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Finding direct partition bijections by two-directional rewriting techniques

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Abstract

One basic activity in combinatorics is to establish combinatorial identities by so-called 'bijective proofs,' which consists in constructing explicit bijections between two types of the combinatorial objects under consideration.

We show how such bijective proofs can be established in a systematic way from the 'lattice properties' of partition ideals, and how the desired bijections are computed by means of multiset rewriting, for a variety of combinatorial problems involving partitions. In particular, we fully characterizes all equinumerous partition ideals with 'disjointly supported' complements. This geometrical characterization is proved to automatically provide the desired bijection between partition ideals but in terms of the minimal elements of the order filters, their complements. As a corollary, a new transparent proof, the 'bijective' one, is given for all equinumerous classes of the partition ideals of order 1 from the classical book "The Theory of Partitions" by G.Andrews.

Establishing the required bijections involves two-directional reductions technique novel in the sense that forward and backward application of rewrite rules heads, respectively, for two different normal forms (representing the two combinatorial types).

It is well-known that non-overlapping multiset rules are confluent. As for termination, it generally fails even for multiset rewriting systems that satisfy certain natural invariant balance conditions. The main technical development of the paper (which is important for establishing that the mapping yielding the combinatorial bijection is functional) is that the restricted two-directional strong normalization holds for the multiset rewriting systems in question.

Keywords: Integer partitions; Partition identities; Multiset rewriting; Confluence; Termination; Strong normalization

1. Motivating examples and summary

The aim of the paper [1] is to demonstrate the possibility of using rewriting techniques (*two-directional* in the sense that forward and backward applications of rewrite rules head for two different normal forms) for the purpose of establishing *explicit bijections* between combinatorial objects of two different types (represented by the normal forms).

The starting point of one of the most intrigue combinatorics—the theory of integer partitions [1,2,13–15], is Euler's Partition Theorem:

Whatever positive integer n we take, the number of partitions of n into odd parts equals the number of partitions of n into distinct parts.

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[1] A preliminary version of this paper appeared in [9].

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A *partition* of n is a *multiset* M consisting of positive integers $m_1, m_2, \dots, m_k$ whose sum is n. Each m_i is called a *part* of the partition.

For example, $\{3,3,1,1,1,1\}$ is a partition of 10 with six odd parts, and $\{6,4\}$ is a partition of 10 into distinct parts.

One approach to the proof of the combinatorial theorems like Euler's partition theorem is to count *separately* the number of partitions of n into odd parts and show that it is the same as the number of partitions of the n into distinct parts (by means of generating functions, the sieve method, etc., see [2,15]).

Another approach to the problem is to find an explicit bijection h that associates with every partition into odd parts a partition into distinct parts, and vice versa (see, for instance, Euler's bijective proof [2,15]). Garsia, Milne, Remmel, and Gordon have developed a unified method for constructing bijections for a large class of partition identities, based on a sophisticated 'ping-pong-ing' back and forth between two specific sets [5,7,12,14]. In all these cases, the explicitness of the bijections produced by their complicated machinery remains debatable.

Later, O'Hara [11] came up with a 'straightforward' algorithm, which works as follows.

We are given two lists of pairwise disjoint multisets of positive integers

$$\mathscr{A} = A_1, A_2, \dots, A_i, \dots$$

and

$$\mathscr{B} = B_1, B_2, \dots, B_i, \dots,$$

such that $\sum_{a\in A_i} a = \sum_{b\in B_i} b$ for all i.

Now given the partition M which contains none of the B_i's, repeat the following until no A_i is contained in M: "Replace some A_i in M by its matched B_i."

"The key to this algorithm is that the mapping that it produces is independent of the order in which the A_i's are chosen —but this requires a good deal of effort to prove" (cited from Wilf's [15]).

Example 1.1 (Wilf [15]). Partitions into odd parts are multisets that do not contain any of the even parts collected in

$$\mathscr{B}_{1.1} = \{2\}, \{4\}, \{6\}, \{8\}, \dots .$$

Partitions into distinct parts are multisets that do not have any of the following list of 'repetition diseases':

$$\mathscr{A}_{1.1} = \{1,1\}, \{2,2\}, \{3,3\}, \{4,4\}, \dots .$$

Each of the following replacement rules is intended to cure an $\mathscr{A}$-disease (but contaminates with a $\mathscr{B}$-disease):

$$\gamma_1 : \{1,1\} \to \{2\}, \quad \gamma_2 : \{2,2\} \to \{4\}, \quad \gamma_3 : \{3,3\} \to \{6\}, \quad \gamma_4 : \{4,4\} \to \{8\}, \dots . \tag{1}$$

It should be pointed out that, from the rewriting point of view, these rules

$$\{i,i\} \to \{2i\},$$

as well as the reversed rules

$$\{2i\} \to \{i,i\},$$

are non-overlapping multiset rules, and, hence, both system (1) and system of the reversed rules, say $(1)^{-1}$, are *confluent*.

Since every rule from (1) contracts the number of parts in a given partition of n, each of the reduction sequences performed by (1), as well as the reduction sequences performed by $(1)^{-1}$, must *terminate* at most in n steps.

Thus, we obviously get a *bijection* between "odd"-normal forms s and "distinct"—normal forms t: if the (unique!) (1)-normal form of s is t then the (unique!) $(1)^{-1}$-normal form of t must be s.

For example,

$$\{3,3,1,1,1,1\} \xrightarrow{\gamma_3} \{6,1,1,1,1\} \xrightarrow{\gamma_1} \{6,2,1,1\} \xrightarrow{\gamma_1} \{6,2,2\} \xrightarrow{\gamma_2} \{6,4\}.$$

But this observation:

> "If both rewriting systems Γ and Γ^{-1} are confluent and terminating, then you get a total bijection between Γ-normal forms and Γ^{-1}-normal forms",

does not apply to more general cases in which termination is more subtle: *it does not hold in general* but for normal forms.

We illustrate this with a 'Church–Rosser' translation (2) from the (unique) representation of n in base 4 into its (unique) binary form:

Example 1.2. The number of partitions of n, in which each part of the form 2^k, if any, is a power of 4, and furthermore, each power of 4 may occur at most thrice, is equal to the number of partitions of the n, in which each power of 2 may occur at most once, and all other positive integers may appear without restriction.

The former partitions, say $\mathscr{B}$-normal forms, are multisets that do not contain any of the list $\mathscr{B}_{1.3}$:

$$\{2\},\{1,1,1,1\},\{8\},\{4,4,4,4\},\{32\},\{16,16,16,16\},\dots\ .$$

The latter partitions, say $\mathscr{A}$-normal forms, are multisets that do not have any of the list $\mathscr{A}_{1.2}$:

$$\{1,1\},\{2,2\},\{4,4\},\{8,8\},\{16,16\},\{32,32\},\dots\ .$$

A bijection between $\mathscr{B}$-normal forms and $\mathscr{A}$-normal forms is expected to be provided by the following 'well-balanced' rules:

$$\gamma_0:\{1,1\}\to\{2\},\quad \gamma_1:\{2,2\}\to\{1,1,1,1\},$$
$$\gamma_2:\{4,4\}\to\{8\},\quad \gamma_3:\{8,8\}\to\{4,4,4,4\},$$
$$\gamma_4:\{16,16\}\to\{32\},\dots\ . \tag{2}$$

Being non-overlapping, both systems (2) and $(2)^{-1}$ are obviously confluent.

The 'balance conditions'—that $\sum_{a\in A_i} a=\sum_{b\in B_i} b$ for all i, provide that every reduction sequence that started from a partition of n may contain only partitions of the same n. Since the number of partitions of n is finite, the *termination problem* seems to be *trivial*, as well.

But each of systems (2) and $(2)^{-1}$ is *not* even *weakly normalizing*.

For example, the partition $\{2,2,2\}$ always generates an infinite 'loop':

$$\{2,2,2\}\overset{\gamma_1}{\to}\underline{\{2,1,1,1,1\}}\overset{\gamma_0}{\to}\{2,2,1,1\}\overset{\gamma_1}{\to}\{1,1,1,1,1,1\}\overset{\gamma_0}{\to}\underline{\{2,1,1,1,1\}}\overset{\gamma_0}{\to}\{2,2,1,1\}\overset{\cdots}{\to}\cdots.$$

In spite of this negative fact, we prove *general theorems* (Theorems 2.1 and 2.2), which guarantee that both systems (2) and $(2)^{-1}$ are *strongly normalizing* in the following restricted but desired sense:

> "Any partition of n, which is the *correct* representation of the n in base 4, e.g., $\{4,4,4,1,1,1\}$, *always* converges to the *correct* binary form $\{8,4,2,1\}$ in this example, and vice versa."

Generally, the classes $\mathscr{C}$ of partitions considered in the literature have the 'local' property that if M is a partition in $\mathscr{C}$ and one part is removed from M to form a new partition M', then M' is also in $\mathscr{C}$ [2]. Such a class $\mathscr{C}$ is called a *partition ideal* [2], or an *order ideal*, of the lattice $\mathscr{P}$ of finite multisets of positive integers, ordered by $\subseteq$.

Andrews [2] has introduced a hierarchy of *partition ideals of order* k. The partitions mentioned above in Examples 1.1 and 1.2 form partition ideals of order 1 (the partition ideals of order 1 are just the *ideals* of $\mathscr{P}$).

In Corollary 3.2 we give a *new proof*, the 'bijective' one, for the Andrews' theorem [2, Theorem 8.4] that fully characterizes the equinumerous partition ideals of order 1. The bijective proof found here allows us to get a broader understanding of the essence of the Andrews' criterion.

We cover much more general class of partition ideals with Theorem 3.1, which *fully characterizes* all equinumerous partition ideals in terms of their 'disjointly supported' complements. The irony of Theorem 3.1 is that matching the minimal elements of the order filter, the complement to one of given ideals, with the minimal elements of another filter, the complement to another ideal, directly guarantees the *most natural bijection* but for the original partition ideals (not the filters themselves).

Example 1.3. This extreme partition identity is taken from Remmel [12]:

The number of partitions of n such that their parts congruent to 1 or 4 mod 5 do not differ by 2 is equal to the number of partitions of the n such that their parts congruent to 1 or 4 mod 5 do not differ by 8.

The former partitions, say $\mathscr{B}$-normal forms, form a partition ideal the minimal elements of its complementary filter, say $B_1,B_2,B_3,\dots$, can be listed as follows:

$$\mathscr{B}_{1.3}:=\{4,6\},\{9,11\},\{14,16\},\dots\ .$$

The latter partitions, say $\mathscr{A}$-normal forms, form a partition ideal the minimal elements of its complementary filter, say $A_1, A_2, A_3, \ldots$, can be listed as

$$\mathscr{A}_{1.3} := \{1,9\}, \{6,14\}, \{11,19\}, \ldots \;.$$

The 'bijective proof' is expected to be provided by the 'well-balanced' matching rules:

$$\gamma_1 : \{1,9\} \to \{4,6\}, \quad \gamma_2 : \{6,14\} \to \{9,11\}, \quad \gamma_3 : \{11,19\} \to \{14,16\}, \ldots \;. \tag{3}$$

Being non-overlapping, both systems (3) and $(3)^{-1}$ are obviously *confluent*.

As for *termination*, it is a good exercise to prove directly (not referring to Theorem 3.1) that both (3) and $(3)^{-1}$ are strongly normalizing, in spite of the 'chaotic' reduction sequences like the following one (wherein one and the same $\mathscr{A}$-disease $\{6,14\}$ appears "persistently" three times!):

$$\{1,1,\underline{6,14},19,19\} \xrightarrow{\gamma_2} \{1,1,9,11,19,19\}$$

$$\xrightarrow{\gamma_1} \{1,4,6,11,19,19\} \xrightarrow{\gamma_3} \{1,4,\underline{6,14},16,19\}$$

$$\xrightarrow{\gamma_2} \{1,4,9,11,16,19\} \xrightarrow{\gamma_1} \{4,4,6,11,16,19\}$$

$$\xrightarrow{\gamma_3} \{4,4,\underline{6,14},16,16\} \xrightarrow{\gamma_2} \{4,4,9,11,16,16\}.$$

In terms of [2], we are dealing here with two partition ideals of order 3 and 9, resp., so they are not within reach of the Andrews' characterization [2, Theorem 8.4] of partition ideals of order 1.

We introduce a two-directional *rewriting* scheme in the following way:
Let Γ be a set of reduction rules

$$\gamma_1 : A_1 \to B_1, \quad \gamma_2 : A_2 \to B_2, \ldots, \quad \gamma_i : A_i \to B_i, \ldots \;.$$

The reversed rules $\gamma_i^{-1} : B_i \to A_i$ form Γ^{-1}

$$\gamma_1^{-1} : B_1 \to A_1, \quad \gamma_2^{-1} : B_2 \to A_2, \ldots, \quad \gamma_i^{-1} : B_i \to A_i, \ldots \;.$$

The *$\mathscr{A}$-normal forms* are the Γ-irreducible forms, i.e. the forms that do not contain any of the following list:

$$\mathscr{A} = A_1, A_2, \ldots, A_i, \ldots$$

and the *$\mathscr{B}$-normal forms* are the Γ^{-1}-irreducible forms, i.e. the forms that do not contain any of the following list:

$$\mathscr{B} = B_1, B_2, \ldots, B_i, \ldots \;.$$

An intended bijection h between $\mathscr{B}$-normal forms and $\mathscr{A}$-normal forms is defined as follows:

$$h(M) := \widetilde{M}, \text{ if } \widetilde{M} \text{ is an } \mathscr{A}\text{-normal form, and}$$
$$M \text{ is } \Gamma\text{-reducible to } \widetilde{M}. \tag{4}$$

Definition 1.1. We will say that Γ is *$\mathscr{B}$-terminating* if every sequence of Γ-reductions must terminate in a finite number of steps, whenever it started from a $\mathscr{B}$-normal form.

Proposition 1.1. *Let both Γ and Γ^{-1} be confluent, and Γ be $\mathscr{B}$-terminating, and Γ^{-1} be $\mathscr{A}$-terminating. Then the above h is a well-defined bijection between $\mathscr{B}$-normal forms and $\mathscr{A}$-normal forms.*

Comment 1.1. As a matter of fact, we need a strong "stratified" version of the conclusion of Proposition 1.1 to supply a bijection between two sets of *partitions* of a fixed n, and to show thereby that the two sets of partitions of the n are equinumerous:

For any fixed n, the above h should be a bijection between $\mathscr{B}$-normal partitions of the n and $\mathscr{A}$-normal partitions of the same n.

The most natural way to guarantee this property is to invoke the following ‘balance conditions’—that for all i,

$$\sum_{a\in A_i} a = \sum_{b\in B_i} b.$$

In this ‘balanced case’ only partitions of one and the same n may appear within any sequence of Γ-reductions.

Section 2 contains the main technical development of the paper.

It is well-known that the non-overlapping multiset rewrite systems Γ are confluent.

We discover here a *new phenomenon* that just the same non-overlapping conditions provide, in addition, the ‘*restricted*’ *loop-freeness*, namely, *there is no repetitions in the intermediate multisets within every sequence of* Γ*-reductions, whenever it started from a* $\mathscr{B}$*-normal form* (Theorem 2.1).

Now, by combining the general non-overlapping conditions with the natural ‘balance conditions’ caused by the nature of a particular combinatorics problem (see Comments 1.1 and 1.2), we obtain the desired *strong* $\mathscr{B}$*-normalization* (Theorems 2.2 and 2.4), which is important for establishing that the mapping h yielding the combinatorial bijection is functional (Theorems 2.3 and 2.5).

Our approach is easily generalized to other combinatorial objects.

Example 1.4 (Cf. Euler’s partition theorem). The number of factorizations of n into non-square integer factors greater than 1 is equal to the number of factorizations of the n into distinct integer factors greater than 1.

To find a bijection, we invoke order ideals, their *complementary filters* and the minimal elements of the filters, as well.

The former factorizations form a partition ideal the minimal elements of its complementary filter can be listed as follows:

$$\mathscr{B}_{1.4} = \{4\}, \{9\}, \{16\}, \ldots\ .$$

The latter factorizations form a partition ideal the minimal elements of its complementary filter can be listed as follows:

$$\mathscr{A}_{1.4} = \{2,2\}, \{3,3\}, \{4,4\}, \ldots\ .$$

A bijection between these two kinds of factorizations is provided by the reduction rules:

$$\gamma_1 : \{2,2\} \to \{4\}, \quad \gamma_2 : \{3,3\} \to \{9\}, \quad \gamma_3 : \{4,4\} \to \{16\},$$

$$\gamma_4 : \{5,5\} \to \{25\}, \quad \gamma_5 : \{6,6\} \to \{36\}, \ldots\ .$$

Comment 1.2. In the case of factorizations, we also need a strong “stratified” version of the conclusion of Proposition 1.1 to supply a bijection between two sets of factorizations of a fixed n, and to show thereby that the two sets of factorizations of the n are equinumerous:

For any fixed n, *the above* h *from Proposition* 1.1 *should be a bijection between* $\mathscr{B}$*-normal factorizations of the* n *and* $\mathscr{A}$*-normal factorizations of the same* n.

The most natural way to guarantee this property is to invoke the following ‘product balance conditions’—that for all i,

$$\prod_{a\in A_i} a = \prod_{b\in B_i} b,$$

which provides that only factorizations of one and the same n may appear within any sequence of Γ-reductions.

Example 1.5 (cf. Euler’s partition theorem). The number of rooted forests of n vertices such that the trees are all different equals the number of rooted forests with no even tree [15].

Furthermore, one can find a `polytime` bijection h that associates with every rooted forest with no even tree a forest whose trees are all different, and vice versa.

(According to [15], if we take two copies of the same rooted tree T and join their two roots together by a new edge, with the new root being one of the original roots, then the resulting tree is an *even tree* G_T.)

The desired bijection is provided by system Γ that consists of all rules γ_T of the form $\gamma_T : \{T, T\} \rightarrow \{G_T\}$ where T is a rooted tree.

2. Termination

Given a multiset M of the form $M = X \cup A \cup Y$, a rewriting rule $\gamma : A \rightarrow B$, replaces A with B, resulting in $M' = X \cup B \cup Y$. This fact is abbreviated as

$$M \stackrel{\gamma}{\rightarrow} M', \text{ or } M' \stackrel{\gamma}{\leftarrow} M, \text{ or } M' = \gamma(M).$$

The latter functional notation is correct because of the following fundamental lemma:

Lemma 2.1. *Assume that* $M \stackrel{\gamma}{\rightarrow} M'$ *and* $M \stackrel{\gamma}{\rightarrow} M''$. *Then* $M' = M''$.

Proof. It follows from the fact that the multiset rewriting rules are rules modulo associativity and commutativity. □

Definition 2.1. Let Γ be a set of rules:

$$\gamma_1 : A_1 \rightarrow B_1, \quad \gamma_2 : A_2 \rightarrow B_2, \ldots, \quad \gamma_i : A_i \rightarrow B_i, \ldots$$

and let $\mathscr{B} := B_1, B_2, B_3, \ldots, B_i, \ldots$.

We say that Γ is *$\mathscr{B}$-loop-free* if whatever sequence of Γ-reductions

$$K_0 \stackrel{\alpha_1}{\rightarrow} K_1 \stackrel{\alpha_2}{\rightarrow} K_2 \stackrel{\alpha_3}{\rightarrow} K_3 \stackrel{\alpha_4}{\rightarrow} \cdots$$

that started from a $\mathscr{B}$-normal form K_0 we take, all these $K_0, K_1, K_2, K_3, \ldots$, are different.

Theorem 2.1 (Loop-freeness). *Let* Γ *be a set of multiset rewriting rules of the form*

$$\gamma_1 : A_1 \rightarrow B_1, \quad \gamma_2 : A_2 \rightarrow B_2, \ldots, \quad \gamma_i : A_i \rightarrow B_i, \ldots,$$

such that $B_i \cap B_j = \emptyset$ *for any* $i \neq j$.

Then Γ *is* $\mathscr{B}$*-loop-free.*

Comment 2.1. Thus the non-overlapping conditions—that

for any $i \neq j$, $B_i \cap B_j = \emptyset$,

provide that

(a) the reversed Γ^{-1} is confluent, and, in addition,
(b) Γ is $\mathscr{B}$-loop-free (Theorem 2.1).

The merely confluence of Γ^{-1} cannot guarantee the desired $\mathscr{B}$-loop-freeness.

Example 2.1. Let Γ consist of two rules:

$$\gamma : \{2\} \rightarrow \{1, 1\}, \quad \gamma' : \{1, 1\} \rightarrow \{1, 1\}.$$

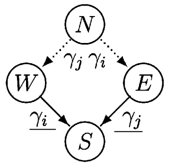
Here $\mathscr{A} = \{2\}, \{1, 1\}$, and $\mathscr{B} = \{1, 1\}, \{1, 1\}$.

Notwithstanding that both Γ and Γ^{-1} are confluent and have the 'balance property', Γ is not $\mathscr{B}$-loop-free: e.g., the $\mathscr{B}$-normal $\{2, 1\}$ yields an infinite stuttering sequence:

$$\{2, 1\} \stackrel{\gamma}{\rightarrow} \{1, 1, 1\} \stackrel{\gamma'}{\rightarrow} \{1, 1, 1\} \stackrel{\gamma'}{\rightarrow} \{1, 1, 1\} \stackrel{\gamma'}{\rightarrow} \cdots.$$

The proof of Theorem 2.1 is based on the following chain of lemmas:

Lemma 2.2 ("Strict" strong confluence). *Let*: $W \xrightarrow{\gamma_i} S \xleftarrow{\gamma_j} E$, *where rewriting rules* γ_i *and* γ_j *are of the form* $\gamma_i : A_i \to B_i$, and $\gamma_j : A_j \to B_j$, *with* B_i *and* B_j *being pairwise disjoint multisets. Then there is an* N *such that* $W \xleftarrow{\gamma_j} N \xrightarrow{\gamma_i} E$.



Proof. The fact that the reversed rules $\gamma_i^{-1} : B_i \to A_i$ and $\gamma_j^{-1} : B_j \to A_j$ are applicable to S means that S, W, and E are to be of the form $S = X \cup B_i \cup B_j$, $W = X \cup A_i \cup B_j$, and $E = X \cup B_i \cup A_j$.

Taking N as $X \cup A_i \cup A_j$, we provide the desired

$$W \xleftarrow{\gamma_j} N \xrightarrow{\gamma_i} E.$$

Lemma 2.3. *Suppose that*

$$M_0 \xrightarrow{\alpha_1} M_1 \xrightarrow{\alpha_2} M_2 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_m} M_m \xrightarrow{\alpha_{m+1}} \cdots \xrightarrow{\alpha_k} M_k \quad \text{and} \quad M_k \xleftarrow{\beta} N_0 \tag{5}$$

and β *differs from each of the* $\alpha_1, \alpha_2, \ldots, \alpha_k$.

Then one can find $N_1, N_2, \ldots, N_k$, *so that*

$$N_0 \xleftarrow{\alpha_k} N_1 \xleftarrow{\alpha_{k-1}} N_2 \xleftarrow{\alpha_{k-2}} \cdots \xleftarrow{\alpha_{k-m}} N_{m+1} \xleftarrow{\alpha_{k-m-1}} \cdots \xleftarrow{\alpha_1} N_k \quad \text{and} \quad N_k \xrightarrow{\beta} M_0. \tag{6}$$

Proof. By repeatedly applying Lemma 2.2, we construct the desired sequence $N_1, N_2, \ldots, N_k$. The case where $k = 3$ is shown in Fig. 1. □

Lemma 2.4. *Let* $N_0 \xrightarrow{\beta} M_0$, *Then every loop*

$$M_0 \overset{\alpha_1\alpha_2\ldots\alpha_k}{- - - - - - \rightarrow} M_0$$

causes, for some permutation π, *a loop of the form*

$$N_0 \overset{\alpha_{\pi(1)}\alpha_{\pi(2)}\cdots\alpha_{\pi(k)}}{- - - - - - \rightarrow} N_0.$$

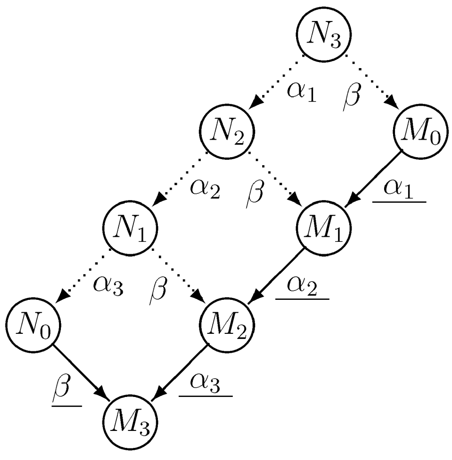


Fig. 1. Lemma 2.3: $N_3 \overset{\alpha_1\alpha_2\alpha_3}{- - - \rightarrow} N_0$

Proof. Assume a computation of the form (5), with $M_k = M_0$.
There are two cases to be considered.

(1) Suppose that rule β differs from each of the rules $\alpha_1, \alpha_2, \ldots, \alpha_k$.
Then Lemma 2.3 provides us with a computation of the form (6).
According to Lemma 2.1,

$$N_k = \beta^{-1}(M_0) = \beta^{-1}(M_k) = N_0,$$

which sums up with the desired 'upper' loop

$$N_0 \overset{\alpha_k}{\leftarrow} N_1 \overset{\alpha_{k-1}}{\leftarrow} N_2 \overset{\alpha_{k-2}}{\leftarrow} \cdots \overset{\alpha_{k-m}}{\leftarrow} N_{m+1} \overset{\alpha_{k-m-1}}{\leftarrow} \cdots \overset{\alpha_1}{\leftarrow} N_0.$$

(2) Suppose that for some m that $\beta = \alpha_m$, but β differs from each of the last rules $\alpha_{m+1}, \alpha_{m+2}, \ldots, \alpha_k$.
Since, in particular,

$$M_m \overset{\alpha_{m+1}}{\rightarrow} M_{m+1} \overset{\alpha_{m+2}}{\rightarrow} M_{m+2} \cdots \overset{\alpha_k}{\rightarrow} M_k \overset{\beta}{\leftarrow} N_0.$$

Lemma 2.3 provides us with $N_1, N_2, \ldots, N_{k-m}$, such that

$$N_0 \overset{\alpha_k}{\leftarrow} N_1 \overset{\alpha_{k-1}}{\leftarrow} N_2 \overset{\alpha_{k-2}}{\leftarrow} \cdots \overset{\alpha_{m+1}}{\leftarrow} N_{k-m} \overset{\beta}{\rightarrow} M_m.$$

According to Lemma 2.1,

$$M_{m-1} = \alpha_m^{-1}(M_m) = \beta^{-1}(M_m) = N_{k-m}.$$

Recalling that

$$M_0 \overset{\alpha_1}{\rightarrow} M_1 \overset{\alpha_2}{\rightarrow} \cdots \overset{\alpha_{m-1}}{\rightarrow} M_{m-1},$$

now we construct the desired loop as follows:

$$N_0 \overset{\beta}{\rightarrow} M_0 \overset{\alpha_1}{\rightarrow} M_1 \overset{\alpha_2}{\rightarrow} \cdots \overset{\alpha_{m-1}}{\rightarrow} M_{m-1} = N_{k-m} \overset{\alpha_{m+1}}{\rightarrow} N_{k-m-1} \overset{\alpha_{m+2}}{\rightarrow} \cdots \overset{\alpha_{k-1}}{\rightarrow} N_1 \overset{\alpha_k}{\rightarrow} N_0.$$

The case where $k = 3$, $m = 2$, is shown in Fig. 2. □

Lemma 2.5. *Let N_0 be Γ-reducible to M_0. Then every loop*

$$M_0 \overset{\alpha_1 \alpha_2 \ldots \alpha_k}{- - - - - - \rightarrow} M_0$$

causes, for some permutation π, a loop of the form

$$N_0 \overset{\alpha_{\pi(1)} \alpha_{\pi(2)} \ldots \alpha_{\pi(k)}}{- - - - - - \rightarrow} N_0.$$

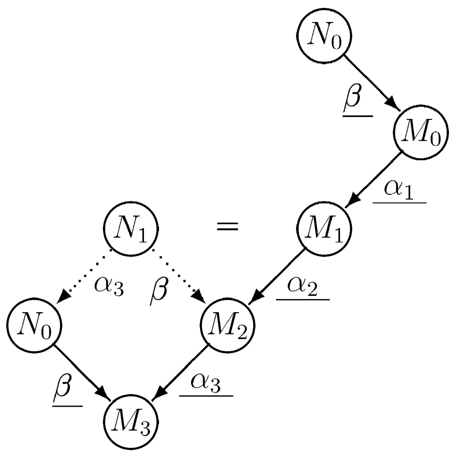


Fig. 2. Lemma 2.4: $N_0 \overset{\alpha_2 \alpha_1 \alpha_3}{- - - \rightarrow} N_0$ where $\beta = \alpha_2$.

Proof. By repeatedly applying Lemma 2.4. □

Proof of Theorem 2.1. Let

$$K_0 \xrightarrow{\beta_1} K_1 \xrightarrow{\beta_2} K_2 \xrightarrow{\beta_3} K_3 \xrightarrow{\beta_4}$$

be a sequence of Γ-reductions, and K_0 be a $\mathscr{B}$-normal form.

According to Lemma 2.5, had any repetition happened within this sequence of the multisets, it would have produced a non-trivial loop of the form

$$K_0 \xrightarrow{\alpha_1\alpha_2\ldots\alpha_k} K_0,$$

communicating thereby some $B_{k'}$ into K_0 by means of rule α_k, which contradicts to the fact that K_0 has no $\mathscr{B}$-diseases. □

Theorem 2.2 (Strong normalization). *Let Γ be a set of multiset rewriting rules of the form*

$$\gamma_1 : A_1 \to B_1, \quad \gamma_2 : A_2 \to B_2, \ldots, \quad \gamma_i : A_i \to B_i, \ldots,$$

such that all B_i's are multisets of positive integers.

If Γ is $\mathscr{B}$-loop-free, and, in addition, the 'balance conditions': $\sum_{a\in A_i} a = \sum_{b\in B_i} b$, hold for all i, then Γ is $\mathscr{B}$-terminating.

Proof. Let $K_0 \xrightarrow{\beta_1} K_1 \xrightarrow{\beta_2} K_2 \xrightarrow{\beta_3} K_3 \xrightarrow{\beta_4} \cdots$ be a sequence of Γ-reductions, and K_0 be a $\mathscr{B}$-normal form.

Multiset K_0 can be conceived of as a partition of n, where

$$n = \sum_{k\in K_0} k.$$

The 'balance property' yields that each of the $K_1, K_2, K_3, \ldots$ is a partition of one and the same n. Since Γ is $\mathscr{B}$-loop-free, all these K_1, $K_2, K_3, \ldots$ must be different. Hence, the length of the sequence cannot exceed $p(n)$, the number of distinct partitions of n. □

Comment 2.2. If we allow 0's, "$\mathscr{B}$-loop-freeness + balance" not necessarily implies $\mathscr{B}$-termination.

E.g., let Γ consist of one rule

$$\gamma : \{0\} \to \{0, 0\}.$$

Here $\mathscr{A} = \{0\}$, and $\mathscr{B} = \{0, 0\}$.

Notwithstanding that Γ is $\mathscr{B}$-loop-free and has the 'balance property', an infinite sequence of reductions happens even if we started with the $\mathscr{B}$-normal form $\{0, 1\}$:

$$\{0,1\} \xrightarrow{\gamma} \{0,0,1\} \xrightarrow{\gamma} \{0,0,0,1\} \xrightarrow{\gamma} \{0,0,0,0,1\} \xrightarrow{\gamma} \cdots$$

Theorem 2.3. *Let Γ be a set of multiset rewriting rules of the form*

$$\gamma_1 : A_1 \to B_1, \quad \gamma_2 : A_2 \to B_2, \ldots, \quad \gamma_i : A_i \to B_i, \ldots,$$

such that for any $i \neq j$, A_i and A_j are pairwise disjoint multisets of positive integers, and B_i and B_j are pairwise disjoint multisets of positive integers, and, in addition, the 'balance conditions': $\sum_{a\in A_i} a = \sum_{b\in B_i} b$, hold for all i.

Define h as follows:

$$h(M) := \widetilde{M}, \text{ if } \widetilde{M} \text{ is an } \mathscr{A}\text{-normal form, and}$$

$$M \text{ is } \Gamma\text{-reducible to } \widetilde{M}.$$

Then for every n, the h is a well-defined bijection between $\mathscr{B}$-normal partitions of the n and $\mathscr{A}$-normal partitions of the same n.

Proof. Both Γ and Γ^{-1} are obviously confluent.

According to Theorems 2.1 and 2.2, both Γ is $\mathscr{B}$-terminating, and Γ^{-1} is $\mathscr{A}$-terminating.

See Proposition 1.1 and Comment 1.1 for the further details. □

Theorem 2.4 (Strong normalization). *Let Γ be a set of multiset rewriting rules of the form*

$$\gamma_1 : A_1 \to B_1, \quad \gamma_2 : A_2 \to B_2, \ldots, \quad \gamma_i : A_i \to B_i, \ldots,$$

such that all B_i's are multisets of integers $\geqslant 2$.

If Γ is $\mathscr{B}$-loop-free, and, in addition, the 'balance conditions': $\prod_{a\in A_i} a = \prod_{b\in B_i} b$, hold for all i, then Γ is $\mathscr{B}$-terminating.

Proof. Let $K_0 \xrightarrow{\beta_1} K_1 \xrightarrow{\beta_2} K_2 \xrightarrow{\beta_3} K_3 \xrightarrow{\beta_4} \cdots$ be a sequence of Γ-reductions, and K_0 be a $\mathscr{B}$-normal form.

Multiset K_0 can be conceived of as a factorization of n, where

$$n = \prod_{k\in K_0}.$$

The 'balance property' yields that each of the $K_1, K_2, K_3, \ldots$ is a factorization of one and the same n. Since Γ is $\mathscr{B}$-loop-free, all these $K_1, K_2, K_3, \ldots$ must be different. Hence, the length of the sequence cannot exceed $\widehat{p}(n)$, the number of factorizations of n into integers $\geqslant 2$. □

Theorem 2.5. *Let Γ be a set of multiset rewriting rules of the form*

$$\gamma_1 : A_1 \to B_1, \quad \gamma_2 : A_2 \to B_2, \ldots, \quad \gamma_i : A_i \to B_i, \ldots,$$

such that for any $i \neq j$, A_i and A_j are pairwise disjoint multisets of integers $\geqslant 2$, and B_i and B_j are pairwise disjoint multisets of integers $\geqslant 2$, and, in addition, the 'balance conditions': $\prod_{a\in A_i} a = \prod_{b\in B_i} b$, hold for all i.

Define h as follows:

$$h(M) := \widetilde{M}, \ \textit{if}\ \widetilde{M} \textit{ is an } \mathscr{A}\textit{-normal form, and}$$

$$M \textit{ is } \Gamma\textit{-reducible to } \widetilde{M}.$$

Then for every n, the h is a well-defined bijection between $\mathscr{B}$-normal factorizations of the n and $\mathscr{A}$-normal factorizations of the same n.

Proof. Both Γ and Γ^{-1} are obviously confluent. According to Theorems 2.1 and 2.4, both Γ is $\mathscr{B}$-terminating, and Γ^{-1} is $\mathscr{A}$-terminating.

See Proposition 1.1 and Comment 1.2 for the further details. □

3. Partition ideals

Definition 3.1 (Andrews [2]). Two classes of partitions $\mathscr{C}_1$ and $\mathscr{C}_2$ are *equivalent*:

$$\mathscr{C}_1 \sim \mathscr{C}_2,$$

if $p(\mathscr{C}_1, n) = p(\mathscr{C}_2, n)$ for all n.

Here $p(\mathscr{C}, n)$ denotes the number of partitions of the n that belong to a given class $\mathscr{C}$.

Generally, the classes $\mathscr{C}$ of partitions considered in the literature have the 'local' property that if M is a partition in $\mathscr{C}$ and one part is removed from M to form a new partition M', then M' is also in $\mathscr{C}$ [2].

Let $\mathscr{P}$ be the lattice $\mathscr{P}$ of finite multisets of positive integers, ordered by $\subseteq$.

In addition, for any finite multiset M, we let a sort of the "norm" by the following:

$$\|M\| := \sum_{m\in M} m.$$

(Another 'working' version is: $\|M\| := \prod_{m\in M} m$, for factorizations.)

Definition 3.2. A class $\mathscr{C} \subseteq \mathscr{P}$ is an *order ideal*, or a *partition ideal* in terms of Andrews [2], if for any M and M' from $\mathscr{P}$ such that $M' \subseteq M \in \mathscr{C}$, necessarily $M' \in \mathscr{C}$.

Dually, a class $\mathscr{F} \subseteq \mathscr{P}$ is an *order filter* if $M' \in \mathscr{F}$, whenever $M \in \mathscr{F}$ and $M \subseteq M'$.

It is readily seen that $\mathscr{C}$ is a partition ideal if and only if its complement $\overline{\mathscr{C}}$ is an order filter.

As for the fundamental problem stated in [2]:

Fully characterize the equivalence classes of partition ideals.

We give a full characterization for a wide class of partition ideals having certain similarities in their lattice structure.

Definition 3.3. M is *minimal* in an order filter $\mathscr{F} \subseteq \mathscr{P}$, if $M \in \mathscr{F}$ and no $M' \in \mathscr{F}$ for a proper submultiset M' of M. The *support* of $\mathscr{F}$, i.e. the set of all its minimal elements, is denoted by $\mu_{\mathscr{F}}$. We say that the support $\mu_{\mathscr{F}}$ is *disjoint*, if $M \cap M' = \emptyset$ for any distinct M and M' in $\mu_{\mathscr{F}}$.

Example 3.1. For example, partitions into distinct parts form a partition ideal, say $\mathscr{D}$. The *support* of its complement, that is $\mu_{\overline{\mathscr{D}}}$, is disjoint

$$\mu_{\overline{\mathscr{D}}} = \{\{1,1\},\{2,2\},\{3,3\},\{4,4\},\ldots\}.$$

Theorem 3.1. *Let $\mathscr{C}$ and $\mathscr{C}'$ be partition ideals such that the support of the order filter $\overline{\mathscr{C}}$,*

$$\mu_{\overline{\mathscr{C}}} = \{A_1, A_2, A_3, \ldots\},$$

is disjoint, and the support of the order filter $\overline{\mathscr{C}'}$,

$$\mu_{\overline{\mathscr{C}'}} = \{B_1, B_2, B_3, \ldots\},$$

is disjoint.

Then $\mathscr{C} \sim \mathscr{C}'$ if and only if the two sequences of integers $\|A_1\|, \|A_2\|, \|A_3\|, \ldots$ and $\|B_1\|, \|B_2\|, \|B_3\|, \ldots$ are merely reorderings of each other.

In addition to that, assuming that the two lists

$$\mathscr{A} := A_1, A_2, \ldots, A_i, \ldots$$

and

$$\mathscr{B} := B_1, B_2, \ldots, B_i, \ldots,$$

are already sorted so that $\|A_i\| = \|B_i\|$ for all i, the desired bijection between $\mathscr{C}'$ and $\mathscr{C}$ is provided by the following set of reduction rules:

$$\gamma_1 : A_1 \to B_1, \quad \gamma_2 : A_2 \to B_2, \ldots, \quad \gamma_i : A_i \to B_i, \ldots \ . \tag{7}$$

Proof. (a) Let the two sequences of $\|A_1\|, \|A_2\|, \|A_3\|, \ldots$ and $\|B_1\|, \|B_2\|, \|B_3\|, \ldots$ be merely reorderings of each other.

We assume that the two lists $\mathscr{A} := A_1, A_2, \ldots$, and $\mathscr{B} := B_1, B_2, \ldots$, have been already sorted so that for all i, $\|A_i\| = \|B_i\|$.

Letting Γ be a set of reduction rules:

$$\gamma_1 : A_1 \to B_1, \quad \gamma_2 : A_2 \to B_2, \ldots, \quad \gamma_i : A_i \to B_i, \ldots,$$

by Theorem 2.3 we construct a function h such that, for every n, the h is a bijection between $\mathscr{B}$-normal partitions of the n and $\mathscr{A}$-normal partitions of the same n.

Taking into account that:

- $M \in \mathscr{C}' \Leftrightarrow M \notin \overline{\mathscr{C}'} \Leftrightarrow B_i \nsubseteq M$ for every $i \Leftrightarrow M$ is a $\mathscr{B}$-normal form, and
- $M \in \mathscr{C} \Leftrightarrow M \notin \overline{\mathscr{C}} \Leftrightarrow A_i \nsubseteq M$ for every $i \Leftrightarrow M$ is a $\mathscr{A}$-normal form,

we can conclude that $\mathscr{C}' \sim \mathscr{C}$.

(b) Suppose that $\mathscr{C} \sim \mathscr{C}'$, and thereby $\overline{\mathscr{C}} \sim \overline{\mathscr{C}'}$.

Let the two lists

$$\mathscr{A} := A_1, A_2, \ldots, A_i, \ldots$$

and

$$\mathscr{B} := B_1, B_2, \ldots, B_i, \ldots$$

be sorted in ascending order of the integers $\|A_i\|$'s and $\|B_i\|$'s, and let a_n be the number of A_i's such that $\|A_i\| = n$, and b_n be the number of B_i's such that $\|B_i\| = n$.

Assume that k is the least positive integer such that $a_k \neq b_k$.

Take i_0 to be the largest index such that $\|A_{i_0}\| = \|B_{i_0}\| < k$, and define $\mathscr{F}_k$ to be the order filter generated by

$$\mathscr{A}_k := A_1, A_2, \ldots, A_{i_0}$$

and $\mathscr{F}'_k$ to be the order filter generated by

$$\mathscr{B}_k := B_1, B_2, \ldots, B_{i_0}.$$

Lemma 3.1.

$$\{M \in \overline{\mathscr{C}} \mid \|M\| = k\} = \{M \in \mathscr{F}_k \mid \|M\| = k\} \cup \{A_i \mid \|A_i\| = k\}$$

and the above union is disjoint.

Proof. (i) If M belongs to the union then $A_i \subseteq M$ for some i, and, hence, $M \in \overline{\mathscr{C}}$.

(ii) Suppose that $\|M\| = k$, and $M \in \overline{\mathscr{C}}$. Then $A_i \subseteq M$ for some i.

If $i \leqslant i_0$ then $M \in \mathscr{F}_k$.

For $i > i_0$, we have $\|A_i\| \geqslant k$, which together with $A_i \subseteq M$ and $\|M\| = k$ yields that $M = A_i$.

(iii) *The above union is disjoint*: Suppose that M is some A_i with $\|A_i\| = k$, and the same M belongs to $\mathscr{F}_k$. Then there is an A_j such that $\|A_j\| < k$, and $A_j \subseteq M = A_i$, which contradicts to the minimality of A_i. □

Lemma 3.2. *For the* k, *the least positive integer such that* $a_k \neq b_k$,

$$p(\overline{\mathscr{C}'}, k) - p(\overline{\mathscr{C}}, k) = b_k - a_k.$$

Proof. Lemma 3.1 shows that $p(\overline{\mathscr{C}}, k) = p(\mathscr{F}_k, k) + a_k$, and, therefore,

$$a_k = p(\overline{\mathscr{C}}, k) - p(\mathscr{F}_k, k).$$

Similarly, $b_k = p(\overline{\mathscr{C}'}, k) - p(\mathscr{F}'_k, k)$.

According to the previous item (a), $\overline{\mathscr{F}_k} \sim \overline{\mathscr{F}'_k}$, and, hence, $\mathscr{F}_k \sim \mathscr{F}'_k$, which implies, in particular, $p(\mathscr{F}_k, k) = p(\mathscr{F}'_k, k)$, with getting the desired result. □

Since $\overline{\mathscr{C}} \sim \overline{\mathscr{C}'}$ is given, Lemma 3.2 yields that $a_k = b_k$, which is a contradiction to the existence of the least positive integer k such that $a_k \neq b_k$.

Thus $a_n = b_n$ for all n, and thereby $\|A_i\| = \|B_i\|$ for all i. □

Comment 3.1. In the case of $\mathscr{C} \nsim \mathscr{C}'$, Lemma 3.2 allows us to detect the first integer n where the partition identity fails and reveal the exact difference

$$p(\mathscr{C}, n) - p(\mathscr{C}', n).$$

Comment 3.2. Along the lines of Theorem 3.1, there is no other choice for a rewriting system providing the 'bijective proof' within the extreme Example 1.3 but

$$\gamma_1 : \{1, 9\} \to \{4, 6\}, \quad \gamma_2 : \{6, 14\} \to \{9, 11\}, \quad \gamma_3 : \{11, 19\} \to \{14, 16\}, \ldots \;.$$

Furthermore, taking advantage of additional 'balance conditions', we automatically obtain more refined bijections.

Corollary 3.1. *For any* n *and* k, *the number of partitions of* n *with* k *parts such that their parts congruent to* 1 *or* $4 \bmod 5$ *do not differ by* 2 *is equal to the number of partitions of the* n *with* k *parts such that their parts congruent to* 1 *or* $4 \bmod 5$ *do not differ by* 8.

Proof. Take a (unique!) system of rewriting rules (3) from Example 1.3:

$$\gamma_1 : \{1, 9\} \to \{4, 6\}, \quad \gamma_2 : \{6, 14\} \to \{9, 11\}, \ldots \;.$$

The 'standard balance conditions' hold for the rules from system (3). Besides, each of the rules from system (3) does not change the number of parts in a given partition.

Hence, for any fixed n and k, the function h provided by Proposition 1.1 will be a bijection between $\mathscr{B}$-normal partitions of the n with k parts and $\mathscr{A}$-normal partitions of the same n with k parts. □

Example 3.2. The number of partitions of n, in which each part of the form 3^k, if any, does not appear together with its double 2×3^k, is equal to the number of partitions of the n that have no part of the form 3^{k+1}.

The former partitions, say $\mathscr{B}$-normal forms, form a partition ideal the minimal elements of its complementary filter, say $B_1, B_2, B_3, \ldots$, can be listed as follows:

$$\mathscr{B}_{3.2} = \{1,2\}, \{3,6\}, \{9,18\}, \{27,54\}, \ldots\ .$$

The latter partitions, say $\mathscr{A}$-normal forms, form a partition ideal the minimal elements of its complementary filter, say $A_1, A_2, A_3, \ldots$, can be listed as follows:

$$\mathscr{A}_{3.2} = \{3\}, \{9\}, \{27\}, \{81\}, \ldots\ .$$

According to Theorem 3.1, the 'bijective proof' is guaranteed by the following (unique!) rewriting system:

$$\gamma_1 : \{1,2\} \to \{3\}, \quad \gamma_2 : \{3,6\} \to \{9\}, \quad \gamma_3 : \{9,18\} \to \{27\}, \quad \gamma_4 : \{27,54\} \to \{81\}, \ldots\ . \tag{8}$$

In terms of [2, Andrews], the latter partition ideal is of order 1, whereas the former partition ideal is of 'infinite order' defined as $1 + \sup_k \{2 \times 3^k - 3^k\} = +\infty$.

So the above partition identity is not within reach of the Andrews' characterization [2, Theorem 8.4] of partition ideals of order 1.

Furthermore, Theorem 3.1 provides a new proof, the 'bijective' one, for the Andrews' theorem [2, Theorem 8.4] that fully characterizes the equivalent partition ideals of order 1.

Let us recall the results from [2] we are dealing with.

Definition 3.4 (Andrews [2]). Any partition M is represented as a sequence $\{f_i\}_{i=1}^{\infty}$ where f_i is the number of occurrences of i in M.

Definition 3.5 (Andrews [2]). A partition ideal $\mathscr{C}$ has order k if k is the least positive integer such that whenever $\{f_i\}_{i=1}^{\infty} \notin \mathscr{C}$, then there exists m such that $\{f_i'\}_{i=1}^{\infty} \notin \mathscr{C}$, where

$$f_i' = \begin{cases} f_i & \text{for } i = m, m+1, \ldots, m+k-1, \\ 0 & \text{otherwise.} \end{cases}$$

For example, Example 1.3 gives two partition ideals of order 3 and 9, respectively.

Proposition 3.1 (Andrews [2]). *A partition ideal $\mathscr{C}$ has order* 1 *if and only if*

$$\mathscr{C} = \{\{f_i\}_{i=1}^{\infty} \mid f_i \leqslant d_i,\ for\ all\ i\},$$

where $d_j := \sup_{\{f_j\} \in \mathscr{C}} f_j$.

Theorem 3.2 (Andrews [2, Theorem 8.4]). *Let $\mathscr{C}$ and $\mathscr{C}'$ be partition ideals of order* 1 *with*

$$d_j = \sup_{\{f_j\} \in \mathscr{C}} f_j \quad \textit{and} \quad d_j' = \sup_{\{f_j\} \in \mathscr{C}'} f_j.$$

Then $\mathscr{C} \sim \mathscr{C}'$ if and only if the two sequences of positive integers

$$\{j(d_j+1)\}_{j=1, d_j<\infty}^{\infty} \quad \textit{and} \quad \{j(d_j'+1)\}_{j=1, d_j'<\infty}^{\infty}$$

are merely reorderings of each other.

The proof proposed in [2] relies heavily upon "a very usable representation of the generating function for $p(\mathscr{C}, n)$ whenever $\mathscr{C}$ is a partition ideal of order 1". The above numbers $\{j(d_j+1)\}$ and $\{j(d_j'+1)\}$ seem to be caused there by pure technical reasons.

Corollary 3.2. *Theorem* 3.2 *follows directly from Theorem* 3.1, *which, in addition, provides a bijective proof of Theorem* 3.2.

Proof. For each $d_j < \infty$, define A_j to be the multiset that consists of exactly $d_j + 1$ copies of the number j:

$$A_j := \{\underbrace{j, j, \ldots, j}_{d_j+1 \ \ times}\}$$

and for each $d'_j < \infty$, define B_j to be the multiset that consists of exactly $d'_j + 1$ copies of the number j:

$$B_j := \{\underbrace{j, j, \ldots, j}_{d'_j+1 \ \ times}\}.$$

Notice that A_j's are pairwise disjoint and

$$\|A_j\| = j(d_j + 1)$$

and B_j's are pairwise disjoint and

$$\|B_j\| = j(d'_j + 1).$$

Proposition 3.1 shows that

$$M \notin \mathscr{C} \Leftrightarrow A_j \subseteq M \ \text{ for some } j \text{ such that } d_j < \infty,$$

which means that the order filter $\overline{\mathscr{C}}$, the complement to $\mathscr{C}$, is exactly generated by the A_j's. Hence, our A_j's are minimal within $\overline{\mathscr{C}}$ and form the disjoint support of $\overline{\mathscr{C}}$.

Similarly, our B_j's are proved to be minimal within $\overline{\mathscr{C}'}$, the complement to $\mathscr{C}'$, and form the disjoint support of $\overline{\mathscr{C}'}$.

It remains to apply Theorem 3.1. □

Thus, the Andrews' theorem [2, Theorem 8.4] has two proofs: the corresponding partition identities has been proven through the use of generating functions [2], but the bijective proof found here allows us to get a broader understanding of the result.

In particular, if the norms $\|A_j\|$ are pairwise distinct, our method yields a *unique relevant bijection* (see all the previous examples). It is remarkable that in many interesting cases our method recovers the Glaisher bijection [6]: the bijection computed from (7) turns out to be the same as the one found by Glaisher [6] in pure 'number theoretical' terms. In a forthcoming paper we will explain why and when the Euler and Glaisher bijections arise in partition identities.

Nevertheless, the following example reveals that even a *continuum number* of relevant bijections is possible.

Example 3.3. The number of partitions of n, in which each odd part may occur at most five times, and no odd multiple of 6 occurs, equals the number of partitions of the n, in which each odd multiple of 2 may occur at most twice, and each odd multiple of 3 may occur at most once.

The former partitions form a partition ideal the minimal elements of its complementary filter, say $B_1, B_2, B_3, \ldots$, are the following:

$$\{1,1,1,1,1,1\}, \{3,3,3,3,3,3\}, \{5,5,5,5,5,5\}, \ldots, \{6\}, \{18\}, \{30\}, \ldots \ . \tag{9}$$

The latter partitions form a partition ideal the minimal elements of its complementary filter, say $A_1, A_2, A_3, \ldots$, are the following:

$$\{2,2,2\}, \{6,6,6\}, \{10,10,10\}, \ldots, \{3,3\}, \{9,9\}, \{15,15\}, \ldots \ . \tag{10}$$

Since the number of possible reordering of (9) with providing $\|A_i\| = \|B_i\|$ is continual, our method produces a continuum number of relevant bijections between the above-partition ideals.

Comment 3.3. It should be pointed out that the class of partition ideals with 'disjointly supported' complements is "orthogonal" to the Andrews' hierarchy by 'order k'. Indeed, our class includes all partition ideals of order 1, and many others of 'unbounded/infinite order' (see Examples 1.3 and 3.2).

The indirect evidence of the size of this class is that it seems problematic to find a usable general representation of the generating functions for the whole variety of the partition ideals with 'disjointly supported' complements, so as to provide a uniform proof of Theorem 3.1.

4. Complexity

In practical cases (e.g., Euler's partition theorem) the reduction sequences converge very fast, which provides `polytime` bijections h between $\mathscr{B}$-normal forms and $\mathscr{A}$-normal forms. Under reasonable hypotheses on the complexity of the lists $\mathscr{A}$ and $\mathscr{B}$, the two-directional rewriting machinery guarantees a `sub-exponential time`, at the very worst:

Corollary 4.1. *Let Γ be a set of multiset rewriting rules of the form*

$$\gamma_1 : A_1 \to B_1, \quad \gamma_2 : A_2 \to B_2, \ldots, \quad \gamma_i : A_i \to B_i, \ldots,$$

such that A_i's and B_i's are recognizable in `polynomial time`, *and for any $i \neq j$, A_i and A_j are pairwise disjoint multisets of positive integers, and B_i and B_j are pairwise disjoint multisets of positive integers, and, in addition, the 'balance conditions': $\sum_{a \in A_i} a = \sum_{b \in B_i} b$, hold for all i.*

Then for every n, Theorem 2.3 *yields a bijection h between $\mathscr{B}$-normal partitions of the n and $\mathscr{A}$-normal partitions of the same n, which runs at most in* `sub-exponential time`.

Proof. It follows from Theorem 2.2, since the asymptotic growth of $p(n)$, the number of partitions of the integer n, is sub-exponential [8]:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \, \mathrm{e}^{\pi\sqrt{2n/3}} \quad (n \to \infty). \qquad \square$$

5. Concluding remarks

The *novelty* of our approach to the combinatorics is in the use of rewriting techniques (*two-directional* in the sense that forward and backward application of rewrite rules heads, respectively, for two different normal forms) for the purpose of establishing *explicit bijections* between combinatorial objects of two different types (represented by these normal forms). For a variety of combinatorial problems involving partitions, we have shown how such bijective proofs can be established, and how the bijections looked for are computed by means of multiset rewriting systems.

Although the non-overlapping multiset rules are obviously *confluent*, the *termination problem* for the combinatorial objects of interest is more subtle: it generally *fails* even for multiset rewriting systems that satisfy certain natural balance conditions. We have proved the '*restricted*' *two-directional strong normalization* for the multiset rewriting systems under consideration which guarantees the desired combinatorial bijections.

As for the fundamental problem stated in [2]:

Fully characterize the equivalence classes of partition ideals.

We have fully characterized a *new* wide class of partition ideals, namely, we have *fully characterized* all equinumerous partition ideals with 'disjointly supported' complements.

A proposed *geometrical characterization* provides the desired bijection between partition ideals but in terms of the minimal elements of the order filters, their complements.

As a corollary, a *new proof*, the 'bijective' one, has been given for all equinumerous classes of the partition ideals of order 1 from the classical book "The Theory of Partitions" by G.Andrews. As compared to the proof through the use of generating functions, the bijective proof suggested here allows us to get a broader understanding of the essence of the result.

We have stated here some results on factorizations and forests to show that the ideas underlying our approach are most likely applicable to many other combinatorial problems.

In a forthcoming paper we will discuss the challenges of the 'overlapping' multiset rewriting systems, and the corresponding 'bijective' proofs of certain 'overlapping' identities related to Fibonacci and Lucas numbers.

Acknowledgements

Since the paper used techniques from the rewriting world (see [3,4,10]) to sort out specific mathematical problems studied completely independently in the world of combinatorics (see [2,15]), selecting and organizing the material has been a real challenge.

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