A property of Ditzian–Totik second order moduli

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A B S T R A C T

For a function \( f \in C^2[0, 1] \), we prove that
\[
\lim_{t \to 0^+} \frac{\omega_2^2(f, t)}{t^2} = \| \varphi^2 f'' \|,
\]
where \( \omega_2^2(f, t) \) denotes a Ditzian–Totik-type modulus of order 2. We apply this result to obtain an asymptotic property for positive linear operators related to Voronovskaja type formulae.

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1. Introduction

A function \( \varphi : [0, 1] \to \mathbb{R} \) is said to be admissible if the following conditions hold: (i) If \( x \in (0, 1) \), then \( \varphi(x) > 0 \). (ii) \( \varphi \) is a continuous function.

For an admissible weight function \( \varphi \), a function \( f \in C[0, 1] \) and \( t \in (0, 1/2] \), the first and the second order modulus of smoothness are defined respectively by (see [1])
\[
\omega_1(f, t) = \sup_{h \in (0, t)} \sup_{x \in I(\varphi, h)} \left| f(x + h\varphi(x)) - f(x - h\varphi(x)) \right|,
\]
where \( I(\varphi, h) = \{ y \in (0, 1) : y \pm h\varphi(y) \in [0, 1] \} \) and
\[
\omega_2^2(f, t) = \sup_{h \in (0, t)} \sup_{x \in J(\varphi, h)} \left| f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x)) \right|
\]
where \( J(\varphi, h) = \{ y \in (0, 1) : y \pm h\varphi(y) \in [0, 1] \} \). The usual modulus of continuity is obtained by setting \( \varphi(x) \equiv 1 \). For the first usual modulus we simply write \( \omega(f, t) \).

In [2, 1] other conditions are assumed for admissible functions, but here we will not need them.

For a function \( f \in C[0, 1] \), \( \| f \| \) denotes the uniform norm. The family of all polynomials of degree not greater than 1 is denoted by \( \Pi_1 \).

Here we will prove that for any function \( f \in C^2[0, 1] \) one has
\[
\lim_{t \to 0^+} \frac{\omega_2^2(f, t)}{t^2} = \| \varphi^2 f'' \|.
\]
This equation will be used to derive some asymptotics related with positive linear operators. In particular, for a sequence of positive linear operators \( \{L_n\} \) satisfying the Voronovskaja type formula
\[
\lim_{n \to \infty} \frac{1}{\alpha_n} \|L_n(f) - f\| = \lambda \|\varphi^2f''\|.
\] (4)
for \( f \in C^2[0, 1] \setminus P_1 \), where \( \alpha_n \) is a sequence of positive real numbers converging to 0, one has
\[
\lim_{n \to \infty} \frac{\|f - L_n(f)\|}{\omega_n^2(f, \sqrt{\alpha_n})} = \lambda.
\] (5)

The preceding formula is useful in the (open) problem of finding the best possible upper constant \( C \) such that
\[
\|f - L_n(f)\| \leq C \omega_n^2(f, \sqrt{\alpha_n}),
\]
for all number \( n \). The conclusion is obvious: \( C \) cannot be smaller than \( \lambda \).

2. The results and examples

**Theorem 1.** Let \( \varphi : [0, 1] \to \mathbb{R} \) be an admissible function. For any function \( f \in C^2[0, 1] \), equality (3) holds.

**Proof.** We can assume that \( \|\varphi^2f''\| \neq 0 \). In fact, since \( \varphi(x) > 0 \) for \( x \in (0, 1) \), if \( \|\varphi^2f''\| = 0 \), then \( f \) is a polynomial of degree no greater than one and \( \omega_n^2(f, t) = 0 \), for \( t > 0 \).

It is sufficient to verify (3) for any sequence \( \{\gamma_n\} \) of positive numbers such that \( \gamma_n \to 0 \). Fix a sequence \( \{\gamma_n\} \) with these properties.

Let \( \varepsilon > 0 \) be arbitrary and chose \( y \in (0, 1) \) such that
\[
\|\varphi^2f''\| < |\varphi^2(y)f''(y)| + \varepsilon.
\]
First notice that, for all \( h \) such that \( y \pm h \in [0, 1] \),
\[
f(y + h) - 2f(y) + f(y - h) = h^2 \int_0^1 \int_0^1 f''(y + h(u + v - 1)) du dv \quad \text{(6)}
\]
and
\[
|\int_0^1 \int_0^1 (f''(y + h(u + v - 1)) - f''(y)) du dv| \leq \int_0^1 \int_0^1 \omega(f'', |h(u + v - 1)|) du dv \leq \omega(f'', h), \quad \text{(7)}
\]
where \( \omega(f'', h) \) denotes the usual first modulus of continuity. Now, from (6) and (7) we obtain
\[
\lim_{n \to \infty} \frac{1}{\gamma_n^2} \int_0^1 \int_0^1 f''(y + \gamma_n \varphi(y)(u + v - 1)) du dv = \varphi^2(y) \lim_{n \to \infty} \left| \int_0^1 \int_0^1 f''(y + \gamma_n \varphi(y)(u + v - 1)) du dv \right|
\]
\[
= \varphi^2(y) \int_0^1 \int_0^1 (f''(y + \gamma_n \varphi(y)(u + v - 1)) - f''(y)) du dv
\]
\[
= \varphi^2(y) |f''(y)| > \|\varphi^2f''\| - \varepsilon.
\]
Since, \( \gamma_n \to 0 \), there exists \( N \) such that, for \( n > N \)
\[
\gamma_n < \max \left\{ \frac{y}{\varphi(y)}, \frac{1 - y}{\varphi(y)} \right\}.
\]
Thus for \( n > N, y \pm \gamma_n \varphi(y) \in [0, 1] \) and then
\[
- \varepsilon + \|\varphi^2f''\| \leq \lim \inf_{n \to \infty} \frac{1}{\gamma_n^2} \omega_n^2(f, \gamma_n). \quad \text{(8)}
\]
In order to estimate the limit superior of the sequence \( \{\omega_n^2(f, \gamma_n)/\gamma_n^2\} \), for each \( n \in \mathbb{N} \), fix points \( x_n \in (0, 1) \) and \( t_n \in (0, \gamma_n) \) such that
\[
\omega_n^2(f, \gamma_n) = |f(x_n + t_n \varphi(x_n)) - 2f(x_n) + f(x_n - t_n \varphi(x_n))|.
\]
Using again (6) and (7), with this choice one has
\[
\omega_{\varphi}^2(f, \gamma_n) = t_n^2 \varphi^2(x_n) \left| \int_0^1 \int_0^1 f''(x_n + t_n \varphi(x_n)(u + v - 1)) \, du \, dv \right|
\leq t_n^2 \varphi^2(x_n) \left| \int_0^1 \int_0^1 (f''(x_n + t_n \varphi(x_n)(u + v - 1)) - f''(x_n)) \, du \, dv \right|
\leq t_n^2 \varphi^2(x_n) \left( \| \varphi^2 f'' \| + \| \varphi \| \omega \left( f'', \gamma_n \varphi \right) \right).
\]

Therefore
\[
\limsup_{n \to \infty} \frac{1}{\gamma_n^2} \omega_{\varphi}^2(f, \gamma_n) \leq \| \varphi^2 f'' \|.
\]
(9)

Since \( \varepsilon > 0 \) is arbitrary, from (8) and (9) we obtain (3). \( \square \)

**Remark 2.** Note that the existence of the limit (3) is part of the result in Theorem 1.

**Corollary 1.** Let \( L_n : C[0, 1] \to C[0, 1] \) be a sequence of operators. Suppose that there exists an admissible function \( \varphi \) and a sequence of positive numbers \( \{ \alpha_n \} \) such that, for each \( f \in C^2[0, 1] \), the Voronovskaja type formula (4) holds. Then, if \( f \in C^2[0, 1] \setminus \Pi_1 \), one has \( (5) \).

**Proof.** If \( f \in C^2[0, 1] \), then from Theorem 1 we have
\[
\lim_{n \to \infty} \frac{\| L_n(f) - f \|}{\omega_{\varphi}^2(f, \sqrt{\alpha_n})} = \lim_{n \to \infty} \frac{\| L_n(f) - f \|}{\alpha_n} \frac{\alpha_n}{\omega_{\varphi}^2(f, \sqrt{\alpha_n})} = \lambda. \quad \square
\]

**Remark 3.** Under the conditions of Corollary 1 and using (11) with \( r = 1 \), we can also compute the asymptotic in terms of the first order Ditzian–Totik modulus, but in this case the limit depends on the function. In fact, from (11) and (4) we obtain
\[
\lim_{n \to \infty} \frac{\| f - L_n(f) \|}{\omega_{\varphi}^2(f, \alpha_n)} = \frac{\lambda \| \varphi^2 f'' \|}{\| f'' \|}.
\]
(12)

Also, using (11) with \( r = 1 \) and \( \varphi \equiv 1 \), we can also write (12) in terms of the usual first modulus of continuity,
\[
\lim_{n \to \infty} \frac{\| f - L_n(f) \|}{\omega(f, \alpha_n)} = \frac{\lambda \| \varphi^2 f'' \|}{\| f'' \|}.
\]
(13)

Finally, using again (11) with \( r = 1 \) and \( f' \) instead of \( f \), one has
\[
\lim_{n \to \infty} \frac{\| f - L_n(f) \|}{\omega(f', \alpha_n)} = \frac{\lambda \| \varphi^2 f'' \|}{\| f'' \|} \leq \lambda \| \varphi^2 \|.
\]
(14)
Several families of positive linear operators satisfying (4) are known. Here we present only three examples.

**Example 1.** Recall that for a function \( f \in C[0, 1] \) and a positive integer \( n \), the Bernstein polynomial \( B_n(f) \) is defined by

\[
B_n(f, x) = \sum_{k=0}^{n} p_{n,k}(x) f \left( \frac{k}{n} \right), \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.
\]

In this case the Voronovskaja type formula (4) is well known. For instance (see [3]),

\[
\left\| n(B_n(f) - f) - \frac{1}{2} \varphi^2 f'' \right\| \leq \omega^2 \left( f'', \frac{1}{3\sqrt{n}} \right),
\]

where \( \varphi(x) = \sqrt{x(1-x)} \) and \( \omega^2(g, t) \) is the usual second order modulus of continuity. That is \( \omega^2(g, t) = \omega^2_\varphi(g, t) \), with \( \psi \equiv 1 \). Then,

\[
\lim_{n \to \infty} n \| B_n(f) - f \| = \frac{1}{2} \| \varphi^2 f'' \|,
\]

and then, from Corollary 1 with \( \alpha_n = 1/n \) and \( \lambda = 1/2 \), we conclude that

\[
\lim_{n \to \infty} \frac{\| B_n(f) - f \|}{\omega^2_\varphi(f, 1/\sqrt{n})} = \frac{1}{2}.
\]

Moreover, from (14), we get

\[
\lim_{n \to \infty} \frac{\| B_n(f) - f \|}{\omega^2_\varphi(f, 1/\sqrt{n})} \leq \frac{1}{8}.
\]

This last inequality improves Corollary 5 of [4]. Note that the constant 1/8 is the best possible. In fact, the above inequality becomes an equality for all functions \( f \in C^2[0, 1] \) such that \( \|f''\| = |f''(1/2)| \).

**Example 2.** For \( n > 2 \), the genuine Bernstein–Durrmeyer operators are defined by

\[
U_n(f, x) = f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + \sum_{k=1}^{n-1} p_{n,k}(x) \int_0^1 p_{n-k,1}(t)f(t)\,dt.
\]

For a quantitative version of the Voronovskaja formula for these operators see [3]. In particular, from Theorem 5.1 in [3], we have that if \( f \in C^2[0, 1] \), then

\[
\lim_{n \to \infty} (n+1)\|U_n(f) - f\| = \|\varphi^2 f''\|,
\]

where \( \varphi(x) = \sqrt{x(1-x)} \). So, applying Corollary 1 with \( \alpha_n = 1/(n+1) \) and \( \lambda = 1 \), we have

\[
\lim_{n \to \infty} \frac{\| U_n(f) - f \|}{\omega^2_\varphi(f, 1/\sqrt{n+1})} = 1.
\]

**Example 3.** For an integer \( r \) and \( n > 2r \), Stancu [5] investigated a family of positive linear operators \( L_n, r : C[0, 1] \to C[0, 1] \) defined by

\[
L_n, r(f, x) = \sum_{k=0}^{n-r} p_{n-r,k}(x) \left[ (1-x)f \left( \frac{k}{n} \right) + xf \left( \frac{k + r}{n} \right) \right].
\]

A direct computation shows that, for all \( x \in (0, 1) \),

\[
\frac{L_n((t-x)^4, x)}{L_n((t-x)^2, x)} \leq \beta_n,
\]

where \( \beta_n = O(1/n) \). Then, applying Theorem 3.2 in [3], it can be proved that

\[
\lim_{n \to \infty} \left\| L_n f - f - \frac{1}{2} \varphi^2 f'' \right\| = 0,
\]

where \( \varphi(x) = \sqrt{x(1-x)} \). So, from Corollary 1, with \( \varphi(x) = \sqrt{x(1-x)} \), \( \alpha_n = 1/n \) and \( \lambda = 1/2 \), we conclude that

\[
\lim_{n \to \infty} \frac{\| L_n(f) - f \|}{\omega^2_\varphi(f, 1/\sqrt{n})} = 1/2.
\]
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