



C^* -algebras of separated graphs

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Received 18 February 2011; accepted 1 July 2011

Available online 28 July 2011

Communicated by S. Vaes

Abstract

The construction of the C^* -algebra associated to a directed graph E is extended to incorporate a family C consisting of partitions of the sets of edges emanating from the vertices of E . These C^* -algebras $C^*(E, C)$ are analyzed in terms of their ideal theory and K -theory, mainly in the case of partitions by finite sets. The groups $K_0(C^*(E, C))$ and $K_1(C^*(E, C))$ are completely described via a map built from an adjacency matrix associated to (E, C) . One application determines the K -theory of the C^* -algebras $U_{m,n}^{\text{nc}}$, confirming a conjecture of McClanahan. A reduced C^* -algebra $C_{\text{red}}^*(E, C)$ is also introduced and studied. A key tool in its construction is the existence of canonical faithful conditional expectations from the C^* -algebra of any row-finite graph to the C^* -subalgebra generated by its vertices. Differences between $C_{\text{red}}^*(E, C)$ and $C^*(E, C)$, such as simplicity versus non-simplicity, are exhibited in various examples, related to some algebras studied by McClanahan.

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Keywords: Graph C^* -algebra; Separated graph; Amalgamated free product; Conditional expectation; Ideal lattice

1. Introduction

Graph C^* -algebras constitute an important class of C^* -algebras, providing models for the classification theory and a rich source of examples and inspiration. Among the most basic examples of graph C^* -algebras are the Cuntz algebras \mathcal{O}_n , initially studied by Cuntz [8,9], and the

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¹ The first-named author was partially supported by DGI MICIIN-FEDER MTM2008-06201-C02-01, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya.

Cuntz–Krieger algebras [10] associated to finite square matrices with entries in $\{0, 1\}$. We refer the reader to [24] for further information on this important class of C^* -algebras.

The present paper addresses the structure of a new class of graph C^* -algebras, associated to *separated graphs* (E, C) , where E is a directed graph and C is a family that gives a partition of the set of edges departing from each vertex of E . (These algebras have also been recently introduced by Duncan [12], with different notation, as *C^* -algebras of edge-labeled graphs*, which should not be confused with the *labeled graph C^* -algebras* developed by Bates and Pask [5]. Our viewpoint, which was developed in [1] for the algebraic case, appears to be more flexible and better adapted to the construction and analysis of these algebras.) It was shown in [1] how to associate to any such separated graph (E, C) a complex $*$ -algebra $L(E, C)$, called the *Leavitt path algebra* of the separated graph (E, C) . We may define the C^* -algebra $C^*(E, C)$ of the separated graph (E, C) as the universal C^* -envelope of $L(E, C)$. We also introduce a reduced version, denoted $C_{\text{red}}^*(E, C)$, in the case that (E, C) is *finitely separated*, meaning that the partitions in C consist of finite sets. To glimpse the differences and similarities between the full and reduced graph C^* -algebras, let us mention the following facts. When we consider a separated graph (E, C) with just one vertex and the sets in the partition C are reduced to singletons, the full graph C^* -algebra $C^*(E, C)$ is just the full group C^* -algebra $C^*(\mathbb{F})$ of a free group \mathbb{F} of rank $|E^1|$, while the reduced graph C^* -algebra is precisely the reduced group C^* -algebra $C_r^*(\mathbb{F})$. On the other hand, when we deal with a *trivially separated graph* (E, C) (meaning that for each non-sink $v \in E^0$, the partition C_v consists of the single set $s^{-1}(v)$), then both the full graph C^* -algebra $C^*(E, C)$ and the reduced graph C^* -algebra $C_{\text{red}}^*(E, C)$ coincide with the usual graph C^* -algebra $C^*(E)$ (Theorem 3.8(2)). In general, the behaviors of the full and reduced graph C^* -algebras are quite different, as suggested by the free group C^* -algebra example above. We consider specific examples in Section 4, for which we show that the reduced graph C^* -algebra is simple, including in particular algebras closely related to the C^* -algebras considered by Brown and McClanahan, see [6,20–22]. Indeed, as we show in Section 6, our examples (both reduced and full) are Morita-equivalent to ones considered in the abovementioned papers.

We also compute, using a result of Thomsen, the K -theory of the full graph C^* -algebras of finitely separated graphs (E, C) , obtaining a formula that very much resembles the one known for ordinary graph C^* -algebras, as stated for instance in [25, Theorem 3.2]. Namely, $K_0(C^*(E, C))$ and $K_1(C^*(E, C))$ are the cokernel and kernel of a map between free abelian groups given by an identity minus an adjacency matrix associated to (E, C) (see Theorem 5.2).

An important ingredient in our work is the construction of a canonical *faithful* conditional expectation $C^*(E) \rightarrow C_0(E^0)$ for any row-finite graph E (see Section 2).

Contents 1.1. We now explain in more detail the contents of this paper. The definitions of a separated graph (E, C) and its Leavitt path algebra $L(E, C)$ and full C^* -algebra $C^*(E, C)$ are given in Subsection 1.2. We construct canonical faithful conditional expectations $\Phi_E : C^*(E) \rightarrow C_0(E^0)$ for all row-finite graphs E in Section 2. The reduced graph C^* -algebras $C_{\text{red}}^*(E, C)$ are introduced in Section 3, based on the conditional expectations constructed in the previous section. Here we make use of the theory of full and reduced amalgamated free products of C^* -algebras (see [31,32]). We show that the Leavitt path algebra $L(E, C)$ embeds in the reduced graph C^* -algebra $C_{\text{red}}^*(E, C)$ (and thus also embeds in the full graph C^* -algebra $C^*(E, C)$), and that, for a trivially separated row-finite graph E , we have $C^*(E) \cong C_{\text{red}}^*(E)$ canonically (Theorem 3.8). We also exhibit a family of closed ideals of $C^*(E, C)$, parametrized by the lattice \mathcal{H} of hereditary C -saturated subsets of E^0 (Corollary 3.12). We show simplicity of the reduced graph C^* -algebras $C_{\text{red}}^*(E, C)$ for various families of finitely separated graphs in Section 4, including

the separated graphs giving rise to C^* -algebras analogous to the ones considered by Brown and McClanahan in [6,20–22]. We also show in Proposition 4.8 that there are examples of finitely separated graphs (E, C) for which $C_{\text{red}}^*(E, C)$ is simple but the lattice of hereditary C -saturated subsets of E^0 has more than two elements, so $C^*(E, C)$ is not simple. This example also shows that the structure of projections in the full and reduced graph C^* -algebras can be quite different. Section 5 is devoted to the computation of K -theory of full graph C^* -algebras. We obtain a quite satisfying formula in Theorem 5.2, using a powerful result of Thomsen [28, Theorem 2.7]. This in particular enables us to confirm a conjecture of McClanahan on the K -theory of the C^* -algebras $U_{m,n}^{\text{nc}}$. The exact relationship of the reduced graph C^* -algebras $C_{\text{red}}^*(E(m, n), C(m, n))$ and the examples considered in [22] is established in Section 6. (See Example 4.5 for the definition of the separated graph $(E(m, n), C(m, n))$.) By using this connection and some results in the literature, we establish that $C_{\text{red}}^*(E(n, n), C(n, n))$, for $n > 1$, is a simple C^* -algebra of stable rank one, with a unique tracial state, and having minimal projections (Corollary 6.3). We end the paper with a discussion of open problems.

Background definitions 1.2. Throughout, all graphs will be directed graphs of the form $E = (E^0, E^1, s, r)$, where E^0 and E^1 denote the sets of vertices and edges of E , respectively, and $s, r : E^1 \rightarrow E^0$ are the source and range maps. No cardinality restrictions are imposed on E^0 and E^1 . We follow the convention of composing paths from left to right – thus, a path in E is given in the form $\alpha = e_1 e_2 \cdots e_n$ where the $e_i \in E^1$ and $r(e_i) = s(e_{i+1})$ for $i < n$. The length of such a path is $|\alpha| := n$. Paths of length 0 are identified with the vertices of E .

Definition 1.3. (See [1, Definition 2.1].) A separated graph is a pair (E, C) where E is a graph, $C = \bigsqcup_{v \in E^0} C_v$, and C_v is a partition of $s^{-1}(v)$ (into pairwise disjoint nonempty subsets) for every vertex v . (In case v is a sink, we take C_v to be the empty family of subsets of $s^{-1}(v)$.)

If all the sets in C are finite, we say that (E, C) is a finitely separated graph. This necessarily holds if E is row-finite.

The set C is a trivial separation of E in case $C_v = \{s^{-1}(v)\}$ for each $v \in E^0 \setminus \text{Sink}(E)$. In that case, (E, C) is called a trivially separated graph or a non-separated graph. Any graph E may be paired with a trivial separation and thus viewed as a trivially separated graph.

The concept of a separated graph is related to that of an edge-colored graph, that is, a pair (E, f) where E is a (directed) graph and $f : E^1 \rightarrow N$ is a function from E^1 to some set N . Given such a pair, set

$$C_v := \{s^{-1}(v) \cap f^{-1}(n) \mid n \in N \text{ and } s^{-1}(v) \cap f^{-1}(n) \neq \emptyset\}$$

for $v \in E^0$ and $C = \bigsqcup_{v \in E^0} C_v$. Then (E, C) is a separated graph. Conversely, given a separated graph (E, C) , the map $f : E^1 \rightarrow C$ such that $e \in f(e)$ for $e \in E^1$ is an edge-coloring of E . The general definition of an edge-coloring allows edges with different sources to receive the same color. However, no relations between such edges are imposed in the C^* -algebras we construct.

Definition 1.4. (See [1, Definition 2.2].) For any separated graph (E, C) , the (complex) Leavitt path algebra of (E, C) is the complex $*$ -algebra $L(E, C)$ with generators $\{v, e \mid v \in E^0, e \in E^1\}$, subject to the following relations:

- (V) $vw = \delta_{v,w}v$ and $v = v^*$ for all $v, w \in E^0$,
- (E) $s(e)e = er(e) = e$ for all $e \in E^1$,

- (SCK1) $e^*f = \delta_{e,fr}(e)$ for all $e, f \in X, X \in C$, and
- (SCK2) $v = \sum_{e \in X} ee^*$ for every finite set $X \in C_v, v \in E^0$.

Definition 1.5. The *graph C*-algebra* of a separated graph (E, C) is the universal C*-algebra $C^*(E, C)$ with generators $\{v, e \mid v \in E^0, e \in E^1\}$, subject to the relations (V), (E), (SCK1), (SCK2). In other words, $C^*(E, C)$ is the enveloping C*-algebra of $L(E, C)$. This C*-algebra exists because the generating set consists of partial isometries.

In case (E, C) is trivially separated, $C^*(E, C)$ is just the classical graph C*-algebra $C^*(E)$.

For $v \in E^0$ and $e \in E^1$, we use the same symbols v and e for the canonical images of v and e in $C^*(E, C)$. This allows us to conveniently abbreviate various expressions – for instance, if $H \subseteq C^*(E, C)$, we can write $E^0 \cap H$ for the set of those $v \in E^0$ whose canonical images in $C^*(E, C)$ lie in H .

By definition, there is a unique *-homomorphism $L(E, C) \rightarrow C^*(E, C)$ sending the generators of $L(E, C)$ to their canonical images in $C^*(E, C)$. This *-homomorphism will be called the *canonical map from $L(E, C)$ to $C^*(E, C)$* .

The $C^*(E, C)$ construction also produces the *C*-algebras of edge-colored graphs* introduced by Duncan [12, Definition 6] (although he only considers edge-colorings with natural number values). Since Duncan allows arrows with different sources to have the same color, his construction can produce the same algebra from many different edge-colorings of a given graph.

In the present paper, we mostly restrict our attention to finitely separated graphs and their C*-algebras.

The natural category of finitely separated graphs is the category **FSGr** defined in [1, Definition 8.4]. Its objects are all finitely separated graphs (E, C) . A morphism from (F, D) to (E, C) in **FSGr** is any graph morphism $\phi : F \rightarrow E$ such that:

- (1) ϕ^0 is injective.
- (2) For each $v \in F^0$ and each $X \in D_v$, there is some $Y \in C_{\phi^0(v)}$ such that ϕ^1 induces a bijection $X \rightarrow Y$.

Condition (2) does not imply that ϕ^1 is injective, since it might map two different sets in D_v to the same member of $C_{\phi^0(v)}$.

A *complete subobject* of an object (E, C) in **FSGr** is any object (F, D) such that F is a subgraph of E and

- (3) $D_v = \{Y \in C_v \mid Y \cap F^1 \neq \emptyset\}$ for all $v \in F^0$. (In particular, this requires that each set in C which meets F^1 must be contained in F^1 .)

(This is the specialization of [1, Definition 3.4] to **FSGr**.) Observe that indeed (F, D) is a complete subobject of (E, C) if and only if F is a subgraph of E and D is a subset of C . In this case, the inclusion $F \rightarrow E$ (that is, the pair of inclusions $(F^0 \rightarrow E^0, F^1 \rightarrow E^1)$) is a morphism in **FSGr**.

Any morphism $\phi : (F, D) \rightarrow (E, C)$ in **FSGr** induces a unique C*-algebra homomorphism $C^*(\phi) : C^*(F, D) \rightarrow C^*(E, C)$ sending

$$v \mapsto \phi^0(v), \quad e \mapsto \phi^1(e) \tag{1.1}$$

for $v \in F^0$ and $e \in F^1$, since the elements $\phi^0(v), \phi^1(e)$ satisfy the defining relations of $C^*(F, D)$. The assignments $(F, D) \mapsto C^*(F, D)$ and $\phi \mapsto C^*(\phi)$ define a functor $C^*(-)$ from **FSGr** to the category $C^*\text{-alg}$ of C^* -algebras. The argument of [1, Proposition 3.6], *mutatis mutandis*, yields the following result:

Proposition 1.6. *The functor $C^*(-) : \mathbf{FSGr} \rightarrow C^*\text{-alg}$ is continuous. \square*

2. The canonical conditional expectation

In this section we define the canonical conditional expectation $\Phi_E : C^*(E) \rightarrow C_0(E^0)$ for a row-finite directed graph E and we show its faithfulness. We will use these conditional expectations (for various subgraphs) to define the reduced graph C^* -algebra of a finitely separated graph (see Section 3). In the following, we identify the C^* -algebra of the edgeless graph (E^0, \emptyset) with the function algebra $C_0(E^0)$ on the discrete set E^0 . Recall that the canonical $*$ -homomorphism $C^*((E^0, \emptyset)) \rightarrow C^*(E)$ is an embedding (e.g., [4, Theorem 2.1]). We thus identify $C_0(E^0)$ with the sub- C^* -algebra of $C^*(E, C)$ generated by E^0 .

Theorem 2.1. *Let E be a row-finite graph. Then there exists a unique conditional expectation*

$$\Phi_E : C^*(E) \longrightarrow C_0(E^0)$$

such that, for all paths γ, ν in E , we have

$$\Phi_E(\gamma\nu^*) = \begin{cases} 0 & (\text{if } \gamma \neq \nu), \\ (\prod_{i=1}^n |s^{-1}s(e_i)|)^{-1}s(\gamma) & (\text{if } \gamma = \nu = e_1e_2 \cdots e_n \text{ for some } e_i \in E^1). \end{cases} \quad (2.1)$$

Moreover the conditional expectation Φ_E is faithful.

Proof. Uniqueness is clear in case of existence.

Let E be a row-finite graph. The map Φ_E will be defined as the composition of three maps: $\Phi_E = \Phi_3 \circ \Phi_2 \circ \Phi_1$, each of which is a faithful conditional expectation. The first of these maps is the canonical conditional expectation $\Phi_1 : C^*(E) \rightarrow C^*(E)^\alpha$, where $\alpha : \mathbb{T} \rightarrow \text{Aut}(C^*(E))$ is the gauge action (e.g., [4, p. 1161]) and $C^*(E)^\alpha$ is the fixed point C^* -algebra, which is the AF-subalgebra of $C^*(E)$ generated by all the paths $\alpha\beta^*$, where α, β are finite paths in E such that $r(\alpha) = r(\beta)$ and $|\alpha| = |\beta|$. The conditional expectation Φ_1 is faithful by [24, Proposition 3.2].

The second conditional expectation Φ_2 appearing in the definition of Φ_E is the unique conditional expectation $\Phi_2 : C^*(E)^\alpha \rightarrow D$ from the AF-algebra $C^*(E)^\alpha$ to its canonical Cartan subalgebra D , where D is the commutative diagonal AF-algebra generated by $\lambda\lambda^*$, $\lambda \in E^*$. Indeed, since every AF-groupoid is amenable (see [26, Remark III.1.2]), it follows from [26, Theorem II.4.15] that D is the image of a unique conditional expectation $\Phi_2 : C^*(E)^\alpha \rightarrow D$, which is faithful. Observe that $\Phi_2(\lambda\nu^*) = 0$ if $|\lambda| = |\nu|$ and $\lambda \neq \nu$. Indeed, since $\lambda\nu^* = (\lambda\lambda^*)(\lambda\nu^*)(\nu\nu^*)$, we have $\Phi_2(\lambda\nu^*) = (\lambda\lambda^*)\Phi_2(\lambda\nu^*)(\nu\nu^*) = 0$ because D is commutative and $\lambda^*\nu = 0$.

Finally, we are going to define the third conditional expectation Φ_3 , from the commutative C^* -algebra D to its C^* -subalgebra $C_0(E^0)$. For this we need an explicit description of D . For $0 \leq r \leq \infty$, let E^r be the set of (forward) paths in E of length r , together with all paths of length $\leq r$ ending in a sink. We have truncation maps $\tau_{r,s} : E^s \rightarrow E^r$, $\gamma \mapsto \gamma[r]$, for $r \leq s \leq \infty$, where

the truncation $\gamma[r]$ of $\gamma = e_1 e_2 \cdots$ is $e_1 e_2 \cdots e_r$ (with $\gamma[0] = s(\gamma)$ and $\gamma[r] = \gamma$ if γ is a path of length $\leq r$ ending in a sink). For $r < \infty$, we put on E^r the discrete topology.

Observe that E^∞ is precisely the projective limit of the inverse system

$$\cdots \xrightarrow{\tau_{r,r+1}} E^r \xrightarrow{\tau_{r-1,r}} E^{r-1} \xrightarrow{\tau_{r-2,r-1}} \cdots \xrightarrow{\tau_{1,2}} E^1 \xrightarrow{\tau_{0,1}} E^0.$$

We put on E^∞ the inverse limit topology. A basis of compact open sets for this topology is provided by the sets

$$U(\lambda) = \{\gamma \in E^\infty : \gamma[r] = \lambda\},$$

for $\lambda \in E^r$, $0 \leq r < \infty$. The maps $\tau_{r,s}$ are continuous, proper, and surjective, and clearly $\tau_{r,s} \tau_{s,t} = \tau_{r,t}$ for $r \leq s \leq t$.

By [18] (see also [17]), $D = C_0(E^\infty)$. We have

$$D = C_0(E^\infty) = \varprojlim C_0(E^r).$$

We next define a positive integer n_λ for each finite path λ in E . If the length of λ is zero, then we set $n_\lambda := 1$. If $\lambda = e_1 \cdots e_t$ is a path of positive length, we set

$$n_\lambda := \prod_{i=1}^{|\lambda|} |s^{-1}s(e_i)|.$$

Let $\Phi^t : C_0(E^t) \rightarrow C_0(E^0)$ be the map defined as follows:

$$\Phi^t(f)(v) = \sum_{\lambda \in E^t, s(\lambda)=v} \frac{1}{n_\lambda} f(\lambda), \tag{2.2}$$

for $f \in C_0(E^t)$ and $v \in E^0$.

Using that $\sum_{\lambda \in E^t, s(\lambda)=v} \frac{1}{n_\lambda} = 1$ for every $0 \leq t < \infty$ and every $v \in E^0$, one can easily check that Φ^t is a positive, contractive linear map, and clearly $\Phi^t(f) = f$ for every $f \in C_0(E^0)$. By Tomiyama’s theorem (see e.g. [7, Theorem 1.5.10]), we get that Φ^t is a conditional expectation for all $t \geq 0$. Note that Φ^t is faithful for all t .

We check now that the conditional expectations Φ^t are compatible with the maps in the inductive system. Let $\iota_{t+1,t} : C_0(E^t) \rightarrow C_0(E^{t+1})$ be the natural inclusion map. For $f \in C_0(E^t)$ and $v \in E^0$, we have

$$\begin{aligned} \Phi^{t+1}(\iota_{t+1,t}(f))(v) &= \sum_{\lambda \in E^{t+1}, s(\lambda)=v} \frac{1}{n_\lambda} f(\lambda[t]) \\ &= \sum_{\substack{\gamma \in E^t, |\gamma|=t \\ s(\gamma)=v, r(\gamma) \notin \text{Sink}(E)}} \frac{|s^{-1}r(\gamma)|}{n_\gamma |s^{-1}r(\gamma)|} f(\gamma) + \sum_{\substack{\gamma \in E^t, |\gamma| \leq t \\ s(\gamma)=v, r(\gamma) \in \text{Sink}(E)}} \frac{1}{n_\gamma} f(\gamma) \\ &= \Phi^t(f)(v), \end{aligned}$$

which proves that $\Phi^{t+1}(\iota_{t+1,t}(f)) = \Phi^t(f)$ for $f \in C_0(E^t)$, as desired.

Since every Φ^t is contractive and positive, we conclude that there is a unique contractive, positive linear map $\Phi_3 : C_0(E^\infty) \rightarrow C_0(E^0)$ extending all Φ^t 's. This map is therefore a conditional expectation from $D = C_0(E^\infty)$ onto $C_0(E^0)$. We now observe that Φ_3 is faithful. Indeed since D is a commutative C^* -algebra of real rank zero, given any positive nonzero element a in D , there are a positive real number ϵ and a nonzero projection p in D such that $\epsilon \cdot p \leq a$. Since $\Phi_3(p) \neq 0$ for all nonzero projections p in D , it follows that Φ_3 is faithful.

In conclusion, we have obtained three faithful conditional expectations $\Phi_i, i = 1, 2, 3$, with

$$C^*(E) \xrightarrow{\Phi_1} C^*(E)^\alpha \xrightarrow{\Phi_2} D \xrightarrow{\Phi_3} C_0(E^0)$$

and so $\Phi_E := \Phi_3 \circ \Phi_2 \circ \Phi_1$ is a faithful conditional expectation from $C^*(E)$ onto $C_0(E^0)$.

It remains to check (2.1). Let γ and ν be two (finite) paths in E with $r(\gamma) = r(\nu)$. If $|\gamma| \neq |\nu|$, then $\Phi_1(\gamma\nu^*) = 0$ and thus $\Phi_E(\gamma\nu^*) = 0$. If $|\gamma| = |\nu|$ but $\gamma \neq \nu$ then

$$\Phi_E(\gamma\nu^*) = \Phi_3(\Phi_2(\Phi_1(\gamma\nu^*))) = \Phi_3(\Phi_2(\gamma\nu^*)) = \Phi_3(0) = 0.$$

Finally, if $\gamma = e_1 \cdots e_t$ is a path of length t in E , then $\gamma\gamma^*$ corresponds to the characteristic function of $\{\gamma\}$ in $C_0(E^t)$, and thus we get from (2.2) that

$$\Phi_E(\gamma\gamma^*) = \Phi_3(\gamma\gamma^*) = \left(\prod_{i=1}^t |s^{-1}s(e_i)| \right)^{-1} s(\gamma),$$

establishing (2.1) also in this case. \square

Definition 2.2. If E is a row-finite graph, we call the conditional expectation Φ_E of Theorem 2.1 the *canonical conditional expectation from $C^*(E)$ to $C_0(E^0)$* .

3. C^* -algebras of separated graphs

Assume that (E, C) is a separated graph. In this section, we develop a characterization of $C^*(E, C)$ as an amalgamated free product of ordinary graph C^* -algebras. This will enable us to define the reduced graph C^* -algebra $C_{\text{red}}^*(E, C)$ when (E, C) is finitely separated. We will show that for a trivially separated row-finite graph E , the reduced graph C^* -algebra agrees with the non-reduced one.

Set $A_0 = C_0(E^0) = C^*(E^0, \emptyset)$. For each $X \in C$, consider the graph C^* -algebra $A_X = C^*(E_X)$, where E_X is the subgraph of E with $(E_X)^0 = E^0$ and $(E_X)^1 = X$. We have natural $*$ -homomorphisms

$$A_0 \longrightarrow A_X \longrightarrow C^*(E, C)$$

arising from the inclusions $(E^0, \emptyset) \rightarrow E_X$ and $(E_X, \{X\}) \rightarrow (E, C)$.

Let \mathcal{C} be a category, and consider an object C_0 in \mathcal{C} and a family $(C_i)_{i \in I}$ of objects in \mathcal{C} , with morphisms $f_i : C_0 \rightarrow C_i$. Then the *amalgamated coproduct of $(C_i)_{i \in I}$ over C_0* is an object C in \mathcal{C} , together with morphisms $g_i : C_i \rightarrow C$ such that $g_i \circ f_i = g_{i'} \circ f_{i'}$ for all $i, i' \in I$, which are universal in the following sense: Given any other family of morphisms $h_i : C_i \rightarrow D$ such that $h_i \circ f_i = h_{i'} \circ f_{i'}$ for all $i, i' \in I$, there is a unique $h : C \rightarrow D$ such that $h_i = h \circ g_i$ for all $i \in I$.

We now show that $C^*(E, C)$ is an amalgamated coproduct of the C^* -algebras $C^*(E_X)$. This is the same idea (and proof) as in Duncan’s Theorem 1 [12], except that we express $C^*(E, C)$ as a coproduct of smaller algebras (but more of them) than Duncan uses.

Proposition 3.1. *Let (E, C) be a separated graph, and consider $A_0 = C_0(E^0)$ and $A_X = C^*(E_X)$ as above. Then $C^*(E, C)$, together with the natural $*$ -homomorphisms $f_X : A_X \rightarrow C^*(E, C)$, is the amalgamated coproduct of the family $(A_X)_{X \in C}$ over the C^* -algebra A_0 in the category C^* -alg.*

Proof. We have to verify the universal property, so for $X \in C$ let $h_X : A_X \rightarrow D$ be a $*$ -homomorphism from A_X to a C^* -algebra D such that all compositions $A_0 \rightarrow A_X \rightarrow D$ give the same map h_0 . We then have a family $(h_0(v))_{v \in E^0}$ of orthogonal projections in D and a family $(h_X(e))_{e \in X}$ of partial isometries in D for each $X \in C$, satisfying the relations (V), (E), (SCK1), (SCK2). By the universal property of $C^*(E, C)$, it follows that there exists a unique $*$ -homomorphism $h : C^*(E, C) \rightarrow D$ such that $h(v) = h_0(v)$ for all $v \in E^0$ and $h(e) = h_X(e)$ for all $e \in X$, for any $X \in C$. It follows that $h \circ f_X = h_X$ for all $X \in C$, and so $C^*(E, C)$ is the amalgamated coproduct of the family $(A_X)_{X \in C}$ over A_0 . \square

Remark 3.2. The same proof as above shows that $L(E, C)$ is the amalgamated coproduct of the family $(L(E_X))_{X \in C}$ over the $*$ -algebra $L_0 = \bigoplus_{v \in E^0} \mathbb{C}v$, in the category of complex $*$ -algebras.

Definition 3.3. Voiculescu defined in [31] the *reduced amalgamated product* of a nonempty family $(A_\iota, \Phi_\iota)_{\iota \in I}$ of unital C^* -algebras containing a unital subalgebra A_0 with conditional expectations $\Phi_\iota : A_\iota \rightarrow A_0$. The reduced amalgamated product (A, Φ) is uniquely determined by the following conditions:

- (1) A is a unital C^* -algebra, and there are unital $*$ -homomorphisms $\sigma_\iota : A_\iota \rightarrow A$ such that $\sigma_\iota|_{A_0} = \sigma_{\iota'}|_{A_0}$ for all $\iota, \iota' \in I$. Moreover the map $\sigma_\iota|_{A_0}$ is injective and we identify A_0 with its image in A through this map.
- (2) A is generated by $\bigcup_{\iota \in I} \sigma_\iota(A_\iota)$.
- (3) $\Phi : A \rightarrow A_0$ is a conditional expectation such that $\Phi \circ \sigma_\iota = \Phi_\iota$ for all $\iota \in I$.
- (4) For $(\iota_1, \dots, \iota_n) \in \Lambda(I)$ and $a_j \in \ker \Phi_{\iota_j}$ we have $\Phi(\sigma_{\iota_1}(a_1) \cdots \sigma_{\iota_n}(a_n)) = 0$. Here, $\Lambda(I)$ denotes the set of all finite tuples $(\iota_1, \dots, \iota_n) \in \bigsqcup_{n=1}^\infty I^n$ such that $\iota_i \neq \iota_{i+1}$ for $i = 1, \dots, n - 1$.
- (5) If $c \in A$ is such that $\Phi(a^*c^*ca) = 0$ for all $a \in A$, then $c = 0$.

The *full amalgamated product* $*_{A_0} A_\iota$ is by definition the amalgamated coproduct of the family $(A_\iota)_{\iota \in I}$ over A_0 in the category of unital C^* -algebras. By (1), there is a unique $*$ -homomorphism $\sigma : *_{A_0} A_\iota \rightarrow A$ such that $\sigma_\iota = \sigma \circ f_\iota$ for all $\iota \in I$, where $f_\iota : A_\iota \rightarrow *_{A_0} A_\iota$ are the canonical maps, and by (2) this map is surjective. We also have a canonical map $*_{A_0}^{\text{alg}} A_\iota \rightarrow *_{A_0} A_\iota$, where $*_{A_0}^{\text{alg}} A_\iota$ denotes the *algebraic* amalgamated free product (which is the amalgamated coproduct of $(A_\iota)_{\iota \in I}$ over A_0 in the category of unital \mathbb{C} -algebras).

We now briefly recall the construction in [31]. Let $M_\iota = L^2(A_\iota, \Phi_\iota)$ be the Hilbert A_0 -bimodule given by the GNS-construction, where the action of A_0 on the left is given by restricting to A_0 the canonical action of A_ι on M_ι . We have $M_\iota = A_0 \oplus M_\iota^0$ as a Hilbert bimodule, and the

Hilbert A_0 -module M is defined by

$$M = A_0 \oplus \bigoplus_{(l_1, \dots, l_n) \in \Lambda(I)} M_{l_1}^0 \otimes_{A_0} \cdots \otimes_{A_0} M_{l_n}^0.$$

There are representations $\lambda_l : A_l \rightarrow \mathcal{L}(M)$ corresponding to the action of A_l on terms with left hand factor M_l , see for instance [31,16], and $*_{A_0}^{\text{red}} A_l$ is defined as the C^* -subalgebra of $\mathcal{L}(M)$ generated by $\bigcup_{l \in I} \lambda_l(A_l)$. We have a cyclic vector $\xi := 1_{A_0}$ in M such that $a \cdot \xi = \hat{a}$ for all $a \in A_l$, where \hat{a} denotes the copy of $a \in A_l$ in $M_l \subseteq M$.

If all the kernels of the GNS representations are 0, then the maps σ_l are isometries, and we can identify each A_l with its image in A .

Preparation 3.4. We are now going to define the reduced graph C^* -algebra $C_{\text{red}}^*(E, C)$ of the finitely separated graph (E, C) . For a C^* -algebra A , we will denote by \tilde{A} the minimal unital C^* -algebra containing A , that is the subalgebra of the multiplier algebra $M(A)$ of A generated by A and $1_{M(A)}$.

Set $B_0 = \tilde{A}_0$, and $B_X = \tilde{A}_X$ for $X \in C$, where, as above, $A_0 = C_0(E^0)$ and $A_X = C^*(E_X)$. Then the canonical conditional expectation $\Phi_X := \Phi_{E_X} : A_X \rightarrow A_0$ constructed in Section 2 extends canonically to a conditional expectation $\Phi_X : B_X \rightarrow B_0$ (see e.g. [7, Proposition 2.2.1]). Since $\Phi_X : A_X \rightarrow A_0$ is faithful, it follows that its extension to B_X is also faithful. Now we consider the reduced amalgamated product (B, Φ) of the family $(B_X, \Phi_X)_{X \in C}$. Since all the conditional expectations Φ_X are faithful, it follows from [16, Theorem 2.1] that the canonical conditional expectation $\Phi : B \rightarrow \tilde{A}_0$ is faithful.

Definition 3.5. Let (E, C) be a finitely separated graph, and let A_0, B_0, A_X, B_X be as defined above, for $X \in C$. Consider the reduced amalgamated product (B, Φ) of the family $(B_X, \Phi_X)_{X \in C}$. Then the *reduced graph C^* -algebra* $C_{\text{red}}^*(E, C)$ is the C^* -subalgebra of B generated by $\bigcup_{X \in C} A_X$ in B (where we identify each A_X with its isometric image in B). Observe that there is a faithful canonical conditional expectation $\Phi : C_{\text{red}}^*(E, C) \rightarrow A_0$, such that $\Phi|_{A_X} = \Phi_X$ for all $X \in C$.

As with $C^*(E, C)$ (cf. Definition 1.5), we use the same symbols to denote vertices and edges of E as for their canonical images in $C_{\text{red}}^*(E, C)$.

We do not address here the question of extending Definition 3.5 to a functor from **FSGr** to C^* -alg. However, several natural maps related to this possible functor will be needed, as follows.

First, given a finitely separated graph (E, C) , observe that the natural images in $C_{\text{red}}^*(E, C)$ of the vertices and edges of E satisfy the defining relations of the \mathbb{C} -algebra $L(E, C)$. Hence, there is a unique $*$ -homomorphism $L(E, C) \rightarrow C_{\text{red}}^*(E, C)$ that sends all vertices and edges of E to their canonical images in $C_{\text{red}}^*(E, C)$. We refer to this map as the *canonical map* from $L(E, C)$ to $C_{\text{red}}^*(E, C)$. For the same reason, we obtain a *canonical map* $C^*(E, C) \rightarrow C_{\text{red}}^*(E, C)$, and the canonical map $L(E, C) \rightarrow C_{\text{red}}^*(E, C)$ is the composition of the canonical maps $L(E, C) \rightarrow C^*(E, C) \rightarrow C_{\text{red}}^*(E, C)$.

Next, suppose that E is a row-finite graph, viewed as a trivially separated graph (E, C) where $C_v = \{s^{-1}(v)\}$ for all $v \in E^0 \setminus \text{Sink}(E)$. We then define $C_{\text{red}}^*(E) := C_{\text{red}}^*(E, C)$. From the previous paragraph, we obtain a *canonical map* $C^*(E) \rightarrow C_{\text{red}}^*(E)$. We prove in Theorem 3.8 that this map is an isomorphism.

The final canonical map we require is given in the following lemma.

Lemma 3.6. *Let (F, D) be a complete subobject of an object (E, C) in **FSGr**, such that $E^0 = F^0$. Then there is a natural embedding of $C_{\text{red}}^*(F, D)$ into $C_{\text{red}}^*(E, C)$ such that $E^0 \cap C_{\text{red}}^*(F, D)$ and $F^1 \cap C_{\text{red}}^*(F, D)$ are sent to their natural images in $E^0 \cap C_{\text{red}}^*(E, C)$ and $E^1 \cap C_{\text{red}}^*(E, C)$.*

Proof. Write $A_0 = C_0(E^0) = C_0(F^0)$, and denote by M and M' the Hilbert A_0 -modules corresponding to (E, C) and (F, D) respectively. For $X \in D$, let $\lambda'_X : \tilde{A}_X \rightarrow \mathcal{L}(M')$ be the canonical representation of \tilde{A}_X on M' , and for $Y \in C$, let $\lambda_Y : \tilde{A}_Y \rightarrow \mathcal{L}(M)$ be the canonical representation of \tilde{A}_Y on M .

Let B be the C^* -subalgebra of $C_{\text{red}}^*(E, C)$ generated by $\bigcup_{X \in D} \lambda_X(\tilde{A}_X)$, and let $\Phi_B : B \rightarrow A_0$ denote the restriction of $\Phi_{(E,C)} : C_{\text{red}}^*(E, C) \rightarrow A_0$ to B . Note that (B, Φ_B) satisfies conditions (1)–(5) of Definition 3.3 with respect to the family $(\tilde{A}_X, \Phi_X)_{X \in D}$ ((5) is trivially satisfied because $\Phi_{(E,C)}$ is faithful). Since these properties determine the reduced amalgamated product, we obtain an isomorphism $\varphi : (C_{\text{red}}^*(F, D))^{\sim} \rightarrow B$ such that $\Phi_B \circ \varphi = \Phi_{(F,D)}$ and $\varphi \lambda'_X = \lambda_X$ for all $X \in D$. It follows that φ restricts to an isomorphism from $C_{\text{red}}^*(F, D)$ to the C^* -subalgebra of $C_{\text{red}}^*(E, C)$ generated by $\bigcup_{X \in D} \lambda_X(A_X)$. \square

The proof of the following lemma is straightforward.

Lemma 3.7. *Assume that (F, D) is a complete subobject of an object (E, C) in **FSGr**, such that $E^1 = F^1$ and $C = D$. Then*

$$C_{\text{red}}^*(E, C) \cong C_{\text{red}}^*(F, D) \times C_0(E^0 \setminus F^0). \quad \square$$

We are now ready to establish one of our main results. In particular, this provides an extension of [29, Theorem 7.3] to finitely separated graphs. It implies that the linear basis of the dense subalgebra $L(E, C)$ explicitly exhibited in [1, Corollary 2.8] is linearly independent in $C^*(E, C)$. Thus, the paths in E are linearly independent in $C^*(E, C)$, and the vertices of E constitute a set of pairwise orthogonal nonzero projections in $C^*(E, C)$.

Theorem 3.8. *Let (E, C) be a finitely separated graph.*

- (1) *The canonical map $L(E, C) \rightarrow C_{\text{red}}^*(E, C)$ is injective, and hence so is the canonical map $L(E, C) \rightarrow C^*(E, C)$.*
- (2) *If E is a (non-separated) row-finite graph, then the canonical map $C^*(E) \rightarrow C_{\text{red}}^*(E)$ is an isomorphism.*

Proof. Throughout, set $A_0 := C^*(E^0, \emptyset) = C_0(E^0)$.

(1) We first consider the case where E^0 is finite. In this case, A_0 is a commutative finite-dimensional C^* -algebra, and $A_0 = L(E^0, \emptyset)$. Let $\psi : L(E, C) \rightarrow C_{\text{red}}^*(E, C)$ be the canonical map, and set $L_X := L(E_X) = A_0 \oplus L_X^0$ for $X \in C$, where $L_X^0 = \ker(\Phi_X)|_{L_X}$. We will denote algebraic tensor products by \odot .

We have

$$L(E, C) \cong *_{A_0}^{\text{alg}} L_X = A_0 \oplus \bigoplus_{(X_1, \dots, X_n) \in \Lambda(C)} L_{X_1}^0 \odot_{A_0} \cdots \odot_{A_0} L_{X_n}^0,$$

and we want to show that ψ embeds each of the terms $L_{X_1}^{\circ} \circlearrowleft_{A_0} \cdots \circlearrowleft_{A_0} L_{X_n}^{\circ}$ into the corresponding $M_{X_1}^{\circ} \otimes_{A_0} \cdots \otimes_{A_0} M_{X_n}^{\circ}$, where $M_X := L^2(C^*(E_X), \Phi_X)$ for all $X \in C$. For $(X_1, \dots, X_n) \in \Lambda(C)$ and $a_i \in L_{X_i}^{\circ}, i = 1, \dots, n$, we have

$$\psi(a_1 \circlearrowleft_{A_0} \cdots \circlearrowleft_{A_0} a_n)1_{A_0} = \hat{a}_1 \otimes_{A_0} \cdots \otimes_{A_0} \hat{a}_n \in M_{X_1}^{\circ} \otimes_{A_0} \cdots \otimes_{A_0} M_{X_n}^{\circ}.$$

Hence, it suffices to show that, for z in the algebraic tensor product $L_{X_1}^{\circ} \circlearrowleft_{\mathbb{C}} \cdots \circlearrowleft_{\mathbb{C}} L_{X_n}^{\circ}$, we have $\langle z, z \rangle = 0$ if and only if z belongs to the kernel K_n of the natural map

$$L_{X_1}^{\circ} \circlearrowleft_{\mathbb{C}} \cdots \circlearrowleft_{\mathbb{C}} L_{X_n}^{\circ} \longrightarrow L_{X_1}^{\circ} \circlearrowleft_{A_0} \cdots \circlearrowleft_{A_0} L_{X_n}^{\circ}, \tag{3.1}$$

cf. the proof of [19, Proposition 4.5]. We proceed by induction on n . If $n = 1$ then the result follows from the fact that Φ_{X_1} is faithful and $L_{X_1} \subseteq C^*(E_{X_1})$, so that $L_{X_1}^{\circ} \subseteq M_{X_1}^{\circ}$. Assume that $n > 1$ and that $L_{X_1}^{\circ} \circlearrowleft_{A_0} \cdots \circlearrowleft_{A_0} L_{X_{n-1}}^{\circ}$ embeds in $M_{X_1}^{\circ} \otimes_{A_0} \cdots \otimes_{A_0} M_{X_{n-1}}^{\circ}$. The map in (3.1) is the composition of the linear maps

$$(L_{X_1}^{\circ} \circlearrowleft_{\mathbb{C}} \cdots \circlearrowleft_{\mathbb{C}} L_{X_{n-1}}^{\circ}) \circlearrowleft_{\mathbb{C}} L_{X_n}^{\circ} \longrightarrow (L_{X_1}^{\circ} \circlearrowleft_{A_0} \cdots \circlearrowleft_{A_0} L_{X_{n-1}}^{\circ}) \circlearrowleft_{\mathbb{C}} L_{X_n}^{\circ} \tag{3.2}$$

and

$$(L_{X_1}^{\circ} \circlearrowleft_{A_0} \cdots \circlearrowleft_{A_0} L_{X_{n-1}}^{\circ}) \circlearrowleft_{\mathbb{C}} L_{X_n}^{\circ} \longrightarrow (L_{X_1}^{\circ} \circlearrowleft_{A_0} \cdots \circlearrowleft_{A_0} L_{X_{n-1}}^{\circ}) \circlearrowleft_{A_0} L_{X_n}^{\circ}. \tag{3.3}$$

Write $N_0 := L_{X_1}^{\circ} \circlearrowleft_{A_0} \cdots \circlearrowleft_{A_0} L_{X_{n-1}}^{\circ}$. By the induction hypothesis, N_0 embeds in the Hilbert A_0 -module $N := M_{X_1}^{\circ} \otimes_{A_0} \cdots \otimes_{A_0} M_{X_{n-1}}^{\circ}$. The Hilbert A_0 -module $M_{X_1}^{\circ} \otimes_{A_0} \cdots \otimes_{A_0} M_{X_n}^{\circ}$ is the interior tensor product $N \otimes_{A_0} M_{X_n}^{\circ}$, so that is the completion of the inner-product module $(N \circlearrowleft_{\mathbb{C}} M_{X_n}^{\circ})/Y$, where

$$Y := \{z \in N \circlearrowleft_{\mathbb{C}} M_{X_n}^{\circ} : \langle z, z \rangle = 0\}$$

and $\langle \cdot, \cdot \rangle$ is the sesquilinear form on $N \circlearrowleft_{\mathbb{C}} M_{X_n}^{\circ}$ defined by

$$\langle n_1 \otimes m_1, n_2 \otimes m_2 \rangle = \langle n_2, \phi((n_1, m_1))m_2 \rangle,$$

for $n_1, n_2 \in N, m_1, m_2 \in M_{X_n}^{\circ}$, where $\phi : A_0 \rightarrow \mathcal{L}(M_{X_n}^{\circ})$ is the map given by the left action of A_0 on $M_{X_n}^{\circ}$.

Now we follow the proof of [19, Proposition 4.5]. Assume that

$$z = \sum_{i=1}^k x_i \otimes y_i \in N_0 \circlearrowleft_{\mathbb{C}} L_{X_n}^{\circ} \subseteq N \circlearrowleft_{\mathbb{C}} M_{X_n}^{\circ}$$

satisfies that $\langle z, z \rangle = 0$. Let $x = (x_1, \dots, x_k) \in N_0^k \subseteq N^k$. As in [19, proof of 4.5], N^k is a Hilbert $M_k(A_0)$ -module and we have

$$\langle z, z \rangle = \langle y, \phi^{(k)}(X)y \rangle,$$

where $y = (y_1, \dots, y_n) \in (L_{X_n}^0)^k$ and $X = (\langle x_i, x_j \rangle) = \langle x, x \rangle_{M_k(A_0)}$. Since $M_k(A_0)$ is a finite-dimensional C^* -algebra, there are a projection E and a positive element B in $M_k(A_0)$ such that

$$BX = E, \quad XE = X.$$

It follows that $x E = x$ and that $\phi^{(k)}(E)y = 0$. This shows that z belongs to the subspace of $N_0 \odot_{\mathbb{C}} L_{X_n}^0$ generated by all elements of the form $na \otimes m - n \otimes \phi(a)m$, $n \in N_0$, $m \in L_{X_n}^0$, $a \in A_0$, that is, to the kernel of the map (3.3).

Finally, assume that $z \in (L_{X_1}^0 \odot_{\mathbb{C}} \dots \odot_{\mathbb{C}} L_{X_{n-1}}^0) \odot_{\mathbb{C}} L_{X_n}^0$ is such that $\langle z, z \rangle = 0$. Let \bar{z} be the image of z under the map (3.2). Then $\langle \bar{z}, \bar{z} \rangle = 0$ and by what we have just proven,

$$\bar{z} = \sum_j (\bar{z}_j a_j \otimes y_j - \bar{z}_j \otimes a_j y_j)$$

for some $z_j \in L_{X_1}^0 \odot_{\mathbb{C}} \dots \odot_{\mathbb{C}} L_{X_{n-1}}^0$, $y_j \in L_{X_n}^0$, $a_j \in A_0$. It follows that

$$z - \sum_j (z_j a_j \otimes y_j - z_j \otimes a_j y_j) \in K_{n-1} \odot_{\mathbb{C}} L_{X_n}^0 \subseteq K_n.$$

Since $z_j a_j \otimes y_j - z_j \otimes a_j y_j \in K_n$ for all j , we conclude that $z \in K_n$, as desired.

This concludes the proof in the case where E^0 is finite. If E^0 is infinite, then by [1, Proposition 3.6] we can write $L(E, C) = \varinjlim L(F, D)$, where (F, D) ranges over all the finite complete subobjects of (E, C) , and all the limit maps $L(F, D) \rightarrow L(E, C)$ are injective.

For a finite complete subobject (F, D) of (E, C) , the canonical map $L(F, D) \rightarrow C_{\text{red}}^*(F, D)$ is injective, as proved above. Let F' be the subgraph of E with $(F')^0 = E^0$ and $(F')^1 = F^1$. Then the canonical map $C_{\text{red}}^*(F, D) \rightarrow C_{\text{red}}^*(E, C)$ is the composition of the canonical maps $C_{\text{red}}^*(F, D) \rightarrow C_{\text{red}}^*(F', D)$ and $C_{\text{red}}^*(F', D) \rightarrow C_{\text{red}}^*(E, C)$. By Lemmas 3.7 and 3.6, both of the latter maps are injective and so the canonical map $L(F, D) \rightarrow C_{\text{red}}^*(E, C)$ is also injective. Since $L(E, C) = \varinjlim L(F, D)$, it follows that the canonical map $L(E, C) \rightarrow C_{\text{red}}^*(E, C)$ is injective, as desired.

(2) Since E is a non-separated graph, we identify C with $E^0 \setminus \text{Sink}(E)$, by corresponding $\{s^{-1}(v)\}$ to v for non-sinks $v \in E^0$. We shall write $E_v = E_{s^{-1}(v)}$ for $v \in C$. Set $n_v := |s^{-1}(v)|$ and $L_v := L(E_v)$, and set $A_v := C^*(E_v)$, and $B_v := \tilde{A}_v$. Let $B = C^*(E)^\sim$ and recall that we have a faithful conditional expectation $\Phi : B \rightarrow \tilde{A}_0$ (Theorem 2.1). To establish the desired isomorphism, it is enough to show that (B, Φ) satisfies conditions (1)–(5) of Definition 3.3, because these conditions characterize completely the reduced amalgamated product of the family $(B_v, \Phi_v)_{v \in C}$. All the conditions are immediate, with the exception of condition (4). To show condition (4), take a sequence of vertices v_1, \dots, v_n in C , with $n \geq 2$, such that $v_i \neq v_{i+1}$ for $i = 1, \dots, n - 1$. We have to show that $\Phi(a_1 a_2 \dots a_n) = 0$ when $a_i \in \ker \Phi_{v_i}$ for $i = 1, \dots, n$. Since $L_{v_i}^0 := L_{v_i} \cap \ker \Phi_{v_i}$ is dense in $\ker \Phi_{v_i}$, it suffices to prove this statement for all choices of $a_i \in L_{v_i}^0$, $i = 1, \dots, n$.

Consider $v \in C$, and note that any path of positive length in E_v consists of either a sequence of loops at v or else a sequence of loops at v followed by one edge from v to a different vertex. In particular, all paths of positive length in E_v start at v . Observe that every element of L_v is a linear combination of terms of the following five types:

- (1) Paths γ in E_v of positive length.
- (2) Paths v^* , where v is a path in E_v of positive length.
- (3) Paths γv^* , where γ and v are distinct paths in E_v of positive length.
- (4) Terms $\gamma(ee^* - n_v^{-1}v)\gamma^*$, where $e \in s^{-1}(v)$ and γ is a path in E_v from v to v .
- (5) Vertices $w \in E^0$.

All terms of types (1)–(4) are in L_v^0 (recall formula (2.1)), whereas $\Phi_v(w) = w$ for $w \in E^0$. Hence, the terms of types (1)–(4) span L_v^0 .

Returning to our previous discussion, we see that it is enough to show that $\Phi(a_1 a_2 \cdots a_n) = 0$ for all choices of $a_i \in L_{v_i}^0$ where each a_i has one of the forms (1)–(4). We may also assume that $a_1 a_2 \cdots a_n \neq 0$. It is easy to verify the following:

- If a_i has one of the forms (1), (3), or (4) and $i > 1$, then a_{i-1} is necessarily of type (1).
- If a_i has one of the forms (2), (3), or (4) and $i < n$, then a_{i+1} is necessarily of type (2).

It follows that at most one a_i can be of type (4). If such a term occurs, then

$$a_1 a_2 \cdots a_n = \gamma(ee^* - n_v^{-1}v)v^*$$

for some $v \in E^0$, some $e \in s^{-1}(v)$, and some paths γ, v in E that end at v . In this case, it is clear that $\Phi(a_1 a_2 \cdots a_n) = 0$. (Consider the cases $\gamma = v$ and $\gamma \neq v$ separately.)

If no a_i is of type (4), then one of the following holds: $a_1 a_2 \cdots a_n = \gamma$ for some path γ in E of positive length; or $a_1 a_2 \cdots a_n = v^*$ for some path v in E of positive length; or

$$a_1 a_2 \cdots a_n = \gamma_1 \gamma_2 \cdots \gamma_j v_k^* v_{k+1}^* \cdots v_n^*$$

where $k = j$ or $k = j + 1$, and each γ_i or v_i is a path of positive length in E_{v_i} . Obviously $\Phi(a_1 a_2 \cdots a_n) = 0$ in the first two cases, and it holds in the third case provided $\gamma_1 \gamma_2 \cdots \gamma_j \neq v_n v_{n-1} \cdots v_k$. Thus, it suffices to assume that the third case obtains, and that $\gamma_1 \gamma_2 \cdots \gamma_j = v_n v_{n-1} \cdots v_k$, and to derive a contradiction.

We cannot have $j = 1$ and $k = n$, since then $n = 2$ while γ_1 and v_2 have different starting vertices. We cannot have $j = 1$ and $k < n$, since γ_1 only changes vertices on its terminal edge, whereas v_n must change vertices once, and the following path v_{n-1} has at least one edge. Thus $j > 1$, and similarly $k < n$. Since $\gamma_1 \gamma_2 \neq 0$, we have $r(\gamma_1) = v_2 \neq v_1 = s(\gamma_1)$, so γ_1 consists of a sequence of loops at v_1 followed by an edge from v_1 to v_2 . Similarly, v_n consists of a sequence of loops at v_n followed by an edge from v_n to v_{n-1} . Thus, since $\gamma_1 \gamma_2 \cdots \gamma_j = v_n v_{n-1} \cdots v_k$, we see that $\gamma_1 = v_n$. Now $\gamma_2 \cdots \gamma_j = v_{n-1} \cdots v_k$, and we can continue in the same manner. We eventually find that $n - k + 1 = j$ and $\gamma_j = v_k$. However, $\gamma_j \neq v_{j+1}$ because these paths have different starting vertices, and $\gamma_j \neq v_j$ (in case $k = j$) by the assumption of type (3) for $a_j = \gamma_j v_j^*$. This provides the desired contradiction. \square

Suppose V is a subset of $E^0 \setminus \text{Sink}(E)$ and $\mathcal{X} : V \rightarrow C$ is a function such that $\mathcal{X}(v) \in C_v$ for every $v \in V$. Define a subgraph $E_{\mathcal{X}}$ of E so that $E_{\mathcal{X}}^0 = E^0$ and $E_{\mathcal{X}}^1 = \bigsqcup_{v \in V} \mathcal{X}(v)$. View $E_{\mathcal{X}}$ as a trivially separated graph, and note that it is a complete subobject of (E, C) .

Corollary 3.9. *For any V and \mathcal{X} as above, the induced map $C^*(E_{\mathcal{X}}) \rightarrow C_{\text{red}}^*(E_{\mathcal{X}}) \rightarrow C_{\text{red}}^*(E, C)$ is injective, and hence so is the canonical map $C^*(E_{\mathcal{X}}) \rightarrow C^*(E, C)$.*

Proof. By Theorem 3.8(2), we have $C^*(E_{\mathcal{X}}) \cong C_{\text{red}}^*(E_{\mathcal{X}})$, and so Lemma 3.6 gives that the canonical map $C^*(E_{\mathcal{X}}) \rightarrow C_{\text{red}}^*(E, C)$ is injective. \square

Remark 3.10. By [7, Corollary 4.5.4], every graph C^* -algebra of a row-finite graph is nuclear. It follows from this and [7, Corollary 4.8.3] that the reduced C^* -algebra $C_{\text{red}}^*(E, C)$ is exact for every finitely separated graph (E, C) . That $C_{\text{red}}^*(E, C)$ is not nuclear in general follows from the example

$$C_{\text{red}}^*(E, C) \cong C_{\text{red}}^*(\mathbb{F}_2),$$

where (E, C) is the separated graph with one vertex v , two edges e_1, e_2 , and $C = \{\{e_1\}, \{e_2\}\}$. (Recall that $C_{\text{red}}^*(\mathbb{F}_2)$ is not nuclear because \mathbb{F}_2 is not amenable [7, Theorem 2.6.8].)

We recall the following definitions, see e.g. [1,29].

Definition 3.11. Let (E, C) be a finitely separated graph. Recall the relation \geq defined on E^0 by setting $v \geq w$ if and only if there is a path μ in E with $s(\mu) = v$ and $r(\mu) = w$. A subset H of E^0 is called *hereditary* if $v \geq w$ and $v \in H$ always imply $w \in H$. The set H is called *saturated* if $r(s^{-1}(v)) \subseteq H$ implies $v \in H$ for any $v \in E^0$ which is not a sink or an infinite emitter. Finally, H is called *C-saturated* if $r(X) \subseteq H$ for some $X \in C_v, v \in E^0$, implies $v \in H$.

Let \mathcal{H} be the lattice of hereditary C -saturated subsets of E^0 . By [1, Theorem 6.11] there is a lattice isomorphism between \mathcal{H} and the lattice $\text{Tr}(A)$ of two-sided ideals of $L(E, C)$ generated by idempotents. In the C^* -algebra case, we are at least able to show that the analogous map $\mathcal{H} \rightarrow \mathcal{L}(C^*(E, C))$ is injective. Here, for any C^* -algebra A , we denote by $\mathcal{L}(A)$ the lattice of closed (two-sided) ideals of A . For a subset X of E^0 we denote by $I(X)$ the closed ideal of $C^*(E, C)$ generated by $X \cap C^*(E, C)$.

Corollary 3.12. *Let (E, C) be a finitely separated graph, and let \mathcal{H} be the lattice of hereditary, C -saturated subsets of E^0 . Then there is an order-embedding $\mathcal{H} \rightarrow \mathcal{L}(C^*(E, C))$, given by $H \mapsto I(H)$.*

Proof. Clearly, it suffices to show that $E^0 \cap I(H) = H$, for any $H \in \mathcal{H}$. Thus, let H be a hereditary C -saturated subset of E^0 .

We construct a finitely separated graph $(E/H, C/H)$ as in [1, Construction 6.8]. Namely, E/H is the quotient graph, that is, the subgraph of E with

$$(E/H)^0 = E^0 \setminus H \quad \text{and} \quad (E/H)^1 = r_E^{-1}(E^0 \setminus H) = E^1/H,$$

and, for $v \in (E/H)^0$, we set

$$(C/H)_v := \{X/H \mid X \in C_v\},$$

which is a partition of $s_{E/H}^{-1}(v)$, and $C/H := \bigsqcup_{v \in E^0 \setminus H} (C/H)_v$. Here, for any $X \subseteq E^1$, we denote by X/H the set $\{e \in X : r(e) \notin H\}$. Observe that $X/H \neq \emptyset$ for all $X \in C_v$ with $v \in E^0 \setminus H$, because H is C -saturated.

Since $r_E^{-1}(H) \cap C^*(E, C) \subseteq I(H)$, the cosets of the elements in $(E/H)^0 \cap C^*(E, C)$ and $(E/H)^1 \cap C^*(E, C)$ generate $C^*(E, C)/I(H)$. It is easily checked (by using the universal property of $C^*(E, C)$) that $C^*(E, C)/I(H)$ is presented by the above generators together with the defining relations of $C^*(E/H, C/H)$. Thus, we obtain an isomorphism

$$C^*(E, C)/I(H) \longrightarrow C^*(E/H, C/H)$$

sending $v + I(H) \mapsto v$ for $v \in (E/H)^0$ and $e + I(H) \mapsto e$ for $e \in (E/H)^1$. Now any vertex $v \in E^0 \setminus H$ is nonzero as an element of $L(E/H, C/H)$ (cf. [1, Corollary 2.8]). Since $L(E/H, C/H)$ embeds naturally in $C^*(E/H, C/H)$ by Theorem 3.8(1), it follows that $v \notin I(H)$. Therefore $E^0 \cap I(H) = H$, as desired. \square

4. Simplicity in $C_{\text{red}}^*(E, C)$

For a finitely separated graph (E, C) , the reduced C^* -algebra $C_{\text{red}}^*(E, C)$ has typically fewer ideals than the full C^* -algebra $C^*(E, C)$. In fact, it can easily happen that $C_{\text{red}}^*(E, C)$ is simple while $C^*(E, C)$ is not. We shall consider the two main examples from [1] and a related one, and we will show that the corresponding reduced graph C^* -algebras are indeed simple. These are somewhat exotic examples of simple C^* -algebras; for instance, one has stable rank one but not real rank zero (see Corollary 6.3). We do not know whether the others are purely infinite or have real rank zero.

We start by taking examples with only one vertex. The main tool is the following result of Avitzour (see [3, Proposition 3.1]). Since we will only use the case of faithful states, we state below the result in this case.

Proposition 4.1. *(See [3, Proposition 3.1].) Let A, B be unital C^* -algebras and ϕ, ψ faithful states on them. Let (D, Φ) be the reduced amalgamated product of (A, ϕ) and (B, ψ) (over \mathbb{C}). Let $a \in \ker \phi$ and $b \in \ker \psi$ be unitaries such that ϕ, ψ are invariant with respect to conjugation by a, b respectively. Let $c \in \ker \psi$ be a unitary such that $\psi(b^*c) = 0$.*

Then for all x in D ,

$$\Phi(x) \in \overline{\text{co}}\{u^*xu : u \text{ unitary}\},$$

where $\overline{\text{co}}$ denotes the norm-closed convex hull. It is enough to take u in the group generated by a, b, c .

It follows readily from this result that in the given situation, D must be simple. Indeed, let J be a nonzero closed ideal of D , and let x be a nonzero positive element of J . Since ϕ and ψ are faithful it follows from [13] or [16] that Φ is faithful and so Proposition 4.1 gives that J contains the invertible element $\Phi(x)$.

We apply now the result to reduced graph C^* -algebras.

As in [22], we will use the following unitaries in $M_n(\mathbb{C})$. Let λ_n be a primitive n -th root of 1, and set:

$$u_n := \text{diag}(1, \lambda_n, \dots, \lambda_n^{n-1}), \quad v_n := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & \dots & & \dots & \\ 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Proposition 4.2. *Let $n, m > 1$, and let (E, C) be the separated graph with one vertex v and with $C_v := \{X, Y\}$, where $|X| = n$ and $|Y| = m$. Then the reduced graph C^* -algebra $C_{\text{red}}^*(E, C)$ is simple.*

Proof. Set $A := \mathcal{O}_n$ and $B := \mathcal{O}_m$, where as usual \mathcal{O}_k denotes the Cuntz algebra, and identify $A = C^*(E_X)$ and $B = C^*(E_Y)$. Then $(C_{\text{red}}^*(E, C), \Phi)$ is the reduced amalgamated product of (\mathcal{O}_n, ϕ_n) and (\mathcal{O}_m, ϕ_m) , where we denote by ϕ_k the canonical faithful state on \mathcal{O}_k . There is a standard copy of $M_n(\mathbb{C})$ in \mathcal{O}_n , namely the linear span of $\{ef^* : e, f \in X\}$, and using this copy we define the unitary $a := v_n$ in A . Notice that ϕ_n is the composition of the canonical conditional expectation from \mathcal{O}_n onto the AF-algebra \mathcal{O}_n^α and the tracial state τ_n on $\mathcal{O}_n^\alpha = \varinjlim M_{n^i}(\mathbb{C})$, where α denotes the gauge action. Using this it is quite easy to show that $\phi_n(axa^*) = \phi_n(x)$ for all $x \in \mathcal{O}_n$. Indeed, ϕ_n is invariant with respect to conjugation by any unitary in \mathcal{O}_n^α . Observe that $\phi_n(a) = \text{trace}(v_n) = 0$.

Similarly, $b := v_m$ and $c := u_m$ are unitaries in $\ker \phi_m$, and ϕ_m is invariant with respect to conjugation by both b and c . Moreover, $\phi_m(b^*c) = 0$. It therefore follows from Proposition 4.1 that $C_{\text{red}}^*(E, C)$ is a simple C^* -algebra. \square

We need for our next examples a slight generalization of Proposition 4.1 for reduced amalgamated products over C^* -algebras different from \mathbb{C} . Other generalizations to this context have been obtained in [22] and [16].

Proposition 4.3. *Let A, B, A_0 be unital C^* -algebras with $A_0 \subseteq A$ and $A_0 \subseteq B$, and let $\phi : A \rightarrow A_0$ and $\psi : B \rightarrow A_0$ be faithful conditional expectations. Let (D, Φ) be the reduced amalgamated product of (A, ϕ) and (B, ψ) , and let $\pi : A *_{A_0}^{\text{alg}} B \rightarrow D$ be the natural map from the algebraic amalgamated product to D .*

*Assume there is a central projection $P \in A_0$ such that $PA_0 = \mathbb{C}P$. Let $a \in P(\ker \phi)P$ and $b \in P(\ker \psi)P$ be unitaries in PAP and PBP respectively, such that $\phi|_{PAP}, \psi|_{PBP}$ are invariant with respect to conjugation by a, b respectively. Let $c \in P(\ker \psi)P$ be a unitary in PBP such that $\psi(b^*c) = 0$. Then for all x in $\pi(P)D\pi(P)$,*

$$\Phi(x) \in \overline{\text{co}}\{u^*xu : u \text{ unitary in } \pi(P)D\pi(P)\}.$$

It is enough to take u in the group generated by a, b, c .

Proof. The proof follows the steps of that of [3, Proposition 3.1]. Let us just mention what are the main steps. Let M be the Hilbert A_0 -module arising in the construction of D (recall Definition 3.3). We identify π with the standard representation $A *_{A_0}^{\text{alg}} B \rightarrow \mathcal{L}_{A_0}(M)$.

Let $W_0 \subseteq PA *_{A_0}^{\text{alg}} B$ be the span of those words starting with an element from $P \ker \phi$ or from the constants $\mathbb{C}P$ or a multiple of b . Let W_1 be the span of those words starting with some b' in

$P \ker \psi$ such that $\psi(b^*b') = 0$. Let

$$H_i = \overline{\pi(W_i)\xi} \subseteq \pi(P)M.$$

Then $\pi(P)M = H_0 \oplus H_1$. Since for $x \in \pi(P)D\pi(P)$ we have

$$\|x\|_{\mathcal{L}_{A_0}(M)} = \|x|_{\pi(P)M}\|_{\mathcal{L}_{A_0}(\pi(P)M)},$$

we can apply the proof of [3, Proposition 3.1] to show that

$$\Phi(x) \in \overline{c0}\{u^*xu : u \text{ unitary in } \pi(P)D\pi(P)\},$$

as desired. \square

Corollary 4.4. *Assume that the conditions of the above proposition hold, and that in addition $\pi(P)$ is a full projection in D . Then D is simple.*

Proof. First, recall from [16, Theorem 2.1] that Φ is faithful. Let J be a nonzero closed ideal of D . Since $\pi(P)$ is a full projection in D , we have that $\pi(P)J\pi(P)$ is nonzero. Let x be a nonzero positive element in $\pi(P)J\pi(P)$. Then $\Phi(x)$ is a nonzero scalar multiple of P , and so it follows from Proposition 4.3 that $\pi(P) \in J$. Since $\pi(P)$ is full in D , we conclude that $J = D$. \square

The next example is related to an example considered by McClanahan in [22, Example 3.12] (see Proposition 6.1 for the precise relationship). However we use in the proof our version of Avitzour’s result (Proposition 4.3), which is simpler than the one used in [22].

Example 4.5. For integers $1 \leq m \leq n$, define the separated graph $(E(m, n), C(m, n))$, where:

- (1) $E(m, n)^0 := \{v, w\}$ (with $v \neq w$).
- (2) $E(m, n)^1 := \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\}$ (with $n + m$ distinct edges).
- (3) $s(\alpha_i) = s(\beta_j) = v$ and $r(\alpha_i) = r(\beta_j) = w$ for all i, j .
- (4) $C(m, n) = C(m, n)_v := \{X, Y\}$, where $X := \{\alpha_1, \dots, \alpha_n\}$ and $Y := \{\beta_1, \dots, \beta_m\}$.

By [1, Proposition 2.12], $L(E(m, n), C(m, n)) \cong M_{n+1}(L(m, n)) \cong M_{m+1}(L(m, n))$, where $L(m, n)$ is the classical Leavitt algebra of type (m, n) . The same argument (by way of universal properties) shows that

$$C^*(E(m, n), C(m, n)) \cong M_{n+1}(U_{m,n}^{nc}) \cong M_{m+1}(U_{m,n}^{nc}), \tag{4.1}$$

where $U_{m,n}^{nc}$ denotes the C^* -algebra generated by the entries of a universal unitary $m \times n$ matrix, as studied by Brown and McClanahan in [6,20–22].

The reduced graph C^* -algebra of $(E(m, n), C(m, n))$ is Morita equivalent to McClanahan’s example, as we will show in Section 6. (Hence, the following proposition can also be obtained as a corollary of McClanahan’s results.)

Proposition 4.6. *Let $1 < m \leq n$, and let $(E, C) := (E(m, n), C(m, n))$ be the separated graph described in Example 4.5. Then the reduced graph C^* -algebra $C_{\text{red}}^*(E(m, n), C(m, n))$ is simple.*

Proof. Set $A_0 := \mathbb{C}v \oplus \mathbb{C}w$, and identify A_0 with \mathbb{C}^2 so that v and w correspond to $(1, 0)$ and $(0, 1)$ respectively. Set $A := C^*(E_X)$ and $B := C^*(E_Y)$, and identify A and B with $M_{n+1}(\mathbb{C})$ and $M_{m+1}(\mathbb{C})$ so that v and w correspond to $\text{diag}(1, \dots, 1, 0)$ and $\text{diag}(0, \dots, 0, 1)$ in each case. The canonical conditional expectations Φ_X and Φ_Y are easily seen to correspond to the maps $\phi : M_{n+1}(\mathbb{C}) \rightarrow \mathbb{C}^2$ and $\psi : M_{m+1}(\mathbb{C}) \rightarrow \mathbb{C}^2$ given by

$$\phi([a_{ij}]) = \left(\frac{1}{n} \sum_{i=1}^n a_{ii}, a_{n+1, n+1} \right), \quad \psi([b_{ij}]) = \left(\frac{1}{m} \sum_{j=1}^m b_{jj}, b_{m+1, m+1} \right).$$

Take $P := v$, and observe that $PA_0 = \mathbb{C}P$ and that P is a full projection in both A and B , so certainly P will be a full projection in $D := A *_A^{\text{red}} B = C_{\text{red}}^*(E, C)$. Consider the unitaries $a := \text{diag}(v_n, 0)$ in $M_{n+1}(\mathbb{C})$ and $b := \text{diag}(v_m, 0)$, $c := \text{diag}(u_m, 0)$ in $M_{m+1}(\mathbb{C})$ respectively; then $a \in PAP$ and $b, c \in PBP$ with the above identifications. We have

$$\phi(a) = \psi(b) = \psi(c) = \psi(b^*c) = 0, \tag{4.2}$$

and moreover $\phi|_{PAP}$, $\psi|_{PBP}$ are invariant with respect to conjugation by a, b respectively, and thus the conditions in Proposition 4.3 are satisfied. It follows from Corollary 4.4 that $C_{\text{red}}^*(E, C)$ is a simple C^* -algebra. \square

Remark 4.7. To fill in the cases not covered by Proposition 4.6, let $n \geq 1$ and consider $(E, C) := (E(1, n), C(1, n))$. If $n > 1$, then $U_{1, n}^{\text{nc}} \cong \mathcal{O}_n$ and (4.1) implies that

$$C^*(E, C) \cong M_2(\mathcal{O}_n) \cong M_{n+1}(\mathcal{O}_n),$$

whence $C^*(E, C)$ is simple. In this case, the full and reduced C^* -algebras of (E, C) coincide, and $C_{\text{red}}^*(E, C)$ is again simple.

Since $U_{1, 1}^{\text{nc}} \cong C(\mathbb{T})$, the case $m = n = 1$ reduces to $C^*(E, C) \cong M_2(C(\mathbb{T}))$ by (4.1). Following the construction in the proof of [1, Proposition 2.12], there is an explicit isomorphism

$$\psi : C^*(E, C) \longrightarrow M_2(C(\mathbb{T})) = M_2(\mathbb{C}) \otimes_{\mathbb{C}} C(\mathbb{T})$$

sending

$$v \longmapsto e_{11}, \quad w \longmapsto e_{22}, \quad \alpha \longmapsto ze_{12}, \quad \beta \longmapsto e_{12},$$

where z is the canonical unitary generator of $C(\mathbb{T})$. We shall use this isomorphism to see that $C^*(E, C) = C_{\text{red}}^*(E, C)$. Thus, the case $m = n = 1$ is the only one for which $C_{\text{red}}^*(E, C)$ is not simple.

Identify $A := C^*(E_X)$ and $B := C^*(E_Y)$ with their canonical images in $C^*(E, C)$, and set $A_0 := \mathbb{C}v \oplus \mathbb{C}w$. There is a faithful conditional expectation

$$\phi \otimes \tau : M_2(\mathbb{C}) \otimes_{\mathbb{C}} C(\mathbb{T}) \longrightarrow \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix},$$

where $\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$ and τ is the canonical faithful trace on $C(\mathbb{T})$. This corresponds to a faithful conditional expectation $\Phi : C^*(E, C) \rightarrow A_0$, because ψ restricts to an isomorphism of A_0 onto $\begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix}$. We claim that $(C^*(E, C), \Phi)$ satisfies the conditions of Definition 3.3 to be the reduced amalgamated product of (A, Φ_X) and (B, Φ_Y) . Conditions (1) and (2) are clear, (3) is easily checked, and (5) follows from the faithfulness of Φ . To check (4), observe that since $\ker \Phi_X = \mathbb{C}\alpha \oplus \mathbb{C}\alpha^*$ and $\ker \Phi_Y = \mathbb{C}\beta \oplus \mathbb{C}\beta^*$, it suffices to show that Φ vanishes on all finite paths of the forms

$$\alpha\beta^*\alpha\beta^*\dots, \quad \alpha^*\beta\alpha^*\beta\dots, \quad \beta\alpha^*\beta\alpha^*\dots, \quad \beta^*\alpha\beta^*\alpha\dots.$$

However, ψ maps these paths to products of the form $z^k e_{ij}$ or $(z^*)^k e_{ij}$ with $k \geq 1$, and $\phi \otimes \tau$ vanishes on such products.

Therefore $(C^*(E, C), \Phi) = (A, \Phi_X) *_{A_0}^{\text{red}} (B, \Phi_Y)$ and so $C^*(E, C) = C_{\text{red}}^*(E, C)$ in this case, as claimed.

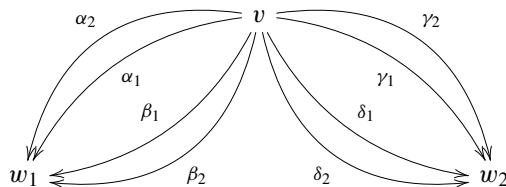
Finally, we show with another example that the structure of hereditary C -saturated subsets of E^0 is not respected in $C_{\text{red}}^*(E, C)$ in general, that is, there can be two different hereditary C -saturated subsets H_1 and H_2 which generate the same ideal of $C_{\text{red}}^*(E, C)$. This heavily contrasts with the situation for the full graph C^* -algebra $C^*(E, C)$.

Let $k, l, m, n \geq 2$ be integers. Consider the separated graph (E, C) , where

- (1) $E^0 := \{v, w_1, w_2\}$ (with 3 distinct vertices).
- (2) $E^1 := \{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l, \gamma_1, \dots, \gamma_m, \delta_1, \dots, \delta_n\}$ (with $k + l + m + n$ distinct edges).
- (3) $s(e) = v$ for all $e \in E^1$, while $r(\alpha_i) = r(\beta_j) = w_1$ for all i, j , and $r(\gamma_i) = r(\delta_j) = w_2$ for all i, j .
- (4) $C = C_v := \{X, Y\}$ where

$$X := \{\alpha_1, \dots, \alpha_k, \gamma_1, \dots, \gamma_m\}, \quad Y := \{\beta_1, \dots, \beta_l, \delta_1, \dots, \delta_n\}.$$

A picture of the graph E for the case $k = l = m = n = 2$ is shown below.



Observe that $H_1 = \{w_1\}$ and $H_2 = \{w_2\}$ are both hereditary C -saturated subsets of E^0 . However, by the next proposition, both H_1 and H_2 generate the full algebra $C_{\text{red}}^*(E, C)$.

Proposition 4.8. *Let (E, C) be the separated graph described above. Then the reduced graph C^* -algebra $C_{\text{red}}^*(E, C)$ is simple.*

$\mathbb{Z}^{E^0 \times E^0}$ where $a(v, w) := |s_E^{-1}(v) \cap r_E^{-1}(w)|$, that is, the number of arrows from v to w in E . Write A_E^t and 1 for the matrices in $\mathbb{Z}^{E^0 \times E^0 \setminus \text{Sink}(E)}$ which result from the transpose of A_E^t and from the identity matrix after removing the columns indexed by sinks. Then the K -theory of $C^*(E)$ is given by the formulas

$$K_0(C^*(E)) \cong \text{coker}(1 - A_E^t : \mathbb{Z}^{(E^0 \setminus \text{Sink}(E))} \longrightarrow \mathbb{Z}^{(E^0)}), \tag{5.2}$$

$$K_1(C^*(E)) \cong \text{ker}(1 - A_E^t : \mathbb{Z}^{(E^0 \setminus \text{Sink}(E))} \longrightarrow \mathbb{Z}^{(E^0)}) \tag{5.3}$$

[25, Theorem 3.2]. Further, the formulation of [25, Theorem 3.2] given in [30, Theorem 2.3.9] shows that the isomorphism of (5.2) sends $[v]$ to the coset of δ_v for all $v \in E^0$, where $(\delta_v)_{v \in E^0}$ denotes the canonical basis of $\mathbb{Z}^{(E^0)}$.

We now present a corresponding result for any finitely separated graph (E, C) . The *adjacency matrix* of (E, C) is the matrix $A_{(E,C)}^t := (a(v, w))_{v,w \in E^0}$ such that the entry $a(v, w)$ is the function $X \mapsto a_X(v, w)$ in \mathbb{Z}^{C_v} where $a_X(v, w)$ equals the number of arrows in X from v to w , for any $v, w \in E^0$ and $X \in C_v$. We denote by $1_C : \mathbb{Z}^{(C)} \rightarrow \mathbb{Z}^{(E^0)}$ and $A_{(E,C)}^t : \mathbb{Z}^{(C)} \rightarrow \mathbb{Z}^{(E^0)}$ the homomorphisms defined by

$$1_C(\delta_X) = \delta_v \quad \text{and} \quad A_{(E,C)}^t(\delta_X) = \sum_{w \in E^0} a_X(v, w)\delta_w \quad (v \in E^0, X \in C_v),$$

where $(\delta_X)_{X \in C}$ denotes the canonical basis of $\mathbb{Z}^{(C)}$.

With this notation, the K -theory of $C^*(E, C)$ has formulas which look very similar to the ones for the non-separated case:

Theorem 5.2. *Let (E, C) be a finitely separated graph, and adopt the notation above. Then the K -theory of $C^*(E, C)$ is given as follows:*

$$K_0(C^*(E, C)) \cong \text{coker}(1_C - A_{(E,C)}^t : \mathbb{Z}^{(C)} \longrightarrow \mathbb{Z}^{(E^0)}), \tag{5.4}$$

$$K_1(C^*(E, C)) \cong \text{ker}(1_C - A_{(E,C)}^t : \mathbb{Z}^{(C)} \longrightarrow \mathbb{Z}^{(E^0)}). \tag{5.5}$$

Further:

$$\text{The isomorphism of (5.4) sends } [v] \text{ to the coset of } \delta_v \text{ for all } v \in E^0. \tag{5.6}$$

Proof. Since K -theory is continuous, we may reduce to the case where E is a finite graph by using Proposition 1.6 and [1, Proposition 3.5 and comments after Definition 8.4]. Set $A_0 := C^*((E^0, \emptyset), \emptyset) = C_0(E^0)$, which is a finite-dimensional commutative C^* -algebra under our current assumption. There is an isomorphism $\kappa : K_0(A_0) \rightarrow \mathbb{Z}^{(E^0)}$ sending $[v] \mapsto \delta_v$ for $v \in E^0$.

For a finite separated graph (E, C) (meaning that E^0, E^1 , and C are all finite), we will show the results by induction on $|C|$. The case where $|C| \leq 1$ follows from the results for non-separated graphs. Assume that $n > 1$ and that the results are true for finite separated graphs (E', C') with $|C'| < n$. Let (E, C) be a finite separated graph with $|C| = n$, and select $X \in C_v$ for some $v \in$

$E^0 \setminus \text{Sink}(E)$. Let $C' := C \setminus \{X\}$, and consider the separated graphs (E_1, C') and $(E_2, \{X\})$, where $(E_1)^0 = (E_2)^0 = E^0$ and $(E_1)^1 = \bigsqcup_{Y \in C'} Y$, $(E_2)^1 = X$. Then we have

$$C^*(E, C) = C^*(E_1, C') *_{A_0} C^*(E_2, \{X\}),$$

relative to the canonical embeddings $i_1 : A_0 \rightarrow C^*(E_1, C')$ and $i_2 : A_0 \rightarrow C^*(E_2, \{X\})$ corresponding to the inclusion morphisms $((E^0, \emptyset), \emptyset) \rightarrow (E_1, C')$ and $((E^0, \emptyset), \emptyset) \rightarrow (E_2, \{X\})$ in **FSGr**. Therefore we can apply Thomsen’s result and the induction hypothesis to compute $K_0(C^*(E, C))$. By induction, there is a commutative diagram as follows, where $\pi_{C'}$ and $\pi_{\{X\}}$ are the obvious quotient maps.

$$\begin{CD} K_0(C^*(E_1, C')) @<i_{1*}<< K_0(A_0) @>i_{2*}>> K_0(C^*(E_2, \{X\})) \\ @V\cong VV @V\kappa VV @V\cong VV \\ \text{coker}(1_{C'} - A^t_{(E_1, C')}) @<\pi_{C'}<< \mathbb{Z}^{(E^0)} @>\pi_{\{X\}}>> \text{coker}(1_{\{X\}} - A^t_{(E_2, \{X\})}) \end{CD}$$

(The diagram is commutative because (5.6) holds for the cases (E_1, C') and $(E_2, \{X\})$.) Since $K_1(A_0) = 0$, it follows from Theorem 5.1 that $K_0(C^*(E, C))$ is isomorphic to the cokernel of the map

$$\mathbb{Z}^{(E^0)} \xrightarrow{(\pi_{C'}, \pi_{\{X\}})} \mathbb{Z}^{(E^0)} / (1_{C'} - A^t_{(E_1, C')})\mathbb{Z}^{(C')} \oplus \mathbb{Z}^{(E^0)} / (1_{\{X\}} - A^t_{(E_2, \{X\})})\mathbb{Z}^{(\{X\})}, \tag{5.7}$$

via an isomorphism that sends $[v]$ to the coset of $(\delta_v + (1_{C'} - A^t_{(E_1, C')})\mathbb{Z}^{(C')}, 0)$ for $v \in E^0$. The cokernel of (5.7) is easily seen to be isomorphic to

$$\mathbb{Z}^{(E^0)} / ((1_{C'} - A^t_{(E_1, C')})\mathbb{Z}^{(C')} + (1_{\{X\}} - A^t_{(E_2, \{X\})})\mathbb{Z}^{(\{X\})}) = \mathbb{Z}^{(E^0)} / ((1_C - A^t_{(E, C)})\mathbb{Z}^{(C)}),$$

in view of the exact sequence

$$\begin{aligned} \mathbb{Z}^{(E^0)} &\xrightarrow{(\pi_{C'}, \pi_{\{X\}})} \text{coker}(1_{C'} - A^t_{(E_1, C')}) \oplus \text{coker}(1_{\{X\}} - A^t_{(E_2, \{X\})}) \\ &\xrightarrow{(q_1, -q_2)} \text{coker}(1_C - A^t_{(E, C)}) \longrightarrow 0, \end{aligned}$$

where q_1 and q_2 are the natural quotient maps. We thus obtain both (5.4) and (5.6).

Now we want to compute $K_1(C^*(E, C))$. From (5.1) and the above observations, we get a short exact sequence:

$$0 \longrightarrow K_1(C^*(E_1, C')) \oplus K_1(C^*(E_2, \{X\})) \longrightarrow K_1(C^*(E, C)) \longrightarrow \ker(\pi_{C'}, \pi_{\{X\}}) \longrightarrow 0. \tag{5.8}$$

Set $A := 1_{C'} - A^t_{(E_1, C')}$ and $B := 1_{\{X\}} - A^t_{(E_2, \{X\})}$.

We distinguish two cases.

Case 1. X consists of a single loop at v . In this case, $B = 0$, and so $\pi_{\{X\}}$ is injective and $\ker(\pi_{C'}, \pi_{\{X\}}) = 0$. By using the induction hypothesis for K_1 , we get

$$K_1(C^*(E, C)) \cong \ker(A) \oplus \mathbb{Z}\delta_X = \ker(1_C - A^t_{(E,C)}),$$

as desired.

Case 2. $|X| > 1$ or X consists of a single edge from v to some different vertex. Then $B : \mathbb{Z}\delta_X \rightarrow \mathbb{Z}^{(E^0)}$ is injective, so that $K_1(C^*(E_2, \{X\})) = \ker(B) = 0$, and we get from (5.8):

$$\begin{aligned} K_1(C^*(E, C)) &\cong K_1(C^*(E_1, C')) \oplus \ker(\pi_{C'}, \pi_{\{X\}}) \\ &\cong \ker(A : \mathbb{Z}^{(C')} \rightarrow \mathbb{Z}^{(E^0)}) \oplus (A(\mathbb{Z}^{(C')}) \cap B(\mathbb{Z}\delta_X)). \end{aligned} \tag{5.9}$$

Now using that $A(\mathbb{Z}^{(C')}) \cap B(\mathbb{Z}\delta_X)$ is cyclic and B is injective, it is straightforward to show that the last direct sum in (5.9) is isomorphic to

$$\ker\left(\begin{pmatrix} A & B \end{pmatrix} : \mathbb{Z}^{(C)} \rightarrow \mathbb{Z}^{(E^0)}\right).$$

Since $\begin{pmatrix} A & B \end{pmatrix} = 1_C - A^t_{(E,C)}$, we get the desired result for $K_1(C^*(E, C))$. \square

As an example, we consider the separated graph $(E(m, n), C(m, n))$ of Example 4.5, for $1 \leq m \leq n$. Now (4.1) and Theorem 5.2 give

$$\begin{aligned} K_0(U_{m,n}^{nc}) &\cong \operatorname{coker}\left(\begin{pmatrix} 1 & 1 \\ -n & -m \end{pmatrix} : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2\right) \cong \begin{cases} \mathbb{Z} & \text{if } n = m, \\ \mathbb{Z}_{n-m} & \text{if } n > m, \end{cases} \\ K_1(U_{m,n}^{nc}) &\cong \ker\left(\begin{pmatrix} 1 & 1 \\ -n & -m \end{pmatrix} : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2\right) \cong \begin{cases} \mathbb{Z} & \text{if } n = m, \\ 0 & \text{if } n > m. \end{cases} \end{aligned}$$

This confirms a conjecture of McClanahan [21, Conjecture, p. 1067], and recovers [20, Corollary 2.4] in the case $n = m$.

6. Relationships with McClanahan’s examples

We show here that the reduced C^* -algebra of the separated graph $(E(m, n), C(m, n))$ of Example 4.5 is Morita-equivalent to the C^* -algebra constructed by McClanahan in [22, Example 3.12]. Let us recall the definition in [22]. Let

$$(\mathcal{B}, \Psi) := (M_{n+m}(\mathbb{C}), \Psi_1) *_{\mathbb{C}^2} (M_2(\mathbb{C}), \Psi_2)$$

be the reduced amalgamated product over \mathbb{C}^2 of the algebras $M_{n+m}(\mathbb{C})$ and $M_2(\mathbb{C})$, with respect to the conditional expectations defined by

$$\begin{aligned} \Psi_1((a_{ij})) &= \left(\frac{1}{n} \sum_{i=1}^n a_{ii}, \frac{1}{m} \sum_{j=1}^m a_{n+j, n+j} \right), \\ \Psi_2\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= (a, d). \end{aligned}$$

Proposition 6.1. *Let $1 < m \leq n$, let $(E, C) := (E(m, n), C(m, n))$ be the separated graph described in Example 4.5, and let*

$$(A, \Phi) := C_{\text{red}}^*(E, C) \equiv (M_{n+1}(\mathbb{C}), \Phi_1) *_C (M_{m+1}(\mathbb{C}), \Phi_2)$$

be the corresponding reduced C^ -algebra. Let $T := vAv$ be the corner of A corresponding to $v \in E^0$, and observe that Φ restricts to a faithful, completely positive conditional expectation $\phi : T \rightarrow \mathbb{C} \cdot 1_T$. Then we have a $*$ -isomorphism*

$$(\mathcal{B}, \Psi) \cong (M_2(\mathbb{C}) \otimes T, \Psi_2 \otimes \phi).$$

Proof. We are going to use again the characterization of the reduced amalgamated product. Let e_{ij} and f_{ij} denote the canonical matrix units in $M_{n+m}(\mathbb{C})$ and $M_2(\mathbb{C})$, respectively. There exist unital $*$ -homomorphisms $\sigma_1 : M_{n+m}(\mathbb{C}) \rightarrow M_2(\mathbb{C}) \otimes T$ and $\sigma_2 : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C}) \otimes T$ such that

$$\sigma_1(e_{ij}) = \begin{cases} f_{11} \otimes \alpha_i \alpha_j^* & (1 \leq i, j \leq n), \\ f_{12} \otimes \alpha_i \beta_{j-n}^* & (1 \leq i \leq n < j \leq n+m), \\ f_{21} \otimes \beta_{i-n} \alpha_j^* & (1 \leq j \leq n < i \leq n+m), \\ f_{22} \otimes \beta_{i-n} \beta_{j-n}^* & (n < i, j \leq n+m), \end{cases}$$

and $\sigma_2(f_{ij}) = f_{ij} \otimes 1_T$ for $i, j = 1, 2$. Now all the conditions in the definition of the reduced amalgamated product are easily verified, with the exception of (4), that needs some work.

Observe that $\ker \Psi_2$ is spanned by $\{f_{12}, f_{21}\}$, while $\ker \Psi_1$ is spanned by the set

$$\mathcal{E} := \{\epsilon_{kl} : 1 \leq k, l \leq n+m, k \neq l\} \cup \{\epsilon_i : 1 \leq i \leq n\} \cup \{\bar{\epsilon}_j : 1 \leq j \leq m\},$$

with $\epsilon_i := e_{ii} - \frac{1}{n} \sum_{t=1}^n e_{it}$ and $\bar{\epsilon}_j := e_{j+n, j+n} - \frac{1}{m} \sum_{s=1}^m e_{s+n, s+n}$. We note that $\sigma_1(\epsilon_k) = f_{11} \otimes \lambda(\alpha_k)$ and $\sigma_1(\bar{\epsilon}_k) = f_{22} \otimes \lambda(\beta_k)$, where $\lambda(\alpha_k) := \alpha_k \alpha_k^* - \frac{1}{n} v$ and $\lambda(\beta_k) := \beta_k \beta_k^* - \frac{1}{m} v$.

For subsets T_1, T_2 of an algebra \mathcal{H} , denote by $\Lambda^0(T_1, T_2)$ the set of all elements of \mathcal{H} of the form $a_1 a_2 \cdots a_r$, where $a_j \in T_{i_j}$ and $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{r-1} \neq i_r$. With this notation, to verify (4) it will be enough to show that $(\Psi_2 \otimes \phi)(a_1 a_2 \cdots a_r) = 0$ for all $a_1 a_2 \cdots a_r$ in $\Lambda^0(\{\sigma_2(f_{12}), \sigma_2(f_{21})\}, \sigma_1(\mathcal{E}))$. This is, of course, clear for $r = 1$. We claim that:

- (I) For $r \geq 2$, any word $a_1 a_2 \cdots a_r \in \Lambda^0(\{\sigma_2(f_{12}), \sigma_2(f_{21})\}, \sigma_1(\mathcal{E}))$ is either zero or has the form $f_{ij} \otimes d$ with $d \in \Lambda^0(T_\alpha, T_\beta)$, where

$$T_\alpha := \{\alpha_k, \alpha_k^*, \lambda(\alpha_k) \mid 1 \leq k \leq n\} \cup \{\alpha_k \alpha_l^* \mid 1 \leq k, l \leq n, k \neq l\},$$

$$T_\beta := \{\beta_k, \beta_k^*, \lambda(\beta_k) \mid 1 \leq k \leq m\} \cup \{\beta_k \beta_l^* \mid 1 \leq k, l \leq m, k \neq l\},$$

and also:

- (1) If $a_r \in \sigma_1(\mathcal{E})$ and $j = 1$, then d ends in one of α_l^* or $\alpha_k \alpha_l^*$ (with $k \neq l$) or $\lambda(\alpha_k)$.
- (2) If $a_r \in \sigma_1(\mathcal{E})$ and $j = 2$, then d ends in one of β_l^* or $\beta_k \beta_l^*$ (with $k \neq l$) or $\lambda(\beta_k)$.

For $c \in \sigma_1(\mathcal{E})$, observe that:

- (1) $\sigma_2(f_{12})c \neq 0$ if and only if $c = f_{21} \otimes \beta_{k-n}\alpha_l^*$; or $c = f_{22} \otimes \beta_{k-n}\beta_{l-n}^*$ with $k \neq l$; or $c = f_{22} \otimes \lambda(\beta_k)$.
- (2) $\sigma_2(f_{21})c \neq 0$ if and only if $c = f_{11} \otimes \alpha_k\alpha_l^*$ with $k \neq l$; or $c = f_{11} \otimes \lambda(\alpha_k)$; or $c = f_{12} \otimes \alpha_k\beta_{l-n}^*$.

With the aid of these observations, the claim is easily established by induction on r .

Since (\mathcal{A}, Φ) is a reduced amalgamated product, $\Lambda^0(T_\alpha, T_\beta) \subseteq \ker \Phi$. Hence, we conclude that $\Lambda^0(\{\sigma_2(f_{12}), \sigma_2(f_{21}), \sigma_1(\mathcal{E})\}) \subseteq \ker(\Psi_2 \otimes \phi)$, as desired. \square

The case $n = m$ is special. In this case, the reduced graph C^* -algebra admits a faithful trace and has minimal projections, as we will see. We establish this by finding a Morita equivalence with a different example of McClanahan’s, namely [22, Example 4.1]. This example is given as follows:

$$(\mathcal{C}, \Psi) := (M_n(\mathbb{C}), \text{tr}_n) *_\mathbb{C} (C(\mathbb{T}), \tau),$$

where tr_n is the normalized matrix trace on $M_n(\mathbb{C})$ and τ is the usual faithful trace on $C(\mathbb{T})$. Then we have:

Proposition 6.2. *Let $n > 1$, let $(E, C) := (E(n, n), C(n, n))$ be the separated graph described in Example 4.5, and let*

$$(\mathcal{A}, \Phi) := C_{\text{red}}^*(E, C) \equiv (M_{n+1}(\mathbb{C}), \Phi_1) *_\mathbb{C}^2 (M_{n+1}(\mathbb{C}), \Phi_2)$$

be the corresponding reduced C^* -algebra. Let $T := vAv$ be the corner of \mathcal{A} corresponding to $v \in E^0$, and observe that Φ restricts to a faithful, completely positive conditional expectation $\phi : T \rightarrow \mathbb{C} \cdot 1_T$. There is a $*$ -isomorphism

$$(\mathcal{C}, \Psi) \cong (T, \phi).$$

Proof. We are going to use again the characterization of the reduced amalgamated product. Let u denote the standard unitary generator of $C(\mathbb{T})$, and let e_{ij} be the canonical matrix units in $M_n(\mathbb{C})$. There exist unital $*$ -homomorphisms $\sigma_1 : M_n(\mathbb{C}) \rightarrow T$ and $\sigma_2 : C(\mathbb{T}) \rightarrow T$ such that $\sigma_1(e_{ij}) = \alpha_i\alpha_j^*$ for all i, j , and

$$\sigma_2(u) = U := \sum_{j=1}^n \beta_j\alpha_j^*.$$

Conditions (1), (2), (5) in the definition of the reduced amalgamated product are easily verified, as is the first part of (3), namely, that $\phi \circ \sigma_1 = \text{tr}_n$.

As in the proof of Proposition 6.1, define

$$T_\alpha := \{\alpha_k, \alpha_k^*, \lambda(\alpha_k) \mid 1 \leq k \leq n\} \cup \{\alpha_k\alpha_l^* \mid 1 \leq k, l \leq n, k \neq l\},$$

$$T_\beta := \{\beta_k, \beta_k^*, \lambda(\beta_k) \mid 1 \leq k \leq n\} \cup \{\beta_k\beta_l^* \mid 1 \leq k, l \leq n, k \neq l\},$$

where $\lambda(\alpha_k) := \alpha_k \alpha_k^* - \frac{1}{n}v$ and $\lambda(\beta_k) := \beta_k \beta_k^* - \frac{1}{n}v$, and observe that $\Lambda^0(T_\alpha, T_\beta) \subseteq \ker \Phi$. Since any nonzero power of U is a linear combination of elements of $\Lambda^0(T_\alpha, T_\beta)$, we see that $\phi \sigma_2(u^t) = \Phi(U^t) = 0$ for all $t \neq 0$. It follows that $\phi \circ \sigma_2 = \tau$, verifying (3).

Observe that $\ker \text{tr}_n$ and $\ker \tau$ are the closed linear spans of the sets

$$\begin{aligned} \mathcal{E} &:= \{e_{kl} : 1 \leq k, l \leq n, k \neq l\} \cup \{\epsilon_i : 1 \leq i \leq n\}, \\ \mathcal{Y} &:= \{u^t : t \in \mathbb{Z} \setminus \{0\}\}, \end{aligned}$$

with $\epsilon_i := e_{ii} - \frac{1}{n} \sum_{t=1}^n e_{tt}$ as before. Also, $\sigma_1(\epsilon_k) = \lambda(\alpha_k)$. To verify condition (4), it is enough to show that $\phi(a_1 a_2 \cdots a_r) = 0$ for all $a_1 a_2 \cdots a_r$ in $\Lambda^0(\sigma_1(\mathcal{E}), \sigma_2(\mathcal{Y}))$. This is clear for $r = 1$.

We claim that:

- (I) Each element $a_1 a_2 \cdots a_r \in \Lambda^0(\sigma_1(\mathcal{E}), \sigma_2(\mathcal{Y}))$ with $r \geq 2$ and $a_r \in \sigma_2(\mathcal{Y})$ can be written as a linear combination of terms $w U^t$ such that $w \in \Lambda^0(T_\alpha, T_\beta)$ and one of the following holds:
 - (a) $t \geq 1$ and w ends in one of α_l^* or $\alpha_k \alpha_l^*$ (with $k \neq l$) or $\lambda(\alpha_k)$;
 - (b) $t \leq 0$ and w ends in one of β_l^* or $\beta_k \beta_l^*$ (with $k \neq l$) or $\lambda(\beta_k)$.

The cases when $r = 2$ and a_2 is a positive power of U are clear. The other $r = 2$ cases follow from the facts that

$$\sigma_1(e_{kl})U^{-1} = \alpha_k \beta_l^*, \quad \sigma_1(\epsilon_k)U^{-1} = \alpha_k \beta_k^* - \frac{1}{n}U^{-1}$$

for all k, l . The remainder of claim (I) is proved by induction on r , with the help of the following observations:

$$\begin{aligned} U \sigma_1(e_{kl})U &= \beta_k \alpha_l^* U, & U \sigma_1(e_{kl})U^{-1} &= \beta_k \beta_l^*, \\ U^{-1} \sigma_1(e_{kl})U &= U^{-1} \alpha_k \alpha_l^* U, & U^{-1} \sigma_1(e_{kl})U^{-1} &= U^{-1} \alpha_k \beta_l^*, \\ U \sigma_1(\epsilon_k)U &= \beta_k \alpha_k^* U - \frac{1}{n}U^2, & U \sigma_1(\epsilon_k)U^{-1} &= \lambda(\beta_k), \\ U^{-1} \sigma_1(\epsilon_k)U &= U^{-1} \lambda(\alpha_k)U, & U^{-1} \sigma_1(\epsilon_k)U^{-1} &= U^{-1} \alpha_k \beta_k^* - \frac{1}{n}U^{-2} \end{aligned}$$

for all k, l .

Finally, we claim that:

- (II) Every element $a_1 a_2 \cdots a_r \in \Lambda^0(\sigma_1(\mathcal{E}), \sigma_2(\mathcal{Y}))$ can be written as a linear combination of elements of $\Lambda^0(T_\alpha, T_\beta)$.

This is clear when $r = 1$, and it follows directly from (I) when $r \geq 2$ and $a_r \in \sigma_2(\mathcal{Y})$, just by expanding the factors U^t . When $r \geq 2$ and $a_r \in \sigma_1(\mathcal{E})$, we obtain (II) from (I) with the help of the facts that

$$U \sigma_1(e_{kl}) = \beta_k \alpha_l^*, \quad U \sigma_1(\epsilon_k) = \beta_k \alpha_k^* - \frac{1}{n}U$$

for all k, l . Since $\Lambda^\circ(T_\alpha, T_\beta) \subseteq \ker \Phi$, we conclude from (II) that $\Lambda^\circ(\sigma_1(\mathcal{E}), \sigma_2(\mathcal{Y})) \subseteq \ker \phi$, as desired. \square

Corollary 6.3. *Let $n > 1$ and $(\mathcal{A}, \Phi) := C_{\text{red}}^*(E, C)$ as in Proposition 6.2. Then \mathcal{A} is a simple C^* -algebra with a faithful trace. It has stable rank 1, but does not have real rank zero.*

Proof. By Proposition 6.2, \mathcal{A} is Morita equivalent to McClanahan’s example \mathcal{C} , where $(\mathcal{C}, \Psi) := (M_n(\mathbb{C}), \text{tr}_n) *_{\mathbb{C}} (C(\mathbb{T}), \tau)$. We show that \mathcal{C} has the described properties. Simplicity follows from either [22, Proposition 3.3] or Proposition 4.6, and Ψ is a trace because tr_n and τ are traces (see [3, Proposition 1.4] or [32, 2.5.3]). It is faithful because tr_n and τ are faithful.

Next, since $C(\mathbb{T})$ is a diffuse abelian algebra with respect to τ , meaning that τ is given by an atomless measure on \mathbb{T} , it follows from [14, Proposition 3.4] that \mathcal{C} has stable rank 1.

Finally, we consider the K -theory of the full free product algebra $\mathcal{C}_{\text{full}} := M_n(\mathbb{C}) * C(\mathbb{T})$. Write e_{ij} for the canonical matrix units in the copies of $M_n(\mathbb{C})$ appearing in the different algebras under consideration. Since $K_0(M_n(\mathbb{C}))$ and $K_0(C(\mathbb{T}))$ are infinite cyclic, with generators $[e_{11}]$ and $[1]$, respectively, it follows from Theorem 5.1 that $K_0(\mathcal{C}_{\text{full}})$ is infinite cyclic, with generator $[e_{11}]$. McClanahan showed in [23, Corollary 8.7] (cf. [22, Example 4.1]) that the natural map $\mathcal{C}_{\text{full}} \rightarrow \mathcal{C}$ induces isomorphisms in K -theory. (This also follows from [15, Theorem 4.1].) Consequently, $K_0(\mathcal{C})$ is infinite cyclic, with generator $[e_{11}]$. The faithful trace Ψ on \mathcal{C} thus takes values in $(1/n)\mathbb{Z}$ on projections, and it follows that e_{11} is a minimal projection in \mathcal{C} . Therefore \mathcal{C} cannot have real rank 0. \square

7. Problems

In this final section, we discuss some open problems which arise naturally in this investigation.

Problem 7.1. Compute the lattices of closed ideals of the full and reduced graph C^* -algebras of a finitely separated graph (E, C) , in terms of graph-theoretic data. In particular, find characterizations of simplicity of $C^*(E, C)$ and/or $C_{\text{red}}^*(E, C)$ in terms of (E, C) .

For the ordinary graph C^* -algebra $C^*(E)$ of a countable graph E , the lattice of gauge-invariant closed ideals was characterized in [4, Theorem 3.6, Corollaries 3.8, 3.10]; this gives the full lattice of closed ideals in case E satisfies condition (K) (see [4], [11, Theorem 3.5]). A characterization of simplicity of $C^*(E)$ was found earlier, in [27, Theorem 12]. For the Leavitt path algebra of a separated graph (E, C) , the lattice of trace ideals was characterized in [1, Theorem 6.11], and necessary and sufficient conditions for “trace-simplicity” of $L(E, C)$ were obtained in [1, Theorem 7.1].

Problem 7.2. Find conditions when the full graph C^* -algebra of a finitely separated graph (E, C) equals the reduced one. Observe that this is always the case for non-separated graphs (Theorem 3.8(2)). A necessary condition for equality is that the full C^* -algebra needs to be exact (Remark 3.10).

Exactness often fails, however, as shown by Duncan [12]. First, if there is a vertex $v \in E^0$ at which there are two loops lying in different members of C_v , then $C^*(E, C)$ is not exact [12, Proposition 6]. Second, if there are vertices $v, w \in E^0$ and three edges from v to w which lie in distinct members of C_v , then $C^*(E, C)$ is not exact [12, Proposition 7].

Problem 7.3. When the reduced graph C^* -algebra of a finitely separated graph is simple and infinite (as in some of the examples in Section 4), is it purely infinite? Does it at least have real rank zero? Both answers are positive in the non-separated case [27, Theorem 18].

Problem 7.4. When the reduced graph C^* -algebra of a finitely separated graph is simple and finite, must have it stable rank one? The answer is positive in the non-separated case [27, Theorem 18]. The corresponding question for real rank zero is answered negatively by Corollary 6.3.

Problem 7.5. For free products of two *nuclear* C^* -algebras with faithful states, Germain proved in [15] that the natural map from the full free product to the reduced one is a *KK*-equivalence and so it induces an isomorphism in *K*-theory. Note that this applies to the examples in Proposition 4.2.

Is there a corresponding result for amalgamated free products? This would apply in particular to all the examples of Section 4, for which we could then compute the *K*-theory of the reduced graph C^* -algebras (thanks to Theorem 5.2).

Problem 7.6. Let (E, C) be a finitely separated graph. Let $M(E, C)$ be the abelian monoid with generators $\{a_v \mid v \in E^0\}$ and relations given by $a_v = \sum_{e \in X} a_{r(e)}$ for all $v \in E^0$ and all $X \in C_v$. It was shown in [1, Theorem 4.3] that there is a natural isomorphism $M(E, C) \rightarrow \mathcal{V}(L(E, C))$, sending a_v to $[v] \in \mathcal{V}(L(E, C))$, where $\mathcal{V}(L(E, C))$ is the abelian monoid of Murray-von Neumann equivalence classes of projections in matrices over $L(E, C)$.

Is the natural map $M(E, C) \rightarrow \mathcal{V}(C^*(E, C))$ also an isomorphism? Equivalently, is the natural induced map $\mathcal{V}(L(E, C)) \rightarrow \mathcal{V}(C^*(E, C))$ an isomorphism?

We conjecture that the answer to this question is positive. This is certainly the case for non-separated graphs (see [2, Theorem 7.1]). If the answer is positive, it would follow, as in [1, Corollary 4.5], that every conical abelian monoid is isomorphic to $\mathcal{V}(C^*(E, C))$ for some finitely separated graph (E, C) .

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