ELSEVIER

Available online at www.sciencedirect.com
ScienceDirect

Discrete Mathematics 306 (2006) 2141 – 2151

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Enumeration of the doubles of the projective plane of order 4 ☆

Veerle Fack[a,*], Svetlana Topalova[b,1], Joost Winne[a,2], Rosen Zlatarski[b]

[a] *Research Group on Combinatorial Algorithms and Algorithmic Graph Theory, Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 281-S9, B-9000 Ghent, Belgium*

[b] *Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, P.O. Box 323, 5000 Veliko Tarnovo, Bulgaria*

Received 3 November 2005; received in revised form 19 April 2006; accepted 4 May 2006

Abstract

A classification of the doubles of the projective plane of order 4 with respect to the order of the automorphism group is presented and it is established that, up to isomorphism, there are 1 746 461 307 doubles. We start with the designs possessing non-trivial automorphisms. Since the designs with automorphisms of odd prime orders have been constructed previously, we are left with the construction of the designs with automorphisms of order 2. Moreover, we establish that a 2-(21, 5, 2) design cannot be reducible in two inequivalent ways. This makes it possible to calculate the number of designs with only the trivial automorphism, and consequently the number of all double designs. Most of the computer results are obtained by two different approaches and implementations.

Keywords: Double design; Projective plane; Construction methods

1. Introduction

Let $V=\{P_i\}_{i=1}^{v}$ be a finite set of *points*, and $\mathscr{B}=\{B_j\}_{j=1}^{b}$ a finite collection of k-element subsets of V, called *blocks*. $D=(V,\mathscr{B})$ is a *design* with parameters t-(v,k,λ) if any t-subset of V is contained in exactly λ blocks of $\mathscr{B}$. For the basic concepts and notations concerning combinatorial designs, refer for instance to [1–3,13].

The *incidence matrix* of a design is a (0,1) matrix with v rows and b columns, where the element of the ith row and jth column is 1 if $P_i \in B_j$ $(i=1,2,\ldots,v;\ j=1,2,\ldots,b)$ and 0 otherwise. The design is completely determined by its incidence matrix.

An *isomorphism* of two designs $D_1=(V_1,\mathscr{B}_1)$ and $D_2=(V_2,\mathscr{B}_2)$ is a bijection between their point sets V_1 and V_2 and their block collections $\mathscr{B}_1$ and $\mathscr{B}_2$, such that the point-block incidence is preserved. In terms of the incidence

☆ Some of the results were partially announced at the Ninth International Workshop on Algebraic and Combinatorial Coding Theory, Kranevo, Bulgaria (June 2004), and some at the Annual Workshop on Coding Theory and Applications, Bankya, Bulgaria (December 2004).

* Corresponding author.

E-mail addresses: Veerle.Fack@UGent.be (V. Fack), svetlana@moi.math.bas.bg (S. Topalova), Joost.Winne@UGent.be (J. Winne), rosen@moi.math.bas.bg (R. Zlatarski).

[1] Partially supported by the Bulgarian National Science Fund under Contract No. I-1301/2003.

[2] Supported by the Fund for Scientific Research-Flanders (Belgium).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2006.05.007

matrices, two designs are isomorphic if their incidence matrices are equivalent, i.e. if the incidence matrix of the second design can be obtained from the incidence matrix of the first one by a permutation of the rows and columns.

An *automorphism* is an isomorphism of the design to itself, i.e. a permutation of the points that preserves the block collection. The set of all automorphisms of a design forms a group called its *full group of automorphisms*. Each subgroup of this group is a group of automorphisms of the design.

Each 2-(v, k, λ) design determines the existence of 2-$(v, k, m\lambda)$ designs (for any integer $m > 1$), which are called *quasimultiples* of a 2-(v, k, λ) design. A quasimultiple 2-$(v, k, m\lambda)$ is *reducible* into m 2-(v, k, λ) designs if there is a partition of its blocks into m subcollections, each of which forms a 2-(v, k, λ) design. This partition is called a *reduction*. For $m = 2$ quasimultiple designs are called *quasidoubles*, and the reducible quasidouble designs are called *doubles*. We shall denote by $(D_1 \cup D_2)$ a double design which can be reduced to the two designs D_1 and D_2. A *reduction* of a double design D with parameters t-$(v, k, 2\lambda)$ can be represented by a set of two collections of blocks, each containing half of the blocks of D, such that each collection of blocks forms a 2-(v, k, λ) design. An obvious reduction of a double design $(D_1 \cup D_2)$ is $\{D_1, D_2\}$; the order in which the constituent designs are listed, is not relevant. We will often use the notation $D_2 = \mu D_1$, in which μ is a point permutation applied to the points of D_1 to obtain D_2. Doubles can be reducible in more than one way. Two reductions $\{D_1, D_2\}$ and $\{D_3, D_4\}$ of a double design are *equivalent* if and only if there exists some point permutation μ such that $D_3 = \mu D_1$ and $D_4 = \mu D_2$, or such that $D_4 = \mu D_1$ and$D_3 = \mu D_2$. A double which has, up to equivalence, only one reduction is *uniquely reducible*.

Reducible 2-(21, 5, 2) designs are the subject of the present note, we will show that they are uniquely reducible. Up to equivalence there is a unique 2-(21, 5, 1) design (the projective plane $PG(2, 4)$ of order 4) and the reducible 2-(21, 5, 2) designs are its doubles. The first lower bound on the number of reducible 2-(21, 5, 2) designs is derived in [11] and it is 10. Lower bounds on the number of doubles of projective planes in general are derived in [5,6]. These bounds are much more powerful for projective planes of bigger orders, but for the doubles of the projective plane of order 4 the bound is 24.

In this paper, we enumerate the reducible 2-(21, 5, 2) designs by constructing those which have non-trivial automorphisms, which allows us to calculate the number of all reducible 2-(21, 5, 2). This is possible, because these designs are made up of two 2-(21, 5, 1) subdesigns. For other examples of enumerating designs which contain incidence structures see for instance [8–10,15].

In [14] all 2-(21, 5, 2) designs with automorphisms of odd prime orders were constructed, their number was determined to be 22 998 and 4170 of them were found to be reducible. This leaves only the reducible 2-(21, 5, 2) designs with automorphisms of order 2 to be constructed. There are two types of such automorphisms, namely those which transform each of the constituent 2-(21, 5, 1) designs into itself and those which transform one of the 2-(21, 5, 1) into the other (and vice versa). We construct 40 485 designs of the first type and 991 957 of the second. We study their automorphism groups. The results coincide with those obtained in [14]. Using this data we calculate that the number of all doubles of the projective plane of order 4 is 1 746 461 307.

2. Doubles of a uniquely reducible design

Below we will consider doubles of designs for which, up to isomorphism, only one design of its parameter set exists. So instead of $(D_1 \cup D_2)$ we will often use the notation $(D \cup \varphi D)$, where the constituent design φD is obtained from D by a permutation φ of its points.

In the rest of this section, D will be a 2-(v, k, λ) design and $(D \cup \varphi D)$ will be a uniquely reducible double of D. By G we denote the full automorphism group of D; by G_φ we denote the intersection of the full automorphism groups of D and φD; by $\widehat{G}_\varphi$ we denote the full automorphism group of the double design $(D \cup \varphi D)$.

The set of all $v!$ permutations φ of the points of D can be partitioned into classes $\mathscr{C}_G(\varphi)$, where $\mathscr{C}_G(\varphi)$ is the set of point permutations ψ having the property that the double $(D \cup \psi D)$ is isomorphic to $(D \cup \varphi D)$. Then obviously

$$v! = \sum_{\mathscr{C}_G(\varphi)} |\mathscr{C}_G(\varphi)|. \tag{1}$$

In the following proposition we determine the size of an isomorphism class $\mathscr{C}_G(\varphi)$ with a given representative point permutation φ.

Proposition 1. *The set $\mathscr{C}_G(\varphi)$ of point permutations ψ having the property that $(D \cup \psi D)$ is isomorphic to $(D \cup \varphi D)$ is given by*

$$\mathscr{C}_G(\varphi) = G\varphi G \cup G\varphi^{-1}G.$$

Moreover, the number of such permutations is given by

$$|\mathscr{C}_G(\varphi)| = \begin{cases} |G|^2/|G_\varphi| & \text{if } G\varphi G = G\varphi^{-1}G, \\ 2|G|^2/|G_\varphi| & \text{otherwise, i.e. } G\varphi G \cap G\varphi^{-1}G = \emptyset. \end{cases} \tag{2}$$

Proof. Suppose $\psi \in G\varphi G$, then $\exists \alpha, \beta \in G : \psi = \beta\varphi\alpha$. Clearly $(D \cup \psi D)$ is isomorphic to $(D \cup \varphi D)$ since $\beta^{-1}(D \cup \beta\varphi\alpha D) = (D \cup \varphi D)$. Similarly, if $\psi \in G\varphi^{-1}G$, then $\exists \alpha, \beta \in G : \psi = \beta\varphi^{-1}\alpha$, and $(D \cup \psi D)$ is isomorphic to $(D \cup \varphi D)$ since $\varphi\beta^{-1}(D \cup \beta\varphi^{-1}\alpha D) = (\varphi D \cup D)$.

Conversely, we now suppose that $(D \cup \psi D)$ is isomorphic to $(D \cup \varphi D)$. Since $(D \cup \varphi D)$ is uniquely reducible, only two cases are possible. In the first case there exists a point permutation μ for which $\mu D = D$ and $\mu\varphi D = \psi D$, which implies that $\mu \in G$ and $\varphi^{-1}\mu^{-1}\psi \in G$, and thus that $\psi \in G\varphi G$. In the second case there exists a point permutation μ such that $\mu D = \psi D$ and $\mu\varphi D = D$, which implies that $\mu^{-1}\psi \in G$ and $\mu\varphi \in G$, and thus $\psi \in \mu G$ and $\mu \in G\varphi^{-1} \Rightarrow \psi \in G\varphi^{-1}G$.

From the theory of double cosets [4, Theorem 2.19] it follows immediately that $|G\varphi G| = |G\varphi^{-1}G| = |G|^2/|G_\varphi|$; moreover, it is known that either $G\varphi G \cap G\varphi^{-1}G = \emptyset$ or $G\varphi G = G\varphi^{-1}G$. Hence $|G\varphi G \cup G\varphi^{-1}G| = 2|G|^2/|G_\varphi|$ when $G\varphi G \cap G\varphi^{-1}G = \emptyset$, and $|G\varphi G \cup G\varphi^{-1}G| = |G|^2/|G_\varphi|$ otherwise. □

Proposition 2. *If $G\varphi G = G\varphi^{-1}G$, then there exists $\omega \in \widehat{G}_\varphi$ such that $(D \cup \varphi D) = (D \cup \omega D)$. This ω transforms D into φD and vice versa. If $|G_\varphi| = 1$, then ω is of order 2.*

Proof. Since $G\varphi G = G\varphi^{-1}G$, it holds that $\varphi^{-1} \in G\varphi G$, hence $\exists \rho, \sigma \in G : \varphi^{-1} = \rho\varphi\sigma$. This means that $\varphi\rho\varphi\sigma = 1$, hence $\varphi\rho\varphi \in G$. Let $\omega = \varphi\rho$.

Then $\omega D = \varphi\rho D = \varphi D$. Since $\omega\varphi \in G$, $\omega\varphi D = D$. Hence $(D \cup \varphi D) = (D \cup \omega D)$ and ω transforms D into φD and vice versa.

Moreover, it follows from $\omega^2 D = D$ and $\omega^2\varphi D = \varphi D$, that $\omega^2 \in G_\varphi$. In case $|G_\varphi| = 1$, this means that $\omega^2 = 1$. □

Corollary 3. *If $|\widehat{G}_\varphi| = 1$, then $|\mathscr{C}_G(\varphi)| = 2|G|^2$.*

Let N_i (resp., N'_i) denote the number of isomorphism classes $\mathscr{C}_G(\varphi)$ for which $|G_\varphi| = i$ and $G\varphi G \cap G\varphi^{-1}G = \emptyset$ (resp., $G\varphi G = G\varphi^{-1}G$). Then, using Eq. (2), Eq. (1) can be rewritten as

$$v! = 2|G|^2 N_1 + |G|^2 N'_1 + \sum_{i>1} \frac{2|G|^2}{i} N_i + \sum_{i>1} \frac{|G|^2}{i} N'_i. \tag{3}$$

Let N be the total number of non-isomorphic doubles of D, then

$$N = N_1 + N'_1 + \sum_{i>1} N_i + \sum_{i>1} N'_i. \tag{4}$$

If we can enumerate the doubles of D with non-trivial automorphisms by means of some construction techniques, i.e. determine the numbers N'_1 as well as N_i and N'_i for all $i > 1$, Eq. (3) can be used to obtain the number N_1 of doubles of D with trivial automorphisms. Eq. (4) can be used to calculate the total number N of doubles of D.

Corollary 4. *A 2-(v, k, λ) design D of which all doubles are uniquely reducible, has at least $v!/(2|G|^2)$ non-isomorphic doubles (with G the full automorphism group of D).*

All quasidoubles of the projective planes of orders 2 and 3 have been known before this work. Up to isomorphism there are 4 doubles of the projective plane of order 2 and 184 doubles of the projective plane of order 3. We established that they are uniquely reducible, and then investigated their automorphisms and checked that they match Eqs. (3) and (4).

We also checked Eqs. (3) and (4) on the doubles of the affine planes of orders 2, 3 and 4. Most related to the main result in this paper are the doubles of the affine plane of order 4, since their unique reducibility follows from the considerations in the next section. All the doubles of the affine plane of order 4 are among the resolvable 2-(16, 4, 2) designs which are constructed in [7] and for which the designs with non-trivial automorphisms are available from the authors' web-page. We determined that 9102 among them are reducible and, using Eqs. (3) and (4), we found that the number of doubles of the affine plane of order 4 is 320 061. We independently constructed all these doubles and obtained the same result.

It follows from Corollary 4 that the number of the non-isomorphic doubles of the projective plane of order 4 is at least 1 745 944 200. To determine their exact number by Eqs. (3) and (4), we have to construct all designs with non-trivial automorphisms.

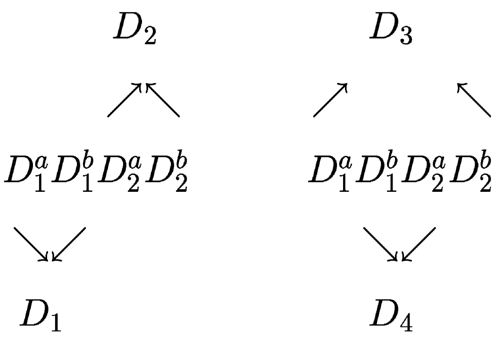
More precisely this means that we have to

- construct the double designs for which $|G_\varphi| \neq 1$ and determine the numbers N_i and N'_i, $i > 1$ (cf. Section 4.1), and
- construct the double designs for which $|G_\varphi| = 1$ and $G\varphi G = G\varphi^{-1}G$, and thus determine the number N'_1 (cf. Section 4.2).

3. On the unique reducibility of 2-(21, 5, 2)

In this section, two reductions $\{D_1, D_2\}$ and $\{D_3, D_4\}$ of a double design are considered *different* if the two sets of collections of blocks are not pairwise equal. In order to prove the unique reducibility of a double 2-(21, 5, 2) design, we will consider all different reductions and show that they are equivalent by a computer-assisted proof.

Consider a double 2-$(v, k, 2\lambda)$ design $(D_1 \cup D_2)$, which is a double of a unique (for its parameter set) 2-(v, k, λ) design. We consider all reductions, different from the obvious reduction $\{D_1, D_2\}$, in the form $\{D_1^a \cup D_2^b, D_2^a \cup D_1^b\}$, where the collection of blocks D_1^a and D_1^b form D_1, the collection of blocks D_2^a and D_2^b form D_2, the collection of blocks D_1^a and D_2^b form D_3 and the collection of blocks D_1^b and D_2^a form D_4, with D_1, D_2, D_3 and D_4 all isomorphic designs:



Without loss of generality, we can restrict ourselves to reductions where the *a* parts have at least as many blocks as the *b* parts. Also, we only consider reductions where D_1^b and D_2^b have no common blocks (i.e. no two blocks, one of D_1^b and one of D_2^b are incident with the same set of points), since such a reduction is not different from the reduction where the equal blocks of the *b* parts are put in the *a* parts.

Proposition 5. *Let n be the number of blocks in* D_1^b (D_2^b), *and* β_i *the number of blocks in* D_1^b (D_2^b) *containing point* i $(i = 1, 2, \ldots, v)$. *The following considerations can be made*:

(a) *Any point is in the same number of blocks of* D_1^a *and* D_2^a $(D_1^b$ *and* $D_2^b)$.
(b) *Any pair of points is in the same number of blocks of* D_1^a *and* D_2^a $(D_1^b$ *and* $D_2^b)$.

(c) *If* D_1 *is a projective plane of order* q (*i.e. a* 2-$(q^2+q+1, q+1, 1)$ *design*), *the following holds*:

$$\beta_i \neq 1, \quad i = 1, 2, \ldots, v, \tag{5}$$

$$\sum_{i=1}^{v} \beta_i = n(q+1), \tag{6}$$

$$\sum_{i=1}^{v} \beta_i^2 = n(n+q), \tag{7}$$

$$n \leqslant \frac{q^2+q}{2}. \tag{8}$$

Proof.

(a) D_1^b and D_1^a (or D_2^a) form a 2-(v, k, λ), and thus point i is in $r-\beta_i$ blocks of D_1^a (D_2^a).
(b) Let the pair of points (*i*,*j*) be in λ_{ij} blocks of D_1^b. Since D_1^b and D_1^a (or D_2^a) form a 2-(v, k, λ), the pair of points (*i*,*j*) is in $\lambda - \lambda_{ij}$ blocks of D_1^a (D_2^a).
(c) In a 2-$(q^2+q+1, q+1, 1)$ design two blocks have exactly one common point. Consider any point i. When we look at an arbitrary subset S_q of q blocks out of the $q+1$ blocks incident with point i, S_q forces the last block (which contains point i) to be incident with all q remaining points which are not in any of the blocks of S_q. So this last block containing point i is fixed by the other q blocks. $\beta_i = 1$ would force $D_1^a(D_2^a)$ to have q blocks containing point i. This will force the block of D_1^b containing point i to be the same as the block of D_2^b containing point i, but we supposed that D_1^b and D_2^b have no common blocks, so (5) follows.

Eq. (6) is obtained by counting the number of ones in the incidence matrix of D_1^b (D_2^b) in two ways.

A 2-$(q^2+q+1, q+1, 1)$ design is symmetric, and from the $\lambda = 1$ condition for the blocks of the D_1^b (D_2^b) part we obtain $\binom{n}{2} = \sum_{i=1}^{v} \binom{\beta_i}{2}$. Using also (6) we get (7).

D_1^b (D_2^b) has at most as many blocks as $D_1^a(D_2^a)$. That is why $n \leqslant \lfloor v/2 \rfloor$, and thus (8) follows. □

Proposition 6. *A reducible* 2-(21, 5, 2) *design is uniquely reducible*.

Proof. For $q = 4$, the set of Eqs. (5)–(8) has solutions only for $n = 6, 8, 9, 10$.

For each case, exhaustive generation is performed in the following way, satisfying (a) and (b) from Proposition 5:

- We generate the set of all non-equivalent D_1^b.
- For each such D_1^b, we generate the set of all non-equivalent D_2^b, taking into account the limitation that D_1^b and D_2^b have no common blocks.
- For each such combination of D_1^b and D_2^b, we generate all non-equivalent *a* parts (D_1^a or D_2^a), and show that all obtained reductions are equivalent.

The unique solution for the values of β_i if $n = 6$ is $(2_{15}, 0_6)$, namely 15 twos and 6 zeroes. There is only one non-equivalent way to choose six blocks for D_1^b matching this pattern, but exhaustive generation shows we cannot construct D_2^b. So $n = 6$ is impossible.

The unique solution for the values of β_i if $n = 8$ is $(4_2, 2_{16}, 0_3)$. There is only one non-equivalent way to choose eight blocks for D_1^b matching this pattern. Given D_1^b, there is only non-equivalent way to construct D_2^b. For the unique combination of D_1^b and D_2^b, we generated 12 non-equivalent D_1^a (D_2^a). For all obtained reductions, one of which is shown in Fig. 1, there exist point permutations φ_a and φ_b such that

- $D_2 = \varphi_b \varphi_a D_1$.
- $D_2^b = \varphi_b D_1^b$, $D_1^a = \varphi_b D_1^a$, $D_2^a = \varphi_b D_2^a$, $(\varphi_b)^2 = 1$.
- $D_2^a = \varphi_a D_1^a$, $D_1^b = \varphi_a D_1^b$, $D_2^b = \varphi_a D_2^b$.

D_1^a				D_1^b		D_2^a				D_2^b	
1	1111	....	....	....	....	1	....	1111	....	....	....
1	....	1111	....	....	....	1	....	....	1111	....	....
1	....	....	1111	....	....	1	1111	....	....	....	....
1	....	....	....	1111	....	1	....	....	....	....	1111
1	....	....	....	....	1111	1	....	....	....	1111	....
.	1...	1...	1...	1...	...1	.	1...	1...	1...	1...	...1
.	.1..	.1..	.1..	.1..	...1	.	.1..	.1..	.1..	.1..	...1
.	..1.	..1.	..1.	..1.	...1	.	..1.	..1.	..1.	..1.	...1
.	...1	...1	...1	...1	...1	.	...1	...1	...1	...1	...1
.	.1..	..1.	...1	1...	..1.	.	.1..	..1.	...1	1...	..1.
.	1...	...1	..1.	.1..	..1.	.	1...	...1	..1.	.1..	..1.
.	...1	1...	.1..	..1.	..1.	.	...1	1...	.1..	..1.	..1.
.	..1.	.1..	1...	...1	..1.	.	..1.	.1..	1...	...1	..1.
.	..1.	...1	.1..	1...	.1..	.	..1.	...1	.1..	1...	.1..
.	...1	..1.	1...	.1..	.1..	.	...1	..1.	1...	.1..	.1..
.	1...	.1..	...1	..1.	.1..	.	1...	.1..	...1	..1.	.1..
.	.1..	1...	..1.	...1	.1..	.	.1..	1...	..1.	...1	.1..
.	...1	.1..	..1.	1...	1...	.	...1	.1..	..1.	1...	1...
.	..1.	1...	...1	.1..	1...	.	..1.	1...	...1	.1..	1...
.	.1..	...1	1...	..1.	1...	.	.1..	...1	1...	..1.	1...
.	1...	..1.	.1..	...1	1...	.	1...	..1.	.1..	...1	1...

Fig. 1. One of the obtained reductions for the $n = 8$ case.

The reduction $\{D_1^a \cup D_2^b, D_2^a \cup D_1^b\}$ is equivalent to the reduction $\{D_1, D_2\}$ because

$$\varphi_b(D_1^a \cup D_2^b) = (D_1^a \cup D_1^b) = D_1, \quad \varphi_b(D_2^a \cup D_1^b) = (D_2^a \cup D_2^b) = D_2.$$

There are 3 solutions for the values of β_i if $n = 9$: $(5_1, 3_4, 2_{14}, 0_2)$, $(4_3, 3_1, 2_{15}, 0_2)$ and $(3_9, 2_9, 0_3)$. None of the subsets of nine blocks of a 2-(21, 5, 1) design matches the first two patterns. There is only one non-equivalent way to choose nine blocks for D_1^b matching pattern $(3_9, 2_9, 0_3)$, but exhaustive generation shows we cannot construct D_2^b.

There are 3 solutions for the values of β_i if $n = 10$: $(5_1, 4_2, 3_3, 2_{14}, 0_1)$, $(4_5, 2_{15}, 0_1)$ and $(4_2, 3_8, 2_9, 0_2)$. None of the subsets of 10 blocks of a 2-(21, 5, 1) design matches the first pattern. There is only one non-equivalent way to choose 10 blocks for D_1^b matching pattern $(4_5, 2_{15}, 0_1)$ or $(4_2, 3_8, 2_9, 0_2)$, but exhaustive generation shows we cannot construct D_2^b.

Thus, we conclude that a 2-(21, 5, 2) design cannot have two inequivalent reductions, i.e. it is uniquely reducible. □

4. Reducible 2-(21, 5, 2) with non-trivial automorphisms

4.1. Automorphisms for which $|G_\varphi| \neq 1$

All 2-(21, 5, 2) designs with automorphisms of odd prime orders are constructed in [14]. It turns out that 4170 of them are reducible and we use these for our classification. Thus, we only have to construct the designs with automorphisms of order 2.

1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1	.	.	.	.	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
1	.	.	.	.	.	.	.	.	1	1	1	1	.	.	.	.	.	.	.	.
1	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	.	.	.	.
1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1
.	1	.	.	.	1	.	.	.	1	.	.	.	1	.	.	.	1	.	.	.
.	1	.	.	.	.	1	.	.	.	1	.	.	.	1	.	.	.	1	.	.
.	1	.	.	.	.	.	1	.	.	.	1	.	.	.	1	.	.	.	1	.
.	1	.	.	.	.	.	.	1	.	.	.	1	.	.	.	1	.	.	.	1
.	.	1	.	.	1	.	.	.	.	1	.	.	.	.	1	.	.	.	.	1
.	.	1	.	.	.	1	.	.	1	.	.	.	.	.	.	1	.	.	1	.
.	.	1	.	.	.	.	1	.	.	.	.	1	1	.	.	.	.	1	.	.
.	.	1	.	.	.	.	.	1	.	.	1	.	.	1	.	.	1	.	.	.
.	.	.	1	.	1	.	.	.	.	.	1	.	.	.	.	1	.	1	.	.
.	.	.	1	.	.	1	.	.	.	.	.	1	.	.	1	.	1	.	.	.
.	.	.	1	.	.	.	1	.	1	.	.	.	.	1	.	.	.	.	.	1
.	.	.	1	.	.	.	.	1	.	1	.	.	1	.	.	.	.	.	1	.
.	.	.	.	1	1	.	.	.	.	.	.	1	.	1	.	.	.	.	1	.
.	.	.	.	1	.	1	.	.	.	.	1	.	1	.	.	.	.	.	.	1
.	.	.	.	1	.	.	1	.	.	1	.	.	.	.	.	1	1	.	.	.
.	.	.	.	1	.	.	.	1	1	.	.	.	.	.	1	.	.	1	.	.

Fig. 2. The design D_1 (5 fixed points).

Consider $(D_1 \cup D_2)$ with a full automorphism group of order 2^s $(s \geqslant 1)$, and $|G_\varphi| \neq 1$. Then D_1 and D_2 have common automorphisms of order 2. The 2-(21, 5, 1) design is known to have automorphisms of order 2 with 5 fixed points and automorphisms of order 2 with 7 fixed points. Their action is illustrated in Figs. 2 and 3. We construct all double designs with such automorphisms of order 2 which are automorphisms of both D_1 and D_2. We consider the two cases, namely

Automorphism of order 2 *with* 5 *fixed points*: Consider the incidence matrix of D_1 in the form presented in Fig. 2 and suppose an automorphism γ which acts on the points of the double as $(1)(2)\cdots(5)(6,7)(8,9)\cdots(20,21)$, and on the blocks as $(1)(2)\cdots(5)(6,7)(8,9)\cdots(20,21)(22)(23)\cdots(26)(27,28)(29,30)\cdots(41,42)$.

Automorphism of order 2 *with* 7 *fixed points*: Consider the incidence matrix of D_1 in the form presented in Fig. 3 and suppose an automorphism δ which acts on the points of the double as $(1)(2)\cdots(7)(8,9)(10,11)\cdots(20,21)$, and on the blocks as $(1)(2)\cdots(7)(8,9)\cdots(20,21)(22)(23)\cdots(28)(29,30)\cdots(41,42)$.

Let $D_2 = \varphi D_1$. Then φ is a permutation of the points which should

(a) transform any fixed point (with respect to γ or δ) into a fixed point,
(b) transform two points of one and the same orbit (with respect to γ or δ) into points which are in one and the same orbit.

We have used two different approaches for the actual construction. The results are the same.

In the first approach we initially leave the fixed points aside and construct the non-trivial orbit part of the incidence matrix of the double design. We generate all possibilities for the non-trivial orbit part of D_2 by applying

1	1	.	1	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
.	1	1	.	1	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.
.	.	1	1	.	1	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.
.	.	.	1	1	.	1	.	.	.	.	.	.	1	1	.	.	.	.	.	.
1	.	.	.	1	1	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.
.	1	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	1	1	.	.
1	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1	1
1	.	.	.	.	.	.	.	.	.	1	1	.	1	.	.	.	1	.	.	.
1	.	.	.	.	.	.	.	.	1	.	.	1	.	1	.	.	.	1	.	.
.	1	.	.	.	.	.	.	.	.	.	.	1	1	.	1	.	.	.	1	.
.	1	.	.	.	.	.	.	.	.	.	1	.	.	1	.	1	.	.	.	1
.	.	1	.	.	.	.	1	.	.	.	.	.	.	1	1	.	1	.	.	.
.	.	1	.	.	.	.	.	1	.	.	.	.	1	.	.	1	.	1	.	.
.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	1	1	.	1	.
.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	1	.	.	1	.	1
.	.	.	.	1	.	.	1	.	.	.	1	.	.	.	.	.	.	1	1	.
.	.	.	.	1	.	.	.	1	.	.	.	1	.	.	.	.	1	.	.	1
.	.	.	.	.	1	.	1	.	1	.	.	.	1	.	.	.	.	.	.	1
.	.	.	.	.	1	.	.	1	.	1	.	.	.	1	.	.	.	.	1	.
.	.	.	.	.	.	1	.	1	1	.	1	.	.	.	1	.	.	.	.	.
.	.	.	.	.	.	1	1	.	.	1	.	1	.	.	.	1	.	.	.	.

Fig. 3. The design D_1 (7 fixed points).

all possible permutations of whole point orbits of D_1 and filtering the equivalent solutions away. For each non-equivalent solution we then generate all possible permutations within the point orbits of D_2 and again filter the equivalent solutions away. Finally, we add the fixed part in all possible ways (the fixed part of D_2 is a permutation of the fixed part of D_1) and check for isomorphism. However, when applying this approach, the equivalence checks in the first and second step should be carried out with great care since a large number of restrictions hold.

In the second approach we first find all automorphisms of D_1. Next we generate all point permutations meeting conditions (a) and (b) in lexicographic order. When generating the current permutation φ, we search for $\alpha, \beta \in G$, such that $\beta\varphi\alpha$ or $\beta\varphi^{-1}\alpha$ is a permutation which is lexicographically smaller than φ (see Proposition 1) and meets conditions (a) and (b). The existence of such a pair $\alpha, \beta \in G$ means that the solution is equivalent to one we have already generated, so we can drop it. Note that conditions (a) and (b) are of such a form that they allow us to prune partial solutions for the permutations, which makes the programme much faster. Since the order of the automorphism group G of the 2-(21, 5, 1) design is 120 960, considering all $120\,960^2$ combinations is too time-consuming, so we only consider α and β among a random part of the elements of the group G. We finally filter away the isomorphic solutions (only a limited number of which happen to turn up) by a full isomorphism test.

In this way we construct 9564 non-isomorphic doubles with an automorphism of order 2 with 5 fixed points, and 31 094 with an automorphism of order 2 with 7 fixed points. This gives a total of 40 485 doubles for this case, because 173 have both an automorphism of order 2 with 5 or 7 fixed points. Of these doubles 305 have also an automorphism of odd prime order, so they were already counted among the 4170 doubles found above.

4.2. Automorphisms of order 2 with $|G_\varphi| = 1$

We generate all designs $(D \cup \varphi D)$, where φ is a permutation of order 2 (see Proposition 2). We use a method similar to the second approach from the previous section. We first find all automorphisms of D and then generate all possible permutations of order 2 in lexicographic order. As the automorphism group of the projective plane of order 4 is doubly transitive, we can fix one non-trivial orbit.

Suppose we have constructed the current permutation φ. Suppose $\exists \alpha, \beta \in G$, such that $\beta\varphi\alpha$ is lexicographically smaller than φ (see Proposition 1, and mind that φ is of order 2, i.e. $\varphi = \varphi^{-1}$). If we have already constructed $\beta\varphi\alpha$, then it is of order 2, namely

$$\beta\varphi\alpha\beta\varphi\alpha = 1 \Rightarrow \varphi\alpha\beta\varphi = \beta^{-1}\alpha^{-1} \Rightarrow \varphi\alpha\beta\varphi \in G.$$

But $\varphi\alpha\beta\varphi$ is also an automorphism of φD. Hence $\varphi\alpha\beta\varphi \in G_\varphi$. If $|G_\varphi| = 1$, then $\alpha\beta = 1 \Rightarrow \beta = \alpha^{-1}$. Since $|G_\varphi| = 1$ for most of the designs constructed this way, for the currently constructed permutation φ, we only search for $\alpha \in G$, such that $\alpha^{-1}\varphi\alpha$ is lexicographically smaller than φ, and we drop the solution if such an α exists. This way most of the isomorphic copies are filtered, the final full isomorphism check does not filter much more. This simpler pruning condition makes the programme much faster, which is important because we cannot prune partial solutions for the permutations in this case.

We checked the results by two different implementations. In one of them we used McKay's program *nauty* [12] for the final isomorphism check. We construct 991 957 non-isomorphic designs which have an automorphism of order 2 transforming the constituent designs into one another. We establish that for 984 549 of them the order of the full group of automorphisms is 2, and $|G_\varphi| = 1$.

5. Classification of the doubles of the projective plane of order 4

Classification results are presented in Table 1. The classification is based on three properties:

- the order of the automorphism group of the doubles (column $|\widehat{G}_\varphi|$),
- the order of the common subgroup of the full automorphism groups of D and φD (column $|G_\varphi|$), and
- whether $G\varphi G = G\varphi^{-1}G$.

The column labeled $N^{(\prime)}_{|G_\varphi|}$ gives the number of non-isomorphic doubles for the given values of the properties. The number[3] of designs isomorphic to one of these doubles among all the 21! possible $(D \cup \psi D)$ is presented in column $|\mathscr{C}_G(\varphi)|/|G|$; this number is determined using Proposition 1. Column $N^{(\prime)}_{|G_\varphi|}$ multiplied by column $|\mathscr{C}_G(\varphi)|/|G|$ gives the last column $N^{(\prime)}_{|G_\varphi|} \times |\mathscr{C}_G(\varphi)|/|G|$.

Having constructed all 1 028 899 doubles which possess non-trivial automorphisms, we use Eqs. (3) and (4) to calculate the number of non-isomorphic designs which possess only the trivial automorphism which turns out to be 1 745 432 408; thus, the first row of Table 1, which is marked, is derived from the other rows.

The number of all 2-(21, 5, 2) doubles is 1 746 461 307, which does not differ very much from the bound obtained by Corollary 4.

Acknowledgments

Part of this work was accomplished during the visit of the second author to Ghent University in September 2004, so she would like to thank the CAAGT group for the invitation, and for their hospitality. We would like to thank professor Kris Coolsaet for his useful advices and for carefully reading the manuscript and suggesting changes which improved a lot its quality.

[3] Note that, to avoid big numbers, the values in the last two columns are divided by $|G| = 120\,960$, i.e. the order of the automorphism group of the projective plane of order 4.

Table 1
Classification of the doubles of the projective plane of order 4

$\lvert\widehat{G}_\varphi\rvert$	$\lvert G_\varphi\rvert$	$G\varphi G \stackrel{?}{=} G\varphi^{-1}G$	$N^{(\prime)}_{\lvert G_\varphi\rvert}$	$\frac{\lvert\mathscr{C}_G(\varphi)\rvert}{\lvert G\rvert}$	$N^{(\prime)}_{\lvert G_\varphi\rvert} \times \frac{\lvert\mathscr{C}_G(\varphi)\rvert}{\lvert G\rvert}$
1	1	No	1 745 432 408	241 920	422 255 008 143 360
2	1	Yes	984 549	120 960	119 091 047 040
2	2	No	33 631	120 960	4 068 005 760
3	3	No	2764	80 640	222 888 960
4	2	Yes	5709	60 480	345 280 320
4	4	No	389	60 480	23 526 720
5	5	No	26	48 384	1 257 984
6	3	Yes	1019	40 320	41 086 080
6	6	No	67	40 320	2 701 440
8	4	Yes	345	30 240	10 432 800
8	8	No	17	30 240	514 080
9	9	No	1	26 880	26 880
10	5	Yes	30	24 192	725 760
12	6	Yes	167	20 160	3 366 720
12	12	No	2	20 160	40 320
14	7	Yes	2	17 280	34 560
14	14	No	1	17 280	17 280
16	8	Yes	55	15 120	831 600
16	16	No	3	15 120	45 360
18	9	Yes	18	13 440	241 920
18	18	No	1	13 440	13 440
21	21	No	1	11 520	11 520
24	12	Yes	24	10 080	241 920
28	14	Yes	2	8640	17 280
30	15	Yes	1	8064	8064
32	16	Yes	20	7560	151 200
32	32	No	1	7560	7560
36	18	Yes	15	6720	100 800
40	20	Yes	1	6048	6048
42	21	Yes	2	5760	11 520
48	24	Yes	3	5040	15 120
54	27	Yes	1	4480	4480
64	32	Yes	7	3780	26 460
96	48	Yes	4	2520	10 080
96	96	No	1	2520	2520
108	54	Yes	2	2240	4480
120	60	Yes	1	2016	2016
128	64	Yes	2	1890	3780
192	96	Yes	4	1260	5040
252	126	Yes	1	960	960
256	128	Yes	1	945	945
384	96	Yes	1	1260	1260
384	192	Yes	1	630	630
480	240	Yes	1	504	504
576	288	Yes	1	420	420
1152	288	Yes	1	420	420
1152	576	Yes	1	210	210
1536	384	Yes	1	315	315
3840	1920	Yes	1	63	63
120 960	120 960	Yes	1	1	1
$\lvert\widehat{G}_\varphi\rvert$	$\lvert G_\varphi\rvert$	$G\varphi G \stackrel{?}{=} G\varphi^{-1}G$	$N^{(\prime)}_{\lvert G_\varphi\rvert}$	$\frac{\lvert\mathscr{C}_G(\varphi)\rvert}{\lvert G\rvert}$	$N^{(\prime)}_{\lvert G_\varphi\rvert} \times \frac{\lvert\mathscr{C}_G(\varphi)\rvert}{\lvert G\rvert}$
All			1 746 461 307		$21!/\lvert G\rvert = 422\,378\,820\,864\,000$

References

[1] Th. Beth, D. Jungnickel, H. Lenz, Design Theory, second ed., Cambridge University Press, Cambridge, 1999.
[2] P.J. Cameron, J.H. van Lint, Designs, Graphs, Codes and their Links, Cambridge University Press, Cambridge, 1991.
[3] C.J. Colbourn, J.H. Dinitz, (Eds.), The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, FL, Berlin, 1996.
[4] B. Huppert, Endliche Gruppen I, Springer, Berlin, 1967.
[5] D. Jungnickel, Quasimultiples of projective and affine planes, J. Geom. 26 (1986) 172–181.
[6] D. Jungnickel, K. Vedder, Simple quasidoubles of projective planes, Aequationes Math. 34 (1987) 96–100.
[7] P. Kaski, P. ÖstergAArd, Miscellaneous classification results for 2-designs, Discrete Math. 280 (2004) 65–75.
[8] P. Kaski, P. ÖstergAArd, S. Topalova, R. Zlatarski, Steiner triple systems of order 19 and 21 with subsystems of order 7, Discrete Math., to appear.
[9] C. Lam, S. Lam, V.D. Tonchev, Bounds on the number of affine, symmetric and Hadamard designs and matrices, J. Combin. Theory Ser. A 92 (2000) 186–196.
[10] C. Lam, S. Lam, V.D. Tonchev, Bounds on the number of Hadamard designs of even order, J. Combin. Des. 9 (2001) 363–378.
[11] R. Mathon, A. Rosa, Some results on the existence and enumeration of BIBD's, Mathematics Report 125, December 1985, Department of Mathematics and Statistics, McMaster University, Hamilton, 1985.
[12] B.D. McKay, *Nauty* user's guide (version 1.5), Technical Report TR-CS-90-02, Computer Science Department, Australian National University, 1990.
[13] V.D. Tonchev, Combinatorial Configurations, Longman Scientific and Technical, New York, 1988.
[14] S. Topalova, Enumeration of 2-(21, 5, 2) designs with automorphisms of an odd prime order, Diskretnii Analiz i Issledovanie Operatsii 5 (N1) (1998) 64–81 (in Russian).
[15] R.M. Wilson, Nonisomorphic Steiner triple systems, Math. Z. 135 (1974) 303–313.