n, L_n, M_n$. Then

$$T > L_n > B_n > F_n \quad \text{or} \quad T > M_n > B_n > F_n,$$

where $L_n$ and $M_n$ are incomparable.

**Lemma 2.8.** Let $T$ be a tree with $n$ vertices and $T'$ a spanning subgraph (resp. a proper spanning subgraph) of $T$ ($T'$ can be disconnected). Then $T \geq T'$ (resp. $T > T'$).

**Lemma 2.9.** Let $T$ be a tree with $n$ vertices which has a perfect matching. Then there exists a tree $T_1$ which is a subgraph of $T$, such that

$$T \geq T_1 \cup kP_2$$

for $0 \leq k \leq \frac{n}{2}$, where $T_1$ is a tree with $n - 2k$ ($0 \leq k \leq \frac{n}{2}$) vertices and with a perfect matching.

**Proof.** We prove this by induction on $n$. It is easy to see that $n$ is even.

If $n = 2$ or $n = 4$, it is trivial to verify the lemma.
We assume inductively that the lemma holds for all trees which have a perfect matching and have the number of vertices less than \( n \). Let \( T = v_0v_1v_2 \ldots v_{l-1}v_l \) be the longest path in \( T \). If \( v_1 \) has other neighbors \( u_1, u_2, \ldots, u_m \) \((m \geq 1)\) different from \( v_0 \) and \( v_2 \). Since \( P \) is a longest path, we know all \( u_i \)'s are pendant vertices of \( T \). Then \( a(T - v_1) = m + 1 \geq 2 \). By Lemma 2.5, this contradicts the fact that \( T \) has a perfect matching. Hence \( d(v_1) = 2 \).

Let \( T' = T - v_0 - v_1 \). So \( V(T') = n - 2 \). Since \( d(v_1) = 2 \), it is easy to see that \( T' \) also has a perfect matching. By induction, \( T' \supseteq T_1 \cup tP_2 \), \( 0 \leq t \leq \frac{n - 2}{2} \), where \( T_1 \) is a tree with \( n - 2 - 2t \) vertices and has a perfect matching. By Lemma 2.8, \( T \supseteq T' \cup P_2 \supseteq T_1 \cup (t + 1)P_2 = T_1 \cup kP_2 \), \( 0 \leq k \leq \frac{n}{2} \).

Hence the lemma holds. \( \square \)

**Lemma 2.10** [13]. Let \( G_1 \) and \( G_2 \) be two \( n \)-vertex graphs. If \( c_k(G_1) \geq c_k(G_2) \) for \( k = 1, 2, \ldots, n - 1 \), then \( \text{LEL}(G_1) \geq \text{LEL}(G_2) \). In particular, if there exists a \( j \) such that \( c_j(G_1) > c_j(G_2) \) for \( 1 \leq j \leq n - 1 \), then \( \text{LEL}(G_1) > \text{LEL}(G_2) \).

**Lemma 2.11**. Suppose that \( T \) is a tree with a \((2, q)\)-bipartition such that \( T \ncong D(2, q) \). Then \( c_k(T) \geq c_k(B(2, q)) \) for \( 0 \leq k \leq n \), with all equalities if and only if \( T \cong B(2, q) \).

**Proof.** By (1.2) and Lemma 2.1, it suffices to prove that \( S(T) \succ S(B(2, q)) \) for any tree \( T \ncong D(2, q) \). \( B(2, q) \) with a \((2, q)\)-bipartition. Note that if \( q = 2 \), then there exists only one tree with a \((2, 2)\)-bipartition. If \( q = 3 \), there exist exactly two trees \( D(2, 3) \) and \( B(2, 3) \), which have a \((2, 3)\)-bipartition. If \( q = 4 \), there exist exactly two trees \( D(2, 4) \) and \( B(2, 4) \), which have a \((2, 4)\)-bipartition. Hence we may assume that \( q \geq 5 \). Since \( T \) is a tree with a \((2, q)\)-bipartition and \( T \ncong D(2, q) \), \( T \ncong B(2, q) \), \( T \) must have the form shown in Fig. 5, where \( a \geq 2, b \geq 2, a + b + 1 = q \).

Let \( (V_1, V_2) \) be the bipartition of \( T \) with \( |V_1| = 2 \) and \( |V_2| = q \). We proceed by induction on \( q \) and assume that the lemma holds if \( |V_2| < q \). For \( S(T) \) (see Fig. 5), by Lemma 2.2,

\[
\phi(S(T)) = \phi(S(T) - uv) - \phi(S(T) - u - v) = (x^2 - 1)\phi(S(T')) - x(x^2 - 1)^{b-1}\phi(T_1),
\]

where \( T' \) is a tree with a \((2, q - 1)\)-bipartition, \( S(T') \) is the subdivision graph of \( T' \), and \( T_1 \) is a tree with \( 2a + 4 \) vertices. By Lemma 2.2,

\[
\phi(S(B(2, q))) = \phi(S(B(2, q)) - wz) - \phi(S(B(2, q)) - w - z) = (x^2 - 1)\phi(S(B(2, q - 1))) - x(x^2 - 1)^{q-3}\phi(P_6).
\]

By Lemma 2.4, it suffices to prove that \( S(T') \succ S(B(2, q - 1)) \) and \( T_1 > (q - b - 2)P_2 \cup P_6 \). We distinguish the following two cases:

1. **Case 1.** \( b = 2 \).

Then \( T' \cong B(2, q - 1), S(T') \cong S(B(2, q - 1)) \). Note that \( a + b + 1 = q \). Then \( q - b - 2 = a - 1 \). It is not difficult to see that \( T_1 \) contains a proper spanning subgraph \((a - 1)P_2 \cup P_6 \). By Lemma 2.8, \( T_1 > (q - b - 2)P_2 \cup P_6 = (a - 1)P_2 \cup P_6 \).
Case 2. \( b > 2 \).

Then \( T' \not\cong B(2,q-1) \) and \( T' \not\cong D(2,q-1) \). By induction, \( S(T') > S(B(2,q-1)) \). Similarly, we can show that \( T_1 > (q-b-2)P_2 \cup P_6 = (a-1)P_2 \cup P_6 \).

Hence the lemma follows. \( \square \)

Now we are in the position to prove the main results.

**Proof of Theorem 1.1.** Obviously, (2) is immediate from (1) and Lemma 2.10. Note that, for any tree \( T \) with \( n \) vertices, the Wiener index \( W(T) = \varphi(T) \). So (1) implies (3). Hence it suffices to verify the first assertion.

Let \( D = D(p,q) \). By Lemma 2.1, it suffices to prove that \( m_k(S(T)) \geq m_k(S(D)) \), with all equalities if and only if \( T = D \). By (1.2), we only need to prove that \( S(T) \geq S(D) \), for any tree \( T \) with a \((p,q)\)-bipartition, with equality if and only if \( T = D \). Without loss of generality, we assume \( p < q \). If \( p = 1 \), then \( T = D(1,q) = K_{1,n} \). So \( S(T) = S(D) \) and the theorem holds. Hence we may now assume that \( 2 \leq p \leq q \), and proceed by induction on \( n = p + q \).

Let \( P = v_1 v_2 \ldots v_l \) be a longest path in \( T \). Let \( (V_1, V_2) \) be the bipartition of \( T \) (see Fig. 6) with \( |V_1| = p \) and \( |V_2| = q \). Let \( V_{11} \) denote the set of pendant vertices incident with \( v_2 \) in \( T \). Assume that \( |V_{11}| = k \). Note that \( v_1 \notin V_{11} \). So \( k \geq 1 \). Without loss of generality, we assume \( v_1 \notin V_2 \). If \( V_{11} \subseteq V_1 \), then \( T = K_{1,p} \), which contradicts the fact that \( q > 2 \). Then \( 1 \leq k < p - 1 \). Let \( T_a = T - (V_2) \cup V_{11} \) be the tree obtained from \( T \) by deleting all vertices in \( V_2 \) \cup V_{11} \).

In the subdivision graph \( S(T) \) (see Fig. 6), the corresponding longest path is \( \hat{\mathcal{T}} = v_1 \hat{v}_1 v_2 \hat{v}_2 \ldots \hat{v}_{l-1} v_1 \), where the vertices \( \hat{v}_i \)’s \((i = 1, 2, \ldots, l - 1)\) are those subdividing the edges of \( T \). By Lemma 2.2,

\[
\phi(S(T)) = \phi(S(T) - \hat{v}_1 v_2) - \phi(S(T) - \hat{v}_1 v_2) = (x^2 - 1)\phi(S(T)) - x(x^2 - 1)^{k-1}\phi(T_1),
\]

where \( T' \) is a tree with a \((p - 1, q)\)-bipartition, \( S(T') \) is the subdivision graph of \( T' \), and \( T_1 = S(T_0) + \hat{v}_1 v_2 \) is a tree with \( 2n - 2k - 2 \) vertices, \( 1 \leq k \leq p - 1 \).

For \( D(p, q) \), let \( V'_1 \) and \( V'_2 \) be the bipartition of \( D(p, q) \) with \( |V'_1| = p \) and \( |V'_2| = q \). Let \( Q = u_1 u_2 u_3 u_4 \) be the longest path in \( D(p, q) \). In the subdivision graph \( S(D) \), the corresponding longest path is \( \hat{Q} = u_1 \hat{u}_1 u_2 \hat{u}_2 u_3 \hat{u}_3 u_4 \). Without loss of generality, we assume \( u_1 \in V'_1 \). Then we have

\[
\phi(S(D)) = \phi(S(D) - \hat{u}_1 u_2) - \phi(S(D) - \hat{u}_1 u_2) = (x^2 - 1)\phi(S(D')) - x(x^2 - 1)^{p-2}\phi(F_{2n-2p})
\]

where \( D' \) is a double star with a \((p - 1, q)\)-bipartition and \( S(D') \) is the subdivision graph of \( D' \). By Lemma 2.4, it suffices to prove that

\[
S(T') \geq S(D') \quad \text{and} \quad T_1 \geq F_{2n-2p} \cup (p - k - 1)P_2,
\]

with all equalities if and only if \( T' \cong D'(p-1,q) \), \( T_1 \cong F_{2n-2p} \cup (p - k - 1)P_2 \).

By induction, \( S(T') \geq S(D') \), with equality if and only if \( T' \cong D'(p-1,q) \). Now we prove \( T_1 \geq F_{2n-2p} \cup (p - k - 1)P_2 \).

By Lemma 2.6, \( T_1 \) has a perfect matching. By Lemma 2.9, there exists a tree \( T_3 \) with \( 2n - 2p \) vertices, which has a perfect matching, such that

\[
T_1 \geq T_3 \cup (p - k - 1)P_2, \quad 0 \leq p - k - 1 < n - k - 1.
\]
By Lemma 2.7, we have $T_3 \succeq F_{2n-2p}$. Hence

$$T_1 \succeq T_3 \cup (p - k - 1)P_2 \succeq F_{2n-2p} \cup (p - k - 1)P_2.$$  

It is easy to see that $T_1 \simeq F_{2n-2p} \cup (p - k - 1)P_2$ and $T' \simeq D'$ if and only if $T \simeq D(p, q)$. Note that $T' \simeq D'$. If $T \simeq D(p, q)$, then $T$ has the structure depicted in Fig. 7, such that $T$ has a $(p, q)$-bipartition and $T \not\simeq D(p, q)$. Hence $S(T)$ has the structure depicted in Fig. 7, where $k = p - 2$. At this time $T_1$ (see Fig. 7) contains a proper spanning subgraph $F_{2n-2p} \cup (p - k - 1)P_2$. By Lemma 2.8, $T_1 \succeq F_{2n-2p} \cup (p - k - 1)P_2$. By Lemma 2.3, if $T \not\simeq D(p, q)$, then $S(T) \succ S(D(p, q))$.

Then the theorem has been proved.

**Proof of Theorem 1.2.** Similarly to the proof of Theorem 1.1, (2) and (3) are immediate from (1). Hence it suffices to show that (1) holds.

Let $B = B(p, q)$. By (1.2) and Lemma 2.1, it suffices to prove that if $T$ is a tree with a $(p, q)$-bipartition such that $T \nless D(p, q)$, then $S(T) \succeq S(B)$ with equality if and only if $T \simeq B(p, q)$.

By Lemma 2.11, the theorem holds if $p = 2$. Hence we may assume that $q \geq p \geq 3$ and proceed by induction on $n = p + q$. If $p = q = 3$, that is $p + q = 6$, there exist exactly three trees $D(3, 3), B(3, 3)$ and $P_6$, each of which has a $(3, 3)$-bipartition. It is easy to see that $S(P_6) \succ S(B(3, 3)) \succ S(D(3, 3))$.

The theorem holds if $p + q = 6$.

Let $P = v_1v_2 \ldots v_1$ be a longest path in $T$. Let $(V_1, V_2)$ be the bipartition of the vertex set of $T$ (see Fig. 6) with $|V_1| = p$ and $|V_2| = q$. In the subdivision graph $S(T)$ (see Fig. 6), the corresponding longest path is $\hat{P} = \hat{v}_1 \hat{v}_2 \hat{v}_3 \ldots \hat{v}_{k-1} \hat{v}_{k-1} v_1$. Let $V_{11}$ denote the set of pendent vertices incident with $v_2$ and $|V_{11}| = k$. Let $T_a = T - \{v_2 \cup V_{11}\}$. Let $V_{12}$ denote the set of pendent vertices incident with $v_{k-1}$ and $|V_{12}| = r$. Let $T_b = T_a - \{v_{k-1} \cup V_{12}\}$. We distinguish the following two cases:

Case 1. $V_1 \subseteq V_1$.

Note that $1 \leq k \leq p - 2$ (If $k = p, T = K_{1,p}$, contradicting the fact that $q \geq 3$. If $k = p - 1, T = D(p, q)$, also a contradiction.). By Lemma 2.2,

$$\phi(S(T)) = \phi(S(T) - \hat{v}_1 v_2) - \phi(S(T) - \hat{v}_1 - v_2) = (x^2 - 1)\phi(S(T')) - x(x^2 - 1)^{k-1}\phi(T_1),$$

where $T'$ is a tree with a $(p - 1, q)$-bipartition, $S(T')$ is the subdivision graph of $T'$, and $T_1 = S(T_a) + \hat{v}_2 v_3$ is a tree with $2n - 2k - 2$ vertices, $1 \leq k \leq p - 2$.

For $B(p, q)$, let $V_1'$ and $V_2'$ be the bipartition of $B(p, q)$ with $|V_1'| = p$ and $|V_2'| = q$. Let $Q = u_1u_2u_3u_4u_5$ be the longest path in $B(p, q)$ (see Fig. 2). In the subdivision graph $S(B)$ (see Fig. 8), the corresponding longest path is $\hat{Q} = u_1 \hat{u}_1 u_2 \hat{u}_2 u_3 \hat{u}_3 u_4 \hat{u}_4 u_5$. Note that $u_1 \in V_1'$.

Then

$$\phi(S(B)) = \phi(S(B) - \hat{u}_1 u_2) - \phi(S(B) - \hat{u}_1 - u_2) = (x^2 - 1)\phi(S(B')) - x(x^2 - 1)^{p-3}\phi(M_{2q+2}),$$

where $B'$ is a tree with a $(p - 1, q)$-bipartition and $S(B')$ is a subdivision graph of $B'$. By Lemma 2.4, it suffices to prove that

$$S(T') \succeq S(B') \quad \text{and} \quad T_1 \succeq M_{2q+2} \cup (p - k - 2)P_2,$$

with all equalities if and only if $T' \simeq B'(p - 1, q), T_1 \simeq M_{2q+2} \cup (p - k - 2)P_2$.

By induction, $S(T') \succeq S(B'),$ with equality if and only if $T' \simeq B'(p - 1, q)$.

Now we prove $T_1 \succeq M_{2q+2} \cup (p - k - 2)P_2$. 

Fig. 7. $T, S(T)$, and $T_1$. 

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By Lemma 2.6, $T_1$ has a perfect matching. Moreover, $T_1 \not\cong F_{2n-2k-2}$. (If $T_1 \cong F_{2n-2k-2}$, then $T = D(p, q)$, a contradiction.) Note that $T_1 = S(T_a)+\hat{v}_2v_3$, i.e., $T_1$ is a tree obtained from the subdivision graph of $T_a$ by adding a pendant edge to some non-subdivision vertex. Hence $T_1 \not\cong B_{2n-2k-2}, L_{2n-2k-2}$.

Lemma 2.6 implies that $T_1$ has a perfect matching. By Lemma 2.7, $T_1 \geq M_{2n-2k-2}$. Note that $M_{2q+2} \cup (p-k-2)P_2$ is a spanning subgraph of $M_{2n-2k-2}$. Hence $M_{2n-2k-2} \geq M_{2q+2} \cup (p-k-2)P_2$ for $k+2 \leq p$. Then

$$T_1 \geq M_{2q+2} \cup (p-k-2)P_2, \quad 0 \leq p-k-2 < n-k-1.$$  

It is not difficult to see that $T_1 \cong M_{2q+2} \cup (p-k-2)P_2$ and $T' \cong B'$ if and only if $T \cong B(p, q)$. Since $T' \cong B'$ and $T \cong B(p, q)$, $T$ has the structure depicted in Fig. 9, such that $T$ has a $(p, q)$-bipartition. Hence $S(T)$ has the structure depicted in Fig. 9, where $k = p - 3$. At this time $T_1$ (see Fig. 9) contains a proper spanning subgraph $M_{2q+2} \cup (p-k-2)P_2$. By Lemma 2.8, $T_1 > M_{2q+2} \cup (p-k-2)P_2$. By Lemma 2.3, $S(T) > S(B(p, q))$ if $T \not\cong B(p, q)$.

Case 2. $v_1 \in V_2$.

If $v_1 \in V_1$, we can use the same method as in the case 1 to prove that if $T \not\cong B(p, q)$, then $S(T) > S(B(p, q))$. Hence we may assume that $v_1 \in V_2$. If $k = q, T = K_{1, q}$, contradicts the fact that $p \geq 3$. If $k = q - 1, T = D(p, q)$, also a contradiction. Hence $1 \leq k \leq q - 2$.

In the subdivision graph $S(B)$ (see Fig. 8), we have $u_6 \in V_2$, then

$$\phi(S(B)) = \phi(S(B)-u_2\hat{u}_6)-\phi(S(B)-u_3\hat{u}_6) = (x^2-1)\phi(S(B'')) - x(x^2-1)^{q-3}\phi(P_4)\phi(F_{2p-2}),$$

where $B''$ is a tree with a $(p, q - 1)$-bipartition and $S(B'')$ is the subdivision graph of $B''$.

It is not difficult to see that $T \not\cong B(p, q), T' \not\cong B'(p, q - 1)$ (If $T \cong B(p, q), T' \cong B'(p, q - 1)$, then $v_1 \in V_2, v_1 \in V_2$, a contradiction). By induction, $S(T') > S(B'')$. Now we prove that $T_1 \geq F_{2p-2} \cup (q-k-2)P_2 \cup P_4$.

Note that, $1 \leq r = |V_{12}| \leq q - 2$. (If $r = q$, we have, $T = K_{1, q}$, contradicts the fact that $p \geq 3$. If $r = q - 1$, we have $T = D(p, q)$, also a contradiction.) Particularly, $k + r \leq q - 1$.

There exists a path $P_4 (R = \hat{v}_1v_1\hat{v}_2v_3) \in T_1$ (see Fig. 10). By Lemma 2.8, it is easy to see that, $T_1 \geq T_3 \cup (r - 1)P_2 \cup P_4$, where $T_3 = S(T_b)+\hat{v}_2v_3$ is a tree with $2n-2r-2k-4$ vertices. By Lemma 2.6, $T_3$ has a perfect matching. Now we need to prove that $T_3 \geq F_{2p-2} \cup (q-k-r-1)P_2$. By Lemma 2.9, there exists a tree $T_4$ with $2p-2$ vertices, which has a perfect matching, such that

$$T_3 \geq T_4 \cup (q-k-r-1)P_2, \quad 0 \leq q-k-r-1 < n-k-r-2.$$
By Lemma 2.7, \( T_4 \succeq F_{2p-2} \). Hence

\[
T_3 \succeq T_4 \cup (q - k - r - 1)P_2 \succeq F_{2n-2q-2} \cup (q - k - r - 1)P_2.
\]

Then

\[
T_1 \succeq F_{2p-2} \cup (q - k - 2)P_2 \cup P_4.
\]

By Lemma 2.4, \( S(T) \succ S(B) \) if \( T \not\cong B(p, q) \). Hence (1) has been proved.

3. Concluding remarks

We characterized the trees with a given bipartition \((p, q)\) which have the minimal and second minimal Laplacian coefficients. One of the referees told us that, in [9], authors proved the following result: Let \( w \) be a vertex of the non-trivial connected graph \( G \) and for non-negative integers \( p \) and \( q \), let \( G(p, q) \) denote the graph obtained from \( G \) by attaching pendent paths \( P = vv_1v_2 \ldots v_p \) and \( Q = uwu_1u_2 \ldots u_q \) of lengths \( p \) and \( q \), respectively, at vertex \( w \). If \( p > q > 1 \), then \( c_k(G(p, q)) \leq c_k(G(p + 1, q - 1)) \) for \( k = 0, 1, \ldots, |V(G(p, q))| \). Note that the transformation \( G(p, q) \rightarrow G(p - 2, q + 2) \) keeps the bipartition sizes the same and simultaneously decreases the Laplacian coefficients. By this result, we can perhaps consider the problem to characterize the tree with a given bipartition which has maximal Laplacian coefficients. But this seems to be difficult. Li and Zhou [10] determined the bipartite unicyclic graph with a given bipartition which has the minimal energy. A similar problem is to consider the Laplacian coefficients of bipartite unicyclic graph with a given bipartition. We leave these problems for future study.

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References


