Journal of Algebra **225,** 698–739 (2000)
doi:10.1006/jabr.1999.8140, available online at http://www.idealibrary.com on IDEAL®

An Explicit Construction of Residual Complexes[1]

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Received January 5, 1996

CONTENTS

Let ϕ: $\mathscr{Y} \to \mathscr{X}$ be a morphism of finite type between locally Noetherian schemes whose fibers have bounded dimensions. Given concretely a residual complex on $\mathscr{X}$, we construct canonically a concrete residual complex on cY.

1. OVERVIEW OF CONSTRUCTIONS

We are seeking an explicit construction of residual complexes on a scheme. Since residual complexes are built up by injective hulls of residue fields of all points on the scheme, we need a model for injective hulls of the residue field of each point in order to explicitly describe the coboundary maps of residual complexes. Let ϕ: $\mathscr{Y} \to \mathscr{X}$ be a morphism of finite type between locally Noetherian schemes whose fibers have bounded dimensions. Given an injective hull $M(\mathfrak{p})$ of the residue field $\kappa(\mathfrak{p})$ of each point $\mathfrak{p}$ in $\mathscr{X}$ and a residual complex $\mathscr{M}^{\bullet}$ on $\mathscr{X}$ – with additional information that how $\mathscr{M}^{\bullet}$ is built up by the $M(\mathfrak{p})$'s, we succeed in explicitly constructing

[1]This paper was partially supported by National Science Council of Republic of China Grant NSC82-0208-M-001-136-T.

0021-8693/00 $35.00

a residual complex $\mathscr{M}^\bullet_{\mathscr{Y}}$ on $\mathscr{Y}$ – with additional information that how $\mathscr{M}^\bullet_{\mathscr{Y}}$ is built up by some injective hulls canonically constructed from the $M(\mathfrak{p})$'s. In particular, for $\mathscr{X}$ being the spectrum of $\mathbb{Z}$ or of a field on which a canonical residual complex exists, we obtain a canonical residual complex on $\mathscr{Y}$ explicitly. Due to the non-uniqueness nature of injective hulls, what we construct is *not* a functor of residual complexes! However our construction can still be regarded as a concrete realization of the functor $\phi^!$ of residual complexes as defined in [4, Chap. VI, Sect. 3] with finer information. Works of related interest are [17] of Sastry and [20, 19] of Yekutielli.

Residues are indispensable for understanding Grothendieck duality theory. There are cohomology residues developed by Lipman et al. (cf. [11, 12, 10, 8, 9, 7]) and residues of differential forms of local fields developed by Parshin et al. (cf. [15, 16, 13, 1, 20]). These residues not only provide tools for understanding Grothendieck duality theory from different angles but also show the richness of the theory. In this article, we will use only the (local cohomology) residue maps for power series rings [6, Chap. 5] to construct residual complexes.

Let $\mathscr{X}$ be a locally Noetherian scheme possessing a residual complex $\mathscr{M}^\bullet$, that is, a complex of quasi-coherent injective $\mathscr{O}_{\mathscr{X}}$-modules, bounded below, with coherent cohomology sheaves, and such that there is an isomorphism

$$\bigoplus_{n\in\mathbb{Z}} \mathscr{M}^n \simeq \bigoplus_{\mathfrak{p}\in\mathscr{X}} J(\mathfrak{p}),$$

where $J(\mathfrak{p})$ is the quasi-coherent $\mathscr{O}_{\mathscr{X}}$-module which is the constant sheaf $M(\mathfrak{p})$, a given injective hull of the residue field $\kappa(\mathfrak{p})$ over the local ring $\mathscr{O}_{\mathscr{X},\mathfrak{p}}$, on $\{\mathfrak{p}\}^-$, and 0 elsewhere [4, p. 304]. The existence of a residual complex gives some constraints on $\mathscr{X}$. For example, the sections on any affine open subset of $\mathscr{X}$ form a universally catenary ring [4]. The residual complex $\mathscr{M}^\bullet$ determines a codimension function $\Delta_{\mathscr{X}}\colon \mathscr{X} \to \mathbb{Z}$, that is, a function $\Delta_{\mathscr{X}}(\mathfrak{q}) = \Delta_{\mathscr{X}}(\mathfrak{p}) + 1$ for every immediate specialization $\mathfrak{q}$ of $\mathfrak{p}$, such that

$$\mathscr{M}^n \simeq \bigoplus_{\Delta_{\mathscr{X}}(\mathfrak{p})=n} J(\mathfrak{p}) \tag{1}$$

(see [4]).

Given a morphism $\phi\colon \mathscr{Y} \to \mathscr{X}$ of finite type and a point $\mathfrak{P}$ in $\mathscr{Y}$, let $\mathfrak{p}$ be the image of $\mathfrak{P}$ under ϕ and let $\kappa(\mathfrak{p})$ (resp. $\kappa(\mathfrak{P})$) be the residue field of $\mathscr{O}_{\mathscr{X},\mathfrak{p}}$ (resp. $\mathscr{O}_{\mathscr{Y},\mathfrak{P}}$). In Section 2, we will give a functor from the category of injective hulls of $\kappa(\mathfrak{p})$ (as $\mathscr{O}_{\mathscr{X},\mathfrak{p}}$-modules) to the category of injective hulls of $\kappa(\mathfrak{P})$ (as $\mathscr{O}_{\mathscr{Y},\mathfrak{P}}$-modules). This functor is essentially the same as that constructed in [6]. Given an injective hull $M(\mathfrak{p})$ of $\kappa(\mathfrak{p})$, applying this functor we get an injective hull of $\kappa(\mathfrak{P})$ which we denote by $M_{\mathscr{Y}}(\mathfrak{P})$.

Note that there is a natural defined codimension function $\Delta_{\mathscr{Y}/\mathscr{X}}$ on $\mathscr{Y}$, given by

$$\Delta_{\mathscr{X}}(\mathfrak{p}) = \Delta_{\mathscr{Y}/\mathscr{X}}(\mathfrak{P}) + \text{ transcendence degree of } \kappa(\mathfrak{P})/\kappa(\mathfrak{p}). \tag{2}$$

(cf. [4, Chap. V, Corollary 8.4]). If $\psi\colon \mathscr{Z} \to \mathscr{Y}$ is a morphism of finite type, the codimension function $\Delta_{\mathscr{Z}/\mathscr{Y}}$ on $\mathscr{Z}$ defined by

$$\Delta_{\mathscr{Y}/\mathscr{X}}(\psi(\mathscr{P})) = \Delta_{\mathscr{Z}/\mathscr{Y}}(\mathscr{P}) + \text{ transcendence degree of } \kappa(\mathscr{P})/\kappa(\psi(\mathscr{P})),$$

for $\mathscr{P} \in \mathscr{Z}$, is the same as $\Delta_{\mathscr{Z}/\mathscr{X}}$. So we will simply write $\Delta_{\mathscr{Z}}$ for $\Delta_{\mathscr{Z}/\mathscr{X}}$.

Assume furthermore that the dimensions of the fibers of ϕ are bounded. Fix an isomorphism (1) for each $n \in \mathbb{Z}$, in this article we give an explicit construction of a residual complex on $\mathscr{Y}$. Our construction provides finer information of the functor $\phi^!$ of residual complexes as defined in [4, Chap. VI, Sect. 3]. For example, we are able to answer the following question.

QUESTION 1.1. *For any point $\mathfrak{P}$ in $\mathscr{Y}$, denote by $J(\mathfrak{P})$ the quasi-coherent $\mathscr{O}_{\mathscr{Y}}$-module which is the constant sheaf $M_{\mathscr{Y}}(\mathfrak{P})$ on $\{\mathfrak{P}\}^-$, and* 0 *elsewhere. How do we define morphism $M_{\mathscr{Y}}(\mathfrak{P}) \to M_{\mathscr{Y}}(\mathfrak{Q})$ for each pair of points $\mathfrak{P}$ and $\mathfrak{Q}$ in $\mathscr{Y}$ so that the induced chain of $\mathscr{O}_{\mathscr{Y}}$-modules*

$$\cdots \to \bigoplus_{\Delta_{\mathscr{Y}}(\mathfrak{P})=n} J(\mathfrak{P}) \to \bigoplus_{\Delta_{\mathscr{Y}}(\mathfrak{Q})=n+1} J(\mathfrak{Q}) \to \cdots$$

is a residual complex?

The answer to this question is not obvious even for $\mathscr{X} = \operatorname{Spec} R$ and $\mathscr{Y} = \operatorname{Spec} R[X]$! Our method to construct residual complexes is down to earth in the sense that it does not involve derived categories and that commutativity of most diagrams is checked directly by chasing the images of elements under various natural maps.

Without loss of generality, we assume that

$$\mathscr{M}^n = \bigoplus_{\Delta_{\mathscr{X}}(\mathfrak{p})=n} J(\mathfrak{p})$$

throughout this article. We now sketch our construction in the affine case based on which the global construction is patched. Assume $\mathscr{X} = \operatorname{Spec} R$, $\mathscr{Y} = \operatorname{Spec} S$, and $\mathscr{M}^\bullet$ is the sheafification of the complex $M^\bullet$ of R-modules of the form

$$\cdots \xrightarrow{\delta^{n-1}} \bigoplus_{\Delta(\mathfrak{p})=n} M(\mathfrak{p}) \xrightarrow{\delta^{n}} \bigoplus_{\Delta(\mathfrak{q})=n+1} M(\mathfrak{q}) \xrightarrow{\delta^{n+1}} \cdots. \tag{3}$$

By our assumption, there exist elements $x_1, \ldots, x_n$ in S such that the R-linear map $R[X_1, \ldots, X_n] \to S$ sending X_i to x_i is surjective. We will construct a residual complex $M_X^\bullet$ on $\operatorname{Spec} R[X]$ (or on $R[X]$ by abusing the notation) from $M^\bullet$ and construct a residual complex $M^\bullet_{X_1,\ldots,X_n}$ on $R[X_1, \ldots, X_n]$ inductively on n from $M^\bullet$. We will then construct a residual complex $M^\bullet_{S/R;x_1,\ldots,x_n}$ on S by taking a subcomplex of $M^\bullet_{X_1,\ldots,X_n}$ isomorphic to $\operatorname{Hom}_{R[X_1,\ldots,X_n]}(S, M^\bullet_{X_1,\ldots,X_n})$. Taking a direct limit of complexes of the form $M^\bullet_{S/R;x_1,\dots,x_n}$, we obtain a residual complex $M^\bullet_{S/R}$ on S independent of the choice of $x_1,\dots,x_n$.

Before giving more details on $M^\bullet_X$, we recall some facts about prime ideals of $R[X]$. Let $\mathfrak{p}$ be a prime ideal of R and $\kappa(\mathfrak{p})$ be the residue field of $R_\mathfrak{p}$. The prime ideals of $R[X]$ lying over $\mathfrak{p}$ correspond to the prime ideals of $\kappa(\mathfrak{p})[X]$. So every irreducible polynomial in $\kappa(\mathfrak{p})[X]$ determines a prime ideal of $R[X]$ lying over $\mathfrak{p}$. Such prime ideals are exactly those which lie over $\mathfrak{p}$ and are immediate specializations of $\mathfrak{p}R[X]$. If $\mathfrak{q}$ is an immediate specialization of $\mathfrak{p}$, then $\mathfrak{q}R[X]$ is also an immediate specialization of $\mathfrak{p}R[X]$. Let $\mathfrak{P}$ (resp. $\mathfrak{Q}$) be a prime ideal of $R[X]$ lying over $\mathfrak{p}$ (resp. $\mathfrak{q}$) but not equal to $\mathfrak{p}R[X]$ (resp. $\mathfrak{q}R[X]$). If $\mathfrak{Q}$ contains $\mathfrak{P}$, then $\mathfrak{Q}$ is an immediate specialization of $\mathfrak{P}$, since $(R/\mathfrak{p})_{\mathfrak{q}/\mathfrak{p}}[X]$ has Krull dimension two. It is also clear that neither $\mathfrak{P}$ contains $\mathfrak{q}R[X]$ nor $\mathfrak{q}R[X]$ contains $\mathfrak{P}$. Note that $\mathfrak{P}$ may be contained in none of the prime ideals of $R[X]$ which lie over $\mathfrak{q}$. It may also exist an immediate specialization $\mathfrak{r}$ of $\mathfrak{q}$ such that $\mathfrak{r}R[X]$ is an immediate specialization of $\mathfrak{P}$. In our assumption R is universally catenary. If $\mathfrak{r}R[X]$ contains $\mathfrak{P}$, then $\mathfrak{r}R[X]$ is an immediate specialization of $\mathfrak{P}$.

For each prime ideal $\mathfrak{p}$ of R and each prime ideal $\mathfrak{P}$ of $R[X]$ lying over $\mathfrak{p}$, we define

$$M_X(\mathfrak{P}) := \begin{cases} \Omega_{R[X]_{\mathfrak{p}R[X]}/R_\mathfrak{p}} \otimes_{R_\mathfrak{p}} M(\mathfrak{p}), & \text{if} \quad \mathfrak{P} = \mathfrak{p}R[X] \\ H^1_\mathfrak{P}(\Omega_{R[X]_\mathfrak{P}/R_\mathfrak{p}} \otimes_{R_\mathfrak{p}} M(\mathfrak{p})), & \text{if} \quad \mathfrak{P} \neq \mathfrak{p}R[X], \end{cases}$$

where $\Omega_{R[X]_{\mathfrak{p}R[X]}/R_\mathfrak{p}}$ (resp. $\Omega_{R[X]_\mathfrak{P}/R_\mathfrak{p}}$) is the module of Kähler differentials of $R[X]_{\mathfrak{p}R[X]}$ (resp. $R[X]_\mathfrak{P}$) over $R_\mathfrak{p}$ and $H^1_\mathfrak{P}(-)$ is the first local cohomology functor supported on $\mathfrak{P}R[X]_\mathfrak{P}$. We denote by Δ the codimension function $\Delta_{\operatorname{Spec} R}$ characterized as in (1). Note that $M_X(\mathfrak{P})$ is an injective hull of $\kappa(\mathfrak{P})$ (cf. [6, Proposition 3.8]). We will construct in Section 3 canonically a short exact sequence

$$0 \longrightarrow \Omega_{R[X]/R} \otimes_R M(\mathfrak{p}) \xrightarrow{i_\mathfrak{p}} M_X(\mathfrak{p}R[X]) \xrightarrow{\oplus\delta_{\mathfrak{p}R[X],\mathfrak{P}}} \oplus M_X(\mathfrak{P}) \longrightarrow 0$$

of $R[X]$-modules, where $\Omega_{R[X]/R}$ is the module of Kähler differentials of $R[X]$ over R and $\mathfrak{P}$ ranges over the prime ideals of $R[X]$ lying over $\mathfrak{p}$ but not equal to $\mathfrak{p}R[X]$. Taking the direct sum of the above short exact sequences for all prime ideals $\mathfrak{p}$ of R with $\Delta(\mathfrak{p}) = n$, we get another short exact sequence

$$0 \longrightarrow \Omega_{R[X]/R} \otimes_R M^n \xrightarrow{\oplus i_\mathfrak{p}} \oplus M_X(\mathfrak{p}R[X]) \xrightarrow{\oplus\delta_{\mathfrak{p}R[X],\mathfrak{P}}} \oplus M_X(\mathfrak{P}) \longrightarrow 0 \ . \quad (4)$$

For prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ with $\Delta(\mathfrak{q}) = \Delta(\mathfrak{p}) + 1$, we will construct in Section 4 canonically an $R[X]$-linear map

$$\delta_{\mathfrak{p}R[X],\mathfrak{q}R[X]}\colon\ M_X(\mathfrak{p}R[X]) \to M_X(\mathfrak{q}R[X])$$

such that the diagram

$$\begin{array}{ccc} \Omega_{R[X]/R}\otimes_R M(\mathfrak{p}) & \xrightarrow{\ i_{\mathfrak{p}}\ } & M_X(\mathfrak{p}R[X]) \\ {\scriptstyle 1\otimes\delta_{\mathfrak{p},\mathfrak{q}}}\big\downarrow & & \big\downarrow{\scriptstyle \delta_{\mathfrak{p}R[X],\mathfrak{q}R[X]}} \\ \Omega_{R[X]/R}\otimes_R M(\mathfrak{q}) & \xrightarrow[\ i_{\mathfrak{q}}\ ]{} & M_X(\mathfrak{q}R[X]) \end{array}$$

is commutative, where $\delta_{\mathfrak{p},\mathfrak{q}}$ is the R-linear map $M(\mathfrak{p})\to M(\mathfrak{q})$ in the complex (3). Taking the direct sum of the above diagram over all prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of R with $\Delta(\mathfrak{p})=\Delta(\mathfrak{q})-1=n$, we get another commutative diagram

$$\begin{array}{ccc} \Omega_{R[X]/R}\otimes_R M^n & \xrightarrow{\ \oplus i_{\mathfrak{p}}\ } & \bigoplus M_X(\mathfrak{p}R[X]) \\ {\scriptstyle 1\otimes\delta^n}\big\downarrow & & \big\downarrow{\scriptstyle \oplus\delta_{\mathfrak{p}R[X],\mathfrak{q}R[X]}} \\ \Omega_{R[X]/R}\otimes_R M^{n+1} & \xrightarrow[\ \oplus i_{\mathfrak{q}}\ ]{} & \bigoplus M_X(\mathfrak{q}R[X]) \end{array}$$

which induces canonically an $R[X]$-linear map

$$\delta_{\mathfrak{P},\mathfrak{Q}}\colon\ M_X(\mathfrak{P})\to M_X(\mathfrak{Q})$$

for each pair of prime ideals $\mathfrak{P}$ and $\mathfrak{Q}$ of $R[X]$ lying over $\mathfrak{p}$ and $\mathfrak{q}$ but not equal to $\mathfrak{p}R[X]$ and $\mathfrak{q}R[X]$, respectively, such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{R[X]/R}\otimes_R M^n & \xrightarrow{\ \oplus i_{\mathfrak{p}}\ } & \oplus M_X(\mathfrak{p}R[X]) & \xrightarrow{\ \oplus\delta_{\mathfrak{p}R[X],\mathfrak{P}}\ } & \oplus M_X(\mathfrak{P}) & \longrightarrow & 0 \\ & & {\scriptstyle 1\otimes\delta^n}\big\downarrow & & {\scriptstyle \oplus\delta_{\mathfrak{p}R[X],\mathfrak{q}R[X]}}\big\downarrow & & {\scriptstyle -\oplus\delta_{\mathfrak{P},\mathfrak{Q}}}\big\downarrow & & \\ 0 & \longrightarrow & \Omega_{R[X]/R}\otimes_R M^{n+1} & \xrightarrow[\ \oplus i_{\mathfrak{q}}\ ]{} & \oplus M_X(\mathfrak{q}R[X]) & \xrightarrow[\ \oplus\delta_{\mathfrak{q}R[X],\mathfrak{Q}}\ ]{} & \oplus M_X(\mathfrak{Q}) & \longrightarrow & 0 \end{array} \tag{5}$$

is commutative.

The composition of $\oplus\delta_{\mathfrak{p}R[X],\mathfrak{q}R[X]}$ followed by $\oplus\delta_{\mathfrak{q}R[X],\mathfrak{r}R[X]}$ in general is not zero. To correct this defect, we define an $R[X]$-linear map

$$\delta_{\mathfrak{P},\mathfrak{r}R[X]}\colon\ M_X(\mathfrak{P})\to M_X(\mathfrak{r}R[X])$$

for each prime ideal $\mathfrak{P}$ of $R[X]$ lying over $\mathfrak{p}$ but not equal to $\mathfrak{p}R[X]$ and each prime ideal $\mathfrak{r}$ of $R[X]$ with $\Delta(\mathfrak{r})=\Delta(\mathfrak{p})+2$ such that the diagram

$$\begin{array}{ccc} \oplus M_X(\mathfrak{p}R[X]) & \xrightarrow{\ \oplus\delta_{\mathfrak{p}R[X],\mathfrak{P}}\ } & \oplus M_X(\mathfrak{P}) \\ {\scriptstyle \oplus\delta_{\mathfrak{p}R[X],\mathfrak{q}R[X]}}\big\downarrow & & \\ \oplus M_X(\mathfrak{q}R[X]) & \swarrow{\scriptstyle -\oplus\delta_{\mathfrak{P},\mathfrak{r}R[X]}} & \\ {\scriptstyle \oplus\delta_{\mathfrak{q}R[X],\mathfrak{r}R[X]}}\big\downarrow & & \\ \oplus M_X(\mathfrak{r}R[X]) & & \end{array} \tag{6}$$

is commutative, where the sums range over all prime ideals $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ with $\Delta(\mathfrak{p})=n$, $\Delta(\mathfrak{q})=n+1$, $\Delta(\mathfrak{r})=n+2$ and all prime ideals $\mathfrak{P}$ lying over $\mathfrak{p}$

but not equal to $\mathfrak{p}R[X]$. The existence and uniqueness of the map $\oplus\delta_{\mathfrak{P},\mathfrak{r}R[X]}$ follows from the exact sequence (4) and from the fact that $\Omega_{R[X]/R} \otimes_R M^\bullet$ is a complex.

Let

$$M_X^n = (\oplus M_X(\mathfrak{q}R[X])) \bigoplus (\oplus M_X(\mathfrak{P})),$$

where $\mathfrak{q}$ ranging over all primes of R with $\Delta(\mathfrak{q}) = n+1$ and $\mathfrak{P}$ ranging over all primes of $R[X]$ with $\Delta(\mathfrak{P} \cap R) = n$ but $\mathfrak{P} \neq (\mathfrak{P} \cap R)R[X]$. From our construction, the maps $\delta_{\mathscr{P},\mathscr{Q}}$, $\mathscr{P}$, $\mathscr{Q} \in \operatorname{Spec} R[X]$, give rise to a complex $M_X^\bullet$ on $R[X]$. The maps $i_\mathfrak{p}$ give rise to a canonical $R[X]$-linear map of complexes

$$\Omega_{R[X]/R} \otimes_R M^\bullet \to M_X^\bullet[-1]$$

which can be checked to be a quasi-isomorphism directly by diagram chasing. Therefore $M_X^\bullet$ has finitely generated cohomology and hence is a residual complex on $R[X]$.

We will use the following conventions and notation: $\mathbf{X}$, X_1, X_2, ... , $\mathbf{Y}$, Y_1, Y_2, ... denote variables and $\mathscr{X}$, $\mathscr{Y}$, ... denote schemes. $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ denote prime ideals of a base ring or points in a base scheme; $\mathfrak{q}$ (resp. $\mathfrak{r}$) is assumed to be an immediately specialization of $\mathfrak{p}$ (resp. $\mathfrak{q}$); $\mathfrak{P}$, $\mathfrak{P}_X$, $\mathfrak{P}_Y$, ... (resp. $\mathfrak{Q}$, $\mathfrak{Q}_X$, $\mathfrak{Q}_Y$, ...) denote some prime ideals or points lying over $\mathfrak{p}$ (resp. $\mathfrak{q}$); $\mathscr{P}$, $\mathscr{Q}$, ... denote arbitrary prime ideals or points, not necessary those prime ideals lying over $\mathfrak{p}$ or $\mathfrak{q}$. When we consider a ring homomorphism ϕ: $R \to S$, an element a in R will also denote its image $\phi(a)$ in S by abusing the notation.

2. CONSTRUCTION OF INJECTIVE HULLS

Let R be a Noetherian ring and $\mathfrak{P}_\mathbf{X}$ be a prime ideal of $R[\mathbf{X}] := R[X_1, \ldots, X_n]$ and $\mathfrak{p}$ be the contraction of $\mathfrak{P}_\mathbf{X}$ in R. The module of Kähler differentials $(\Omega_{R[\mathbf{X}]_{\mathfrak{P}_\mathbf{X}}/R_\mathfrak{p}}, d)$ of $R[\mathbf{X}]_{\mathfrak{P}_\mathbf{X}}$ over $R_\mathfrak{p}$ is free with basis $dX_1, \ldots, dX_n$. For an injective hull $M(\mathfrak{p})$ of $\kappa(\mathfrak{p})$, we define an $R[\mathbf{X}]_{\mathfrak{P}_\mathbf{X}}$-module

$$\det_{\mathfrak{P}_\mathbf{X}} M(\mathfrak{p}) := \left(\wedge^n \Omega_{R[\mathbf{X}]_{\mathfrak{P}_\mathbf{X}}/R_\mathfrak{p}}\right) \otimes_{R_\mathfrak{p}} M(\mathfrak{p})$$

which is isomorphic to $R[\mathbf{X}]_{\mathfrak{P}_\mathbf{X}} \otimes_{R_\mathfrak{p}} M(\mathfrak{p})$ non-canonically. The $R[\mathbf{X}]_{\mathfrak{P}_\mathbf{X}}$-module

$$H^{\text{top}}_{\mathfrak{P}_\mathbf{X}}(\det_{\mathfrak{P}_\mathbf{X}} M(\mathfrak{p})) := H^\ell_{\mathfrak{P}_\mathbf{X}}(\det_{\mathfrak{P}_\mathbf{X}} M(\mathfrak{p}))$$

is an injective hull of $\kappa(\mathfrak{P}_\mathbf{X})$ [6, Proposition 3.8], where ℓ is the relative dimension of $R[\mathbf{X}]_{\mathfrak{P}_\mathbf{X}}$ over $R_\mathfrak{p}$ and $H^\ell_{\mathfrak{P}_\mathbf{X}}(-)$ is the ℓth local cohomology functor supported on $\mathfrak{P}_{\mathbf{X}}R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}$. Elements in $H^{\mathrm{top}}_{\mathfrak{P}_{\mathbf{X}}}(\det_{\mathfrak{P}_{\mathbf{X}}} M(\mathfrak{p}))$ can be described using generalized fractions [6, 2.1 and 2.2] or [11, Sect. 7]

$$\begin{bmatrix} k\,dX_1\cdots dX_n\otimes\alpha \\ f_1,\ldots,f_\ell \end{bmatrix},$$

where $\alpha\in M(\mathfrak{p})$ $f_1,\ldots,f_\ell\in R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}$ is a relative system of parameters of $R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}$ over $R_{\mathfrak{p}}$, and k is an element in $R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}$. If ℓ is a positive number, we may assume that $k,f_1,\ldots,f_\ell\in R[\mathbf{X}]$ by changing k and the relative system of parameters $f_1,\ldots,f_\ell$. Generalized fractions have the following properties:

Property 2.1 (linearity law). For $\omega_1,\omega_2\in\det_{\mathfrak{P}_{\mathbf{X}}} M(\mathfrak{p})$ and $k_1,k_2\in R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}$,

$$\begin{bmatrix} k_1\omega_1+k_2\omega_2 \\ f_1,\cdots,f_\ell \end{bmatrix}=k_1\begin{bmatrix} \omega_1 \\ f_1,\cdots,f_\ell \end{bmatrix}+k_2\begin{bmatrix} \omega_2 \\ f_1,\cdots,f_\ell \end{bmatrix}.$$

Property 2.2 (vanishing law) [6, 2.3.i] or [11, 7.2.a]. Assume that $\ell>0$. For $\omega\in\det_{\mathfrak{P}_{\mathbf{X}}} M(\mathfrak{p})$,

$$\begin{bmatrix} \omega \\ f_1,\ldots,f_\ell \end{bmatrix}=0$$

if and only if $(f_1\cdots f_\ell)^s\omega\in(f_1^{s+1},\ldots,f_\ell^{s+1})\det_{\mathfrak{P}_{\mathbf{X}}} M(\mathfrak{p})$ for some $s\geq 0$.

In particular, for any $1\leq i\leq\ell$,

$$\begin{bmatrix} f_i\omega \\ f_1,\ldots,f_\ell \end{bmatrix}=0.$$

If ℓ is positive, $g_1,\ldots,g_\ell\in R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}$, and g_i is a unit for some i, we define

$$\begin{bmatrix} \omega \\ g_1,\cdots,g_\ell \end{bmatrix}:=0.$$

Property 2.3 (transformation law) [6, 2.3.ii] or [11, 7.2.b]. For relative system of parameters $f'_1,\ldots,f'_\ell$ of $R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}$ over $R_{\mathfrak{p}}$ and $\omega\in\det_{\mathfrak{P}_{\mathbf{X}}} M(\mathfrak{p})$,

$$\begin{bmatrix} \omega \\ f_1,\ldots,f_\ell \end{bmatrix}=\begin{bmatrix} \det(r_{i,j})\,\omega \\ f'_1,\ldots,f'_\ell \end{bmatrix},$$

if $f'_i=\sum_{j=1}^{\ell} r_{i,j}f_j$ for $i=1,\ldots,\ell$.

Let S be an R-algebra and $\mathfrak{P}$ be a prime ideal of S lying over a prime ideal $\mathfrak{p}$ of R. Assume that there exist elements $x_1,\ldots,x_n$ in S such that the R-linear map $R[\mathbf{X}]\to S$ sending X_i to x_i induces a surjection $R[X]_{\mathfrak{P}_{\mathbf{X}}}\to S_{\mathfrak{P}}$, where $\mathfrak{P}_{\mathbf{X}}$ is the preimage of $\mathfrak{P}$ in $R[\mathbf{X}]$. The functor $\mathrm{Hom}_{R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}}(S_{\mathfrak{P}},-)$ from the category of $R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}$-modules to the category of $S_{\mathfrak{P}}$-modules preserves injective hulls of the residue fields.

DEFINITION 2.4. $M_{S/R;x_1,\dots,x_n}(\mathfrak{P})$ is defined as the subset of $H^{\text{top}}_{\mathfrak{P}_{\mathbf{X}}}(\det_{\mathfrak{P}_{\mathbf{X}}} M(\mathfrak{p}))$ consisting of all elements annihilated by the kernel of the R-linear map $R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}} \to S_{\mathfrak{P}}$ sending X_i to x_i. The $S_{\mathfrak{P}}$-module structure on $M_{S/R;x_1,\dots,x_n}(\mathfrak{P})$ is given via the canonical $R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}$-isomorphism

$$M_{S/R;x_1,\dots,x_n}(\mathfrak{P}) \simeq \operatorname{Hom}_{R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}}\left(S_{\mathfrak{P}}, H^{\text{top}}_{\mathfrak{P}_{\mathbf{X}}}(\det_{\mathfrak{P}_{\mathbf{X}}} M(\mathfrak{p}))\right).$$

We will write $M_{x_1,\dots,x_n}(\mathfrak{P})$ for $M_{S/R;x_1,\dots,x_n}(\mathfrak{P})$ if it is clear from the context that we are working on the R-algebra S. Note that

$$M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}}) = H^{\text{top}}_{\mathfrak{P}_{\mathbf{X}}}(\det_{\mathfrak{P}_{\mathbf{X}}} M(\mathfrak{p})).$$

As an injective hull of $\kappa(\mathfrak{P}_{\mathbf{X}})$, every element of $M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})$ is annihilated by a power of $\mathfrak{P}_{\mathbf{X}} R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}$ (see, for example, [14, Theorem 18.4]). Given $\omega \in M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})$ and $\hat{f} \in R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}$, the $R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}$-module structure on $M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})$ is defined by

$$\hat{f}\omega := f\omega,$$

where $f \in R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}$ is chosen such that $\hat{f} - f \in \mathfrak{P}^m_{\mathbf{X}} R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}$ for some $m \in \mathbb{N}$ with $\mathfrak{P}^m_{\mathbf{X}} R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}} \omega = 0$.

Let $R[\mathbf{X}, \mathbf{Y}] := R[X_1, \dots, X_n, Y_1, \dots, Y_m] \to S$ be an R-linear map extending $R[\mathbf{X}] \to S$. Denote by $\mathfrak{P}_{\mathbf{X},\mathbf{Y}}$ the preimage of $\mathfrak{P}$ in $R[\mathbf{X}, \mathbf{Y}]$. The induced map $R[\mathbf{X}, \mathbf{Y}]_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}} \to \mathbf{S}_{\mathfrak{P}}$ is still surjective. The canonical isomorphisms

$$\det_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}} M(\mathfrak{p}) \simeq (\wedge^m \Omega_{R[\mathbf{X},\mathbf{Y}]_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}/\mathbf{R}[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}}) \otimes_{R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}} \left(\det_{\mathfrak{P}_{\mathbf{X}}} M(\mathfrak{p})\right)$$

and

$$H^{\text{top}}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}(\det_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}} M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})) \tilde{\to} H^{m+\ell}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}\left((\wedge^m \Omega_{R[\mathbf{X},\mathbf{Y}]_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}/\mathbf{R}[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}}) \otimes_{R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}} \left(\det_{\mathfrak{P}_{\mathbf{X}}} M(\mathfrak{p})\right)\right)$$

[6, (2.5) and (2.6)] give rise to a canonical isomorphism

$$H^{\text{top}}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}(\det_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}} M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})) \tilde{\to} M_{\mathbf{X},\mathbf{Y}}(\mathfrak{P}_{\mathbf{X},\mathbf{Y}}) \tag{7}$$

explicitly described by generalized fractions:

$$\begin{bmatrix} k\, dY_1 \cdots dY_m \otimes \begin{bmatrix} dX_1 \cdots dX_n \otimes \alpha \\ f_1^{j_1}, \dots, f_\ell^{j_\ell} \end{bmatrix} \\ (Y_1 - g_1)^{i_1}, \dots, (Y_m - g_m)^{i_m} \end{bmatrix} \mapsto \begin{bmatrix} k\, dY_1 \cdots dY_m dX_1 \cdots dX_n \otimes \alpha \\ (Y_1 - g_1)^{i_1}, \dots, (Y_m - g_m)^{i_m}, f_1^{j_1}, \dots, f_\ell^{j_\ell} \end{bmatrix},$$

where $k \in R[\mathbf{X},\mathbf{Y}]_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}$ and $g_1,\dots,g_m$ are elements in $R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}$ mapping to the images of $Y_1,\dots,Y_m$ in $S_{\mathfrak{P}}$, respectively. Note that $Y_1-g_1,\dots,Y_m-g_m$ is a regular system of parameters of $R[\mathbf{X},\mathbf{Y}]_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}$ (resp. $R[\mathbf{X},\mathbf{Y}]^{\wedge}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}$) over $R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}$ (resp. $R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}$). Note also that the universal separated differential module $\widetilde{\Omega}_{R[\mathbf{X},\mathbf{Y}]^{\wedge}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}/R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}}$ of $R[\mathbf{X},\mathbf{Y}]^{\wedge}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}$ over $R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}$ is free with basis $dY_1,\dots,dY_m$. We define a bijective map

$$H^{\mathrm{top}}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}(\det\nolimits_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}} M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})) \to H^m_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}\big(\wedge^m \widetilde{\Omega}_{R[\mathbf{X},\mathbf{Y}]^{\wedge}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}/R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}} \otimes_{R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}} M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})\big)$$

by

$$\begin{bmatrix} k\,dY_1\cdots dY_m \otimes \begin{bmatrix} dX_1\cdots dX_n\otimes\alpha \\ f_1^{j_1},\dots,f_\ell^{j_\ell}\end{bmatrix} \\ (Y_1-g_1)^{i_1},\dots,(Y_m-g_m)^{i_m}\end{bmatrix} \mapsto \begin{bmatrix} k\,dY_1\cdots dY_m \otimes \begin{bmatrix} dX_1\cdots dX_n\otimes\alpha \\ f_1^{j_1},\dots,f_\ell^{j_\ell}\end{bmatrix} \\ (Y_1-g_1)^{i_1},\dots,(Y_m-g_m)^{i_m}\end{bmatrix}.$$

The canonical map

$$H^m_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}\big(\wedge^m \widetilde{\Omega}_{R[\mathbf{X},\mathbf{Y}]^{\wedge}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}/R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}} \otimes M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})\big) \to M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})$$

defined to be the residue map, in the sense of [6, Chap. V] for the power series ring $R[\mathbf{X},\mathbf{Y}]^{\wedge}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}$ over $R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}$, has the form:
For $k \in R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}$,

$$\begin{bmatrix} k\,dY_1\cdots dY_m \otimes \begin{bmatrix} dX_1\cdots dX_n\otimes\alpha \\ f_1^{j_1},\dots,f_\ell^{j_\ell}\end{bmatrix} \\ (Y_1-g_1)^{i_1},\dots,(Y_m-g_m)^{i_m}\end{bmatrix} \mapsto \begin{cases} \begin{bmatrix} k\,dX_1\cdots dX_n\otimes\alpha \\ f_1^{j_1},\dots,f_\ell^{j_\ell}\end{bmatrix}, & \text{if } i_1=\cdots=i_m=1; \\ 0, & \text{otherwise.}\end{cases}$$

We now define a map, still called residue map in this article,

$$\mathrm{res}\colon\ M_{\mathbf{X},\mathbf{Y}}(\mathfrak{P}_{\mathbf{X},\mathbf{Y}}) \to M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}}) \tag{8}$$

so that the diagram

$$\begin{array}{ccc} H^{\mathrm{top}}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}(\det\nolimits_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}} M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})) & \longrightarrow & M_{\mathbf{X},\mathbf{Y}}(\mathfrak{P}_{\mathbf{X},\mathbf{Y}}) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ H^m_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}\big(\wedge^m \widetilde{\Omega}_{R[\mathbf{X},\mathbf{Y}]^{\wedge}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}/R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}} \otimes M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})\big) & \longrightarrow & M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}}) \end{array}$$

is commutative.

Property 2.5 (transitivity law). For variables $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, the diagram

$$\begin{array}{ccc} M_{\mathbf{X},\mathbf{Y},\mathbf{Z}}(\mathfrak{P}_{\mathbf{X},\mathbf{Y},\mathbf{Z}}) & & \\ {\scriptstyle \mathrm{res}}\downarrow & \searrow{\scriptstyle \mathrm{res}} & \\ M_{\mathbf{X},\mathbf{Y}}(\mathfrak{P}_{\mathbf{X},\mathbf{Y}}) & \xrightarrow[\mathrm{res}]{} & M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}}) \end{array}$$

is commutative.

The transitivity law can be easily verified using the following formula: For $k \in R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}$,

$$\mathrm{res}\begin{bmatrix} k\, dY_1 \cdots dY_m dX_1 \cdots dX_n \otimes \alpha \\ (Y_1 - g_1)^{i_1}, \ldots, (Y_m - g_m)^{i_m}, f_1^{j_1}, \ldots, f_\ell^{j_\ell} \end{bmatrix}$$
$$= \begin{cases} \begin{bmatrix} k\, dX_1 \cdots dX_n \otimes \alpha \\ f_1^{j_1}, \ldots, f_\ell^{j_\ell} \end{bmatrix}, & \text{if } i_1 = \cdots = i_{m=1}; \\ 0, & \text{otherwise.} \end{cases}$$

The above formula determines the residue map (8) since every element in $M_{\mathbf{X},\mathbf{Y}}(\mathfrak{P}_{\mathbf{X},\mathbf{Y}})$ is a finite sum of elements of the form

$$\begin{bmatrix} k\, dY_1 \cdots dY_m dX_1 \cdots dX_n \otimes \alpha \\ (Y_1 - g_1)^{i_1}, \ldots, (Y_m - g_m)^{i_m}, f_1^{j_1}, \ldots, f_\ell^{j_\ell} \end{bmatrix},$$

where $k \in R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}$. Denote by $\mathrm{Hom}^{\mathrm{c}}_{R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}}(R[\mathbf{X},\mathbf{Y}]^{\wedge}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}, M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}}))$ (resp. $\mathrm{Hom}^{\mathrm{c}}_{R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}}(R[\mathbf{X},\mathbf{Y}]_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}, M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}}))$) the $R[\mathbf{X},\mathbf{Y}]^{\wedge}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}$ (resp. $R[\mathbf{X},\mathbf{Y}]_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}$)-module consisting of all continuous homomorphisms from $R[\mathbf{X},\mathbf{Y}]^{\wedge}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}$ (resp. $R[\mathbf{X},\mathbf{Y}]_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}$) to $M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})$ (that is, the homomorphisms annihilated by some power of $\mathfrak{P}_{\mathbf{X},\mathbf{Y}}$). Using the local duality [6, Theorem 5.9]

$$H^m_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}(\wedge^m \widetilde{\Omega}_{R[\mathbf{X},\mathbf{Y}]^{\wedge}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}/R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}} \otimes M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}}))$$
$$\tilde{\to} \mathrm{Hom}^{\mathrm{c}}_{R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}}(R[\mathbf{X},\mathbf{Y}]^{\wedge}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}, M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}}))$$

and the canonical isomorphism

$$\mathrm{Hom}^{\mathrm{c}}_{R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}}(R[\mathbf{X},\mathbf{Y}]^{\wedge}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}, M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})) \tilde{\to} \mathrm{Hom}^{\mathrm{c}}_{R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}}(R[\mathbf{X},\mathbf{Y}]_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}, M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})),$$

we define

$$M_{\mathbf{X},\mathbf{Y}}(\mathfrak{P}_{\mathbf{X},\mathbf{Y}}) \to \mathrm{Hom}^{\mathrm{c}}_{R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}}(R[\mathbf{X},\mathbf{Y}]_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}, M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})) \tag{9}$$

to be the isomorphism which makes the diagram

$$\begin{array}{ccc}
H^{\mathrm{top}}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}(\det_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}} M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})) & \longrightarrow & M_{\mathbf{X},\mathbf{Y}}(\mathfrak{P}_{\mathbf{X},\mathbf{Y}}) \\
\downarrow & & \\
H^{m}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}(\wedge^m \widetilde{\Omega}_{R[\mathbf{X},\mathbf{Y}]^{\wedge}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}/R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}} \otimes M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})) & & \downarrow \\
\downarrow & & \\
\mathrm{Hom}^{\mathrm{c}}_{R[\mathbf{X}]^{\wedge}_{\mathfrak{P}_{\mathbf{X}}}}(R[\mathbf{X},\mathbf{Y}]^{\wedge}_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}, M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})) & \longrightarrow & \mathrm{Hom}^{\mathrm{c}}_{R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}}(R[\mathbf{X},\mathbf{Y}]_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}, M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}}))
\end{array}$$

commutative. In terms of the residue map, the map (9) sends each element ω in $M_{\mathbf{X},\mathbf{Y}}(\mathfrak{P}_{\mathbf{X},\mathbf{Y}})$ to the continuous homomorphism $f \mapsto \mathrm{res}(f\omega)$.

Denote by $\mathbf{y} := y_1, \dots, y_m$ the images of $Y_1, \dots, Y_m$ in S. Using the canonical isomorphism

$$\begin{aligned}
&\mathrm{Hom}_{R[\mathbf{X},\mathbf{Y}]_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}}(S_{\mathfrak{P}}, \mathrm{Hom}^{\mathrm{c}}_{R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}}(R[\mathbf{X},\mathbf{Y}]_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}, M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})))\\
&\qquad \tilde{\to} \mathrm{Hom}_{R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}}(S_{\mathfrak{P}}, M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}}))
\end{aligned}$$

[6, 4.4., iii], we define

$$\mathrm{res}^{-1}\colon\ M_{\mathbf{x}}(\mathfrak{P}) \to M_{\mathbf{x},\mathbf{y}}(\mathfrak{P})$$

to be the isomorphism which makes the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}}(S_{\mathfrak{P}}, M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})) & \longrightarrow & M_{\mathbf{x}}(\mathfrak{P}) \\
\uparrow & & \\
\mathrm{Hom}_{R[\mathbf{X},\mathbf{Y}]_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}}(S_{\mathfrak{P}}, \mathrm{Hom}^{\mathrm{c}}_{R[\mathbf{X}]_{\mathfrak{P}_{\mathbf{X}}}}(R[\mathbf{X},\mathbf{Y}]_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}, M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}}))) & & \downarrow \mathrm{res}^{-1} \\
\uparrow & & \\
\mathrm{Hom}_{R[\mathbf{X},\mathbf{Y}]_{\mathfrak{P}_{\mathbf{X},\mathbf{Y}}}}(S_{\mathfrak{P}}, M_{\mathbf{X},\mathbf{Y}}(\mathfrak{P}_{\mathbf{X},\mathbf{Y}})) & \longrightarrow & M_{\mathbf{x},\mathbf{y}}(\mathfrak{P})
\end{array}$$

commutative. The notation "res^{-1}" is justified by the fact that the inverse of res^{-1} is the restriction of the residue map $M_{\mathbf{X},\mathbf{Y}}(\mathfrak{P}_{\mathbf{X},\mathbf{Y}}) \to M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})$ to $M_{\mathbf{x},\mathbf{y}}(\mathfrak{P})$.

For elements $x_1, \dots, x_n \in S$, we define $\mathfrak{P}_{\mathbf{x}}$ to be $R[x_1, \dots, x_n] \cap \mathfrak{P}$. Consider the directed set

$$\mathcal{I}_{S/R,\mathfrak{P}} = \{\mathbf{x} \mid \mathbf{x} = \{x_1, \dots, x_n\} \subseteq S,\ R[x_1, \dots, x_n]_{\mathfrak{P}_{\mathbf{x}}} = S_{\mathfrak{P}}\}$$

with the order defined by inclusion. For each $\mathbf{x}$ in $\mathcal{I}_{S/R,\mathfrak{P}}$, we choose variables $\mathbf{X}$ and define $M_{\mathbf{x}}(\mathfrak{P})$ as above. Using the transitivity law (2.5), it is easy to see that the system $\{M_{\mathbf{x}}(\mathfrak{P})\}$ of $S_{\mathfrak{P}}$-modules is compatible with the isomorphisms res^{-1}.

DEFINITION 2.6. $M_{S/R}(\mathfrak{P}) := \lim\limits_{\to} M_{S/R,\mathbf{x}}(\mathfrak{P})$.

The canonical map $M_{S/R,\mathbf{x}}(\mathfrak{P}) \to M_{S/R}(\mathfrak{P})$ is an isomorphism. Therefore $M_{S/R}(\mathfrak{P})$ is an injective hull of $\kappa(\mathfrak{P})$. Note also that not only any two direct limits (objects with the universal property) are canonically isomorphic, there is also a canonical choice among all direct limits. $M_{S/R}(\mathfrak{P})$ is defined as the one canonically chosen as done for example in [6, p. 32].

Let $\mathscr{X}$ and $\mathscr{Y}$ be locally Noetherian schemes and $\phi\colon \mathscr{Y} \to \mathscr{X}$ be a morphism of finite type. Given a point $\mathfrak{P}$ in $\mathscr{Y}$, we denote by $\mathfrak{p}$ its image under ϕ. Let $M(\mathfrak{p})$ be an injective hull of $\kappa(\mathfrak{p})$, we want to define canonically an injective hull of $\kappa(\mathfrak{P})$. Consider the directed set

$$\mathscr{J}_{\mathscr{Y}/\mathscr{X},\mathfrak{P}} = \{(V, U) \mid \mathfrak{P} \in V,\ \phi(V) \subseteq U\}$$

of affine open subsets V of $\mathscr{Y}$ and U of $\mathscr{X}$ with the order defined by

$$(V_1, U_1) \le (V_2, U_2) \quad \text{if and only if} \quad V_1 \supseteq V_2 \ \text{ and } \ U_1 \supseteq U_2.$$

Let (V_1, U_1) and (V_2, U_2) be elements in $\mathscr{J}_{\mathscr{Y}/\mathscr{X},\mathfrak{P}}$ with $(V_1, U_1) \le (V_2, U_2)$. Given $\mathbf{x}$ in $\mathscr{I}_{\Gamma(V_1,\mathscr{O}_\mathscr{Y})/\Gamma(U_1,\mathscr{O}_\mathscr{X}),\mathfrak{P}}$, we denote by $\overline{\mathbf{x}}$ the restrictions of $\mathbf{x}$ in $\Gamma(V_2, \mathscr{O}_\mathscr{Y})$, then $\overline{\mathbf{x}}$ is in $\mathscr{I}_{\Gamma(V_2,\mathscr{O}_\mathscr{Y})/\Gamma(U_2,\mathscr{O}_\mathscr{X}),\mathfrak{P}}$. So we may define $M_{\Gamma(V_1,\mathscr{O}_\mathscr{Y})/\Gamma(U_1,\mathscr{O}_\mathscr{X}),\mathbf{x}}(\mathfrak{P})$, as well as $M_{\Gamma(V_2,\mathscr{O}_\mathscr{Y})/\Gamma(U_2,\mathscr{O}_\mathscr{X}),\overline{\mathbf{x}}}(\mathfrak{P})$. Let $\mathbf{X}$ (resp. $\overline{\mathbf{X}}$) be variables chosen for the elements $\mathbf{x}$ (resp. $\overline{\mathbf{x}}$). Let $\mathfrak{P}_\mathbf{X}$ (resp. $\mathfrak{P}_{\overline{\mathbf{X}}}$) be the preimage of $\mathfrak{P}$ under the canonical map $\Gamma(U_1, \mathscr{O}_\mathscr{X})[\mathbf{X}] \to \Gamma(V_1, \mathscr{O}_\mathscr{Y})$ (resp. $\Gamma(U_2, \mathscr{O}_\mathscr{X})[\overline{\mathbf{X}}] \to \Gamma(V_2, \mathscr{O}_\mathscr{Y})$) sending X_i (resp. $\overline{X}_i$) to x_i (resp. $\overline{x}_i$). There is a bijective map

$$\zeta_{\mathfrak{P}_\mathbf{X}}\colon\ M_{\Gamma(U_1,\mathscr{O}_\mathscr{X})[\mathbf{X}]/\Gamma(U_1,\mathscr{O}_\mathscr{X}),\mathbf{X}}(\mathfrak{P}_\mathbf{X}) \to M_{\Gamma(U_2,\mathscr{O}_\mathscr{X})[\overline{\mathbf{X}}]/\Gamma(U_2,\mathscr{O}_\mathscr{X}),\overline{\mathbf{X}}}(\mathfrak{P}_{\overline{\mathbf{X}}})$$

given by

$$\begin{bmatrix} k\,dX_1\cdots dX_n \otimes \alpha \\ f_1,\ldots,f_\ell \end{bmatrix} \mapsto \begin{bmatrix} \overline{k}\,d\overline{X}_1\cdots d\overline{X}_n \otimes \alpha \\ \overline{f}_1,\ldots,\overline{f}_\ell \end{bmatrix},$$

where $\overline{k}, \overline{f}_1, \ldots, \overline{f}_\ell$ are the images of $k, f_1, \ldots, f_\ell$ under the canonical isomorphism

$$\Gamma(U_1, \mathscr{O}_\mathscr{X})[\mathbf{X}]_{\mathfrak{P}_\mathbf{X}} \to \Gamma(U_2, \mathscr{O}_\mathscr{X})[\overline{\mathbf{X}}]_{\mathfrak{P}_{\overline{\mathbf{X}}}}.$$

Given an element g in $\Gamma(U_1, \mathscr{O}_\mathscr{X})[\mathbf{X}]^\wedge_{\mathfrak{P}_\mathbf{X}}$, let $\overline{g}$ be the image of g under the canonical isomorphism

$$\Gamma(U_1, \mathscr{O}_\mathscr{X})[\mathbf{X}]^\wedge_{\mathfrak{P}_\mathbf{X}} \to \Gamma(U_2, \mathscr{O}_\mathscr{X})[\overline{\mathbf{X}}]^\wedge_{\mathfrak{P}_{\overline{\mathbf{X}}}},$$

it is easy to see that

$$\zeta_{\mathfrak{P}_\mathbf{X}}\left(g\begin{bmatrix} k\,dX_1\cdots dX_n \otimes \alpha \\ f_1,\ldots,f_\ell \end{bmatrix}\right) = \overline{g}\left(\zeta_{\mathfrak{P}_\mathbf{X}}\begin{bmatrix} k\,dX_1\cdots dX_n \otimes \alpha \\ f_1,\ldots,f_\ell \end{bmatrix}\right).$$

The restriction of the map $\zeta_{\mathfrak{P}_{\mathbf{x}}}$ to $M_{\Gamma(V_1,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_1,\mathscr{O}_{\mathscr{X}}),\mathbf{x}}(\mathfrak{P})$ has image equal to $M_{\Gamma(V_2,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_2,\mathscr{O}_{\mathscr{X}}),\bar{\mathbf{x}}}(\mathfrak{P})$. So we get a map

$$M_{\Gamma(V_1,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_1,\mathscr{O}_{\mathscr{X}}),\mathbf{x}}(\mathfrak{P}) \to M_{\Gamma(V_2,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_2,\mathscr{O}_{\mathscr{X}}),\bar{\mathbf{x}}}(\mathfrak{P}) \tag{10}$$

which is easily seen to be an $\mathscr{O}_{\mathscr{Y},\mathfrak{P}}$-isomorphism. The isomorphisms (10) for different $\mathbf{x} \in \mathscr{I}_{\Gamma(V_1,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_1,\mathscr{O}_{\mathscr{X}}),\mathfrak{P}}$ are compatible with the residue maps in the following sense: Given $\mathbf{x}$ and $\mathbf{y}$ in $\mathscr{I}_{\Gamma(V_1,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_1,\mathscr{O}_{\mathscr{X}}),\mathfrak{P}}$ with $\mathbf{x} \le \mathbf{y}$, the diagram

$$\begin{array}{ccc}
M_{\Gamma(V_1,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_1,\mathscr{O}_{\mathscr{X}}),\mathbf{x}}(\mathfrak{P}) & \longrightarrow & M_{\Gamma(V_2,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_2,\mathscr{O}_{\mathscr{X}}),\bar{\mathbf{x}}}(\mathfrak{P}) \\
{\scriptstyle \mathrm{res}^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}^{-1}} \\
M_{\Gamma(V_1,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_1,\mathscr{O}_{\mathscr{X}}),\mathbf{y}}(\mathfrak{P}) & \longrightarrow & M_{\Gamma(V_2,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_2,\mathscr{O}_{\mathscr{X}}),\bar{\mathbf{y}}}(\mathfrak{P})
\end{array}$$

is commutative, where $\bar{\mathbf{x}}$ (resp. $\bar{\mathbf{y}}$) are the restrictions of $\mathbf{x}$ (resp. $\mathbf{y}$) in $\Gamma(V_2, \mathscr{O}_{\mathscr{Y}})$. Therefore the isomorphism

$$M_{\Gamma(V_1,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_1,\mathscr{O}_{\mathscr{X}})}(\mathfrak{P}) \to M_{\Gamma(V_2,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_2,\mathscr{O}_{\mathscr{X}})}(\mathfrak{P})$$

defined as the map making the diagram

$$\begin{array}{ccc}
M_{\Gamma(V_1,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_1,\mathscr{O}_{\mathscr{X}}),\mathbf{x}}(\mathfrak{P}) & \longrightarrow & M_{\Gamma(V_2,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_2,\mathscr{O}_{\mathscr{X}}),\bar{\mathbf{x}}}(\mathfrak{P}) \\
\big\downarrow & & \big\downarrow \\
M_{\Gamma(V_1,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_1,\mathscr{O}_{\mathscr{X}})}(\mathfrak{P}) & \longrightarrow & M_{\Gamma(V_2,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_2,\mathscr{O}_{\mathscr{X}})}(\mathfrak{P})
\end{array}$$

commutative is independent of the choice of $\mathbf{x} \in \mathscr{I}_{\Gamma(V_1,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_1,\mathscr{O}_{\mathscr{X}}),\mathfrak{P}}$. Let (V_3, U_3) be an element in $\mathscr{J}_{\mathscr{Y}/\mathscr{X},\mathfrak{P}}$ with $(V_2, U_2) \le (V_3, U_3)$. It is easy to see that the maps defined above form a commutative diagram

$$\begin{array}{ccc}
M_{\Gamma(V_1,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_1,\mathscr{O}_{\mathscr{X}})}(\mathfrak{P}) & \longrightarrow & M_{\Gamma(V_2,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_2,\mathscr{O}_{\mathscr{X}})}(\mathfrak{P}) \\
& \searrow & \big\downarrow \\
& & M_{\Gamma(V_3,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_3,\mathscr{O}_{\mathscr{X}})}(\mathfrak{P}).
\end{array}$$

DEFINITION 2.7. $M_{\mathscr{Y}}(\mathfrak{P}) := \varinjlim M_{\Gamma(V_i,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_i,\mathscr{O}_{\mathscr{X}})}(\mathfrak{P})$.

The canonical map $M_{\Gamma(V_i,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_i,\mathscr{O}_{\mathscr{X}})}(\mathfrak{P}) \to M_{\mathscr{Y}}(\mathfrak{P})$ is an isomorphism. Therefore $M_{\mathscr{Y}}(\mathfrak{P})$ is an injective hull of $\kappa(\mathfrak{P})$. Note that $M_{\mathscr{Y}}(\mathfrak{P})$ depends only on $M(\mathfrak{p})$ and the morphism ϕ.

3. CONSTRUCTIONS OF $i_{\mathfrak{p}}$ AND $\delta_{\mathfrak{p}R[X],\mathfrak{P}}$

The maps which we are going to construct in this section are trivial in formalism. Let $\mathfrak{p}$ be a prime ideal of R and let $M(\mathfrak{p})$ be an injective hull of $\kappa(\mathfrak{p})$.

DEFINITION 3.1. The map

$$i_{\mathfrak{p}}\colon\ \Omega_{R[X]/R} \otimes_R M(\mathfrak{p}) \to \Omega_{R[X]_{\mathfrak{p}R[X]}/R_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p})$$

is defined as the identity map $1_{M(\mathfrak{p})}$ tensorized by the functorial map

$$\Omega_{R[X]/R} \to \Omega_{R[X]_{\mathfrak{p}R[X]}/R_{\mathfrak{p}}}.$$

THEOREM 3.2. *The map $i_{\mathfrak{p}}$ is injective.*

Proof. Assume that $X^s\ dX \otimes \alpha_s + \cdots + X\ dX \otimes \alpha_1 + dX \otimes \alpha_0$ ($\alpha_i \in M(\mathfrak{p})$) is in the kernel of $i_{\mathfrak{p}}$ (that is, it is annihilated by an element $a_tX^t + \cdots + a_0 \in R[X] \setminus \mathfrak{p}R[X]$). In terms of system of equations in $M(\mathfrak{p})$,

$$\begin{pmatrix} a_t & 0 & 0 & \cdots & 0 & 0 \\ a_{t-1} & a_t & 0 & \cdots & 0 & 0 \\ \cdot & a_{t-1} & a_t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_0 & a_1 & \cdot & \cdots & a_t & 0 \\ 0 & a_0 & a_1 & \cdots & a_{t-1} & a_t \\ 0 & 0 & a_0 & \cdots & \cdot & a_{t-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 & a_1 \\ 0 & 0 & 0 & \cdots & 0 & a_0 \end{pmatrix} \begin{pmatrix} \alpha_s \\ \alpha_{s-1} \\ \vdots \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Assume that $a_i \notin \mathfrak{p}$ and $a_j \in \mathfrak{p}$ for $j > i$. Using row operations, we get

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_i & 0 & \cdots & 0 & 0 \\ 0 & a_{i,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{i,s-1} & 0 \\ 0 & 0 & \cdots & 0 & a_{i,s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_s \\ \alpha_{s-1} \\ \vdots \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

for some units $a_{i,j}$ in $R_{\mathfrak{p}}$. Therefore

$$X^s\ dX \otimes \alpha_s + \cdots + X\ dX \otimes \alpha_1 + dX \otimes \alpha_0 = 0.$$

∎

DEFINITION 3.3. (cf. [2, 1.1.2.]). Let $\mathfrak{P}$ be a prime ideal of $R[X]$ lying over $\mathfrak{p}$ but not equal to $\mathfrak{p}R[X]$. Define the map

$$\delta_{\mathfrak{p}R[X],\mathfrak{P}}\colon\ \Omega_{R[X]_{\mathfrak{p}R[X]}/R_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p}) \to H^1_{\mathfrak{P}}(\Omega_{R[X]_{\mathfrak{P}}/R_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p}))$$

by

$$\delta_{\mathfrak{p}R[X],\mathfrak{P}}\left(\frac{k}{f}dX \otimes \alpha\right) = \begin{bmatrix} k\,dX \otimes \alpha \\ f \end{bmatrix}$$

for all $k \in R[X]$, $f \in R[X]\backslash \mathfrak{p}R[X]$, and $\alpha \in M(\mathfrak{p})$. Define

$$\delta_{\mathfrak{p}R[X]}\colon\ \Omega_{R[X]_{\mathfrak{p}R[X]}/R_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p}) \to \oplus H^1_{\mathfrak{P}}(\Omega_{R[X]_{\mathfrak{P}}/R_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p}))$$

by $\delta_{\mathfrak{p}R[X]} = \oplus \delta_{\mathfrak{p}R[X],\mathfrak{P}}$, where $\mathfrak{P}$ ranges over the prime ideals of $R[X]$ lying over $\mathfrak{p}$ but not equal to $\mathfrak{p}R[X]$.

It is easy to check that $\delta_{\mathfrak{p}R[X],\mathfrak{P}}$ is well-defined using the linearity law (2.1) and transformation law (2.3). We leave the details to the reader. The fact that $\delta_{\mathfrak{p}R[X]}$ is well-defined follows from the next proposition which provides "Cousin data" for elements in $\Omega_{R[X]_{\mathfrak{p}R[X]}/R_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p})$ and can be viewed as a generalized version of the decomposition into partial fractions. It is also easy to see that $\delta_{\mathfrak{p}R[X]} \circ i_{\mathfrak{p}} = 0$.

PROPOSITION 3.4. *Every element in* $\Omega_{R[X]_{\mathfrak{p}R[X]}/R_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p})$ *of the form* $(k/f)dX \otimes \alpha$, *where* $\alpha \in M(\mathfrak{p})$, $k \in R[X]$, *and* $f \in R[X] \setminus \mathfrak{p}R[X]$, *can be written as*

$$\frac{k}{f}dX \otimes \alpha = k_0\ dX \otimes \alpha_0 + \frac{k_1}{f_1}dX \otimes \alpha_1 + \cdots + \frac{k_s}{f_s}dX \otimes \alpha_s$$

for some $\alpha_i \in M(\mathfrak{p})$ *and* $k_i, f_i \in R[X]$ *satisfying the following conditions*:

- $\deg k_i < \deg f_i$.
- *The leading coefficient of* f_i *is not in* $\mathfrak{p}$.
- *The image of* f_i *in* $\kappa(\mathfrak{p})[X]$ *is a power of an irreducible polynomial.*
- *The images of* f_i *and* f_j *in* $\kappa(\mathfrak{p})[X]$ *are relatively prime if* $i \neq j$.

Since $\Omega_{R[X]_{\mathfrak{p}R[X]}/R_{\mathfrak{p}}}$ is free of rank 1 with generator dX, the proposition is a direct consequence of the following two lemmas.

LEMMA 3.5. *Let* $k \in R[X]$, $f, f_1, f_2 \in R[X] \setminus \mathfrak{p}R[X]$, *and* $\alpha \in M(\mathfrak{p})$. *Assume* $f - f_1 f_2 \in \mathfrak{p}R[X]$ *and the images of* f_1, f_2 *in* $\kappa(\mathfrak{p})[X]$ *are relatively prime*. *Then there exist* $k_1, k_2 \in R[X]$, $r \in R \setminus \mathfrak{p}$, *and* $n \in \mathbb{N}$ *such that*

$$\frac{k}{f} \otimes \alpha = \frac{k_1}{f_1^n} \otimes \frac{\alpha}{r} + \frac{k_2}{f_2^n} \otimes \frac{\alpha}{r}$$

in $R[X]_{\mathfrak{p}R[X]} \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p})$.

Proof. There exists $n \in \mathbb{N}$ such that $\mathfrak{p}^n \alpha = 0$. Since $(f - f_1 f_2)^n \otimes \alpha = 0$,

$$(f_1 f_2)^n \otimes \alpha = \left(\binom{n}{1}(f_1 f_2)^{n-1} f - \binom{n}{2}(f_1 f_2)^{n-2} f^2 + \cdots + (-1)^{n+1}\binom{n}{n} f^n\right) \otimes \alpha$$

and hence

$$\frac{k}{f} \otimes \alpha = \frac{k(f_1 f_2)^n}{f(f_1 f_2)^n} \otimes \alpha = \frac{\binom{n}{1}(f_1 f_2)^{n-1} k - \binom{n}{2}(f_1 f_2)^{n-2} f k + \cdots + (-1)^{n+1}\binom{n}{n} f^{n-1} k}{(f_1 f_2)^n} \otimes \alpha.$$

So we may assume $f = f_1 f_2$. Choose $h_1, h_2 \in R[X]$ and $r \in R \setminus \mathfrak{p}$ such that

$$h_1 f_1 + h_2 f_2 - r \in \mathfrak{p} R[X],$$

then

$$\frac{k(h_1 f_1 + h_2 f_2 - r)^n}{f_1 f_2} \otimes \frac{\alpha}{r^n} = 0.$$

Hence there exist $k_1, k_2 \in R[X]$ such that

$$\frac{k}{f} \otimes \alpha = \frac{k r^n}{f_1 f_2} \otimes \frac{\alpha}{r^n} = \left(\frac{k_1}{f_1} + \frac{k_2}{f_2}\right) \otimes \frac{\alpha}{r^n}.$$

∎

Here is a special case: For any $k \in R[X]$, $f \in R[X] \setminus \mathfrak{p} R[X]$, and $\alpha \in M(\mathfrak{p})$, choose $f_1 \in R[X] \setminus \mathfrak{p} R[X]$ whose leading coefficient is not in $\mathfrak{p}$ such that $f - f_1 \in \mathfrak{p} R[X]$. Then applying the above lemma for $f_2 = 1$, there exist $k_1, k_2 \in R[X]$ and $r \in R \setminus \mathfrak{p}$ such that

$$\frac{k}{f} \otimes \alpha = \frac{k_1}{f_1^n} \otimes \frac{\alpha}{r} + k_2 \otimes \frac{\alpha}{r}.$$

LEMMA 3.6. *Let $k, f \in R[X]$ and $\alpha \in M(\mathfrak{p})$. Assume that the leading coefficient of f is not in $\mathfrak{p}$. Then there exist $k', k'' \in R[X]$, $r \in R \setminus \mathfrak{p}$ such that*

$$\frac{k}{f} \otimes \alpha = k' \otimes \frac{\alpha}{r} + \frac{k''}{f} \otimes \frac{\alpha}{r}$$

in $R[X]_{\mathfrak{p} R[X]} \otimes_{R_\mathfrak{p}} M(\mathfrak{p})$ and $\deg k'' < \deg f$.

The lemma is proved by induction on the degree of k. We leave the details to the reader. Another main result in this section is as follows.

THEOREM 3.7. *The map $\delta_{\mathfrak{p}R[X]}$ is surjective with kernel equal to the image of $i_{\mathfrak{p}}$.*

Proof. Let $\mathfrak{P}$ be a prime ideal of $R[X]$ lying over $\mathfrak{p}$ but not equal to $\mathfrak{p}R[X]$. Any element in $H^1_{\mathfrak{P}}(\Omega_{R[X]_{\mathfrak{P}}/R_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p}))$ can be written as a sum of elements of the form

$$\begin{bmatrix} k\,dX \otimes \alpha \\ f \end{bmatrix},$$

for some $\alpha \in M(\mathfrak{p})$, $k \in R[X]$, and $f \in R[X] \setminus \mathfrak{p}R[X]$. Choose k_i, f_i, and α_i as in Proposition 3.4 such that

$$\frac{k}{f} dX \otimes \alpha = k_0 dX \otimes \alpha_0 + \frac{k_1}{f_1} dX \otimes \alpha_1 + \cdots + \frac{k_s}{f_s} dX \otimes \alpha_s$$

in $\Omega_{R[X]_{\mathfrak{p}R[X]}/R_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p})$. If $f_i \notin \mathfrak{P}$, then $\delta_{\mathfrak{p}R[X],\mathfrak{P}}((k_i/f_i)dX \otimes \alpha_i) = 0$. Assume that

$$\begin{bmatrix} k\,dX \otimes \alpha \\ f \end{bmatrix} \neq 0.$$

Then $\mathfrak{P}$ contains exactly one of f_i, say f_1, and

$$\begin{aligned} \delta_{\mathfrak{p}R[X]}\left(\frac{k_1}{f_1} dX \otimes \alpha_1\right) &= \delta_{\mathfrak{p}R[X],\mathfrak{P}}\left(\frac{k_1}{f_1} dX \otimes \alpha_1\right) \\ &= \delta_{\mathfrak{p}R[X],\mathfrak{P}}\left(\frac{k}{f} dX \otimes \alpha_1\right) \\ &= \begin{bmatrix} k\,dX \otimes \alpha \\ f \end{bmatrix}, \end{aligned}$$

whence the surjectivity. The kernel of $\delta_{\mathfrak{p}R[X]}$ is equal to the image of $i_{\mathfrak{p}}$ by Proposition 3.4 and the following lemma. ∎

LEMMA 3.8. *Given $k_i, f \in R[X]$ and $\alpha_i \in M(\mathfrak{p})$. Assume that*

- $\deg k_i < \deg f$;
- *the leading coefficient of f is not in $\mathfrak{p}$; and*
- *the image of f in $\kappa(\mathfrak{p})[X]$ is a power of an irreducible polynomial.*

Let $\mathfrak{P}$ be the unique prime ideal of $R[X]$ lying over $\mathfrak{p}$ and containing f. If $\delta_{\mathfrak{p}R[X],\mathfrak{P}}(\sum(k_i/f)dX \otimes \alpha_i) = 0$, then $\sum(k_i/f)dX \otimes \alpha_i = 0$.

Proof. By the vanishing law (2.2),

$$\sum k_i f^s \, dX \otimes \alpha_i \in f^{s+1} \Omega_{R[X]_{\mathfrak{P}}/R_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p})$$

for some $s \geq 0$, equivalently

$$\sum k_i f^s \otimes \alpha_i \in f^{s+1} R[X]_{\mathfrak{P}} \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p}).$$

$R[X]_{\mathfrak{P}}$ is a localization of $R_{\mathfrak{p}}[X]$, so there exists $h \in R[X] \setminus \mathfrak{P}$ such that

$$\sum hk_i f^s \otimes \alpha_i \in f^{s+1} R_{\mathfrak{p}}[X] \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p}).$$

The images of f and h in $\kappa(\mathfrak{p})[X]$ are relatively prime, so the map

$$\mu_h\colon \frac{\kappa(\mathfrak{p})[X]}{f^{s+1}\kappa(\mathfrak{p})[X]} \to \frac{\kappa(\mathfrak{p})[X]}{f^{s+1}\kappa(\mathfrak{p})[X]},$$

multiplication by h, is an isomorphism. By Nakayama's lemma, the map

$$\mu_h\colon \frac{R_{\mathfrak{p}}[X]}{f^{s+1}R_{\mathfrak{p}}[X]} \to \frac{R_{\mathfrak{p}}[X]}{f^{s+1}R_{\mathfrak{p}}[X]}$$

is also an isomorphism. Hence so is the map

$$\mu_h\colon \frac{R_{\mathfrak{p}}[X] \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p})}{f^{s+1}R_{\mathfrak{p}}[X] \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p})} \to \frac{R_{\mathfrak{p}}[X] \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p})}{f^{s+1}R_{\mathfrak{p}}[X] \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p})}.$$

Therefore

$$\sum k_i f^s \otimes \alpha_i \in f^{s+1} R_{\mathfrak{p}}[X] \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p}).$$

Since $\deg k_i f^s < \deg f^{s+1}$ for each i, $\sum k_i f^s \otimes \alpha_i$ vanishes in $R_{\mathfrak{p}}[X] \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p})$. In the localization $R[X]_{\mathfrak{p}R[X]} \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p})$ of $R_{\mathfrak{p}}[X] \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p})$,

$$\sum \frac{k_i}{f} \otimes \alpha_i = \sum \frac{k_i f^s}{f^{s+1}} \otimes \alpha_i = 0,$$

and therefore $\sum (k_i/f)dX \otimes \alpha_i = 0$. ∎

4. CONSTRUCTIONS OF $\delta_{\mathfrak{p}R[X],\mathfrak{q}R[X]}$, $\delta_{\mathfrak{P},\mathfrak{Q}}$, AND $\delta_{\mathfrak{P},\mathfrak{r}R[X]}$

In this section, we will construct coboundary maps which are not trivial in formalism.

PROPOSITION 4.1. *Let R be a Noetherian ring and $\mathfrak{q}$ be an immediate specialization of $\mathfrak{p}$ in Spec R. Then, for any $n \in \mathbb{N}$ and $f \in R[X] \setminus \mathfrak{p}R[X]$, there exist $a \in \mathfrak{q} \setminus \mathfrak{p}$, $g \in R[X]^{\wedge}_{\mathfrak{q}R[X]}$ and $h \in R[X]^{\wedge}_{\mathfrak{q}R[X]} \setminus \mathfrak{q}R[X]^{\wedge}_{\mathfrak{q}R[X]}$ such that $fg - ah \in \mathfrak{p}^n R[X]^{\wedge}_{\mathfrak{q}R[X]}$.*

Proof. Let $\mathfrak{r}_1 \cap \mathfrak{r}_2 \cap \cdots$ be a primary decomposition of $\mathfrak{p}R^{\wedge}_{\mathfrak{q}}$ and $\mathfrak{p}_i$ be the associated prime of $\mathfrak{r}_i$. Since the image of any non-zero element of $(R/\mathfrak{p})_{\mathfrak{q}}$ in $(R/\mathfrak{p})^{\wedge}_{\mathfrak{q}}$ is a non-zero-divisor, all $\mathfrak{p}_i$'s are minimal over $\mathfrak{p}R^{\wedge}_{\mathfrak{q}}$ and the image $\overline{f}$ of f in $(R^{\wedge}_{\mathfrak{q}}/\mathfrak{p}_i)[X]$ is not zero. By Cohen's structure theorem, for each $\mathfrak{p}_i$ there exist a complete regular local ring $A_i \subseteq R^{\wedge}_{\mathfrak{q}}/\mathfrak{p}_i$ and $a_i \in \mathfrak{q} \setminus \mathfrak{p}_i$ such that $R^{\wedge}_{\mathfrak{q}}/\mathfrak{p}_i$ is finite over A_i and the maximal ideal $\mathfrak{m}_i$ of A_i is generated by the image $\overline{a}_i$ of a_i. The ring $(R^\wedge_{\mathfrak{q}}/\mathfrak{p}_i)[X]$ is finite over $A_i[X]$, so there exist $\overline{h}_{i,1}, \ldots, \overline{h}_{i,s_i-1}, \overline{h}'_{i,s_i} \in A_i[X]$ with $\overline{h}'_{i,s_i} \neq 0$ such that

$$\overline{f}^{s_i} + \overline{h}_{i,1}\overline{f}^{s_i-1} + \cdots + \overline{h}_{i,s_i-1}\overline{f} + \overline{h}'_{i,s_i} = 0$$

in $(R^\wedge_{\mathfrak{q}}/\mathfrak{p}_i)[X]$. The polynomial $\overline{h}'_{i,s_i}$ can be written as $\overline{a}_i^{n_i}\overline{h}_{i,s_i}$ for some $\overline{h}_{i,s_i} \in A_i[X] \setminus \mathfrak{m}_i A_i[X]$ and $n_i \geq 0$. So we can find $h_{i,1}, \ldots, h_{i,s_i} \in R^\wedge_{\mathfrak{q}}[X]$ with $h_{i,s_i} \notin \mathfrak{q}R^\wedge_{\mathfrak{q}}[X]$ such that

$$f^{s_i} + h_{i,1}f^{s_i-1} + \cdots + h_{i,s_i-1}f + a_i^{n_i}h_{i,s_i} \in \mathfrak{p}_i R^\wedge_{\mathfrak{q}}[X].$$

By raising to suitable power if necessary, we may assume

$$f^{s_i} + h_{i,1}f^{s_i-1} + \cdots + h_{i,s_i-1}f + a_i^{n_i}h_{i,s_i} \in \mathfrak{r}_i R^\wedge_{\mathfrak{q}}[X],$$

for each i. Multiply the above element for various i, we get

$$f^s + h_1 f^{s-1} + \cdots + h_{s-1}f + a_0 h_s \in (\mathfrak{r}_1 \cap \mathfrak{r}_2 \cap \cdots)R^\wedge_{\mathfrak{q}}[X] = \mathfrak{p}R^\wedge_{\mathfrak{q}}[X],$$

for some $a_0 \in \mathfrak{q} \setminus \mathfrak{p}$, $h_1, \ldots, h_{s-1} \in R^\wedge_{\mathfrak{q}}[X]$, and $h_s \in R^\wedge_{\mathfrak{q}}[X] \setminus \mathfrak{q}R^\wedge_{\mathfrak{q}}[X]$. Raising to the nth power, we may assume

$$f^s + h_1 f^{s-1} + \cdots + h_{s-1}f + a_0 h_s \in \mathfrak{p}^n R^\wedge_{\mathfrak{q}}[X].$$

Let $a := -a_0$ and g (resp. h) be the image of $f^{s-1} + h_1 f^{s-2} + \cdots + h_{s-1}$ (resp. h_s) in $R[X]^\wedge_{\mathfrak{q}R[X]}$, then $h \notin \mathfrak{q}R[X]^\wedge_{\mathfrak{q}R[X]}$ and $fg - ah \in \mathfrak{p}^n R[X]^\wedge_{\mathfrak{q}R[X]}$. ∎

Now we can talk about "division."

DEFINITION 4.2. Let R be a Noetherian ring, $\mathfrak{q}$ be an immediate specialization of $\mathfrak{p}$ in $\operatorname{Spec} R$, $M(\mathfrak{q})$ (resp. $M(\mathfrak{p})$) be an injective hull of $\kappa(\mathfrak{q})$ (resp. $\kappa(\mathfrak{p})$), and $\delta_{\mathfrak{p},\mathfrak{q}}\colon M(\mathfrak{p}) \to M(\mathfrak{q})$ be an R-linear map. Given $k \in R[X]$, $f \in R[X] \setminus \mathfrak{p}R[X]$, and $\alpha \in M(\mathfrak{p})$, choose a, g, h as in Proposition 4.1 for some $n \in \mathbb{N}$ satisfying $\mathfrak{p}^n\alpha = 0$, we define

$$\langle k, f, \alpha\rangle := \frac{g}{h}\left(k\, dX \otimes \delta_{\mathfrak{p},\mathfrak{q}}\left(\frac{\alpha}{a}\right)\right).$$

Since $f\langle k, f, \alpha\rangle = k\, dX \otimes \delta_{\mathfrak{p},\mathfrak{q}}(\alpha)$, $\langle k, f, \alpha\rangle$ is meant to be $k\, dX \otimes \delta_{\mathfrak{p},\mathfrak{q}}(\alpha)$ "divided" by f. The "division" is independent of the choice of n, a, g, and h as shown in the next proposition.

PROPOSITION 4.3. *The notations and assumptions are as above. If* $n_i \in \mathbb{N}$, $a_i \in \mathfrak{q} \setminus \mathfrak{p}$, $g_i \in R[X]^\wedge_{\mathfrak{q}R[X]}$, $h_i \in R[X]^\wedge_{\mathfrak{q}R[X]} \setminus \mathfrak{q}R[X]^\wedge_{\mathfrak{q}R[X]}$ *satisfy* $\mathfrak{p}^{n_i}\alpha = 0$ *and* $fg_i - a_i h_i \in \mathfrak{p}^{n_i}R[X]^\wedge_{\mathfrak{q}R[X]}$ *for* $i = 1, 2$, *then*

$$\frac{g_1}{h_1}\left(k\, dX \otimes \delta_{\mathfrak{p},\mathfrak{q}}\left(\frac{\alpha}{a_1}\right)\right) = \frac{g_2}{h_2}\left(k\, dX \otimes \delta_{\mathfrak{p},\mathfrak{q}}\left(\frac{\alpha}{a_2}\right)\right).$$

Proof. Let $n = \min(n_1, n_2)$. Then $a_2h_2g_1 - a_1h_1g_2 \in \mathfrak{p}^n R[X]^{\wedge}_{\mathfrak{q}R[X]}$ and

$$\frac{a_2h_2g_1 - a_1h_1g_2}{h_1h_2}\left(k\,dX \otimes \delta_{\mathfrak{p},\mathfrak{q}}\left(\frac{\alpha}{a_1a_2}\right)\right) = 0.$$

Therefore

$$\begin{aligned} \frac{g_1}{h_1}\left(k\,dX \otimes \delta_{\mathfrak{p},\mathfrak{q}}\left(\frac{\alpha}{a_1}\right)\right) &= \frac{a_2h_2g_1}{h_1h_2}\left(k\,dX \otimes \delta_{\mathfrak{p},\mathfrak{q}}\left(\frac{\alpha}{a_1a_2}\right)\right) \\ &= \frac{a_1h_1g_2}{h_1h_2}\left(k\,dX \otimes \delta_{\mathfrak{p},\mathfrak{q}}\left(\frac{\alpha}{a_1a_2}\right)\right) \\ &= \frac{g_2}{h_2}\left(k\,dX \otimes \delta_{\mathfrak{p},\mathfrak{q}}\left(\frac{\alpha}{a_2}\right)\right). \end{aligned}$$

∎

To compute $\langle k, f, \alpha\rangle$, elements a, g, h can be taken from a larger class.

PROPOSITION 4.4. *The notations and assumptions are as above. If* $a \in \mathfrak{q} \setminus \mathfrak{p}$, $g \in R[X]^{\wedge}_{\mathfrak{q}R[X]}$, $h \in R[X]^{\wedge}_{\mathfrak{q}R[X]} \setminus \mathfrak{q}R[X]^{\wedge}_{\mathfrak{q}R[X]}$ *satisfy*

$$(fg - ah)\left(k\,dX \otimes \delta_{\mathfrak{p},\mathfrak{q}}\left(\frac{\alpha}{b}\right)\right) = 0$$

for all $b \in \mathfrak{q} \setminus \mathfrak{p}$, *then*

$$\langle k, f, \alpha\rangle = \frac{g}{h}\left(k\,dX \otimes \delta_{\mathfrak{p},\mathfrak{q}}\left(\frac{\alpha}{a}\right)\right).$$

The proof is the same as that of Proposition 4.3 and is left to the reader.

As a corollary of the next proposition, the “division" can be extended to all the module $\Omega_{R[X]_{\mathfrak{p}R[X]}/R_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p})$.

PROPOSITION 4.5. *Given* $b \in R$, $c \in R \setminus \mathfrak{p}$, $\alpha, \alpha_1, \alpha_2 \in M(\mathfrak{p})$, $k, k_1, k_2 \in R[X]$, *and* $f, f_1, f_2 \in R[X] \setminus \mathfrak{p}R[X]$, *then*

- $\langle k_1f_2 + k_2f_1, f_1f_2, \alpha\rangle = \langle k_1, f_1, \alpha\rangle + \langle k_2, f_2, \alpha\rangle$,
- $\langle k, f, \alpha_1 + \alpha_2\rangle = \langle k, f, \alpha_1\rangle + \langle k, f, \alpha_2\rangle$,
- $\langle bk, cf, \alpha\rangle = \left\langle k, f, \dfrac{b}{c}\alpha\right\rangle$.

If $k_1f_2 = k_2f_1$, *then* $\langle k_1, f_1, \alpha\rangle = \langle k_2, f_2, \alpha\rangle$.

It is straightforward to verify the proposition. The details are left to the reader.

DEFINITION 4.6. Let R be a Noetherian ring, $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R, $M(\mathfrak{q})$ (resp. $M(\mathfrak{p})$) be an injective hull of $\kappa(\mathfrak{q})$ (resp. $\kappa(\mathfrak{p})$), and $\delta_{\mathfrak{p},\mathfrak{q}}\colon M(\mathfrak{p}) \to M(\mathfrak{q})$ be an R-linear map. If $\mathfrak{q}$ is an immediate specialization of $\mathfrak{p}$, we define

$$\delta_{\mathfrak{p}R[X],\mathfrak{q}R[X]}\colon \Omega_{R[X]_{\mathfrak{p}R[X]}/R_\mathfrak{p}} \otimes_{R_\mathfrak{p}} M(\mathfrak{p}) \to \Omega_{R[X]_{\mathfrak{q}R[X]}/R_\mathfrak{q}} \otimes_{R_\mathfrak{q}} M(\mathfrak{q})$$

to be the unique $R[X]$-linear map satisfying

$$\delta_{\mathfrak{p}R[X],\mathfrak{q}R[X]}\left(\frac{k}{f}dX \otimes \alpha\right) = \langle k, f, \alpha\rangle.$$

If $\mathfrak{q}$ is not an immediate specialization of $\mathfrak{p}$, we define $\delta_{\mathfrak{p}R[X],\mathfrak{q}R[X]}$ to be zero.

It is easy to see that $\delta_{\mathfrak{p}R[X],\mathfrak{q}R[X]} \circ i_\mathfrak{p} = i_\mathfrak{q} \circ (1 \otimes \delta_{\mathfrak{p},\mathfrak{q}})$. So there exists a unique $R[X]$-linear map

$$\oplus H^1_{\mathfrak{m}_\mathfrak{P}}(\Omega_{R[X]_\mathfrak{P}/R_\mathfrak{p}} \otimes_{R_\mathfrak{p}} M(\mathfrak{p})) \xrightarrow{\oplus\delta_{\mathfrak{P},\mathfrak{Q}}} \oplus H^1_{\mathfrak{m}_\mathfrak{Q}}(\Omega_{R[X]_\mathfrak{Q}/R_\mathfrak{q}} \otimes_{R_\mathfrak{q}} M(\mathfrak{q})) \tag{11}$$

such that

$$(\oplus\delta_{\mathfrak{P},\mathfrak{Q}}) \circ \delta_{\mathfrak{p}R[X]} = -\delta_{\mathfrak{q}R[X]} \circ \delta_{\mathfrak{p}R[X],\mathfrak{q}R[X]}$$

(see diagram (5)), where $\mathfrak{P}$ (resp. $\mathfrak{Q}$) ranges over all prime ideals of $R[X]$ lying over $\mathfrak{p}$ (resp. $\mathfrak{q}$) but not equal to $\mathfrak{p}R[X]$ (resp. $\mathfrak{q}R[X]$). Note that, if $\mathfrak{P} \not\subseteq \mathfrak{Q}$, there is an element in $R[X]$ whose image in $R[X]_\mathfrak{Q}$ is a unit and whose image in $R[X]_\mathfrak{P}$ is in the maximal ideal. This implies that $\delta_{\mathfrak{P},\mathfrak{Q}} = 0$. Therefore $\delta_{\mathfrak{P},\mathfrak{Q}} \neq 0$ only if $\mathfrak{Q}$ is an immediate specialization of $\mathfrak{P}$.

Assume that R possesses a residual complex

$$\cdots \xrightarrow{\delta^{n-1}} \bigoplus_{\Delta(\mathfrak{p})=n} M(\mathfrak{p}) \xrightarrow{\delta^n} \bigoplus_{\Delta(\mathfrak{q})=n+1} M(\mathfrak{q}) \xrightarrow{\delta^{n+1}} \cdots,$$

where $M(\mathfrak{p})$ is an injective hull of the residue field $\kappa(\mathfrak{p})$ of the point $\mathfrak{p}$ and Δ is a codimension function on R. Given prime ideals $\mathfrak{p}$ and $\mathfrak{r}$ of R such that $\Delta(\mathfrak{r}) = \Delta(\mathfrak{p}) + 2$, there exists a unique $R[X]$-linear map

$$\oplus H^1_{\mathfrak{m}_\mathfrak{P}}(\Omega_{R[X]_\mathfrak{P}/R_\mathfrak{p}} \otimes_{R_\mathfrak{p}} M(\mathfrak{p})) \xrightarrow{\oplus\delta_{\mathfrak{P},\mathfrak{r}R[X]}} \Omega_{R[X]_{\mathfrak{r}R[X]}/R_\mathfrak{r}} \otimes_{R_\mathfrak{r}} M(\mathfrak{r}) \tag{12}$$

such that

$$(-\oplus\delta_{\mathfrak{P},\mathfrak{r}R[X]}) \circ (\oplus\delta_{\mathfrak{p}R[X],\mathfrak{P}}) = (\oplus\delta_{\mathfrak{q}R[X],\mathfrak{r}R[X]}) \circ (\oplus\delta_{\mathfrak{p}R[X],\mathfrak{q}R[X]})$$

(see diagram (6)), where $\mathfrak{q}$ ranges over all prime ideals of R properly between $\mathfrak{p}$ and $\mathfrak{r}$; and $\mathfrak{P}$ ranges over all prime ideals of $R[X]$ lying over $\mathfrak{p}$ but not equal to $\mathfrak{p}R[X]$. Note that $\delta_{\mathfrak{P},\mathfrak{r}R[X]} = 0$ if $\mathfrak{P} \not\subseteq \mathfrak{r}R[X]$. Since R is universally catenary, $\delta_{\mathfrak{P},\mathfrak{r}R[X]} \neq 0$ only if $\mathfrak{r}R[X]$ is an immediate specialization of $\mathfrak{P}$.

Given prime ideals $\mathscr{P}$ immediately specializing to $\mathscr{Q}$ in $R[X]$, we summarize the map $\delta_{\mathscr{P},\mathscr{Q}}$ defined in the previous and the present sections.

• If the contraction $\mathfrak{q}$ of $\mathscr{Q}$ in R is the same as that of $\mathscr{P}$, then $\mathscr{P} = \mathfrak{q}R[X]$,

$$M_X(\mathscr{P}) = \Omega_{R[X]_{\mathfrak{q}R[X]}/R_\mathfrak{q}} \otimes_{R_\mathfrak{q}} M(\mathfrak{q}),$$
$$M_X(\mathscr{Q}) = H^1_{\mathscr{Q}}(\Omega_{R[X]_{\mathscr{Q}}/R_\mathfrak{p}} \otimes_{R_\mathfrak{p}} M(\mathfrak{q})),$$

and $\delta_{\mathscr{P},\mathscr{Q}}$ defined in Definition 3.3 satisfies

$$\delta_{\mathscr{P},\mathscr{Q}}\left(\frac{k}{f}\, dX \otimes \alpha\right) = \begin{bmatrix} k\, dX \otimes \alpha \\ f \end{bmatrix},$$

where $k \in R[X]$, $f \in R[X] \setminus \mathfrak{q}R[X]$, and $\alpha \in M(\mathfrak{q})$. In this case $\delta_{\mathscr{P},\mathscr{Q}}$ is called a coboundary map of type $(0, 1)$, where 0 (resp. 1) refers to the relative dimension of $R[X]_{\mathscr{P}}$ (resp. $R[X]_{\mathscr{Q}}$) over $R_\mathfrak{p}$ (resp. $R_\mathfrak{q}$).

• If the contraction $\mathfrak{q}$ of $\mathscr{Q}$ in R is an immediate specialization of the contraction $\mathfrak{p}$ of $\mathscr{P}$ in R and $\mathscr{P} = \mathfrak{p}R[X]$, then $\mathscr{Q} = \mathfrak{q}R[X]$,

$$M_X(\mathscr{P}) = \Omega_{R[X]_{\mathfrak{p}R[X]}/R_\mathfrak{p}} \otimes_{R_\mathfrak{p}} M(\mathfrak{p}),$$
$$M_X(\mathscr{Q}) = \Omega_{R[X]_{\mathfrak{q}R[X]}/R_\mathfrak{q}} \otimes_{R_\mathfrak{q}} M(\mathfrak{q}),$$

and $\delta_{\mathscr{P},\mathscr{Q}}$ defined in Definition 4.6 satisfies

$$\delta_{\mathscr{P},\mathscr{Q}}\left(\frac{k}{f}\, dX \otimes \alpha\right) = \frac{g}{h}\left(k\, dX \otimes \delta_{\mathfrak{p},\mathfrak{q}}\left(\frac{\alpha}{a}\right)\right),$$

where $k \in R[X]$, $f \in R[X] \setminus \mathfrak{p}R[X]$, $\alpha \in M(\mathfrak{p})$, and a, g, h are chosen as in Proposition 4.1 for $n \in \mathbb{N}$ satisfying $\mathfrak{p}^n\alpha = 0$. In this case $\delta_{\mathscr{P},\mathscr{Q}}$ is called a coboundary map of type $(0, 0)$.

• If the contraction $\mathfrak{q}$ of $\mathscr{Q}$ in R is an immediate specialization of the contraction $\mathfrak{p}$ of $\mathscr{P}$ in R but $\mathscr{P} \neq \mathfrak{p}R[X]$, then

$$M_X(\mathscr{P}) = H^1_{\mathscr{P}}(\Omega_{R[X]_{\mathscr{P}}/R_\mathfrak{p}} \otimes_{R_\mathfrak{p}} M(\mathfrak{p})),$$
$$M_X(\mathscr{Q}) = H^1_{\mathscr{Q}}(\Omega_{R[X]_{\mathscr{Q}}/R_\mathfrak{q}} \otimes_{R_\mathfrak{q}} M(\mathfrak{q})).$$

Elements in $M_X(\mathscr{P})$ can be written as a sum of elements of the form

$$\begin{bmatrix} k\, dX \otimes \alpha \\ f \end{bmatrix},$$

where $k, f \in R[X]$, $\alpha \in M(\mathfrak{p})$, and the image of f in $\kappa(\mathfrak{p})[X]$ is a power of an irreducible polynomial (see the proof of Theorem 3.7). We choose a, g, h as in Proposition 4.1 for some $n \in \mathbb{N}$ satisfying $\mathfrak{p}^n\alpha = 0$ and choose $g' \in R[X]$ and $h' \in R[X] \setminus \mathfrak{q}R[X]$ such that

$$\left(\frac{g}{h} - \frac{g'}{h'}\right)\left(dX \otimes \delta_{\mathfrak{p},\mathfrak{q}}\left(\frac{\alpha}{a}\right)\right) = 0$$

in $\Omega_{R[X]_{\mathfrak{q}R[X]}/R_{\mathfrak{q}}} \otimes_{R_{\mathfrak{q}}} M(\mathfrak{q})$, then $\delta_{\mathscr{P},\mathscr{Q}}$ defined in (11) satisfies

$$\delta_{\mathscr{P},\mathscr{Q}}\left(\begin{bmatrix} k\,dX \otimes \alpha \\ f \end{bmatrix}\right) = -\begin{bmatrix} g'k\,dX \otimes \delta_{\mathfrak{p},\mathfrak{q}}\left(\dfrac{\alpha}{a}\right) \\ h' \end{bmatrix}.$$

In this case $\delta_{\mathscr{P},\mathscr{Q}}$ is called a coboundary map of type $(1,1)$.

• If the contraction $\mathfrak{r}$ of $\mathscr{Q}$ in R is not equal to nor an immediate specialization of the contraction $\mathfrak{p}$ of $\mathscr{P}$ in R, then $\dim(R/\mathfrak{p})_{\mathfrak{r}/\mathfrak{p}} = 2$, $\mathscr{Q} = \mathfrak{r}R[X]$,

$$\begin{aligned} M_X(\mathscr{P}) &= H^1_{\mathscr{P}}(\Omega_{R[X]_{\mathscr{P}}/R_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} M(\mathfrak{p})), \\ M_X(\mathscr{Q}) &= \Omega_{R[X]_{\mathfrak{r}R[X]}/R_{\mathfrak{r}}} \otimes_{R_{\mathfrak{r}}} M(\mathfrak{r}). \end{aligned}$$

Given an element

$$\begin{bmatrix} k\,dX \otimes \alpha \\ f \end{bmatrix}$$

in $M_X(\mathfrak{P}_X)$, where $k, f \in R[X]$, $\alpha \in M(\mathfrak{p})$, and the image of f in $\kappa(\mathfrak{p})[X]$ is a power of an irreducible polynomial, for each $\mathfrak{q} \in \operatorname{Spec} R$ properly between $\mathfrak{p}$ and $\mathfrak{r}$, we choose $a_{\mathfrak{q}} \in \mathfrak{q} \setminus \mathfrak{p}$, $g_{\mathfrak{q}} \in R[X]^{\wedge}_{\mathfrak{q}R[X]}$, $h_{\mathfrak{q}} \in R[X]^{\wedge}_{\mathfrak{q}R[X]} \setminus \mathfrak{q}R[X]^{\wedge}_{\mathfrak{q}R[X]}$ as in Proposition 4.1 for some $n_{\mathfrak{q}} \in \mathbb{N}$ with $\mathfrak{p}^{n_{\mathfrak{q}}}\alpha = 0$ such that $fg_{\mathfrak{q}} - a_{\mathfrak{q}}h_{\mathfrak{q}} \in \mathfrak{p}^{n_{\mathfrak{q}}}R[X]^{\wedge}_{\mathfrak{q}R[X]}$; choose $g'_{\mathfrak{q}} \in R[X]$ and $h'_{\mathfrak{q}} \in R[X] \setminus \mathfrak{q}R[X]$ such that

$$\left(\frac{g_{\mathfrak{q}}}{h_{\mathfrak{q}}} - \frac{g'_{\mathfrak{q}}}{h'_{\mathfrak{q}}}\right)\left(dX \otimes \delta_{\mathfrak{p},\mathfrak{q}}\left(\frac{\alpha}{a_{\mathfrak{q}}}\right)\right) = 0$$

in $\Omega_{R[X]_{\mathfrak{q}R[X]}/R_{\mathfrak{q}}} \otimes_{R_{\mathfrak{q}}} M(\mathfrak{q})$; and choose $a'_{\mathfrak{q}} \in \mathfrak{r} \setminus \mathfrak{q}$, $g''_{\mathfrak{q}} \in R[x]^{\wedge}_{\mathfrak{r}R[X]}$ and $h''_{\mathfrak{q}} \in R[x]^{\wedge}_{\mathfrak{r}R[X]} \setminus \mathfrak{r}R[x]^{\wedge}_{\mathfrak{r}R[X]}$ as in Proposition 4.1 for some $n'_{\mathfrak{q}} \in \mathbb{N}$ with $\mathfrak{q}^{n'_{\mathfrak{q}}}(\alpha/a_{\mathfrak{q}}) = 0$ such that $h'_{\mathfrak{q}}g''_{\mathfrak{q}} - a'_{\mathfrak{q}}h''_{\mathfrak{q}} \in \mathfrak{q}^{n'_{\mathfrak{q}}}R[x]^{\wedge}_{\mathfrak{r}R[X]}$, then $\delta_{\mathscr{P},\mathscr{Q}}$ defined in (12) satisfies

$$\delta_{\mathscr{P},\mathscr{Q}}\left(\begin{bmatrix} k\,dX \otimes \alpha \\ f \end{bmatrix}\right) = -\sum_{\mathfrak{q}} \frac{g''_{\mathfrak{q}}}{h''_{\mathfrak{q}}}\left(g'_{\mathfrak{q}}k\,dX \otimes (\delta_{\mathfrak{q},\mathfrak{r}} \circ \delta_{\mathfrak{p},\mathfrak{q}})\left(\frac{\alpha}{a_{\mathfrak{q}}a'_{\mathfrak{q}}}\right)\right),$$

where $\mathfrak{q}$ ranges over the prime ideals of R properly between $\mathfrak{p}$ and $\mathfrak{r}$. In this case $\delta_{\mathscr{P},\mathscr{Q}}$ is called a coboundary map of type $(1,0)$.

5. CONSTRUCTION OF $M^{\bullet}_{X_1,\ldots,X_n}$

Let R be a Noetherian ring possessing a residual complex

$$\cdots \xrightarrow{\delta^{n-1}} \bigoplus_{\Delta(\mathfrak{p})=n} M(\mathfrak{p}) \xrightarrow{\delta^{n}} \bigoplus_{\Delta(\mathfrak{q})=n+1} M(\mathfrak{q}) \xrightarrow{\delta^{n+1}} \cdots,$$

where $M(\mathfrak{p})$ is an injective hull of the residue field $\kappa(\mathfrak{p})$ of the point $\mathfrak{p}$ and Δ is a codimension function on R. Let $\Delta_{X_1,\dots,X_n}$ be the codimension function on $R[X_1,\dots,X_n]$ as characterized in (2). First we construct $M_X^\bullet$. Let $\mathscr{P}$ and $\mathscr{Q}$ be two prime ideals of $R[X]$ such that

$$\Delta_X(\mathscr{P}) = \Delta_X(\mathscr{Q}) - 1.$$

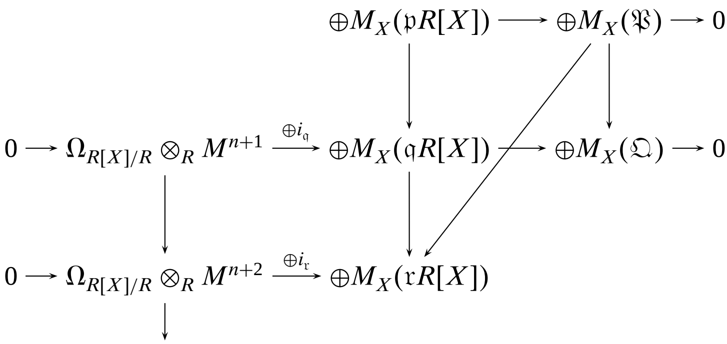
If $\mathscr{Q}$ is an immediate specialization of $\mathscr{P}$, $\delta_{X;\mathscr{P},\mathscr{Q}} := \delta_{\mathscr{P},\mathscr{Q}}$ was defined in Sections 3 and 4. If $\mathscr{Q}$ is not an immediate specialization of $\mathscr{P}$, $\delta_{X;\mathscr{P},\mathscr{Q}}$ is defined to be 0.

THEOREM 5.1. *Define $M_X^n := \oplus M_X(\mathscr{P})$, where $\mathscr{P}$ ranges over the prime ideals with $\Delta_X(\mathscr{P}) = n$; and define $\delta_X^n := \oplus \delta_{X;\mathscr{P},\mathscr{Q}}$, where $\mathscr{P}$ and $\mathscr{Q}$ range over the prime ideals with $\Delta_X(\mathscr{P}) = n$ and $\Delta_X(\mathscr{Q}) = n+1$. Then the chain $M_X^\bullet$ of $R[X]$-modules defined by*

$$\cdots \xrightarrow{\delta_X^{n-1}} M_X^n \xrightarrow{\delta_X^n} M_X^{n+1} \xrightarrow{\delta_X^{n+1}} \cdots \tag{13}$$

is a residual complex quasi-isomorphic to $\Omega_{R[X]/R} \otimes_R M^\bullet[1]$.

The fact that the chain (13) is a complex follows directly from our construction. To see that the chain (13) is a residual complex quasi-isomorphic to $\Omega_{R[X]/R} \otimes_R M^\bullet[1]$, we chase elements in the following diagram



whose rows are exact. The details are left to the reader.

Coboundary maps of type $(1,0)$ and $(1,1)$ are determined by coboundary maps of type $(0,0)$ and $(0,1)$ in the following sense.

PROPOSITION 5.2. *Let $\delta'_{X;\mathscr{P},\mathscr{Q}}\colon M_X(\mathscr{P}) \to M_X(\mathscr{Q})$ be $R[X]$-linear maps $(\mathscr{P},\mathscr{Q} \in \operatorname{Spec} R[X])$ such that*

$$\cdots \longrightarrow \bigoplus_{\Delta_X(\mathscr{P})=n} M_X(\mathscr{P}) \xrightarrow{\oplus\delta'_{X;\mathscr{P},\mathscr{Q}}} \bigoplus_{\Delta_X(\mathscr{Q})=n+1} M_X(\mathscr{Q}) \longrightarrow \cdots \tag{14}$$

is a complex. If $\delta'_{X;\mathscr{P},\mathscr{Q}} = \delta_{X;\mathscr{P},\mathscr{Q}}$, for all $\mathscr{P}$ such that $\mathscr{P} = (\mathscr{P} \cap R)R[X]$, then $\delta'_{X;\mathscr{P},\mathscr{Q}} = \delta_{X;\mathscr{P},\mathscr{Q}}$ for all $\mathscr{P}$ and $\mathscr{Q}$.

Proof. Given prime ideals $\mathscr{P}$ and $\mathscr{Q}$ with $\Delta_X(\mathscr{Q}) = \Delta_X(\mathscr{P}) + 1$, if $\mathscr{P} \subseteq \mathscr{Q}$, then $\mathscr{Q}$ is an immediate specialization of $\mathscr{P}$; if $\mathscr{P} \not\subseteq \mathscr{Q}$, then $\delta'_{X;\mathscr{P},\mathscr{Q}} = \delta_{X;\mathscr{P},\mathscr{Q}} = 0$. Let $\mathfrak{P}$ be a prime ideal of $R[X]$ which does not equal to the extension $\mathfrak{p}R[X]$ of its contraction $\mathfrak{p}$ in R. To prove the proposition, it suffices to prove $\delta'_{X;\mathfrak{P},\mathscr{Q}} = \delta_{X;\mathfrak{P},\mathscr{Q}}$ for all immediate specialization $\mathscr{Q}$ of $\mathfrak{P}$. Given an element Θ in $M_X(\mathfrak{P})$, by Theorem 3.7, there is an element Υ in $M_X(\mathfrak{p}R[X])$ such that

$$\delta'_{X;\mathfrak{p}R[X],\mathscr{P}}(\Upsilon) = \delta_{X;\mathfrak{p}R[X],\mathscr{P}}(\Upsilon) = \begin{cases} \Theta, & \text{if } \mathscr{P} = \mathfrak{P}; \\ 0, & \text{if } \mathscr{P} \cap R = \mathfrak{p} \text{ but } \mathscr{P} \neq \mathfrak{P} \text{ or } \mathfrak{p}R[X]. \end{cases}$$

Since (13) and (14) are complexes, for $\mathscr{P}$ ranging over all prime ideals of $R[X]$ except $\mathfrak{P}$ with $\Delta_X(\mathscr{P}) = \Delta(\mathfrak{p})$ and $\mathfrak{q}$ ranging over all prime ideals of R with $\Delta(\mathfrak{q}) = \Delta(\mathfrak{p}) + 1$, we have

$$\begin{aligned}
\delta'_{X;\mathfrak{P},\mathscr{Q}}(\Theta) &= \left(\delta'_{X;\mathfrak{P},\mathscr{Q}} \circ \delta'_{X;\mathfrak{p}R[X],\mathfrak{P}}\right)(\Upsilon) \\
&= -\bigoplus_{\mathscr{P}} \left(\delta'_{X;\mathscr{P},\mathscr{Q}} \circ \delta'_{X;\mathfrak{p}R[X],\mathscr{P}}\right)(\Upsilon) \\
&= -\bigoplus_{\mathfrak{q}} \left(\delta'_{X;\mathfrak{q}R[X],\mathscr{Q}} \circ \delta'_{X;\mathfrak{p}R[X],\mathfrak{q}R[X]}\right)(\Upsilon) \\
&= -\bigoplus_{\mathfrak{q}} \left(\delta_{X;\mathfrak{q}R[X],\mathscr{Q}} \circ \delta_{X;\mathfrak{p}R[X],\mathfrak{q}R[X]}\right)(\Upsilon) \\
&= -\bigoplus_{\mathscr{P}} \left(\delta_{X;\mathscr{P},\mathscr{Q}} \circ \delta_{X;\mathfrak{p}R[X],\mathscr{P}}\right)(\Upsilon) \\
&= \left(\delta_{X;\mathfrak{P},\mathscr{Q}} \circ \delta_{X;\mathfrak{p}R[X],\mathfrak{P}}\right)(\Upsilon) \\
&= \delta_{X;\mathfrak{P},\mathscr{Q}}(\Theta).
\end{aligned}$$

∎

Using the same method on the polynomial ring $R[X, Y]$ over $R[X]$, we construct a residual complex

$$\cdots \longrightarrow \bigoplus_{\Delta_{X,Y}(\mathscr{P})=n} M_Y(\mathscr{P}) \xrightarrow{\delta^n_{X,Y/X}} \bigoplus_{\Delta_{X,Y}(\mathscr{Q})=n+1} M_Y(\mathscr{Q}) \longrightarrow \cdots$$

on $R[X, Y]$ from $M_X^\bullet$. We remind the reader that $M_Y(\mathscr{P})$ is the $R[X, Y]$-module constructed from the $R[X]$-module $M_X(\mathscr{P} \cap R[X])$ (Section 2). Using the canonical isomorphism $M_Y(\mathscr{P}) \tilde{\to} M_{X,Y}(\mathscr{P})$, see (7), we get a residual complex

$$\cdots \longrightarrow \bigoplus_{\Delta_{X,Y}(\mathscr{P})=n} M_{X,Y}(\mathscr{P}) \xrightarrow{\delta^n_{X,Y}} \bigoplus_{\Delta_{X,Y}(\mathscr{Q})=n+1} M_{X,Y}(\mathscr{Q}) \longrightarrow \cdots ,$$

denoted by $M^\bullet_{X,Y}$, where $\delta^n_{X,Y}$ is the map making the diagram

$$\begin{array}{ccc} \bigoplus_{\Delta_{X,Y}(\mathscr{P})=n} M_Y(\mathscr{P}) & \xrightarrow{\delta^n_{X,Y/X}} & \bigoplus_{\Delta_{X,Y}(\mathscr{Q})=n+1} M_Y(\mathscr{Q}) \\ \downarrow{\scriptstyle \text{can.}} & & \downarrow{\scriptstyle \text{can.}} \\ \bigoplus_{\Delta_{X,Y}(\mathscr{P})=n} M_{X,Y}(\mathscr{P}) & \xrightarrow[\delta^n_{X,Y}]{} & \bigoplus_{\Delta_{X,Y}(\mathscr{Q})=n+1} M_{X,Y}(\mathscr{Q}) \end{array}$$

commutative. In $M^\bullet_{X,Y}$, the definitions of $\Delta_{X,Y}$ and $M_{X,Y}(\mathscr{P})$ are independent of the order of X and Y, but $\delta^n_{X,Y}$ depends on the order of X and Y *a priori*.

THEOREM 5.3. *The construction of $M^\bullet_{X,Y}$ is independent of the order of X and Y.*

Proof. Let $\mathfrak{P}$ be a prime ideal of $R[X,Y]$ and let $\mathfrak{P}_X$ (resp. $\mathfrak{P}_Y$ and $\mathfrak{p}$) be its contraction in $R[X]$ (resp. $R[Y]$ and R). Assume $\Delta_{X,Y}(\mathfrak{P}) = n$. Let $\mathscr{Q}$ be a prime ideal of $R[X,Y]$ with $\Delta_{X,Y}(\mathscr{Q}) = n+1$ and let $\delta_{X,Y;\mathfrak{P},\mathscr{Q}}$ (resp. $\delta_{Y,X;\mathfrak{P},\mathscr{Q}}$) be the map $M_{X,Y}(\mathfrak{P}) \to M_{X,Y}(\mathscr{Q})$ in the complex $M^\bullet_{X,Y}$ (resp. $M^\bullet_{Y,X}$). We need to show $\delta_{X,Y;\mathfrak{P},\mathscr{Q}} = \delta_{Y,X;\mathfrak{P},\mathscr{Q}}$. Let $\mathscr{Q}_X$ (resp. $\mathscr{Q}_Y$) be the contraction of $\mathscr{Q}$ in $R[X]$ (resp. $R[Y]$). If $\mathfrak{p} \not\subseteq \mathscr{Q} \cap R$, then both $\delta_{X,Y;\mathfrak{P},\mathscr{Q}}$ and $\delta_{Y,X;\mathfrak{P},\mathscr{Q}}$ are zero. So we may assume $\mathfrak{p} \subseteq \mathscr{Q} \cap R$. By Proposition 5.2, we may also assume $\mathfrak{P} = \mathfrak{P}_X R[X,Y]$. We consider the following cases and subcases

- $\mathfrak{P}_X = \mathfrak{p}R[X]$
  — $\mathscr{Q} \cap R = \mathfrak{p}$
  — $\mathscr{Q} \cap R \neq \mathfrak{p}$
- $\mathfrak{P}_X \neq \mathfrak{p}R[X]$
  — $\mathscr{Q} \cap R = \mathfrak{p}$
  — $\mathscr{Q} \cap R$ is an immediate specialization of $\mathfrak{p}$
  — $\mathscr{Q} \cap R$ is neither equal to nor an immediate specialization of $\mathfrak{p}$

and continue our proof.

- First we treat the case $\mathfrak{P}_X = \mathfrak{p}R[X]$. In this case $\mathfrak{P} = \mathfrak{p}R[X,Y]$ and

$$M_{X,Y}(\mathfrak{P}) = \det_{\mathfrak{P}}.$$

If $\mathscr{Q} \cap R = \mathfrak{p}$, then

$$M_{X,Y}(\mathscr{Q}) = H^1_{\mathscr{Q}}(\det_{\mathscr{Q}} M(\mathfrak{p})).$$

Let $(k/f)\, dX\, dY \otimes \alpha$ be an element in $M_{X,Y}(\mathfrak{P})$, where $k \in R[X,Y]$, $f \in R[X,Y] \setminus \mathfrak{p}R[X,Y]$, and $\alpha \in M(\mathfrak{p})$. We claim that

$$\delta_{X,Y;\mathfrak{P},\mathscr{Q}}\left(\frac{k}{f}\, dX\, dY \otimes \alpha\right) = \left[\begin{array}{c} k\, dX\, dY \otimes \alpha \\ f \end{array}\right].$$

This is obviously true if $\mathscr{Q}_X = \mathfrak{p}R[X]$. If $\mathscr{Q}_X \neq \mathfrak{p}R[X]$,

$$\delta_{X,Y;\mathfrak{P},\mathscr{Q}}\left(\frac{k}{f}\,dX\,dY \otimes \alpha\right) = \frac{G}{H}\begin{bmatrix} k\,dX\,dY \otimes \alpha \\ A \end{bmatrix},$$

where $G \in R[X,Y]^{\wedge}_{\mathscr{Q}_X R[X,Y]}$, $H \in R[X,Y]^{\wedge}_{\mathscr{Q}_X R[X,Y]} \setminus \mathscr{Q}_X R[X,Y]^{\wedge}_{\mathscr{Q}_X R[X,Y]}$, and $A \in \mathscr{Q}_X \setminus \mathfrak{p}R[X]$ are chosen such that $fG - AH \in \mathfrak{p}^n R[X,Y]^{\wedge}_{\mathscr{Q}_X R[X,Y]}$ for some $n \in \mathbb{N}$ with $\mathfrak{p}^n\alpha = 0$ (hence $\mathfrak{p}^n(\,dX \otimes \alpha) = 0$),

$$\begin{aligned}\frac{G}{H}\begin{bmatrix} k\,dX\,dY \otimes \alpha \\ A \end{bmatrix} &= \frac{fG}{H}\begin{bmatrix} k\,dX\,dY \otimes \alpha \\ fA \end{bmatrix} \\ &= \frac{AH}{H}\begin{bmatrix} k\,dX\,dY \otimes \alpha \\ fA \end{bmatrix} \\ &= \begin{bmatrix} k\,dX\,dY \otimes \alpha \\ f \end{bmatrix}.\end{aligned}$$

This proves the claim and hence $\delta_{X,Y;\mathfrak{P},\mathscr{Q}}$ is independent of the order of X and Y.

If $\mathscr{Q} \cap R \neq \mathfrak{p}$, we denote $\mathscr{Q} \cap R$ by $\mathfrak{q}$. In this case $\mathscr{Q} = \mathfrak{q}R[X,Y]$ and

$$M_{X,Y}(\mathscr{Q}) = \det{}_{\mathscr{Q}} M(\mathfrak{q}).$$

Let $(k/f)\,dX\,dY \otimes \alpha$ be an element in $M_{X,Y}(\mathfrak{P})$, where $k \in R[X,Y]$, $f \in R[X,Y] \setminus \mathfrak{P}$, and $\alpha \in M(\mathfrak{p})$. We choose $G' \in R[X,Y]^{\wedge}_{\mathfrak{q}R[X,Y]}$, $H' \in R[X,Y]^{\wedge}_{\mathfrak{q}R[X,Y]} \setminus \mathfrak{q}R[X,Y]^{\wedge}_{\mathfrak{q}R[X,Y]}$, and $A \in \mathfrak{q}R[X] \setminus \mathfrak{p}R[X]$ such that $fG' - AH' \in \mathfrak{p}^n R[X,Y]^{\wedge}_{\mathfrak{q}R[X,Y]}$ for some $n \in \mathbb{N}$ with $\mathfrak{p}^n\alpha = 0$, and choose $g \in R[X]^{\wedge}_{\mathfrak{q}R[X]}$, $h \in R[X]^{\wedge}_{\mathfrak{q}R[X]} \setminus \mathfrak{q}R[X]^{\wedge}_{\mathfrak{q}R[X]}$, and $a \in \mathfrak{q} \setminus \mathfrak{p}$ such that $Ag - ah \in \mathfrak{p}^n R[X]^{\wedge}_{\mathfrak{q}R[X]}$. Let $G = G'g$ (more precisely the product of G' and the image of g in $R[X,Y]^{\wedge}_{\mathfrak{q}R[X,Y]}$) and $H = H'h$. Then $H \notin \mathfrak{q}R[X,Y]^{\wedge}_{\mathfrak{q}R[X,Y]}$, $fG - aH \in \mathfrak{p}^n R[X,Y]^{\wedge}_{\mathfrak{q}R[X,Y]}$, and

$$\delta_{X,Y;\mathfrak{P},\mathscr{Q}}\left(\frac{k}{f}dX\,dY \otimes \alpha\right) = \frac{G}{H}\left(k\,dX\,dY \otimes \delta_{\mathfrak{p},\mathfrak{q}}\left(\frac{\alpha}{a}\right)\right).$$

This formula for $\delta_{X,Y;\mathfrak{P},\mathscr{Q}}$ implies that $\delta_{X,Y;\mathfrak{P},\mathscr{Q}}$ is independent of the order of X and Y (see the proof of Proposition 4.3).

- Now we treat the case $\mathfrak{P}_X \neq \mathfrak{p}R[X]$. In this case

$$M_{X,Y}(\mathfrak{P}) = H^1_{\mathfrak{P}}(\det{}_{\mathfrak{P}} M(\mathfrak{p})),$$

$\mathfrak{P}_Y = \mathfrak{p}R[Y]$, but $\mathfrak{P} \neq \mathfrak{P}_Y R[X,Y]$. Elements in $M_{X,Y}(\mathfrak{P})$ can be written as a sum of elements of the form

$$\begin{bmatrix} \dfrac{k}{f_1}\,dX\,dY \otimes \alpha \\ f_2 \end{bmatrix},$$

where $k \in R[X, Y]$, $f_1 \in R[X, Y] \setminus \mathfrak{P}$, $\alpha \in M(\mathfrak{p})$, and f_2 being a relative system of parameters of $R[X, Y]_{\mathfrak{P}}$ over $R_{\mathfrak{p}}$ is chosen to be an element in $\mathfrak{P}_X$ whose image $\overline{f}_2$ in $\kappa(\mathfrak{p})[X, Y]$ is a power of an irreducible polynomial.

If $\mathcal{Q} \cap R = \mathfrak{p}$, then

$$M_{X,Y}(\mathcal{Q}) = H^2_{\mathcal{Q}}(\det_{\mathcal{Q}} M(\mathfrak{p})).$$

Note that either f_1 is a unit in $R[X, Y]_{\mathcal{Q}}$ or the sequence f_1, f_2 forms a relative system of parameters of $R[X, Y]_{\mathcal{Q}}$ over $R_{\mathfrak{p}}$. It is easy to see that

$$\delta_{X,Y;\mathfrak{P},\mathcal{Q}}\left(\begin{bmatrix} \dfrac{k}{f_1}\, dX\, dY \otimes \alpha \\ f_2 \end{bmatrix}\right) = \begin{bmatrix} k\, dX\, dY \otimes \alpha \\ f_1, f_2 \end{bmatrix}.$$

Now we compute $\delta_{Y,X;\mathfrak{P},\mathcal{Q}}$: First we compute the special case that f_1 is in $R[Y]$ and its image $\overline{f}_1$ in $\kappa(\mathfrak{p})[X, Y]$ is a power of an irreducible polynomial. Besides $\mathfrak{P}$, there is at most one prime ideal $\mathcal{P}$ of $R[X, Y]$ properly between $\mathfrak{p}R[X, Y]$ and $\mathcal{Q}$ such that

$$\delta_{Y,X;\mathfrak{p}R[X,Y],\mathcal{P}}\left(\frac{k}{f_1 f_2} dX\, dY \otimes \alpha\right) \neq 0,$$

namely the preimage $\mathcal{P}_0$ of the radical of $\overline{f}_1$ under the canonical map $R[X, Y] \to \kappa(\mathfrak{p})[X, Y]$. As

$$\delta_{Y,X;\mathfrak{p}R[X,Y],\mathfrak{P}}\left(\frac{k}{f_1 f_2} dX\, dY \otimes \alpha\right) = \begin{bmatrix} \dfrac{k}{f_1}\, dX\, dY \otimes \alpha \\ f_2 \end{bmatrix},$$

we have

$$\begin{aligned} \delta_{Y,X;\mathfrak{P},\mathcal{Q}} & \begin{bmatrix} \dfrac{k}{f_1}\, dX\, dY \otimes \alpha \\ f_2 \end{bmatrix} \\ &= -\delta_{Y,X;\mathcal{P}_0,\mathcal{Q}} \circ \delta_{Y,X;\mathfrak{p}R[X,Y],\mathcal{P}_0}\left(\frac{k}{f_1 f_2} dX\, dY \otimes \alpha\right) \\ &= -\delta_{Y,X;\mathcal{P}_0,\mathcal{Q}}\left(\begin{bmatrix} k\, dX\, dY \otimes \alpha \\ f_2 f_1 \end{bmatrix}\right) \\ &= -\begin{bmatrix} k\, dX\, dY \otimes \alpha \\ f_2, f_1 \end{bmatrix} \\ &= \begin{bmatrix} k\, dX\, dY \otimes \alpha \\ f_1, f_2 \end{bmatrix}. \end{aligned}$$

Now we consider the general case. By the Gauss lemma, the image of f_2 in $\kappa(\mathfrak{P}_Y)[X]$ is also a power of an irreducible polynomial. The images of

f_1 and f_2 in $\kappa(\mathfrak{P}_Y)[X]$ are relatively prime, so there exist $g_1, g_2 \in R[X, Y]$ and $r \in R[Y] \setminus \mathfrak{P}_Y$ such that

$$g_1 f_1 + g_2 f_2 - r \in \mathfrak{p}R[X, Y].$$

Choose $n \in \mathbb{Z}$ such that $\mathfrak{p}^n \alpha = 0$. For prime ideals $\mathcal{P}$ of $R[X, Y]$ lying over $\mathfrak{P}_Y$ but not equal to $\mathfrak{P}_Y R[X, Y]$, we have

$$\delta_{X,Y;\mathfrak{p}R[X,Y],\mathcal{P}}\left(\sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}\frac{kr^{n-i}g_1^i f_1^{i-1}}{r^n f_2}\,dX\,dY \otimes \alpha\right) = \begin{cases} \begin{bmatrix} \dfrac{k}{f_1}\,dX\,dY \otimes \alpha \\ f_2 \end{bmatrix}, & \text{if } \mathcal{P} = \mathfrak{P}; \\ 0, & \text{if } \mathcal{P} \neq \mathfrak{P}. \end{cases}$$

By Proposition 3.4, there exist $\alpha_i \in M(\mathfrak{p})$ and $k_i, h_i \in R[Y]$ such that the image $\overline{h}_i$ of h_i in $\kappa(\mathfrak{p})[Y]$ is a power of an irreducible polynomial and such that

$$\frac{1}{r^n} dY \otimes \alpha = k_0 dY \otimes \alpha_0 + \frac{k_1}{h_1} dY \otimes \alpha_1 + \cdots + \frac{k_s}{h_s} dY \otimes \alpha_s$$

in $M_Y(\mathfrak{p}R[Y])$. Define

$$\Upsilon_0 := \sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}\frac{k_0 k r^{n-i} g_1^i f_1^{i-1}}{f_2}\,dX\,dY \otimes \alpha$$

and

$$\Upsilon_j := \sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}\frac{k_j k r^{n-i} g_1^i f_1^{i-1}}{h_j f_2}\,dX\,dY \otimes \alpha$$

for $1 \leq j \leq s$. Then

$$\delta_{X,Y;\mathfrak{p}R[X,Y],\mathfrak{P}}(\Upsilon_0 + \Upsilon_1 + \cdots + \Upsilon_s) = \begin{bmatrix} \dfrac{k}{f_1}\,dX\,dY \otimes \alpha \\ f_2 \end{bmatrix}.$$

For $1 \leq j \leq s$, let $\mathcal{P}_j$ be the prime ideal of $R[X, Y]$ which is the preimage of the radical of $\overline{h}_j$ under the canonical map $R[X, Y] \to \kappa(\mathfrak{p})[X, Y]$. Besides $\mathfrak{P}$, there are at most s prime ideals $\mathcal{P}$ properly between $\mathfrak{p}R[X, Y]$ and $\mathcal{Q}$ such that

$$\delta_{X,Y;\mathfrak{p}R[X,Y],\mathcal{P}}(\Upsilon_0 + \Upsilon_1 + \cdots + \Upsilon_s) \neq 0,$$

namely $\mathcal{P}_1, \ldots, \mathcal{P}_s$. For $1 \leq j \leq s$ and $0 \leq \ell \leq s$,

$$\delta_{X,Y;\mathfrak{p}R[X,Y],\mathcal{P}_j}(\Upsilon_\ell) = 0,$$

if $j \neq \ell$. By the proof in the special case above,

$$\delta_{X,Y;\mathcal{P}_j,\mathcal{Q}} \circ \delta_{X,Y;\mathfrak{p}R[X,Y],\mathcal{P}_j}(\Upsilon_j) = \delta_{Y,X;\mathcal{P}_j,\mathcal{Q}} \circ \delta_{X,Y;\mathfrak{p}R[X,Y],\mathcal{P}_j}(\Upsilon_j).$$

Therefore

$$\begin{aligned}
&\delta_{Y,X;\mathfrak{P},\mathcal{Q}}\left(\begin{bmatrix} \dfrac{k}{f_1}\,dX\,dY \otimes \alpha \\ f_2 \end{bmatrix}\right) \\
&= -\bigoplus_{j=1}^{s} \delta_{Y,X;\mathcal{P}_j,\mathcal{Q}} \circ \delta_{Y,X;\mathfrak{p}R[X,Y],\mathcal{P}_j}\left(\sum_{\ell=0}^{s} \Upsilon_\ell\right) \\
&= -\bigoplus_{j=1}^{s} \delta_{X,Y;\mathcal{P}_j,\mathcal{Q}} \circ \delta_{X,Y;\mathfrak{p}R[X,Y],\mathcal{P}_j}\left(\sum_{\ell=0}^{s} \Upsilon_\ell\right) \\
&= \delta_{X,Y;\mathfrak{P},\mathcal{Q}}\left(\begin{bmatrix} \dfrac{k}{f_1}\,dX\,dY \otimes \alpha \\ f_2 \end{bmatrix}\right).
\end{aligned}$$

If $\mathcal{Q} \cap R =: \mathfrak{q}$ is an immediate specialization of $\mathfrak{p}$, then

$$M_{X,Y}(\mathcal{Q}) = H^1_{\mathcal{Q}}(\det_{\mathcal{Q}} M(\mathfrak{q})),$$

$\mathcal{Q} = \mathcal{Q}_X R[X, Y]$, but $\mathcal{Q} \neq \mathcal{Q}_Y R[X, Y]$. Given $\Theta \in M_{X,Y}(\mathfrak{P})$, there exists $\Upsilon \in M_{X,Y}(\mathfrak{p}R[X, Y])$ such that, for prime ideals $\mathcal{P}$ lying over $\mathfrak{p}R[Y]$ but not equal to $\mathfrak{p}R[X, Y]$,

$$\delta_{X,Y;\mathfrak{p}R[X,Y],\mathcal{P}}(\Upsilon) = \begin{cases} \Theta, & \text{if } \mathcal{P} = \mathfrak{P}; \\ 0, & \text{otherwise.} \end{cases}$$

Besides $\mathfrak{q}R[X, Y]$, prime ideals properly between $\mathcal{Q}$ and $\mathfrak{p}R[X, Y]$ are exactly those prime ideals lying over $\mathfrak{p}R[Y]$ but not equal to $\mathfrak{p}R[X, Y]$. Hence

$$\begin{aligned}
\delta_{Y,X;\mathfrak{P},\mathcal{Q}}(\Theta) &= -\left(\delta_{X,Y;\mathfrak{q}R[X,Y],\mathcal{Q}} \circ \delta_{X,Y;\mathfrak{p}R[X,Y],\mathfrak{q}R[X,Y]}\right)(\Upsilon) \\
&= \delta_{X,Y;\mathfrak{P},\mathcal{Q}}(\Theta).
\end{aligned}$$

If $\mathcal{Q} \cap R =: \mathfrak{r}$ is neither equal to nor an immediate specialization of $\mathfrak{p}$, then $\mathcal{Q} = \mathfrak{r}R[X, Y]$ and

$$M_{X,Y}(\mathcal{Q}) = \det_{\mathcal{Q}} M(\mathfrak{r}).$$

First we consider the special case that f_1 is in $R[Y]$ and its image $\overline{f}_1$ in $\kappa(\mathfrak{p})[X, Y]$ is a power of an irreducible polynomial. Choose $n \in \mathbb{N}$ such that $\mathfrak{p}^n \alpha = 0$. We claim that in this special case there exist $G_1, G_2 \in R[X, Y]^{\wedge}_{\mathfrak{r}R[X,Y]}$, $H \in R[X, Y]^{\wedge}_{\mathfrak{r}R[X,Y]} \setminus \mathfrak{r}R[X, Y]^{\wedge}_{\mathfrak{r}R[X,Y]}$, and $a \in \mathfrak{r} \setminus \mathfrak{p}$ such that

$$f_1 G_1 + f_2 G_2 - aH \in \mathfrak{p}^n R[X, Y]^{\wedge}_{\mathfrak{r}R[X,Y]}.$$

Since $f_2 \in \mathfrak{P}_X \setminus \mathfrak{p}R[X]$, there exist $r \in R \setminus \mathfrak{p}$ and $m \in \mathbb{N}$ such that

$$r\mathfrak{P}_X^m \in f_2R[X] + \mathfrak{p}R[X].$$

By Proposition 4.1, there exist $A \in \mathfrak{r}R[X] \setminus \mathfrak{P}_X$, $G_1 \in R[X, Y]^\wedge_{\mathfrak{r}R[X,Y]}$, and $H \in R[X, Y]^\wedge_{\mathfrak{r}R[X,Y]} \setminus \mathfrak{r}R[X, Y]^\wedge_{\mathfrak{r}R[X,Y]}$, such that

$$f_1G_1 - AH \in \mathfrak{P}_X^m R[X, Y]^\wedge_{\mathfrak{r}R[X,Y]}.$$

By multiplying G_1 and A by r, we may assume that

$$f_1G_1 - AH \in f_2R[X, Y]^\wedge_{\mathfrak{r}R[X,Y]} + \mathfrak{p}R[X, Y]^\wedge_{\mathfrak{r}R[X,Y]}.$$

Since the images of A and f_2 in $\kappa(\mathfrak{p})[X]$ are relatively prime, there exist $B, C \in R[X]$ and $a \in R \setminus \mathfrak{p}$ such that

$$AB + Cf_2 - a \in \mathfrak{p}R[X].$$

By multiplying G_1 and A by B, we may assume that $B = 1$. Hence

$$f_1G_1 - aH \in f_2R[X, Y]^\wedge_{\mathfrak{r}R[X,Y]} + \mathfrak{p}R[X, Y]^\wedge_{\mathfrak{r}R[X,Y]}$$

and there exist $G_2 \in R[X, Y]^\wedge_{\mathfrak{r}R[X,Y]}$ such that

$$f_1G_1 + f_2G_2 - aH \in \mathfrak{p}R[X, Y]^\wedge_{\mathfrak{r}R[X,Y]}.$$

By raising the above element to the nth power and replacing a (resp. H) by a^n (resp. H^n), we may assume that

$$f_1G_1 + f_2G_2 - aH \in \mathfrak{p}^n R[X, Y]^\wedge_{\mathfrak{r}R[X,Y]}.$$

Now we compute $\delta_{X,Y;\mathfrak{P},\mathcal{Q}}$ and $\delta_{Y,X;\mathfrak{P},\mathcal{Q}}$. Since

$$(f_1G_1 - aH)\left(kdY \otimes \delta_{\mathfrak{P}_X,\mathfrak{r}R[X]}\begin{bmatrix} \frac{1}{b}dX \otimes \alpha \\ f_2 \end{bmatrix}\right) = 0$$

for all $b \in \mathfrak{r}R[X] \setminus \mathfrak{P}_X$, by Proposition 4.4,

$$\begin{aligned}
&\delta_{X,Y;\mathfrak{P},\mathcal{Q}}\left(\begin{bmatrix} \frac{k}{f_1}\,dX\,dY \otimes \alpha \\ f_2 \end{bmatrix}\right) \\
&\quad = \frac{G_1}{H}\left(\oplus\delta_{X,Y;\mathfrak{q}R[X,Y],\mathcal{Q}} \circ \delta_{X,Y;\mathfrak{p}R[X,Y],\mathfrak{q}R[X,Y]}\frac{k}{f_2}dY\,dX \otimes \frac{\alpha}{a}\right) \\
&\quad = \frac{f_1G_1}{H}\left(\oplus\delta_{X,Y;\mathfrak{q}R[X,Y],\mathcal{Q}} \circ \delta_{X,Y;\mathfrak{p}R[X,Y],\mathfrak{q}R[X,Y]}\frac{k}{f_1f_2}dY\,dX \otimes \frac{\alpha}{a}\right),
\end{aligned}$$

where $\mathfrak{q}$ runs over the prime ideals of R properly between $\mathfrak{p}$ and $\mathfrak{r}$. Besides $\mathfrak{P}$, there is at most one prime ideal $\mathscr{P}$ lying over $\mathfrak{p}$ and properly between $\mathfrak{p}R[X, Y]$ and $\mathscr{Q}$ such that

$$\delta_{Y,X;\mathfrak{p}R[X,Y],\mathscr{P}}\left(\frac{k}{f_1 f_2} dY\, dX \otimes \alpha\right) \neq 0,$$

namely the preimage $\mathscr{P}_0$ of the radical of $\overline{f}_1$ under the canonical map $R[X, Y] \to \kappa(\mathfrak{p})[X, Y]$. Therefore

$$\begin{aligned}
&\delta_{Y,X;\mathfrak{P},\mathscr{Q}}\left(\begin{bmatrix} \dfrac{k}{f_1} dX\, dY \otimes \alpha \\ f_2 \end{bmatrix}\right) \\
&\quad = -\delta_{Y,X;\mathscr{P}_0,\mathscr{Q}}\left(\begin{bmatrix} \dfrac{k}{f_2} dX\, dY \otimes \alpha \\ f_1 \end{bmatrix}\right) \\
&\qquad - \oplus\, \delta_{X,Y;\mathfrak{q}R[X,Y],\mathscr{Q}} \circ \delta_{X,Y;\mathfrak{p}R[X,Y],\mathfrak{q}R[X,Y]} \frac{k}{f_1 f_2} dX\, dY \otimes \alpha.
\end{aligned}$$

Then

$$\begin{aligned}
&\delta_{Y,X;\mathscr{P}_0,\mathscr{Q}}\left(\begin{bmatrix} \dfrac{k}{f_2} dX\, dY \otimes \alpha \\ f_1 \end{bmatrix}\right) \\
&\quad = \frac{f_2 G_2}{H}\left(\oplus \delta_{X,Y;\mathfrak{q}R[X,Y],\mathscr{Q}} \circ \delta_{X,Y;\mathfrak{p}R[X,Y],\mathfrak{q}R[X,Y]} \frac{k}{f_1 f_2} dY\, dX \otimes \frac{\alpha}{a}\right),
\end{aligned}$$

where $\mathfrak{q}$ runs over prime ideals of R properly between $\mathfrak{p}$ and $\mathfrak{r}$. Hence

$$\begin{aligned}
&\delta_{Y,X;\mathfrak{P},\mathscr{Q}}\left(\begin{bmatrix} \dfrac{k}{f_1} dX\, dY \otimes \alpha \\ f_2 \end{bmatrix}\right) \\
&\quad = -\frac{f_2 G_2}{H}\left(\oplus \delta_{X,Y;\mathfrak{q}R[X,Y],\mathscr{Q}} \circ \delta_{X,Y;\mathfrak{p}R[X,Y],\mathfrak{q}R[X,Y]} \frac{k}{f_1 f_2} dY\, dX \otimes \frac{\alpha}{a}\right) \\
&\qquad + a\left(\oplus \delta_{X,Y;\mathfrak{q}R[X,Y],\mathscr{Q}} \circ \delta_{X,Y;\mathfrak{p}R[X,Y],\mathfrak{q}R[X,Y]} \frac{k}{f_1 f_2} dY\, dX \otimes \frac{\alpha}{a}\right) \\
&\quad = \delta_{X,Y;\mathfrak{P},\mathscr{Q}}\left(\begin{bmatrix} \dfrac{k}{f_1} dX\, dY \otimes \alpha \\ f_2 \end{bmatrix}\right) - \frac{f_1 G_1 + f_2 G_2 - aH}{H} \\
&\qquad \left(\oplus \delta_{X,Y;\mathfrak{q}R[X,Y],\mathscr{Q}} \circ \delta_{X,Y;\mathfrak{p}R[X,Y],\mathfrak{q}R[X,Y]} \frac{k}{f_1 f_2} dY\, dX \otimes \frac{\alpha}{a}\right) \\
&\quad = \delta_{X,Y;\mathfrak{P},\mathscr{Q}}\left(\begin{bmatrix} \dfrac{k}{f_1} dX\, dY \otimes \alpha \\ f_2 \end{bmatrix}\right).
\end{aligned}$$

Now we consider the general case. Recall that there exist elements $\Upsilon_0, \Upsilon_1, \dots, \Upsilon_s \in M_{X,Y}(\mathfrak{p}R[X,Y])$ of the form

$$\Upsilon_0 = \frac{K_0}{f_2} dX\, dY \otimes \alpha$$

and

$$\Upsilon_j = \frac{K_j}{h_j f_2} dX\, dY \otimes \alpha$$

for $1 \le j \le s$, where $K_0, \dots, K_s \in R[X,Y]$ and $h_1, \dots, h_s \in R[Y]$, such that the image $\overline{h}_j$ of each h_j in $\kappa(\mathfrak{p})[X,Y]$ is a power of an irreducible polynomial and such that

$$\delta_{X,Y;\mathfrak{p}R[X,Y],\mathfrak{P}}(\Upsilon_0 + \Upsilon_1 + \cdots + \Upsilon_s) = \begin{bmatrix} \dfrac{k}{f_1} dX\, dY \otimes \alpha \\ f_2 \end{bmatrix}.$$

For $1 \le j \le s$, let $\mathscr{P}_j$ be the prime ideal of $R[X,Y]$ which is the preimage of the radical of $\overline{h}_j$ under the canonical map $R[X,Y] \to \kappa(\mathfrak{p})[X,Y]$. Besides $\mathfrak{P}$, there are at most s prime ideals $\mathscr{P}$ lying over $\mathfrak{p}$ and properly between $\mathfrak{p}R[X,Y]$ and $\mathscr{Q}$ such that

$$\delta_{X,Y;\mathfrak{p}R[X,Y],\mathscr{P}}(\Upsilon_0 + \Upsilon_1 + \cdots + \Upsilon_s) \neq 0,$$

namely $\mathscr{P}_1, \dots, \mathscr{P}_s$. For $1 \le j \le s$ and $0 \le \ell \le s$,

$$\delta_{X,Y;\mathfrak{p}R[X,Y],\mathscr{P}_j}(\Upsilon_\ell) = 0,$$

if $j \neq \ell$. Proved in the special case above,

$$\delta_{X,Y;\mathscr{P}_j,\mathscr{Q}} \circ \delta_{X,Y;\mathfrak{p}R[X,Y],\mathscr{P}_j}(\Upsilon_j) = \delta_{Y,X;\mathscr{P}_j,\mathscr{Q}} \circ \delta_{X,Y;\mathfrak{p}R[X,Y],\mathscr{P}_j}(\Upsilon_j).$$

Therefore

$$\begin{aligned}
&\delta_{Y,X;\mathfrak{P},\mathscr{Q}}\left(\begin{bmatrix} \dfrac{k}{f_1} dX\, dY \otimes \alpha \\ f_2 \end{bmatrix}\right) \\
&\qquad = -\bigoplus_{j=1}^{s} \delta_{Y,X;\mathscr{P}_j,\mathscr{Q}} \circ \delta_{Y,X;\mathfrak{p}R[X,Y],\mathscr{P}_j}\left(\sum_{\ell=0}^{s} \Upsilon_\ell\right) \\
&\qquad\quad - \bigoplus \delta_{Y,X;\mathfrak{q}R[X,Y],\mathscr{Q}} \circ \delta_{Y,X;\mathfrak{p}R[X,Y],\mathfrak{q}R[X,Y]}\left(\sum_{\ell=0}^{s} \Upsilon_\ell\right) \\
&\qquad = -\bigoplus_{j=1}^{s} \delta_{X,Y;\mathscr{P}_j,\mathscr{Q}} \circ \delta_{X,Y;\mathfrak{p}R[X,Y],\mathscr{P}_j}\left(\sum_{\ell=0}^{s} \Upsilon_\ell\right)
\end{aligned}$$

$$-\bigoplus \delta_{X,Y;\mathfrak{q}R[X,Y],\mathscr{Q}} \circ \delta_{X,Y;\mathfrak{p}R[X,Y],\mathfrak{q}R[X,Y]}\left(\sum_{\ell=0}^{s} \Upsilon_\ell\right)$$
$$= \delta_{X,Y;\mathfrak{P},\mathscr{Q}}\left(\left[\begin{array}{c} \dfrac{k}{f_1}\, dX\, dY \otimes \alpha \\ f_2 \end{array}\right]\right),$$

where $\mathfrak{q}$ runs over the prime ideals of R properly between $\mathfrak{p}$ and $\mathfrak{r}$. ∎

Assume that the complex $M^\bullet_{X_1,\dots,X_{n-1}}$ has been defined on $R[X_1,\dots, X_{n-1}]$ for $n>2$. We define a complex

$$\cdots \longrightarrow \bigoplus_{\Delta_{X_1,\dots,X_n}(\mathscr{P})=n} M_{X_1,\dots,X_n}(\mathscr{P}) \xrightarrow{\delta^n_{X_1,\dots,X_n}} \bigoplus_{\Delta_{X_1,\dots,X_n}(\mathscr{Q})=n+1} M_{X_1,\dots,X_n}(\mathscr{Q}) \longrightarrow \cdots,$$

denoted by $M^\bullet_{X_1,\dots,X_n}$, on $R[X_1,\cdots,X_n]$ such that the diagram

$$\begin{array}{ccc}
\bigoplus_{\Delta_{X_1,\dots,X_n}(\mathscr{P})=n} M_{X_n}(\mathscr{P}) & \xrightarrow{\delta^n_{X_1,\dots,X_n/X_1,\dots,X_{n-1}}} & \bigoplus_{\Delta_{X_1,\dots,X_n}(\mathscr{Q})=n+1} M_{X_n}(\mathscr{Q}) \\
\Big\downarrow \text{can.} & & \Big\downarrow \text{can.} \\
\bigoplus_{\Delta_{X_1,\dots,X_n}(\mathscr{P})=n} M_{X_1,\dots,X_n}(\mathscr{P}) & \xrightarrow[\delta^n_{X_1,\dots,X_n}]{} & \bigoplus_{\Delta_{X_1,\dots,X_n}(\mathscr{Q})=n+1} M_{X_1,\dots,X_n}(\mathscr{Q})
\end{array}$$

is commutative, where

$$\cdots \longrightarrow \bigoplus_{\Delta_{X_1,\dots,X_n}(\mathscr{P})=n} M_{X_n}(\mathscr{P}) \xrightarrow{\delta^n_{X_1,\dots,X_n/X_1,\dots,X_{n-1}}} \bigoplus_{\Delta_{X_1,\dots,X_n}(\mathscr{Q})=n+1} M_{X_n}(\mathscr{Q}) \longrightarrow \cdots$$

is the complex on $R[X_1,\cdots,X_n]$ constructed from $M^\bullet_{X_1,\cdots,X_{n-1}}$ using the above method.

COROLLARY 5.4. *The residual complex $M^\bullet_{X_1,\dots,X_n}$ obtained inductively on n from $M^\bullet$ is independent of the order of $X_1,\dots,X_n$.*

We denote by $\delta_{X_1,\dots,X_n;\mathscr{P},\mathscr{Q}}$ the $R[X_1,\dots,X_n]$-linear map $M_{X_1,\dots,X_n}(\mathscr{P}) \to M_{X_1,\dots,X_n}(\mathscr{Q})$ in the complex $M^\bullet_{X_1,\dots,X_n}$.

6. CONSTRUCTION OF $M^\bullet_{S/R}$

Let S be a finitely generated R-algebra. Assume that elements $x_1,\dots,x_n$ generate S as an R-algebra. Then $\{x_1,\dots,x_n\} \in \mathscr{I}_{S/R,\mathfrak{P}}$ for every prime ideal $\mathfrak{P}$ of S. Write $\mathbf{x}$ for the elements $x_1,\dots,x_n$ and $\mathbf{X}$ for variables $X_1,\dots,X_n$ chosen for $\mathbf{x}$. Let I be the kernel of the R-linear map $R[\mathbf{X}] \to S$ sending X_i to x_i. For any prime ideal $\mathfrak{P}_{\mathbf{X}}$ of $R[\mathbf{X}]$, let $\mathfrak{P}$ be its image in S, there are canonical isomorphisms

$$\mathrm{Hom}_{R[\mathbf{X}]}(S, M_{\mathbf{X}}(\mathfrak{P}_{\mathbf{X}})) \simeq \begin{cases} M_{S/R,\mathbf{x}}(\mathfrak{P}), & \text{if } I \subseteq \mathfrak{P}_{\mathbf{X}}, \\ 0, & \text{if } I \not\subseteq \mathfrak{P}_{\mathbf{X}}. \end{cases}$$

Let Δ_S be the codimension function on $\operatorname{Spec} S$ as characterized in (2). If $\mathfrak{P}_{\mathbf{X}} \supset I$, then $\Delta_S(\mathfrak{P}) = \Delta_{R[\mathbf{X}]}(\mathfrak{P}_{\mathbf{X}})$. For prime ideals $\mathscr{P}$ and $\mathscr{Q}$ of S with $\Delta_S(\mathscr{Q}) = \Delta_S(\mathscr{P}) + 1$, let $\mathscr{P}_{\mathbf{X}}$ and $\mathscr{Q}_{\mathbf{X}}$ be their preimages in $R[\mathbf{X}]$, we define

$$\delta_{S/R,\mathbf{x};\mathscr{P},\mathscr{Q}}\colon M_{S/R,\mathbf{x}}(\mathscr{P}) \to M_{S/R,\mathbf{x}}(\mathscr{Q})$$

to be the restriction of $\delta_{\mathbf{X};\mathscr{P},\mathscr{Q}}\colon M_{\mathbf{X}}(\mathscr{P}_{\mathbf{X}}) \to M_{\mathbf{X}}(\mathscr{Q}_{\mathbf{X}})$ on $M_{S/R,\mathbf{x}}(\mathscr{P})$. It is easy to see that

$$\cdots \longrightarrow \bigoplus_{\Delta_S(\mathscr{P})=n} M_{S/R,\mathbf{x}}(\mathscr{P}) \xrightarrow{\oplus\delta_{S/R,\mathbf{x};\mathscr{P},\mathscr{Q}}} \bigoplus_{\Delta_S(\mathscr{Q})=n+1} M_{S/R,\mathbf{x}}(\mathscr{Q}) \longrightarrow \cdots$$

denoted by $M^\bullet_{S/R,\mathbf{x}}$, is a complex which is isomorphic to the bounded below complex $\operatorname{Hom}_{R[\mathbf{X}]}(S, M^\bullet)$. The complex $M^\bullet_{S/R,\mathbf{x}}$ has finitely generated cohomology easily seen from the spectral sequence

$$\operatorname{Ext}^p_{R[\mathbf{X}]}(S, H^q(M^\bullet)) \Rightarrow H^{p+q}(M^\bullet_{S/R,\mathbf{x}}),$$

and hence is a residual complex.

Let $y_1, \dots, y_m$ be elements in S and $Y_1, \dots, Y_m$ be the variables chosen for them. Then the R-linear map $R[\mathbf{X}, Y_1, \dots, Y_m] \to S$ extending $R[\mathbf{X}] \to S$ and sending Y_i to y_i is also surjective. The coboundary maps of $M^\bullet_{S/R,\mathbf{x}}$ are compatible with the residue maps in the following sense.

PROPOSITION 6.1. *With the notation above, write* $\mathbf{y}$ *for the elements* $y_1, \dots, y_m$. *The diagram*

$$\begin{array}{ccc} M_{S/R,\mathbf{x}}(\mathscr{P}) & \xrightarrow{\delta_{S/R,\mathbf{x};\mathscr{P},\mathscr{Q}}} & M_{S/R,\mathbf{x}}(\mathscr{Q}) \\ \Big\downarrow{\scriptstyle \mathrm{res}^{-1}} & & \Big\downarrow{\scriptstyle \mathrm{res}^{-1}} \\ M_{S/R,\mathbf{x},\mathbf{y}}(\mathscr{P}) & \xrightarrow[\delta_{S/R,\mathbf{x},\mathbf{y};\mathscr{P},\mathscr{Q}}]{} & M_{S/R,\mathbf{x},\mathbf{y}}(\mathscr{Q}) \end{array}$$

is commutative.

Proof. It suffices to show that the diagram

$$\begin{array}{ccc} M_{\mathbf{X}}(\mathscr{P}_{\mathbf{X}}) & \xrightarrow{\delta_{\mathbf{X};\mathscr{P},\mathscr{Q}}} & M_{\mathbf{X}}(\mathscr{Q}_{\mathbf{X}}) \\ \Big\uparrow{\scriptstyle \mathrm{res}} & & \Big\uparrow{\scriptstyle \mathrm{res}} \\ M_{\mathbf{X},\mathbf{Y}}(\mathscr{P}_{\mathbf{X},\mathbf{Y}}) & \xrightarrow[\delta_{\mathbf{X},\mathbf{Y};\mathscr{P},\mathscr{Q}}]{} & M_{\mathbf{X},\mathbf{Y}}(\mathscr{Q}_{\mathbf{X},\mathbf{Y}}) \end{array}$$

is commutative since the maps in the proposition are the restrictions or inverses of the restrictions of the maps above. By the transitivity law (2.5) of the residue maps, we may also assume $m = 1$. Let

$$\begin{bmatrix} k\,dY_1\,dX_1 \cdots dX_n \otimes \alpha \\ (Y_1 - g_1)^{i_1}, f_1^{j_1}, \dots, f_\ell^{j_\ell} \end{bmatrix}$$

be an element in $M_{\mathbf{X},\mathbf{Y}}(\mathscr{P}_{\mathbf{X},\mathbf{Y}})$, where $k \in R[\mathbf{X}]_{\mathscr{P}_{\mathbf{X}}}$ and $g_1 \in R[\mathbf{X}]$ maps to the image of Y_1 in S. Write

$$\delta_{\mathbf{X};\mathscr{P},\mathscr{Q}}\left(\begin{bmatrix} k\,dX_1\cdots dX_n \otimes \alpha \\ f_1^{j_1},\ldots,f_\ell^{j_\ell}\end{bmatrix}\right) = \sum_j \begin{bmatrix} k_j\,dX_1\cdots dX_n \otimes \alpha_j \\ f'_1,\ldots,f'_{\ell'}\end{bmatrix},$$

where $f'_1,\ldots,f'_{\ell'}$ is a relative system of parameters of $R[\mathbf{X}]_{\mathscr{Q}_{\mathbf{X}}}$ over $R_{\mathscr{Q}\cap R}$, $k_j \in R[\mathbf{X}]_{\mathscr{Q}_{\mathbf{X}}}$, and $\alpha_j \in M(\mathscr{Q}\cap R)$. Note that $\delta_{\mathbf{X},Y_1;\mathscr{P},\mathscr{Q}}$ is a coboundary map of type (1, 1) obtained from $\delta_{\mathbf{X};\mathscr{P},\mathscr{Q}}$. Since the image of $Y_1 - g_1$ in $\kappa(\mathscr{P}_{\mathbf{X}})[Y_1]$ generates a maximal ideal and the image of $Y_1 - g_1$ in $R[\mathbf{X},Y_1]^{\wedge}_{\mathscr{Q}_{\mathbf{X}}R[\mathbf{X},Y_1]}$ is invertible,

$$\delta_{\mathbf{X},\mathbf{Y};\mathscr{P},\mathscr{Q}}\left(\begin{bmatrix} k\,dY_1\,dX_1\cdots dX_n \otimes \alpha \\ (Y_1-g_1)^{i_1}, f_1^{j_1},\ldots,f_\ell^{j_\ell}\end{bmatrix}\right) = \sum_j \begin{bmatrix} k_j\,dY_1\,dX_1\cdots dX_n \otimes \alpha_j \\ (Y_1-g_1)^{i_1}, f'_1,\ldots,f'_{\ell'}\end{bmatrix}.$$

If $i_1 = 1$,

$$\begin{aligned}(\mathrm{res}\circ\delta_{\mathbf{X},\mathbf{Y};\mathscr{P},\mathscr{Q}})&\left(\begin{bmatrix} k\,dY_1\,dX_1\cdots dX_n \otimes \alpha \\ (Y_1-g_1)^{i_1}, f_1^{j_1},\ldots,f_\ell^{j_\ell}\end{bmatrix}\right)\\ &= \sum_j \begin{bmatrix} k_j\,dX_1\cdots dX_n \otimes \alpha_j \\ f'_1,\ldots,f'_{\ell'}\end{bmatrix}\\ &= \delta_{\mathbf{X};\mathscr{P},\mathscr{Q}}\left(\begin{bmatrix} k\,dX_1\cdots dX_n \otimes \alpha \\ f_1^{j_1},\ldots,f_\ell^{j_\ell}\end{bmatrix}\right)\\ &= (\delta_{\mathbf{X};\mathscr{P},\mathscr{Q}}\circ\mathrm{res})\left(\begin{bmatrix} k\,dY_1\,dX_1\cdots dX_n \otimes \alpha \\ (Y_1-g_1)^{i_1}, f_1^{j_1},\ldots,f_\ell^{j_\ell}\end{bmatrix}\right).\end{aligned}$$

If $i_1 \neq 1$,

$$\begin{aligned}(\mathrm{res}\circ\delta_{\mathbf{X},\mathbf{Y};\mathscr{P},\mathscr{Q}})&\left(\begin{bmatrix} k\,dY_1\,dX_1\cdots dX_n \otimes \alpha \\ (Y_1-g_1)^{i_1}, f_1^{j_1},\ldots,f_\ell^{j_\ell}\end{bmatrix}\right)\\ &= 0\\ &= (\delta_{\mathbf{X};\mathscr{P},\mathscr{Q}}\circ\mathrm{res})\left(\begin{bmatrix} k\,dY_1\,dX_1\cdots dX_n \otimes \alpha \\ (Y_1-g_1)^{i_1}, f_1^{j_1},\ldots,f_\ell^{j_\ell}\end{bmatrix}\right).\end{aligned}$$

∎

The S-linear map

$$\delta_{S/R;\mathscr{P},\mathscr{Q}}\colon\ M_{S/R}(\mathscr{P}) \to M_{S/R}(\mathscr{Q})$$

making the diagram

$$\begin{array}{ccc} M_{S/R,\mathbf{x}}(\mathscr{P}) & \xrightarrow{\ \delta_{S/R,\mathbf{x};\mathscr{P},\mathscr{Q}}\ } & M_{S/R,\mathbf{x}}(\mathscr{Q}) \\ \downarrow & & \downarrow \\ M_{S/R}(\mathscr{P}) & \xrightarrow[\ \delta_{S/R;\mathscr{P},\mathscr{Q}}\ ]{} & M_{S/R}(\mathscr{Q}) \end{array}$$

commutative is independent of the choice of $\mathbf{x}$. Since $M_{S/R,\mathbf{x}}(\mathscr{P}) \to M_{S/R}(\mathscr{P})$ is an isomorphism for every prime ideal $\mathscr{P}$ of S, the chain of S-modules

$$\cdots \longrightarrow \bigoplus_{\Delta_S(\mathscr{P})=n} M_{S/R}(\mathscr{P}) \xrightarrow{\oplus\delta_{S/R;\mathscr{P},\mathscr{Q}}} \bigoplus_{\Delta_S(\mathscr{Q})=n+1} M_{S/R}(\mathscr{Q}) \longrightarrow \cdots,$$

denoted by $M^{\bullet}_{S/R}$, being isomorphic to $M^{\bullet}_{S/R,\mathbf{x}}$ is a residual complex on S.

7. GLOBAL CONSTRUCTION

Let $\mathscr{M}^{\bullet}$ be a residual complex on a locally Noetherian scheme $\mathscr{X}$. Assume that

$$\mathscr{M}^n = \bigoplus_{\Delta_{\mathscr{X}}(\mathfrak{p})=n} J(\mathfrak{p})$$

(see the notation in Section 1). Let $\phi\colon \mathscr{Y} \to \mathscr{X}$ be a morphism of finite type whose fibers have bounded dimensions. Given points $\mathscr{P}$ and $\mathscr{Q}$ of $\mathscr{Y}$ with $\Delta_{\mathscr{Y}}(\mathscr{Q}) = \Delta_{\mathscr{Y}}(\mathscr{P}) + 1$, we define an $\mathscr{O}_{\mathscr{Y}}$-morphism $\delta_{\mathscr{P},\mathscr{Q}}\colon M_{\mathscr{Y}}(\mathscr{P}) \to M_{\mathscr{Y}}(\mathscr{Q})$ as follows: If $\mathscr{Q}$ is not an immediate specialization of $\mathscr{P}$, $\delta_{\mathscr{P},\mathscr{Q}}$ is defined to be zero. If $\mathscr{Q}$ is an immediate specialization of $\mathscr{P}$, we choose $(V, U) \in \mathscr{J}_{\mathscr{Y}/\mathscr{X},\mathscr{Q}}$ (see the notation in Section 2), $\delta_{\mathscr{P},\mathscr{Q}}$ is defined to be the map making the diagram

$$\begin{array}{ccc}
M_{\Gamma(V,\mathscr{O}_{\mathscr{Y}})/\Gamma(U,\mathscr{O}_{\mathscr{X}})}(\mathscr{P}) & \xrightarrow{\delta_{\Gamma(V,\mathscr{O}_{\mathscr{Y}})/\Gamma(U,\mathscr{O}_{\mathscr{X}});\mathscr{P},\mathscr{Q}}} & M_{\Gamma(V,\mathscr{O}_{\mathscr{Y}})/\Gamma(U,\mathscr{O}_{\mathscr{X}})}(\mathscr{Q}) \\
\downarrow & & \downarrow \\
M_{\mathscr{Y}}(\mathscr{P}) & \xrightarrow[\delta_{\mathscr{P},\mathscr{Q}}]{} & M_{\mathscr{Y}}(\mathscr{Q})
\end{array}$$

commutative.

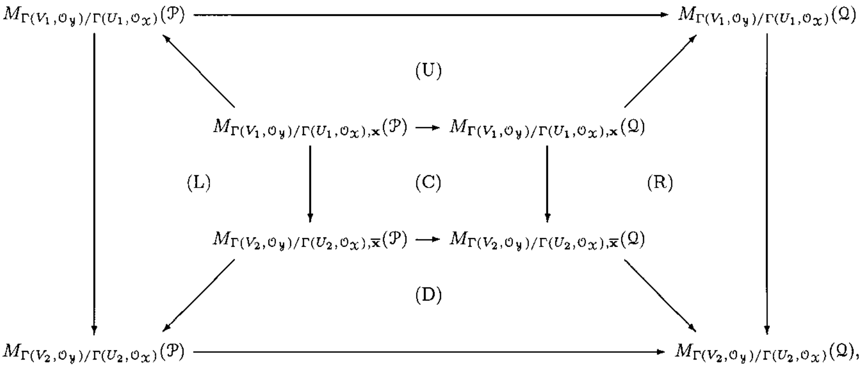
PROPOSITION 7.1. *The map $\delta_{\mathscr{P},\mathscr{Q}}$ is independent of the choice of V and U.*

Proof. It suffices to show that the diagram

$$\begin{array}{ccc}
M_{\Gamma(V_1,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_1,\mathscr{O}_{\mathscr{X}})}(\mathscr{P}) & \xrightarrow{\delta_{\Gamma(V_1,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_1,\mathscr{O}_{\mathscr{X}});\mathscr{P},\mathscr{Q}}} & M_{\Gamma(V_1,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_1,\mathscr{O}_{\mathscr{X}})}(\mathscr{Q}) \\
\downarrow & & \downarrow \\
M_{\Gamma(V_2,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_2,\mathscr{O}_{\mathscr{X}})}(\mathscr{P}) & \xrightarrow[\delta_{\Gamma(V_2,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_2,\mathscr{O}_{\mathscr{X}});\mathscr{P},\mathscr{Q}}]{} & M_{\Gamma(V_2,\mathscr{O}_{\mathscr{Y}})/\Gamma(U_2,\mathscr{O}_{\mathscr{X}})}(\mathscr{Q})
\end{array}$$

is commutative for (V_1, U_1) and (V_2, U_2) in $\mathscr{J}_{\mathscr{Y}/\mathscr{X},\mathscr{Q}}$ with $(V_1, U_1) \leq (V_2, U_2)$.*

Expand the above diagram as follows:



where $\mathbf{x} = \{x_1, \dots, x_n\}$ generate $\Gamma(V_1, \mathcal{O}_\mathcal{Y})$ as an $\Gamma(U_1, \mathcal{O}_\mathcal{X})$-algebra and $\overline{\mathbf{x}} = \{\overline{x}_1, \dots, \overline{x}_n\}$ are the restrictions of $\mathbf{x}$ in $\Gamma(V_2, \mathcal{O}_\mathcal{Y})$. The subdiagrams (U), (L), and (R) are commutative by definitions. The subdiagram (D) is also commutative easily seen from the proof of Proposition 6.1. Since the diagonal maps in the above diagram are isomorphisms, to prove the proposition, it suffices to prove that the subdiagram (C) is commutative. Let $\mathbf{X} = \{X_1, \dots, X_n\}$ (resp. $\overline{\mathbf{X}} = \{\overline{\mathbf{X}}_1, \dots, \overline{\mathbf{X}}_n\}$) be the variables chosen for the elements $\mathbf{x}$ (resp. $\overline{\mathbf{x}}$). The maps of subdiagram (C) are the restrictions of the maps of the diagram

$$\begin{array}{ccc}
M_{\Gamma(U_1,\mathcal{O}_\mathcal{X})[\mathbf{X}]/\Gamma(U_1,\mathcal{O}_\mathcal{X}),\mathbf{X}}(\mathcal{P}_\mathbf{X}) & \xrightarrow{\delta_{\mathbf{X};\mathcal{P}_\mathbf{X},\mathcal{Q}_\mathbf{X}}} & M_{\Gamma(U_1,\mathcal{O}_\mathcal{X})[\mathbf{X}]/\Gamma(U_1,\mathcal{O}_\mathcal{X}),\mathbf{X}}(\mathcal{Q}_\mathbf{X}) \\
\downarrow{\scriptstyle \zeta_{\mathcal{P}_\mathbf{X}}} & & \downarrow{\scriptstyle \zeta_{\mathcal{Q}_\mathbf{X}}} \\
M_{\Gamma(U_2,\mathcal{O}_\mathcal{X})[\overline{\mathbf{X}}]/\Gamma(U_2,\mathcal{O}_\mathcal{X}),\overline{\mathbf{X}}}(\mathcal{P}_{\overline{\mathbf{X}}}) & \xrightarrow[\delta_{\overline{\mathbf{X}};\mathcal{P}_{\overline{\mathbf{X}}},\mathcal{Q}_{\overline{\mathbf{X}}}}]{} & M_{\Gamma(U_2,\mathcal{O}_\mathcal{X})[\overline{\mathbf{X}}]/\Gamma(U_2,\mathcal{O}_\mathcal{X}),\overline{\mathbf{X}}}(\mathcal{Q}_{\overline{\mathbf{X}}}),
\end{array}$$

where $\mathcal{P}_\mathbf{X}$ (resp. $\mathcal{Q}_\mathbf{X}$) is the preimage of $\mathcal{P}$ (resp. $\mathcal{Q}$) in $\Gamma(U_1, \mathcal{O}_\mathcal{X})[\mathbf{X}]$ (resp. $\Gamma(U_2, \mathcal{O}_\mathcal{X})[\mathbf{X}]$). So it suffices to show that the above diagram is commutative. If $\delta_{\mathbf{X};\mathcal{P}_\mathbf{X},\mathcal{Q}_\mathbf{X}}$ is constructed from $M^\bullet_{X_1,\dots,X_{n-1}}$ by a coboundary map of type (i, j), then $\delta_{\overline{\mathbf{X}};\mathcal{P}_{\overline{\mathbf{X}}},\mathcal{Q}_{\overline{\mathbf{X}}}}$ is constructed from $M^\bullet_{\overline{X}_1,\dots,\overline{X}_{n-1}}$ by a coboundary map also of type (i, j). It is not difficult to see that $\oplus\zeta_{\mathcal{Q}_\mathbf{X}} \circ \delta_{\mathbf{X};\mathcal{P}_\mathbf{X},\mathcal{Q}_\mathbf{X}} \circ \zeta^{-1}_{\mathcal{P}_\mathbf{X}}$ gives rise to a complex

$$\cdots \to \oplus M_{\Gamma(U_2,\mathcal{O}_\mathcal{X})[\overline{\mathbf{X}}]/\Gamma(U_2,\mathcal{O}_\mathcal{X}),\overline{\mathbf{X}}}(\mathcal{P}_{\overline{\mathbf{X}}}) \to \oplus M_{\Gamma(U_2,\mathcal{O}_\mathcal{X})[\overline{\mathbf{X}}]/\Gamma(U_2,\mathcal{O}_\mathcal{X}),\overline{\mathbf{X}}}(\mathcal{Q}_{\overline{\mathbf{X}}}) \to \cdots.$$

Since coboundary maps are determined by coboundary maps of type $(0, 1)$ and $(0, 0)$ (Proposition 5.2), it suffices to prove the proposition for coboundary maps of type $(0, 1)$ and $(0, 0)$. In what follows, the case $(0, j)$ means that $\delta_{\mathbf{X};\mathcal{P}_\mathbf{X},\mathcal{Q}_\mathbf{X}}$ is constructed by a coboundary map of type $(0, j)$. Let $\mathscr{P}_{n-1}$ (resp. $\mathscr{P}_{\overline{n-1}}$) be the preimage of $\mathscr{P}_{\mathbf{X}}$ (resp. $\mathscr{P}_{\overline{\mathbf{X}}}$) in $\Gamma(U_1, \mathscr{O}_{\mathscr{X}})[X_1, \ldots, X_{n-1}]$ (resp. $\Gamma(U_2, \mathscr{O}_{\mathscr{X}})[\overline{X}_1, \ldots, \overline{X}_{n-1}]$); let $\mathscr{Q}_{n-1}$ (resp. $\mathscr{Q}_{\overline{n-1}}$) be the preimage of $\mathscr{Q}_{\mathbf{X}}$ (resp. $\mathscr{Q}_{\overline{\mathbf{X}}}$) in $\Gamma(U_1, \mathscr{O}_{\mathscr{X}})[X_1, \ldots, X_{n-1}]$ (resp. $\Gamma(U_2, \mathscr{O}_{\mathscr{X}})[\overline{X}_1, \ldots, \overline{X}_{n-1}]$); and let $\mathfrak{p} = \phi(\mathscr{P})$.

Case $(0, 1)$. Elements in $M_{\Gamma(U_1,\mathscr{O}_{\mathscr{X}})[\mathbf{X}]/\Gamma(U_1,\mathscr{O}_{\mathscr{X}}),\mathbf{X}}(\mathscr{P}_{\mathbf{X}})$ can be written as a sum of elements of the form

$$\begin{bmatrix} \dfrac{k}{f_0}\, dX_1 \cdots dX_n \otimes \alpha \\ f_1, \ldots, f_\ell \end{bmatrix}, \tag{15}$$

where $k/f_0 \in \Gamma(U_1, \mathscr{O}_{\mathscr{X}})[\mathbf{X}]_{\mathscr{P}_{\mathbf{X}}}$, $\alpha \in M(\mathfrak{p})$, and $f_1, \ldots, f_\ell$ form a relative system of parameters of $\Gamma(U_1, \mathscr{O}_{\mathscr{X}})[X_1, \ldots, X_{n-1}]_{\mathscr{P}_{n-1}}$ over $\Gamma(U_1, \mathscr{O}_{\mathscr{X}})_{\mathfrak{p}}$. Note that $f_1, \ldots, f_\ell$ also form a relative system of parameters of $\Gamma(U_1, \mathscr{O}_{\mathscr{X}})[\mathbf{X}]_{\mathscr{P}_{\mathbf{X}}}$ over $\Gamma(U_1, \mathscr{O}_{\mathscr{X}})_{\mathfrak{p}}$. The images $\overline{f}_1, \ldots, \overline{f}_\ell$ of $f_1, \ldots, f_\ell$ in $\Gamma(U_2, \mathscr{O}_{\mathscr{X}})[\overline{\mathbf{X}}]_{\mathscr{P}_{\overline{\mathbf{X}}}}$ form a relative system of parameters of $\Gamma(U_2, \mathscr{O}_{\mathscr{X}})[\overline{\mathbf{X}}]_{\mathscr{P}_{\overline{\mathbf{X}}}}$ over $\Gamma(U_2, \mathscr{O}_{\mathscr{X}})_{\mathfrak{p}}$.

$$\begin{aligned}
&\zeta_{\mathscr{Q}_{\mathbf{X}}} \circ \delta_{\mathbf{X};\mathscr{P}_{\mathbf{X}},\mathscr{Q}_{\mathbf{X}}} \begin{bmatrix} \dfrac{k}{f_0}\, dX_1 \cdots dX_n \otimes \alpha \\ f_1, \ldots, f_\ell \end{bmatrix} \\
&\quad = \zeta_{\mathscr{Q}_{\mathbf{X}}} \begin{bmatrix} k\, dX_1 \cdots dX_n \otimes \alpha \\ f_0, f_1, \ldots, f_\ell \end{bmatrix} \\
&\quad = \begin{bmatrix} \overline{k}\, d\overline{X}_1 \cdots d\overline{X}_n \otimes \alpha \\ \overline{f}_0, \overline{f}_1, \ldots, \overline{f}_\ell \end{bmatrix} \\
&\quad = \delta_{\overline{\mathbf{X}};\mathscr{P}_{\overline{\mathbf{X}}},\mathscr{Q}_{\overline{\mathbf{X}}}} \begin{bmatrix} \dfrac{\overline{k}}{\overline{f}_0}\, d\overline{X}_1 \cdots d\overline{X}_n \otimes \alpha \\ \overline{f}_1, \ldots, \overline{f}_\ell \end{bmatrix} \\
&\quad = \delta_{\overline{\mathbf{X}};\mathscr{P}_{\overline{\mathbf{X}}},\mathscr{Q}_{\overline{\mathbf{X}}}} \circ \zeta_{\mathscr{P}_{\mathbf{X}}} \begin{bmatrix} \dfrac{k}{f_0}\, dX_1 \cdots dX_n \otimes \alpha \\ f_1, \ldots, f_\ell \end{bmatrix}.
\end{aligned}$$

Case $(0, 0)$. Let

$$\begin{bmatrix} \dfrac{k}{f_0}\, dX_1 \cdots dX_n \otimes \alpha \\ f_1, \ldots, f_\ell \end{bmatrix}$$

be an element of $M_{\Gamma(U_1,\mathscr{O}_{\mathscr{X}})[\mathbf{X}]/\Gamma(U_1,\mathscr{O}_{\mathscr{X}}),\mathbf{x}}(\mathscr{P}_{\mathbf{x}})$ as in (15). Choose elements $g \in \Gamma(U_1, \mathscr{O}_{\mathscr{X}})[\mathbf{X}]^{\wedge}_{\mathscr{Q}_{\mathbf{x}}}$, $h \in \Gamma(U_1, \mathscr{O}_{\mathscr{X}})[\mathbf{X}]^{\wedge}_{\mathscr{Q}_{\mathbf{x}}} \setminus \mathscr{Q}_{\mathbf{x}}\Gamma(U_1, \mathscr{O}_{\mathscr{X}})[\mathbf{X}]^{\wedge}_{\mathscr{Q}_{\mathbf{x}}}$, and $a \in \mathscr{Q}_{n-1} \setminus \mathscr{P}_{n-1}$ such that $f_0 g - ah \in \mathscr{P}^m_{n-1}\Gamma(U_1, \mathscr{O}_{\mathscr{X}})[\mathbf{X}]^{\wedge}_{\mathscr{Q}_{\mathbf{x}}}$ for some $m \in \mathbb{N}$ satisfying

$$\mathscr{P}^m_{n-1}\begin{bmatrix} dX_1 \cdots dX_{n-1} \otimes \alpha \\ f_1, \ldots, f_\ell \end{bmatrix} = 0,$$

then

$$\delta_{\mathbf{X};\mathscr{P}_{\mathbf{x}},\mathscr{Q}_{\mathbf{x}}}\begin{bmatrix} \dfrac{k}{f_0}\, dX_1 \cdots dX_n \otimes \alpha \\ f_1, \ldots, f_\ell \end{bmatrix} = \frac{g}{h}\begin{bmatrix} \dfrac{k}{a}\, dX_1 \cdots dX_n \otimes \alpha \\ f_1, \ldots, f_\ell \end{bmatrix}.$$

Let $\overline{g}$, $\overline{h}$, and $\overline{a}$ be the images of g, h, and a in $\Gamma(U_2, \mathscr{O}_{\mathscr{X}})[\overline{\mathbf{X}}]^{\wedge}_{\mathscr{Q}_{\overline{\mathbf{x}}}}$, respectively. Since $\overline{h} \notin \mathscr{Q}_{\mathbf{x}}\Gamma(U_2, \mathscr{O}_{\mathscr{X}})[\mathbf{X}]^{\wedge}_{\mathscr{Q}_{\mathbf{x}}}$, $\overline{a} \in \overline{\mathscr{Q}_{n-1}} \setminus \overline{\mathscr{P}_{n-1}}$, $\overline{f}_0\overline{g} - \overline{a}\overline{h} \in (\overline{\mathscr{P}_{n-1}})^m\Gamma(U_2, \mathscr{O}_{\mathscr{X}})[\mathbf{X}]^{\wedge}_{\mathscr{Q}_{\mathbf{x}}}$, and

$$(\overline{\mathscr{P}_{n-1}})^m\begin{bmatrix} d\overline{X}_1 \cdots d\overline{X}_{n-1} \otimes \alpha \\ \overline{f}_1, \ldots, \overline{f}_\ell \end{bmatrix} = 0,$$

we have

$$\delta_{\overline{\mathbf{X}};\mathscr{P}_{\overline{\mathbf{x}}},\mathscr{Q}_{\overline{\mathbf{x}}}}\begin{bmatrix} \dfrac{\overline{k}}{\overline{f}_0} d\overline{X}_1 \cdots d\overline{X}_n \otimes \alpha \\ \overline{f}_1, \ldots, \overline{f}_\ell \end{bmatrix} = \frac{\overline{g}}{\overline{h}}\begin{bmatrix} \dfrac{\overline{k}}{\overline{a}} d\overline{X}_1 \cdots d\overline{X}_n \otimes \alpha \\ \overline{f}_1, \ldots, \overline{f}_\ell \end{bmatrix}.$$

Therefore

$$\begin{aligned}
\zeta_{\mathscr{Q}_{\mathbf{x}}} \circ \delta_{\mathbf{X};\mathscr{P}_{\mathbf{x}},\mathscr{Q}_{\mathbf{x}}} &\begin{bmatrix} \dfrac{k}{f_0}\, dX_1 \cdots dX_n \otimes \alpha \\ f_1, \ldots, f_\ell \end{bmatrix} \\
&= \zeta_{\mathscr{Q}_{\mathbf{x}}}\left(\frac{g}{h}\begin{bmatrix} \dfrac{k}{a}\, dX_1 \cdots dX_n \otimes \alpha \\ f_1, \ldots, f_\ell \end{bmatrix}\right) \\
&= \frac{\overline{g}}{\overline{h}}\begin{bmatrix} \dfrac{\overline{k}}{\overline{a}}\, d\overline{X}_1 \cdots d\overline{X}_n \otimes \alpha \\ \overline{f}_0, \overline{f}_1, \ldots, \overline{f}_\ell \end{bmatrix} \\
&= \delta_{\overline{\mathbf{X}};\mathscr{P}_{\overline{\mathbf{x}}},\mathscr{Q}_{\overline{\mathbf{x}}}}\begin{bmatrix} \dfrac{\overline{k}}{\overline{f}_0}\, d\overline{X}_1 \cdots d\overline{X}_n \otimes \alpha \\ \overline{f}_1, \ldots, \overline{f}_\ell \end{bmatrix}
\end{aligned}$$

$$= \delta_{\overline{\mathbf{X}};\mathscr{P}_{\overline{\mathbf{X}}},\mathscr{Q}_{\overline{\mathbf{X}}}} \circ \zeta_{\mathscr{P}_{\mathbf{X}}} \begin{bmatrix} \dfrac{k}{f_0} dX_1 \cdots dX_n \otimes \alpha \\ f_1, \ldots, f_\ell \end{bmatrix}.$$

∎

The map $\delta_{\mathscr{P},\mathscr{Q}}$ induces an $\mathscr{O}_{\mathscr{Y}}$-linear map $J(\mathscr{P}) \to J(\mathscr{Q})$, denoted still by $\delta_{\mathscr{P},\mathscr{Q}}$ by abusing the notation.

THEOREM 7.2. *The chain of $\mathscr{O}_{\mathscr{Y}}$-modules*

$$\cdots \longrightarrow \bigoplus_{\Delta_{\mathscr{Y}}(\mathscr{P})=n} J(\mathscr{P}) \xrightarrow{\oplus \delta_{\mathscr{P},\mathscr{Q}}} \bigoplus_{\Delta_{\mathscr{Y}}(\mathscr{Q})=n+1} J(\mathscr{Q}) \longrightarrow \cdots,$$

denoted by $\mathscr{M}^{\bullet}_{\mathscr{Y}}$, is a residual complex.

Proof. From our construction, $\mathscr{M}^n_{\mathscr{Y}}$ is a quasi-coherent injective $\mathscr{O}_{\mathscr{Y}}$-module for each n and there is an isomorphism

$$\bigoplus_{n\in\mathbb{Z}} \mathscr{M}^n \simeq \bigoplus_{\mathscr{P}\in\mathscr{Y}} J(\mathscr{P}).$$

The dimensions of fibers of ϕ are bounded, hence $\mathscr{M}^{\bullet}_{\mathscr{Y}}$ is bounded below. To show that $\mathscr{M}^{\bullet}_{\mathscr{Y}}$ is a complex having coherent cohomology, it suffices to show that the restriction of $\mathscr{M}^{\bullet}_{\mathscr{Y}}$ on any affine open subsets of $\mathscr{Y}$ is a complex having coherent cohomology. This was proved in the previous section. ∎

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