On a Theorem of Dubins*

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Dubins [1] has proved that if $E$ is a real linear space, $K$ is a convex subset of $E$ which is linearly closed and linearly bounded, and $M$ is a flat of finite codimension $n$ in $E$, then each extreme point of $K \cap M$ is a convex combination of $n + 1$ extreme points of $K$. (Here and later, the $n + 1$ points are not required to be distinct.) His proof, designed to include other material as well, is rather circuitous. We record here a more direct proof which applies also to unbounded sets and to faces of arbitrary finite dimension.

Recall that by “factoring out” the largest flat contained in a linearly closed convex set $C$, the extremal structure of $C$ can be described in terms of the extremal structure of a linearly closed convex set $K$ which contains no line (1.2 of [2]). One of our tools is the following result (2.5 of [2]).

(1) If a finite-dimensional linearly closed convex set $K$ contains no line, then $K = \text{conv} (\text{ext } K \cup \text{rext } K)$; that is, $K$ is the convex hull of its extreme points together with its extreme rays.

Another tool is a well-known theorem of Carathéodory (p. 35 of [3]).

(2) If $X$ is a subset of an $n$-dimensional space, then each point of the convex hull of $X$ is a convex combination of $n + 1$ points of $X$.

Combining (1) and (2), we obtain a result which is familiar when $K$ is linearly bounded or is a convex cone.

(3) If $K$ is an $n$-dimensional convex set which is linearly closed and contains no line, then each point $p$ of $K$ is

- a convex combination of $n + 1$ extreme points of $K$ or
- a convex combination of $n$ points of $K$, each an extreme point or in an extreme ray of $K$.

PROOF: We assume without loss of generality that $K$ lies in an $n$-dimen-
sional linear space $E$ and that $p = 0$, the origin of $E$. By (1) and (2) the point $p$ can be expressed in the form

$$ p = \sum_{0}^{m} \alpha_{i} x_{i} \quad \text{with} \quad m \leq n, \sum_{0}^{m} \alpha_{i} = 1, \alpha_{i} > 0, $$

and

$$ x_{i} \in \text{ext } K \quad \text{for} \quad 0 \leq i < k \quad \text{while} \quad x_{i} \in \text{ext } K \quad \text{for} \quad k \leq i \leq m. $$

We may assume that among all such expressions for $p$, the representation (*) has the smallest value of $m$, and that among those with this value of $m$, (*) has the smallest value of $k$. If $m < n$ the representation (*) is of the second sort described in (3), while if $k = 0$ it is of the first sort. Thus we may assume that $m = n$ and $k \geq 1$, while $x_{0} = u + \beta y$ where $\beta > 0$ and the ray $u + [0, \infty][y$ is an extreme ray of $K$. Then $u \in \text{ext } K$ and we have

$$ p = 0 = \alpha_{0}(u + \beta y) + \sum_{1}^{n} \alpha_{i} x_{i}. $$

Let $N$ be the set of all points of the form $\sum_{1}^{n} \mu_{i} x_{i}$ with $\mu_{i} < 0$. From (2) in conjunction with the minimality of $m(= n)$ it follows that the set $\{x_{0}, \ldots, x_{n}\}$ does not lie in any $(n - 1)$-dimensional subspace of $E$. Hence the points $x_{1}, \ldots, x_{n}$ form a basis for $E$ and the set $N$ is open. Clearly $u + \beta y \in N$. If $u \in N$ then $0 \in \text{conv } \{u, x_{1}, \ldots, x_{n}\}$, contradicting the minimality of $k$. If $u \notin N$, then the boundary of $N$ includes the point $u + \gamma y$ for some $\gamma \in ]0, \beta[,$ and $u + \gamma y$ is a positive combination of some $n - 1$ of the points $x_{1}, \ldots, x_{n}$. But then $0$ is a convex combination of these $n - 1$ extreme points together with the point $u + \gamma y \in \text{ext } K$, and the minimality of $m$ is contradicted. This completes the proof.

For a point $p$ of a convex set $K$, let $K_{p}$ denote the union of all open segments in $K$ which have $p$ as an inner point. Then $K_{p}$ is a convex subset of $K$, and is in fact the smallest face of $K$ which includes $p$ (see [1]). The face $K_{p}$ is zero-dimensional if and only if $p$ is an extreme point of $K$, and $K_{p}$ is one-dimensional when $p$ is an inner point of an extreme ray of $K$.

**Theorem.** Suppose $E$ is a real linear space, $K$ is a convex subset of $E$ which is linearly closed and contains no line, $M$ is a flat of finite codimension $n$ in $E$, and $p$ is a point of $K \cap M$ for which the face $(K \cap M)_{p}$ has finite dimension $j$. Then $p$ is

- a convex combination of $n + j + 1$ extreme points of $K$ or
- a convex combination of $n + j$ points of $K$, each an extreme point or in an extreme ray of $K$. 
Proof: We assume without loss of generality that \( p = 0 \), whence \( M \) is a linear subspace of \( E \). Let \( S \) denote the union of all lines \( T \) through 0 for which 0 is an inner point of \( K \cap T \). Then \( S \) is a linear subspace of \( E \) and \( S \cap M \) is the linear hull of the face \( (K \cap M) \). Since this face is of dimension \( j \) while \( M \) is of codimension \( n \), the subspace \( S \) is of dimension \( \leq n + j \). The set \( K \cap S \) is linearly closed and contains no line, whence (3) applies to yield an expression of \( p(=0) \) in the form \( 0 = \sum_{i=1}^{m} \alpha_i x_i \) with \( \sum_{i=1}^{m} \alpha_i = 1 \), \( \alpha_i > 0 \), and \( m = n + j + 1 \) with all the \( x_i \)'s in \( \text{ext} (K \cap S) \) or \( m = n + j \) with all the \( x_i \)'s in \( \text{ext} (K \cap S) \cup \text{ext} (K \cap S) \). Now let

\[
y_i = \left( -\alpha_i/(1 - \alpha_i) \right) x_i = \sum_{h \neq i} \left( \alpha_h/(1 - \alpha_i) \right) x_h \in K,
\]

and suppose \( x_i \) lies in an open segment \( [u, v] \) joining two points \( u \) and \( v \) of \( K \). Since \( y_i \) is a negative multiple of \( x_i \), elementary computation produces points \( u' \in y_i, v' \in y_i \) such that \( 0 \in [u', u] \) and \( 0 \in [v', v] \). But then \( u, v \in S \) and \( [u, v] \subseteq K \cap S \). This completes the proof, for it shows that if \( x_i \) is an extreme point (resp. in an extreme ray) of \( K \cap S \), then \( x_i \) has the same relationship to \( K \).

When \( j = 0 \) and \( K \) is linearly bounded, the above result reduces to Dubins’s theorem [1]. This special case (as in [1]) does not require the lemma (3), but only the well known special case of that lemma for bounded \( K \).

References