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A Central Limit Theorem for Generalized Multilinear Forms

PETER DE JONG

Courseware Europe b.v., Ebbehout 1, 1507 EA Zaandam, The Netherlands

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Let $X_1, ..., X_n$ be independent random variables and define for each finite subset $I \subset \{1, ..., n\}$ the σ-algebra $\mathscr{F}_I = \sigma\{X_i : i \in I\}$. In this paper $\mathscr{F}_I$-measurable random variables W_I are considered, subject to the centering condition $E(W_I \mid \mathscr{F}_J) = 0$ a.s. unless $I \subset J$. A central limit theorem is proven for d-homogeneous sums $W(n) = \sum_{|I|=d} W_I$, with var $W(n) = 1$, where the summation extends over all $\binom{n}{d}$ subsets $I \subset \{1, ..., n\}$ of size $|I| = d$, under the condition that the normed fourth moment of $W(n)$ tends to 3. Under some extra conditions the condition is also necessary. © 1990 Academic Press, Inc.

1. INTRODUCTION AND SUMMARY

We start with a sketch of the general setting. Consider independent random variables $X_1, ..., X_n$ on the probability space $(\Omega, \mathscr{F}, P)$. Define for each finite subset $I \subset \{1, ..., n\}$ the σ-algebra $\mathscr{F}_I = \sigma\{X_i : i \in I\}$ (with $\mathscr{F}_\varnothing$ the trivial σ-algebra) and let W_I denote an $\mathscr{F}_I$-measurable random variable. (Throughout this paper the random variables W_I may depend on n, $W_I = W_{In}$; the parameter n will be suppressed where possible.) We assume the random variables W_I to be centered, square integrable, and uncorrelated:

$$EW_I = 0, \qquad EW_I^2 = \sigma_I^2 < \infty, \qquad EW_I W_J = 0 \qquad \text{if} \quad I \neq J.$$

We are interested in conditions that ensure asymptotic normality for *d-homogeneous sums*,

$$W(n) = \sum_{|I|=d} W_I,$$

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where the summation extends over all $\binom{n}{d}$ subsets $I \subset \{1, ..., n\}$ of size $|I| = d$. Without loss of generality we may assume the sum $W(n)$ to be normalized so as to have unit variance:

$$\operatorname{var} W(n) = \sum_{|I|=d} \sigma_I^2 = 1.$$

The following condition will play a crucial role in the theory

$$EW(n)^4 \to 3 \qquad \text{for} \quad n \to \infty,$$

with 3 being just the fourth moment of the standard normal distribution.

This assumption does not suffice for a central limit theorem; $W(n)$ may converge (even in the simple case $d = 1$) to any centered random variable with unit variance and fourth moment equal to 3. What is needed, is a negligibility condition which forces the total contribution of the random variables W_I which depend on X_i to the variance of $W(n)$ to be small for each $i = 1, ..., n$:

$$\max_i \sum_{I \ni i} \sigma_I^2 \to 0 \qquad \text{for} \quad n \to \infty.$$

In the important special case of homogeneous multilinear forms in independent centered random variables,

$$W_I = a_I \prod_{i \in I} X_i,$$

the above assumptions imply asymptotic normality for $W(n)$. This will be shown in Theorem 1 below. The multilinearity in itself is not the essential property which we need for asymptotic normality. What seems to be crucial is that, as in the case of martingales, certain conditional expectations should vanish. This will be made more precise.

Any square integrable $\mathscr{F}_{\{1, ..., n\}}$-measurable random variable $Z(n)$ can be written as

$$Z(n) = \sum_{I \subset \{1, ..., n\}} W_I,$$

where the components W_I are uniquely determined by the conditions

(a) W_I is $\mathscr{F}_I$-measurable;

(b) $E(W_I \mid \mathscr{F}_J) = 0$ a.s. unless $I \subset J$. (1.1)

This follows from the expressions

$$W_\varnothing = EW_\varnothing = E\Big(Z(n) - \sum_{J \neq \varnothing} W_J\Big) = EZ(n)$$

and

$$W_I = E\left(Z(n) - \sum_{J \neq I} W_J \mid \mathscr{F}_I\right) = E\left(Z(n) - \sum_{J \subsetneq I} W_J \mid \mathscr{F}_I\right) \qquad \text{a.s.}$$

E.g., in case of a multilinear form, $W_I = a_I \prod_{i \in I} X_i$, condition (b) is satisfied if $EX_i = 0$, $i \in I$.

The above decomposition was used in Hoeffding [7] to obtain central limit theorems for $Z(n)$, $Z(n)$ being approximately a sum of independent random variables. We shall refer to (1.1) as the *Hoeffding decomposition* (see Van Zwet [14]).

For d-homogeneous sums in the Hoeffding decomposition $W(n) = \sum_{|I|=d} W_I$—we shall reserve the notation $W(n)$ for homogeneous sums—we have the following central limit theorem:

THEOREM 1. *Let $W(1)$, $W(2)$, ... be d-homogeneous sums in the Hoeffding decomposition, $W(n) = \sum_{|I|=d} W_I$, for fixed d with* var $W(n) = 1$, *for $n = 1, 2, \ldots$. Suppose*

(a) $\max_i \sum_{I \ni i} \sigma_I^2 \to 0$ *for* $n \to \infty$,

(b) $EW(n)^4 \to 3$ *for* $n \to \infty$.

Then

$$W(n) \xrightarrow{d} N(0, 1) \qquad \textit{for} \quad n \to \infty.$$

The proof of Theorem 1 partly rests on algebraic methods. This is reflected in the rather severe fourth moment condition. An algebraic approach is more obvious in case $d = 2$. The quadratic form in i.i.d. $N(0, 1)$ random variables

$$W(n) = \sum_{i \neq j} a_{ij} X_i X_j$$

can be decomposed orthogonally as

$$W(n) = \sum_i \mu_i Y_i^2,$$

with μ_i the eigenvalues of the matrix (a_{ij}) and $Y_i N(0, 1)$ distributed, orthogonal, and hence independent. Clearly, $W(n)$ has a normal limit distribution iff max $|\mu_i|$ vanishes for $n \to \infty$. Straightforward calculation yields that the latter condition is equivalent with condition (b) of Theorem 1 (see De Jong [3]).

The above example illustrates two important aspects. On the one hand, a central limit theorem with conditions expressed in terms of absolute $|W_I|$ cannot be sharp: One can construct a sequence of matrices $(a_{ijn} = (\pm)n^{-1})$ such that the maximal eigenvalue vanishes, whereas the matrix $(|a_{ijn}|)$ has only one non-zero eigenvalue. On the other hand, only conditions on the tails of the random variables W_I will not be sufficient to ensure asymptotic normality.

For asymptotic normality the interaction of the random variables W_I has to be taken into account. The fourth moment condition (b) of Theorem 1 expresses this (lack of) interaction neatly. Under an extra condition, only imposing a further restriction on the tails of the summands W_I, it can be shown that condition (b) is equivalent to asymptotic normality. (For proofs of the results indicated by A and B, see De Jong [4].)

THEOREM A (Theorem 3.2.5 in De Jong [4]). *Let $W(1)$, $W(2)$, ... be a sequence of d-homogeneous sums in the Hoeffding decomposition, $W(n) = \sum_{|I|=d} W_I$, with* var $W(n) = 1$, *for $n = 1, 2, \ldots$. Suppose*

(a) $\max_i \sum_{I \ni i} \sigma_I^2 \to 0, \quad n \to \infty,$

(b) $\max_I EW_I^4/\sigma_I^4 \leqslant D$, *$D$ not depending on n.*

Then the following two statements are equivalent:

(1) $EW(n)^4 \to 3, \qquad n \to \infty,$

(2) $W(n) \to^d N(0, 1), \quad n \to \infty.$

Condition (b) of Theorem 1 is restrictive as far as the tail distribution of the summands W_I is concerned. In this respect improvement can be gained by truncation techniques. (See De Jong [4].) However, this is not our main concern. We shall concentrate on the interplay of the summands W_I as expressed in the fourth moment.

One remark on Theorem 1 remains: The condition of homogeneity cannot be removed from Theorem 1. Thus the theorem is not valid for a non-homogeneous sum of two homogeneous sums.

In the remainder of the introduction we refer to some literature related to our work. Quadratic forms in i.i.d. $N(0, 1)$ random variables are treated exhaustively in Sevast'yanov [13]. In Rotar' [11] this approach is generalized to independent centered square integrable random variables.

By the method of moments Jammalamadaka and Janson [8] obtain a central limit theorem in the non-homogeneous case for $d = 2$. Hall [5] also obtains a central limit theorem for this case, using a martingale central limit theorem. Weber [15] uses backward martingales to obtain a central limit theorem.

Central limit theorems for sums of random variables indexed by subsets of the integers with three or more elements are more scarce. See, e.g., Noether [10] and Barbour and Eagleson [1]. Both papers give results for dissociated random variables. (The concept of dissociatedness was introduced in McGinley and Sibson [9].) The random variables W_I defined above are examples of dissociated random variables. Rotar' [12] considers multilinear forms.

2. Homogeneous Sums in the Hoeffding Decomposition

In this section we are concerned with properties of homogeneous sums in the Hoeffding decomposition that are valid irrespective of the assumptions (a) or (b) of Theorem 1. The Hoeffding decomposition is orthogonal. Suppose $J\backslash I\neq\varnothing$, then $EW_I W_J = EW_I E(W_J \mid \mathscr{F}_I) = 0$. Similarly, we have:

Lemma 2. *Let $W_{I_1}, ..., W_{I_q}$ be components in the Hoeffding decomposition and suppose $I_1 \cap (I_2 \cup \cdots \cup I_q) \neq I_1$. ($I_1$ is called a* free index *of the q-tuple $(I_1, ..., I_q)$.) Then $EW_{I_1} \cdots W_{I_q} = 0$ (provided the expectation exists).*

Proof. By the defining relation 1.1.b we have

$$EW_{I_1} \cdots W_{I_q} = EW_{I_2} \cdots W_{I_q} E(W_{I_1} \mid \mathscr{F}_{I_2 \cup \cdots \cup I_q}) = 0.$$

Remark. In fact, we have shown more, namely, that

$$E(W_{I_1} \cdots W_{I_q} \mid \mathscr{F}_A) = 0 \text{ a.s.,} \qquad \text{if} \quad I_1 \cap (I_2 \cup \cdots \cup I_q \cup A) \neq I_1.$$

In the rest of this section we shall concentrate on the several partial sums that form the fourth moment $EW(n)^4 = E(\sum_{|I|=d} W_I)^4 = \sum_{(I,J,K,L)} EW_I W_J W_K W_L$. The collection of quadruples (I, J, K, L) is split into the following three sets:

$\mathscr{F}$ the collection of quadruples with a free index (see Lemma 2),

$\mathscr{B}$ the collection of quadruples (I, J, K, L) for which each element in the union $I \cup J \cup K \cup L$ lies in exactly two of the sets I, J, K, L. This is the collection of *bifold* quadruples:

$$1_I + 1_J + 1_K + 1_L = 2 \cdot 1_{I \cup J \cup K \cup L},$$

$\mathscr{T}$ the rest $\mathscr{F}^c \backslash \mathscr{B}$; a quadruple in $\mathscr{T}$ has no free index and at least one element in the union $I \cup J \cup K \cup L$ is in three or more sets:

$$1_I + 1_J + 1_K + 1_L \gneqq 2 \cdot 1_{I \cup J \cup K \cup L}.$$

In Lemma 2 it is shown that the set $\mathscr{F}$ contributes nothing to the fourth moment $EW(n)^4$. For any subset $\mathscr{F}^* \subset \mathscr{F}$ we have

$$\sum_{(I, J, K, L) \in \mathscr{F}^*} EW_I W_J W_K W_L = 0.$$

The quantities τ and τ', defined below, will play an important role in this paper:

$$\tau' = \sum_{(I, J, K, L) \in \mathscr{T}} EW_I W_J W_K W_L,$$

$$\tau = \sum_{(I, J, K, L) \in \mathscr{T}} \sigma_I \sigma_J \sigma_K \sigma_L.$$

Without proof we give the following estimate for τ:

LEMMA B. *$\tau \leqslant C_d(\max_i \sum_{I \ni i} \sigma_I^2)^{1/2}$, with C_d a constant depending on d, but not on n.*

For a proof which is elementary, we refer to De Jong [4]. Note that Lemma B together with assumption (a) of Theorem 1 above implies that τ vanishes. (See Proposition 6(a) below.)

The remainder of the section is concerned with the bifold quadruples. The collection of bifold quadruples is split into $(d+1)(d+2)/2$ subsets $\mathscr{B}(e, f)$, $0 \leqslant f \leqslant e \leqslant d$, where

$$\mathscr{B}(e, f) = \{(I, J, K, L) \in \mathscr{B} : |I \cap J| = e, |I \cap K| = f\}.$$

Given the numbers $e = |I \cap J|$ and $f = |I \cap K|$, the size of each intersection of two indices is known: $|I \cap L| = d - e - f$ and $|I \cap J| = |K \cap L|$, etc. Put

$$S(e, f) = \sum_{(I, J, K, L) \in \mathscr{B}(e, f)} EW_I W_J W_K W_L.$$

Since the value of $EW_I W_J W_K W_L$ is not changed by a permutation of (W_I, W_J, W_K, W_L), we have

$$S(e, f) = S(f, e) = S(e, d - e - f).$$

Put

$$S = \sum_{1 \leqslant e \leqslant d-2} \sum_{1 \leqslant f \leqslant d-e-1} S(e, f),$$

$$S_0 = \sum_{1 \leqslant e \leqslant d-1} S(e, 0).$$

It is helpful to envisage the quantities $S(e, f)$ as being indexed by the points of an equilateral triangle. There is a sixfold symmetry. The three vertices $S(0, 0)$, $S(0, d)$, and $S(d, 0)$ are equal. S is the sum over the interior points and S_0 the sum over one side, exclusive of the endpoints. We have

$$\begin{aligned} EW(n)^4 &= \sum_{(I, J, K, L) \in \mathscr{B}} EW_I W_J W_K W_L + \sum_{(I, J, K, L) \in \mathscr{T}} EW_I W_J W_K W_L \\ &= S + 3S_0 + 3S(0, 0) + \tau'. \end{aligned} \tag{2.1}$$

The following lemma will be needed below.

LEMMA 3. *Let W_I, W_J be measurable with respect to the σ-algebras $\mathscr{F}_I$ and $\mathscr{F}_J$, respectively. Then*

$$E[E(W_I W_J \mid \mathscr{F}_{I \Delta J})]^2 \leqslant \sigma_I^2 \sigma_J^2.$$

Proof. Recall that $I \Delta J = (I \backslash J) \cup (J \backslash I)$. By the conditional version of the Cauchy–Schwarz inequality we have

$$\begin{aligned} E[E(W_I W_J \mid \mathscr{F}_{I \Delta J})]^2 &\leqslant E[E(W_I^2 \mid \mathscr{F}_{I \Delta J})\, E(W_J^2 \mid \mathscr{F}_{I \Delta J})] \\ &= E[E(W_I^2 \mid \mathscr{F}_{I \backslash J})\, E(W_J^2 \mid \mathscr{F}_{J \backslash I})] \\ &= \sigma_I^2 \sigma_J^2, \end{aligned}$$

where the two equalities follow by the independence of the underlying random variables X_i. (For the first equality see Chung [2, Theorem 9.2.1].) This proves Lemma 3.

Using the independence of the underlying random variables X_i in the same way as above, we obtain for bifold quadruples an equality.

LEMMA 4. *Let W_I, W_J, W_K, W_L be measurable with respect to the σ-algebras $\mathscr{F}_I$, $\mathscr{F}_J$, $\mathscr{F}_K$ and $\mathscr{F}_L$, respectively. Then for a bifold quadruple (I, J, K, L) we have*

$$EW_I W_J W_K W_L = E[E(W_I W_J \mid \mathscr{F}_{I \Delta J})\, E(W_K W_L \mid \mathscr{F}_{K \Delta L})].$$

Proof.

$$\begin{aligned} EW_I W_J W_K W_L &= E[W_I W_J E(W_K W_L \mid \mathscr{F}_{I \cup J})] \\ &\overset{(1)}{=} E[W_I W_J E(W_K W_L \mid \mathscr{F}_{(I \cup J) \cap (K \cup L)})] \\ &\overset{(2)}{=} E[E(W_I W_J \mid \mathscr{F}_{I \Delta J})\, E(W_K W_L \mid \mathscr{F}_{K \Delta L})], \end{aligned}$$

where equality (1) follows by the independence of the random variables (see [2, Theorem 9.2.1]). Equality (2) follows since we have in a bifold quadruple $(I \cup J) \cap (K \cup L) = I \Delta J = K \Delta L$.

Remark. Notice that it is not necessary in Lemma 3 or 4 to assume that the random variables W_I, W_J, W_K, W_L are components in the Hoeffding decomposition.

The main part of the proof of Theorem 1 rests on the above four lemmas, for the Lemmas 2–4 imply the technical results on bifold quadruples of Proposition 5 below. With the latter results, the proof of Theorem 1 can be established by taking a closer look at certain partial sums over the sets $\mathscr{B}$ and $\mathscr{T}$. The first two assertions of Proposition 5 establish an almost non-negativity for $2S_0 + S$ and S_0, respectively. Both terms do not exceed $-\tau$ (which vanishes under assumption (a) of Theorem 1).

PROPOSITION 5. *Let $(W_I)_{|I|=d}$ be homogeneous components in the Hoeffding decomposition. Then*

(a) $2S_0 + S \geqslant -\tau$,

(b) $S_0 \geqslant -\tau$,

(c) $2S_0 + S \leqslant \binom{2d}{d}(S_0 + \tau) + \tau$.

Proof. To study the bifold quadruples we introduce the auxiliary random variable

$$A = \sum_{I,J} E(W_I W_J \mid \mathscr{F}_{I \Delta J}).$$

For a fixed subset $C \subset \{1, ..., n\}$ the random variable

$$Y_C = \sum_{I \Delta J = C} E(W_I W_J \mid \mathscr{F}_{I \Delta J})$$

satisfies $E(Y_C \mid \mathscr{F}_B) = 0$ a.s. if $C \backslash B \neq \varnothing$ (Lemma 2, Remark). Hence there is a Hoeffding decomposition for A with d-homogeneous sums

$$A^{(e)} = \sum_{|C| = 2d-2e} \sum_{I \Delta J = C} E(W_I W_J \mid \mathscr{F}_{I \Delta J}).$$

We shall prove assertion (a) by showing that $2S_0 + S$ equals (up to a remainder term not exceeding τ) the variance of $\sum_{1 \leqslant e < d} A^{(e)}$. By the orthogonality of the Hoeffding decomposition we have

$$\operatorname{var} \sum_{1 \leqslant e < d} A^{(e)} = \sum_{1 \leqslant e < d} \operatorname{var} A^{(e)},$$

and

$$\begin{aligned}
\operatorname{var} A^{(e)} &\overset{(1)}{=} \sum_{|C|=2d-2e} E\bigg(\sum_{I\Delta J=C} E(W_I W_J \mid \mathscr{F}_{I\Delta J})\bigg)^2 \\
&= \sum_{(I,J,K,L),\, I\Delta J = K\Delta L,\, |I\Delta J| = 2d-2e} \\
&\quad \times E[E(W_I W_J \mid \mathscr{F}_{I\Delta J})\, E(W_K W_L \mid \mathscr{F}_{K\Delta L})] \\
&\overset{(2)}{=} \sum_{\mathscr{B},\, |I\cap J| = e} EW_I W_J W_K W_L \\
&\quad + \sum_{\mathscr{T},\, |I\cap J| = e} E[E(W_I W_J \mid \mathscr{F}_{I\Delta J})\, E(W_K W_L \mid \mathscr{F}_{K\Delta L})] \\
&= \sum_{0\leqslant f\leqslant d-e} S(e,f) + R_{1e},
\end{aligned} \tag{2.2}$$

where equality (1) rests on the orthogonality of the Hoeffding decomposition. Equality (2) rests on Lemma 4. By the Cauchy–Schwarz inequality and Lemma 3 we have

$$R_{1e} \leqslant \sum_{\mathscr{T},\, |I\cap J| = e} \sigma_I \sigma_J \sigma_K \sigma_L$$

and hence $|R_{11} + \cdots + R_{1d-1}| \leqslant \tau$.

The proof of assertions (b) and (c) is similar. For a fixed subset C with $|C| = 2d-2e$ we have

$$\begin{aligned}
&\sum_{I\Delta J = C} E(W_I W_J \mid \mathscr{F}_{I\Delta J}) \\
&\quad = \sum_{B,\, B'\subset C,\, |B| = |B'| = d-e,\, B\cap B' = \varnothing} \; \sum_{I\setminus J = B,\, J\setminus I = B'} E(W_I W_J \mid \mathscr{F}_{I\Delta J}).
\end{aligned} \tag{2.3}$$

Assertion (b) follows by

$$\begin{aligned}
&\sum_{|C| = 2d-2e} \; \sum_{B,\, B'\subset C,\, |B| = |B'| = d-e,\, B\cap B' = \varnothing} \\
&\qquad \times E\bigg[\sum_{I\setminus J = B,\, J\setminus I = B'} E(W_I W_J \mid \mathscr{F}_{I\Delta J})\bigg]^2 \\
&\quad = \sum_{\mathscr{B},\, |I\cap J| = e,\, I\setminus J = L\setminus K,\, J\setminus I = K\setminus L} EW_I W_J W_K W_L + R_{2e} \\
&\quad = S(e, 0) + R_{2e},
\end{aligned}$$

with the remainder term $|\sum R_{2e}| \leqslant \tau$ (by Lemma 3). The final equality follows, since for bifold quadruples the equality $I\setminus J = L\setminus K$ implies $I\cap K = \varnothing$. (Recall $S_0 = S(1, 0) + \cdots + S(d-1, 0)$.)

Assertion (c) follows by the Cauchy–Schwarz inequality. Using (2.3) we obtain

$$\begin{aligned}\operatorname{var} A^{(e)} &= \sum_{|C|=2d-2e} E\Bigg[\sum_{B,\,B'\subset C,\,|B|=|B'|=d-e,\,B\cap B'=\varnothing} \\ &\quad\times\Bigg(\sum_{I\setminus J=B,\,J\setminus I=B'} E(W_I W_J \mid \mathscr{F}_{I\Delta J})\Bigg)\Bigg]^2 \\ &\leqslant \sum_{|C|=2d-2e}\Bigg(\Bigg[\sum_{B,\,B'\subset C,\,|B|=|B'|=d-e,\,B\cap B'=\varnothing} 1^2\Bigg] \\ &\quad\times\Bigg[\sum_{B,\,B'\subset C,\,|B|=|B'|=d-e,\,B\cap B'=\varnothing} \\ &\quad\times E\Bigg(\sum_{I\setminus J=B,\,J\setminus I=B'} E(W_I W_J \mid \mathscr{F}_{I\Delta J})\Bigg)^2\Bigg]\Bigg) \\ &= \binom{2d-2e}{d-e}(S(e,0)+R_{2e}).\end{aligned}$$

Assertion (c) follows, since by (2.2) we have

$$2S_0 + S \leqslant \sum_{1\leqslant e<d} \operatorname{var} A^{(e)} + \tau \leqslant \max_e \binom{2d-2e}{d-e}(S_0+\tau)+\tau.$$

This proves Proposition 5.

3. Proof of Theorem 1

Under the conditions of Theorem 1 most of the quantities defined above vanish. In order to make this clear we introduce the auxiliary random variable

$$Z(n) = \sum_{I\cap J\neq\varnothing} W_I W_J.$$

Proposition 6. *Under the conditions of Theorem 1 we have*

(a) $\tau \to 0, \quad n\to\infty,$

(b) $S_0 \to 0, \quad n\to\infty,$

(c) $\tau' \to 0, \quad n\to\infty,$

(d) $S \to 0, \quad n\to\infty.$

Proof. Assertion (a) follows from condition (a) of Theorem 1 by Lemma B. Assertion (d) follows from assertion (b) and assertion (a) by Proposition 5(c). In order to prove assertion (b) consider the decomposition of $W(n)^2$,

$$W(n)^2 = \sum_{I,J} W_I W_J = Z(n) + \sum_{I \cap J = \varnothing} W_I W_J.$$

Note that $EZ(n) = \sum \sigma_I^2 = 1$ and that the two terms on the right-hand side are orthogonal, since (I, J, K, L) has a free index if $I \cap J = \varnothing$ and $K \cap L \neq \varnothing$. We have

$$E\Big(\sum_{I \cap J = \varnothing} W_I W_J\Big)^2 \overset{(1)}{=} 2 \sum_{I \cap J = \varnothing} \sigma_I^2 \sigma_J^2 + \sum_{I \cap J = \varnothing,\, I \neq I \cap K \neq \varnothing} EW_I W_J W_K W_L$$

$$\overset{(2)}{=} 2 - 2 \sum_{I \cap J \neq \varnothing} \sigma_I^2 \sigma_J^2 + S_0,$$

where equality (1) follows, since $I \cap J = \varnothing = I \cap K$ implies $I = L$ for a bifold quadruple. Equality (2) rests on

$$1 = \Big(\sum_{|I| = d} \sigma_I^2\Big)^2 = S(0, 0) + \sum_{I \cap J \neq \varnothing} \sigma_I^2 \sigma_J^2,$$

where the final equality rests on the independence of W_I and W_J if $I \cap J = \varnothing$. By the orthogonality of the decomposition of $W(n)^2$ and (2.1), respectively, we have

$$EW(n)^4 - 3 \overset{(1)}{=} \operatorname{var} Z(n) + S_0 - 2 \sum_{I \cap J \neq \varnothing} \sigma_I^2 \sigma_J^2$$

$$\overset{(2)}{=} \tau' + S + 3S_0 - 3 \sum_{I \cap J \neq \varnothing} \sigma_I^2 \sigma_J^2. \tag{3.1}$$

The final terms in the equalities (1) and (2) vanish, since τ does. Since $S_0 \geqslant -\tau$ by Proposition 5(b), the right-hand side of (1) equals the sum of two non-negative quantities (up to a vanishing remainder term). Hence, both $\operatorname{var} Z(n)$ and S_0 vanish, since the left-hand side vanishes by condition (b) of Theorem 1. This proves assertion (b). Assertion (c) follows, since by the assertions (a), (b), and (d) all other right-hand-side terms in the final expression above vanish and thus also τ'. This proves Proposition 6.

The first equality of (3.1) yields:

COROLLARY 7. *Under the conditions of Theorem* 1,

$$Z(n) \xrightarrow{L^2} 1, \qquad n \to \infty.$$

Proof of Theorem 1. We write $W(n)$ as a sum of martingale differences:

$$W(n) = \sum_{1 \leqslant k \leqslant n} U_k, \qquad \text{with} \quad U_k = \sum_{I,\, \max I = k} W_I,$$

with respect to the sequence of increasing σ-algebras $\mathscr{F}_{\{1, \ldots, k\}}$. Theorem 1 will follow from the martingale central limit theorem of Heyde and Brown [6] if the following two conditions are satisfied:

I. $\sum_{1 \leqslant k \leqslant n} EU_k^4 \to 0, \quad n \to \infty,$

II. $\sum_{1 \leqslant k \leqslant n} U_k^2 \to^{L^2} 1, \quad n \to \infty.$

Remark. The above conditions are by far not the sharpest conditions known. However, they suit particularly well in the proof of Theorem 1. Write

$$Z(n) = \sum_{1 \leqslant k \leqslant n} V_k, \qquad \text{with} \quad V_k = \sum_{I \cap J \neq \varnothing,\, \max I \cup J = k} W_I W_J.$$

The conditions I and II follow from the following three assertions:

(a) $\operatorname{var}(Z(n) - \sum_{1 \leqslant k \leqslant n} U_k^2) \to 0, \quad n \to \infty,$

(b) $\operatorname{var}(Z(n) - \sum_{1 \leqslant k \leqslant n} U_k^2) = \sum_{1 \leqslant k \leqslant n} E(V_k - U_k^2)^2,$

(c) $\sum_{1 \leqslant k \leqslant n} EV_k^2 \to 0, \quad n \to \infty.$

Assertion (a) implies, in combination with Corollary 7, condition II; the conditions (b) and (c) together imply condition I.

Proof of the Assertions (a), (b), *and* (c). Note that $EV_k = EU_k^2$.

$$\begin{aligned}
E\Big(Z(n) - \sum_{1 \leqslant k \leqslant n} U_k^2\Big)^2
&= E\Big(\sum_{I \cap J \neq \varnothing} W_I W_J - \sum_{\max I = \max J} W_I W_J\Big)^2 \\
&= E\Big(\sum_{I \cap J \neq \varnothing,\, \max I \cup J \in I \Delta J} W_I W_J\Big)^2 \\
&= E\Big(2 \sum_{I \cap J \neq \varnothing,\, \max I \cup J \in I \backslash J} W_I W_J\Big)^2 \\
&= 4 \sum_{I \cap J \neq \varnothing \neq K \cap L,\, \max I \cup J \cup K \cup L \in I \cap K \backslash (J \cup L)} EW_I W_J W_K W_L \\
&= 4 \sum_{\mathscr{T},\, \max I \cup J \cup K \cup L \in I \cap K \backslash (J \cup L)} EW_I W_J W_K W_L \\
&\quad + 4 \sum_{\mathscr{B},\, I \cap J \neq \varnothing,\, \max I \cup J \cup K \cup L \in I \cap K} EW_I W_J W_K W_L, \qquad (3.2)
\end{aligned}$$

where the final equality follows from the definitions of the sets $\mathscr{T}$ and $\mathscr{B}$: for a quadruple in $\mathscr{T}$ each pair has a non-empty intersection and for a quadruple in $\mathscr{B}$ we have $|I \cap J| = |K \cap L|$, since $I \Delta J = K \Delta L$.

We claim that both the bifold and non-bifold part in the above sum vanish. The bifold part will be shown to be equal to $(2/3)S + S_0$, which vanishes according to Proposition 6. By symmetry we have

$$\begin{aligned} S + 3S_0 &= \sum_{\mathscr{B},\, I \neq J,\, I \neq K,\, I \neq L} EW_I W_J W_K W_L \\ &= 6 \sum_{\mathscr{B},\, I \neq J,\, I \neq K,\, I \neq L,\, \max I \cup J \cup K \cup L \in I \cap K} EW_I W_J W_K W_L \\ &\overset{(1)}{=} 6 \sum_{\mathscr{B},\, I \neq K,\, \max I \cup J \cup K \cup L \in I \cap K} EW_I W_J W_K W_L \\ &= 6 \Bigg(\sum_{\mathscr{B},\, I \cap J \neq \varnothing \neq I \cap L,\, \max I \cup J \cup K \cup L \in I \cap K} EW_I W_J W_K W_L \\ &\quad + 2 \sum_{\mathscr{B},\, I \cap J \neq \varnothing,\, I \cap L = \varnothing,\, \max I \cup J \cup K \cup L \in I \cap K} EW_I W_J W_K W_L \Bigg), \end{aligned}$$

where equality (1) follows, since the quadruples are bifold. Thus we have shown

$$2/3S + S_0 = 4 \sum_{\mathscr{B},\, I \cap J \neq \varnothing,\, \max I \cup J \cup K \cup L \in I \cap K} EW_I W_J W_K W_L.$$

It remains to show that the non-bifold quadruples in (3.2) vanish. Suppose that $(I, J, K, L) \in \mathscr{T}$, and let $m = \max I \cup J \cup K \cup L$. The point m lies either in all four sets I, J, K, and L, in exactly three of these sets or in exactly two. By symmetry we find

$$\begin{aligned} \tau' = &\sum_{\max I = \max J = \max K = \max L} EW_I W_J W_K W_L \\ &+ 4 \sum_{\max I \cup J \cup K \cup L \in I \cap J \cap K \setminus L} EW_I W_J W_K W_L \\ &+ 6 \sum_{\mathscr{T},\, \max I \cup J \cup K \cup L \in I \cap K \setminus (J \cup L)} EW_I W_J W_K W_L. \end{aligned}$$

Note that the first term equals

$$\sum_{1 \leqslant k \leqslant n} EU_k^4 = \sum_{1 \leqslant k \leqslant n} \sum_{\max I = \max J = \max K = \max L = k} EW_I W_J W_K W_L.$$

A similar decomposition yields

$$EV_k^2 = EU_k^4 + 4 \sum_{\max I\cup J\cup K\cup L = k\in I\cap J\cap K\setminus L} EW_I W_J W_K W_L$$
$$+4 \sum_{\max I\cup J\cup K\cup L = k\in I\cap J\cap K\setminus L,\, I\cap J\neq\varnothing\neq K\cap L} EW_I W_J W_K W_L.$$

Thus, as in the equality (2.3), we find

$$\tau' + 4 \sum_{\mathscr{B},\, I\cap J\neq\varnothing,\, \max I\cup J\cup K\cup L\in I\cap K} EW_I W_J W_K W_L$$
$$= \sum_{1\leqslant k\leqslant n} EV_k^2 + 2 \sum_{\mathscr{T},\, \max I\cup J\cup K\cup L\in I\cap K\setminus (J\cup L)} EW_I W_J W_K W_L.$$

Note that the final term in the above equality equals half the non-bifold part of the right-hand side of (3.2). Since the left-hand side of (3.2) is non-negative, the final term above is non-negative up to a vanishing remainder term. Since the two terms of the left-hand side vanish and the right-hand side is a sum of non-negative terms (the final one up to a vanishing remainder term), all terms on the right-hand side vanish. This proves the assertions (a) and (c). Finally, assertion (b), for $l > k$

$$E(V_k - U_k^2)(V_l - U_l^2) = 0,$$

since each quadruple in the expectation has a free index containing l. This completes the proof of Theorem 1.

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