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A Fixed-Point Theorem for a Class of Integral Operators

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INTRODUCTION

In this paper we develop a fixed-point theorem for a general class of nonlinear operators which are frequently encountered in the theory of Ordinary Differential Equations. The theorem is then applied to prove the existence of a periodic solution of a system of quasi-linear ordinary differential equations. As is often done, the problem of showing that a fixed point exists is reduced to proving that a certain compact convex subset of a Banach space B is mapped into itself by the operator and then applying the Schauder fixed-point theorem (see e.g. [1], Chapt. 12). However, we do not make the *a priori* assumption of the existence of such a set, but instead postulate the existence of a closed invariant subset $\mathscr{M}$ (which need not be convex or compact) in a Banach space $\tilde{B}$ which contains B. The existence of an invariant compact convex subset of B is then deduced by applying a result used in the Theory of Optimal Control. This result is due to D. Blackwell [2]. In Theorem 1.2 we show that under certain relatively weak assumptions our result can be used to acertain the existence of fixed points which would not be immediately obtainable using Schauder's theorem since the requisite convexity conditions are absent.

1. So as to keep to a minimum the introduction of definitions in the course of proving Theorems 1.1 and 1.2 we shall first set down some notational conventions, definitions and three hypotheses.

(1) $U(t)$ will always stand for a compact subset of E^n which may possibly be dependent on some parameter t. The norm in E^n will be denoted by the symbol $\|\cdot\|$.

(2) The symbol $L_2[0, 1]$ will represent the set of all vector-valued measurable functions from $[0, 1] \to E^n$ whose Euclidean norms are square integrable. The norm in this Banach space will be denoted by $\|\cdot\|_{L_2}$.

(3) The symbol $C[0, 1]$ will represent the set of all continuous vector-

valued functions from $[0, 1] \to E^n$. The norm of this Banach space will be denoted by $\|\cdot\|_C$ and defined as follows:

$$\|\phi\|_C = \sup\{\|\varphi(t)\| : 0 \leqslant t \leqslant 1\}.$$

(4) Let $\mathscr{M}$ denote the family of all vector valued measurable functions m satisfying the condition that $m(t) \in U(t)$ a.e. for $t \in [0, 1]$. C will denote the family of all continuous vector valued functions from $[0, 1] \to U(t)$.

(5) Let $\mathscr{K}$ be any family of functions in $C[0, 1]$. Then $\bar{\mathscr{K}}$ will stand for the closure of $\mathscr{K}$ in the topology of $C[0, 1]$.

(6) Let E be any measurable subset in $[0, 1]$. Then $\mu(E)$ will denote the Lebesgue measure of E.

(7) $U(t)$ may be considered as a mapping from $[0, 1]$ into the collection τ of all compact subsets of E^n. This collection can be made into a metric space (τ, d) by means of the Hausdorff metric (see e.g. [3], p. 131). We shall make the assumption that $U(t)$ is a continuous mapping from $[0, 1]$ into (τ, d).

REMARK 1.1. The set $U = \bigcup_{0 \leqslant t \leqslant 1} U(t)$ is compact in the usual topology of E^n.

PROOF. First of all notice that the function (diameter $U(t)$) is a continuous function of t. Hence U is a bounded subset of E^n. Since U is bounded in E^n, to prove that it is compact, it is only necessary to show that every Cauchy sequence of points $\{y_n\}$ in U has its limit in U. Thus let $\{y_n\} \subset U$ converge to some point y_0. Each subscript n corresponds to some t_n in $[0, 1]$ such that $y_n \in U(t_n)$. We select a subsequence $\{t_{n_1}\}$ which converges to t_0 in $[0, 1]$. But this implies by the continuity of $U(t)$ that $\{U(t_{n_1})\} \to U(t_0)$ in the space (τ, d). Hence $y_0 \in U(t_0) \subset U$ for otherwise $U(t)$ would not be continuous at t_0.

Let $g : [0, 1] \times E^n \to E^n$ and $f : [0, 1] \to E^n$ be continuous. Let A denote the mapping from $L_2[0, 1] \to C[0, 1]$ defined by the integral equation

$$z(t) = f(t) + \int_0^t g(s, \varphi(s))\, ds, \qquad \phi \in L_2[0, 1], \qquad 0 \leqslant t \leqslant 1. \tag{1.1}$$

Because of the assumptions made concerning g, f, and $U(t)$ it is evident that A is a completely continuous operator from $\mathscr{M}$ in $L_2[0, 1]$ into $C[0, 1]$.

H_1. A maps $\mathscr{M}$ into itself.

H_2. $U(t)$ is a fixed compact set for all t in $[0, 1]$ and this set is homeomorphic to a compact convex set K in E^n.

REMARK 1.2. Since $\mathscr{M}$ is closed and bounded in $L_2[0, 1]$, A is completely continuous on $\mathscr{M}$ and U is compact in E^n we can conclude that

$A(\mathscr{M}) \subset \overline{A(\mathscr{M})} \subset \mathscr{M}$. Moreover $\overline{A(\mathscr{M})}$ is by the Ascoli-Arzela theorem compact in $C\,[0, 1]$ since $A(\mathscr{M})$ is precompact (see e.g. [4]).

LEMMA 1.1. *If H_1 holds then $\overline{A(\mathscr{M})}$ is a compact convex subset of* $C[0, 1]$.

PROOF. The compactness was justified in Remark 1.2. Thus we need prove only convexity. For any $m \in \mathscr{M}$ define the mapping $Q : \mathscr{M} \to C[0, 1]$ which is given by the integral equation

$$Q(m)\,(t) = w(t) = \int_0^t g(s, m(s))\, ds, \qquad m \in \mathscr{M}, \qquad 0 \leqslant t \leqslant 1. \tag{1.2}$$

Notice that $w(0) = 0$ for each $w \in Q(\mathscr{M})$.

Let $0 \leqslant t_1 \leqslant t_2 \leqslant 1$ and define the set of points

$$R(t_1\,, t_2) = \left\{\int_{t_1}^{t_2} g(s, m(s))\, ds \colon m \in \mathscr{M}\right\}. \tag{1.3}$$

By a result of Blackwell [2] $R(t_1\,, t_2)$ is compact and convex for each pair $(t_1\,, t_2)$. In particular this means that given any two points r_1 and r_2 in $R(t_1\,, t_2)$ and any $\alpha \in [0, 1]$, then there exists an $m \in \mathscr{M}$ such that

$$\int_{t_1}^{t_2} g(s, m(s))\, ds = (1 - \alpha)\, r_1 + \alpha r_2\,.$$

Consider the dense set of points $\mathscr{D}$ in [0, 1] which is given by the relation

$$\mathscr{D} = \{i2^{-n}\}, \qquad i = 0, 1,..., 2^n, \qquad n = 1, 2,..., \tag{1.4}$$

and for each n the intervals

$$J_i^{\,n} = [(i - 1)\, 2^{-n}, i2^{-n}), \qquad i = 1,..., 2^n - 1$$

and

$$J_{2^n}^n = [(2^n - 1)\, 2^{-n}, 1]. \tag{1.5}$$

Notice that for n fixed the sets $J_i^{\,n}$, $i = 1,..., 2^n$, form a disjoint partition of [0, 1] and if $n \geqslant \bar{n}$ then the sets $\{J_i^{\,n}\}$ are refinements of $\{J_i^{\,\bar{n}}\}$.

Let $\bar{w} = Q(\bar{m})$ and $\bar{\bar{w}} = Q(\bar{\bar{m}})$. Let $\alpha \in [0, 1]$ and $0 \leqslant t_0 \leqslant t_1 \leqslant 1$. Then according to Blackwell's result there exists an $m \in \mathscr{M}$ such that

$$\alpha \int_{t_0}^{t_1} g(s, \bar{m}(s))\, ds + (1 - \alpha) \int_{t_0}^{t_1} g(s, \bar{\bar{m}}(s))\, ds$$

$$= \int_{t_0}^{t_1} g(s, m(s))\, ds = w(t_1) - w(t_0).$$

Hence for each integer n there exists a function m_n in $\mathscr{M}$ such that

$$\alpha\bar{w}(i2^{-n}) + (1-\alpha)\,\bar{\bar{w}}(i2^{-n}) - \{\alpha\bar{w}((i-1)\,2^{-n}) + (1-\alpha)\,\bar{\bar{w}}((i-1)\,2^{-n})\}$$

$$= w_n(i^{2-n}) - w_n((i-1)\,2^{-n}) = \int_{(i-1)2^{-n}}^{i_2^{-n}} g(s, m_n(s))\,ds, \qquad i = 1, 2, \ldots, 2^n. \tag{1.6}$$

For $i = 1$ Eq. 1.6 implies that

$$\alpha\bar{w}(2^{-n}) + (1-\alpha)\,\bar{\bar{w}}(2^{-n}) = w_n(2^{-n}).$$

Using this fact as the first step in an induction argument we deduce that for each n

$$\alpha\bar{w}(i2^{-n}) + (1-\alpha)\,\bar{\bar{w}}(i2^{-n}) = w_n(i2^{-n}), \qquad i = 1, 2, \ldots, 2^n. \tag{1.7}$$

Since $Q(\mathscr{M}) = A(\mathscr{M}) - \{f\}$ it follows that $Q(\mathscr{M})$ is a uniformly equicontinuous family of functions because the same is true of $A(\mathscr{M})$. Thus we can select a subsequence $\{w_{n_1}\} \subset \{w_n\}$ which converges in $C[0, 1]$ to a continuous function w in $\overline{Q(\mathscr{M})}$. We claim that at each point of $\mathscr{D}$ the function w satisfies the equation $w(i2^n) = \alpha\bar{w}(i2^{-n}) + (1-\alpha)\,\bar{\bar{w}}(i2^{-n})$. This is consequence of (1.7) and the fact that by our construction

$$w_{n_1}(i2^{-n}) = w_n(i2^{-n}) \qquad \text{if} \qquad n_1 \geqslant n.$$

Since the points $\{i2^{-n}\}$ are dense in $[0, 1]$ it follows from continuity that

$$w(t) = \alpha\bar{w}(t) + (1-\alpha)\,\bar{\bar{w}}(t), \qquad 0 \leqslant t \leqslant 1. \tag{1.8}$$

We now select $\bar{w}$ and $\bar{\bar{w}}$ in $\overline{Q(\mathscr{M})}$ and $\alpha \in [0, 1]$. By virtue of what has just been shown we can find for each integer n functions $\bar{w}_n$ and $\bar{\bar{w}}_n$ in $Q(\mathscr{M})$ and w_n in $\overline{Q(\mathscr{M})}$ such that

$$\alpha\bar{w}_n(t) + (1-\alpha)\,\bar{\bar{w}}_n(t) = w_n(t) \tag{1.9}$$

and

$$\|\bar{w}_n - \bar{w}\|_2 < \frac{1}{n}, \qquad \|\bar{\bar{w}}_n - \bar{\bar{w}}\|_2 < \frac{1}{n}.$$

The sequences $\{\bar{w}_n\}$ and $\{\bar{\bar{w}}_n\}$ are Cauchy sequences in $C[0, 1]$. Hence given any $\epsilon > 0$ there exists $n_0(\epsilon)$ such that $\|\bar{\bar{w}}_m - \bar{\bar{w}}_n\|_2 < \epsilon$ and $\|\bar{w}_m - \bar{w}_n\|_2 < \epsilon$ for $n, m \geqslant n_0(\epsilon)$. From this and Eq. (1.9) it follows that $\{w_n\}$ is a Cauchy sequence in $C[0, 1]$ with a limit in $\overline{Q(\mathscr{M})}$ and this limit satisfies the inequality

$$\| w - (1-\alpha)\bar{\bar{w}} - \alpha\bar{w} \|_2 \leqslant \| w - w_n \|_2 + \| w_n - (1-\alpha)\bar{\bar{w}}_n - \alpha\bar{w}_n \|_2$$
$$- \| (1-\alpha)\bar{\bar{w}}_n + \alpha\bar{w}_n - (1-\alpha)\bar{\bar{w}} - \alpha\bar{w} \|_2$$
$$\leqslant \| w - w_n \|_2 + (1-\alpha) \| \bar{\bar{w}}_n - \bar{\bar{w}} \|_2$$
$$+ \alpha \| \bar{w} - \bar{w}_n \|_2 .$$

This shows that

$$w = (1-\alpha)\bar{\bar{w}} + \alpha\bar{w} \tag{1.10}$$

and hence that $\overline{Q(\mathscr{M})}$ is convex. But $\overline{A(\mathscr{M})} = \{f\} + \overline{Q(\mathscr{M})}$, which proves that $\overline{A(\mathscr{M})}$ is also convex.

THEOREM 1.1. *The integral equation* (1.1), *subject to the hypotheses of Lemma 1.1 has a fixed point in* $\mathscr{M}$.

PROOF. $\mathscr{M}$ is closed in $L_2[0, 1]$ and $A(\mathscr{M}) \subset \mathscr{M}$ by H_1. Hence the closure of $A(\mathscr{M})$ in $\overline{C[0, 1]}$ is also in $\mathscr{M}$, i.e., $\overline{A(\mathscr{M})} \subset \mathscr{M}$. This leads to the inclusion relation

$$\overline{A(\overline{A(\mathscr{M})})} \subset A(\mathscr{M}) \subset \overline{A(\mathscr{M})}. \tag{1.11}$$

Thus the mapping A restricted to $\overline{A(\mathscr{M})}$ satisfies the conditions of Schauder's theorem and hence there exists at least one fixed point of A in $\overline{A(\mathscr{M})}$.

The hypothesis H_1 is unfortunate in that it is harder to verify than one which left the set of continuous functions from $[0, 1] \to U$ invariant. However, this difficulty can be overcome if we assume that $U(t)$ satisfies H_2.

LEMMA 1.2. *Assume $U(t)$ is a fixed compact set U in E^n which is homeomorphic to a compact convex set K is E^n. Then C is dense in $\mathscr{M}$ in terms of the metric* $\|\cdot\|_{L_2}$.

PROOF. The proof of this lemma rests on a result of Dugundji [5]. We paraphrase it for our proof:

Let φ be a continuous function from a closed set F on $[0, 1]$ into E^n. Then there exists a continuous extension of φ, ϕ, such that $\phi([0, 1]) \subset$ convex hull of $\varphi(F)$.

Let φ be a continuous function from a closed set F in $[0, 1]$ into U, and let q be a homeomorphism of $U \to K$. Then the composite function $\psi = q \cdot \phi : F \to K$ is continuous. Applying Dugundji's result we can find a function Ψ which extends ψ continuously to $[0, 1]$ and whose range is in K. But then the function ϕ given by the composition $q^{-1} \cdot \Psi$ is a continuous extension of φ to $[0, 1]$ whose range is in U.

Now let $m \in \mathscr{M}$. Then by Lusin's theorem (see e.g. [6]) given any $\epsilon > 0$ there exists a closed set F in $[0, 1]$ such that $\mu([0, 1] - F) < \epsilon$ and m restricted to F is continuous. By what we have just shown we can extend m on $[0, 1]$ to a continuous function M_ϵ whose range is in U. This function satisfies the inequality

$$\| m - M_\epsilon \|_1^2 = \int_{[0,1]-F} \| m(t) - M_\epsilon(t) \|^2 \, dt$$
$$\leqslant 2\epsilon\{\sup \| x \|^2 : x \in U\}.$$

But U is compact in E^n and hence the above supremum is finite. Since ϵ is arbitrary this shows that C is dense in $\mathscr{M}$ and concludes the proof.

THEOREM 1.2. *If* H_2 *is satisfied then the hypotheses* H_1 *can be weakened to the statement that* $A(C) \subset C$. *Under these conditions Eq.* (1.1) *will have a fixed point in* C.

PROOF. To prove the theorem we only need to show that $A(\mathscr{M}) \subset \mathscr{M}$ and apply Theorem 1.1. To do this notice that because of Lemma 1.2 given any $m \in \mathscr{M}$ there exists a sequence $\{\varphi_n\} \subset C$ which converges in the mean square norm to m. Hence $\{\varphi_n\}$ converges in measure to m (see, e.g., [6]). Since $[0, 1]$ has finite measure this implies there exists a subsequence $\{\varphi_{n_1}\} \subset \{\varphi_n\}$ which converges a.e. to m on $[0, 1]$. Then since g is continuous

$$\{g(s, \varphi_{n_1}(s))\} \to g(s, m(s))$$

a.e. on $[0, 1]$. This in turn implies, because of the Lebesgue dominated convergence theorem, that

$$\left\{ f(t) + \int_0^t g(s, \varphi_{n_1}(s)) \, ds \right\} \to f(t) + \int_0^t g(s, m(s)) \, ds$$

pointwise on $[0, 1]$. The set U is compact and by H_1 for each t in $[0, 1]$ $f(t) + \int_0^t g(s, \varphi_n(s)) \, ds$ is in U, hence it follows that $A(m) \in \mathscr{M}$.

2. We shall now apply Theorem 1.1 to prove the existence of periodic solutions for a class of quasilinear differential equations.

Consider the equation

$$\dot{x} = A(t)\, x + g(x, t), \tag{2.1}$$

where $A(t)$ is a continuous matrix periodic of period one. x is an n-vector and $g : E^n \times E^1 \to E^n$ is continuous and periodic of period one in t. Related to system (2.1) are the two systems

$$\dot{y} = A(t)\, y + g(m(t), t), \tag{2.2}$$

where $m \in \mathcal{M}$, and the homogeneous system

$$\dot{y} = A(t)\, y. \tag{2.3}$$

System (2.3) has a fundamental solution $\Lambda(t)$ with $\Lambda(0) = I$ the $n \times n$ identity matrix. In terms of this fundamental solution all solutions of (2.2) can be expressed in the form

$$y(t) = \Lambda(t) \left[y(0) + \int_0^t \Lambda^{-1}(s)\, g(m(s), s)\, ds \right] \tag{2.4}$$

H_3. Assume the monodromy matrix $\Lambda(1)$ associated with the system (2.3) has no characteristic roots equal to one.

For each $m \in \mathcal{M}$ consider the unique solution of (2.2) given by the equation

$$z(t) = \Lambda(t) \Big[(I - \Lambda(1))^{-1} \Lambda(1) \int_0^1 \Lambda^{-1}(s)\, g(m(s), s)\, ds + \int_0^t \Lambda^{-1}(s)\, g(m(s), s)\, ds \Big]. \tag{2.5}$$

Equation (2.5) defines a nonlinear continuous mapping $H : \mathcal{M} \to C[0, 1]$.
Some properties of H:

PROPERTY 1. For $m \in \mathcal{M}$ the image $g = H(m)$ satisfies the relation $z(0) = z(1)$.

PROOF. By direct substitution in Eq. (2.5).

PROPERTY 2. If $m \in \mathcal{M}$ is periodic of period one then $z = H(m)$ is also periodic of period one.

PROOF. Again by substitution in (2.5) we have

$$\begin{aligned} z(t + 1) &= \Lambda(t) \left[z(1) + \int_0^1 \Lambda^{-1}(s)\, g(m(s), s)\, ds \right] \\ &= \Lambda(t) \left[z(0) + \int_0^t \Lambda^{-1}(s)\, g(m(s), s)\, ds \right] = z(t) \end{aligned}$$

since by property one $z(0) = z(1)$.

PROPERTY 3. Let $m \in \mathcal{M}$. Then $H(m) = z$ is of bounded variation on any finite interval.

PROOF. $H(m) = z$ is absolutely continuous on any finite interval and such functions are of bounded variation.

Let $P[0, 1]$ denote the closed subspace of $C[0, 1]$ consisting of all functions periodic of period one. A consequence of property two above is that if $\varphi \in P[0, 1]$ is substituted into Eq. (2.5) in place of $m \in \mathscr{M}$ then $H(\varphi)$ is periodic of period one and hence $H : P[0, 1] \to P[0, 1]$. A fixed point of this mapping will be equivalent to the existence of a periodic solution with period one of Eq. (2.1).

Let α and β denote the sequences of continuous vector-valued mappings from $E^1 \to E^n$ defined as follows:

$$\begin{aligned} a_{i_1 \dots i_n}(t) &= (\sin 2\pi i_1 t, \dots, \sin 2\pi i_n t), \\ i_j &= 1, 2, \dots, j = 1, \dots, n. \\ b_{i_1 \dots i_n}(t) &= (\cos 2\pi i\,(t), \dots, \cos 2\pi i_n(t)), \\ i_j &= 0, 1, 2, \dots, j = 1, 2, \dots, n. \end{aligned} \tag{2.6}$$

It is evident that the set $\alpha \cup \beta$ defined by (2.6) forms a Schauder basis for the Banach space $L_2[0, 1]$. That is each $\varphi \in L_2[0, 1]$ has a unique representation $\hat{\varphi}$ in terms of the basis (2.6). By a result of Carleson [7] $\hat{\varphi}(t) = \varphi(t)$ a.e. on $[0,1]$ and $\hat{\varphi}(t + 1) = \hat{\varphi}(t)$ for all t in E^1. Moreover if in Eq. (2.5) we replace φ by its periodic extension $\hat{\varphi}$ then because of Carlson's result $H(\varphi) = H(\hat{\varphi})$ on $[0, 1]$ and by property two $H(\hat{\varphi})$ is a periodic function of period one.

Thus we have uniquely extended the functions $H(\mathscr{M})$ to periodic functions of period one. Denote this extension by $\hat{H}(\hat{\mathscr{M}})$.

H_4. Assume that $\mathscr{M}$ is invariant under H.

THEOREM 2.1. *If the hypotheses* H_3 *and* H_4 *hold then there exists a periodic solution of* (2.1) *of period one.*

PROOF. It is evident from the definition of the family $\mathscr{M}$ and the form of Eq. (2.5) that $H(\mathscr{M})$ is a uniformly equicontinuous set of functions in $C[0, 1]$. Hence by the Ascoli-Arzela theorem [4] $\overline{H(\mathscr{M})}$ is compact. If $m \in \mathscr{M}$ is a fixed point of H it follows that $m \in C[0, 1]$. Also $H(m) = H(\hat{m}) = m$ on the interval $[0, 1]$, and by property 2 the extension $\hat{H}(\hat{m})$ of $H(m)$ is periodic of period one on E^1. We claim that if m is a fixed point in $\mathscr{M}$ of the mapping H then $\hat{H}(\hat{m}) = \hat{m}$ and hence $\hat{m}$ will be a periodic solution of (2.1).

To see this notice that by property two $\hat{m}$ is periodic, and by property three m is of bounded variation on $[0, 1]$. Hence using a result of Dirichelet (see, e.g., Hobson [8] Section 446) $\hat{m}$ is a continuous periodic function of period one which coincides with m on $[0, 1]$. Since any t in E^1 can be represented uniquely in the form $t = \tau + n$, $0 \leqslant \tau < 1$, $n = \pm 1, \pm 2, \dots$, we can write the following equation

$$\hat{H}(\hat{m}(t)) = \hat{H}(\hat{m}(\tau + n)) = H(m(\tau)) = m(\tau) = \hat{m}(\tau + n) = \hat{m}(t), \tag{2.7}$$

which shows that $\hat{m}$ is a fixed point of $\hat{H}$ and hence a periodic solution of Eq. (2.1)).

To prove the existence of a fixed point of H in $\mathscr{M}$ we first show that $\overline{H(\mathscr{M})}$ is convex and compact. Let $F(\mathscr{M})$ denote the mapping from $\mathscr{M} \to C[0, 1]$ defined by the equation

$$w(t) = \int_0^t A^{-1}(s)\, g(s, \varphi(s))\, ds, \qquad \varphi \in \mathscr{M}. \tag{2.8}$$

The mapping F satisfies the conditions of Lemma 1.1. Hence $\overline{F(\mathscr{M})}$ is a closed convex set in $C[0, 1]$. Consider the two continuous linear mappings from $C[0, 1]$ into itself which are defined by the equations:

$$z(t) = A(t)\,[(I - A(1))^{-1} A(1)\, w(1) + w(t)] \tag{2.9}$$

and

$$w(t) = A^{-1}(t)\, z(t) - z(1). \tag{2.10}$$

Denote (2.9) by $z = W(w)$ and (2.10) by $w = Z(z)$. By direct substitution it is clear that $WZ = I = ZW$ (here I is the identity mapping in $C[0, 1]$), and $W(F(\mathscr{M})) = H(\mathscr{M})$. Hence, since W is a closed mapping (being linear and a homeomorphism onto), $W(\overline{F(\mathscr{M})}) = \overline{H(\mathscr{M})}$.

Because $\overline{F(\mathscr{M})}$ is a compact convex set in $C[0, 1]$, $\overline{H(\mathscr{M})}$ will be also, due to the fact that W is linear and continuous.

The proof used in Theorem 1.1 can now be applied to show that H has a fixed point in $\mathscr{M}$ which from the above discussion is equivalent to proving the existence of a periodic solution of (2.1).

REMARK 2.1. Carleson's result was used in the proof of Theorem 2.1 to show that the Fourier expansion of a function on [0, 1] coincides with the function a.e. on [0, 1]. Hence if $g(t, y)$ is continuous in both variables and $\hat{\phi}(t)$ represents the Fourier expansion of $\phi(t)$ on [0, 1] then

$$g(s, \phi(s)) = g(s, \hat{\phi}(s))$$

a.e. on [0, 1] and

$$\int_0^t A^{-1}(s)\, g(s, \phi(s))\, ds = \int_0^t A^{-1}(s)\, g(s, \hat{\phi}(s))\, ds.$$

THEOERM 2.2. *Suppose* H_3 *holds and that* H_4 *is replaced by* H_2 *and the statement that* $H(C) \subset C$, *then Eq.* (2.1) *has a periodic solution of period one in* C.

PROOF. The proof is a consequence of the fact that if H_2 holds then by Lemma 1.2 the family C is dense in $\mathscr{M}$ in terms of the $\|\cdot\|_{L_2}$ topology. From this fact we can show as was done in the proof of Theorem 1.2 that $H(\mathscr{M}) \subset \mathscr{M}$, i.e., that H_4 holds, and then we apply Theorem 2.1.

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