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Graph homology: Koszul and Verdier duality ☆

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Abstract

We show that Verdier duality for certain sheaves on the moduli spaces of graphs associated to differential graded operads corresponds to the cobar-duality of operads (which specializes to Koszul duality for Koszul operads). This in particular gives a conceptual explanation of the appearance of graph cohomology of both the commutative and Lie types in computations of the cohomology of the outer automorphism group of a free group. Another consequence is an explicit computation of dualizing sheaves on spaces of metric graphs, thus characterizing to which extent these spaces are different from oriented orbifolds. We also provide a relation between the cohomology of the space of metric ribbon graphs, known to be homotopy equivalent to the moduli space of Riemann surfaces, and the cohomology of a certain sheaf on the space of usual metric graphs.

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0. Introduction

The popularity of graph homology owes largely to the fact that the cohomology of two important spaces in mathematics, the classifying space Y_n of the outer automorphism group of the free group on n generators and the (decorated) moduli space $\mathscr{M}_{g,n}$ of Riemann surfaces of genus g

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with n punctures, even though generally intractable, may be computed via a deceptively simple combinatorial construction, called *graph homology*, see M. Culler and K. Vogtmann [4] and R.C. Penner [15]. These results, combined with further study of graph homology by M. Kontsevich [12], rendered the following identifications:

$$\begin{aligned}
H_\bullet(Y_n, k) &\cong H^{3n-4-\bullet}_{\Gamma\mathcal{L}ie}(n),\\
H^\bullet_c(Y_n, k) &\cong \widetilde{H}^\bullet_{\Gamma\mathcal{C}omm}(n),\\
H_\bullet(\mathcal{M}_{g,n}, k) &\cong H^{6g+3n-7-\bullet}_{\Gamma\mathcal{A}ss}(g, n),\\
H^\bullet_c(\mathcal{M}_{g,n}, k) &\cong \widetilde{H}^\bullet_{\Gamma\mathcal{A}ss}(g, n),
\end{aligned}$$

where k is a coefficient field of characteristic zero, $H^\bullet_c$ denotes cohomology with compact supports, and in the right-hand side, we have graph cohomology of various flavors, Lie, commutative, and associative, with trivial or twisted coefficients.

The appearance of Koszul-dual operads in the right-hand side as corresponding to the homology vs. cohomology with compact supports in the left-hand side is quite suggestive: it hints on a relationship between some kind of Poincaré duality for spaces and Koszul duality for operads.

In this paper we show that this relationship indeed takes place and in fact prove more general results, Theorems 3.9 and 4.3, which show that up to an orientation twist, Verdier duality on the moduli space of graphs transfers a certain constructible sheaf corresponding to an operad $\mathcal{O}$ to the sheaf corresponding to the dg-dual operad $D\mathcal{O}$, which is quasi-isomorphic to the Koszul-dual operad $\mathcal{O}^!$, if $\mathcal{O}$ happens to be Koszul. The idea of a relationship between the two dualities originates from the paper [7] by Ginzburg and Kapranov, who noticed that Verdier duality for sheaves on buildings (contractible spaces of metric trees) provided a sheaf-theoretic interpretation of Koszul duality for operads. Koszulity was thus interpreted in terms of vanishing of higher cohomology for corresponding sheaves, while in our paper it translates into a duality statement between highly nontrivial cohomology groups of spaces of metric graphs.

Our results concern non-compact moduli spaces. As pointed out by the referee of this paper, stronger results must hold for certain compactifications of our moduli spaces; cyclic operads need to be replaced with modular operads in this more general setting.

Finally we mention that the relationship between Koszul and Verdier dualities (in a different context) was also observed in the paper [17].

Notation. Throughout this paper we work with vector spaces, graded vector spaces, and dg-vector spaces or complexes—all finite-dimensional in each degree and bounded, over a ground field k, which is assumed to be of characteristic zero with the exception of Section 1. We consider chain complexes $V_\bullet = \bigoplus_{i\in\mathbb{Z}} V_i$ with a differential $d: V_i \to V_{i-1}$ and cochain complexes $V^\bullet = \bigoplus_{i\in\mathbb{Z}} V^i$ with a differential $d: V^i \to V^{i+1}$.

The (*degree*) *shift* $V[1]$ of a complex V has components $(V[1])_i = V_{i+1}$ in the category of chain complexes and $(V[1])^i = V^{i+1}$ in the category of cochain complexes. For chain complexes the degree shift is also known as desuspension.

The functor $V \mapsto V^*$ of taking the linear dual acts within each of the two categories:

$$\begin{aligned}
(V^*)_i &= (V_{-i})^*, \quad d^*: (V^*)_i \to (V^*)_{i-1},\\
(V^*)^i &= (V^{-i})^*, \quad d^*: (V^*)^i \to (V^*)^{i+1},
\end{aligned}$$

while another functor, $V \mapsto V^\vee$, takes the category of chain complexes to that of cochain ones:

$$\left(V^\vee\right)^i = (V_i)^*, \quad d^\vee : \left(V^\vee\right)^i \to \left(V^\vee\right)^{i+1}.$$

Note that $(V[1])^* \cong V^*[-1]$ and $(V[1])^\vee \cong V^\vee[1]$. The double dual V^{**} of a chain complex V is naturally isomorphic to V, while $V^{*\vee} \cong V^{\vee *}$ and the functor $V \mapsto V^{\vee *}$ is an equivalence of categories of chain and cochain complexes. Clearly $(V^{\vee *})^i \cong V_{-i}$.

An ungraded vector space V could be assumed to lie in degree 0, and it will be clear from the context whether this (trivial) grading is considered homological or cohomological. If $\dim V = n$ we will call the *determinant* of V the one-dimensional graded vector space $\mathrm{Det}(V) = S^n(V[-1]) = \Lambda^n(V)[-n]$, concentrated in degree n. Note that $\mathrm{Det}(V)^*[-2n] \cong \mathrm{Det}(V^*)$. We will use negative powers of one-dimensional graded vector spaces for the corresponding positive tensor powers of their $*$-duals, so that

$$\mathrm{Det}^{-p}(V) = \left((\mathrm{Det}\, V)^*\right)^{\otimes p}.$$

For a finite collection $\{V_\alpha \mid \alpha \in I\}$ of finite-dimensional vector spaces, we have a natural identification

$$\bigotimes_{\alpha \in I} V_\alpha[-1] \cong \mathrm{Det}(I) \otimes \bigotimes_{\alpha \in I} V_\alpha.$$

If S is a finite set, let $\mathrm{Det}(S) := \mathrm{Det}(k^S)$. Since there is a canonical isomorphism $(k^S)^* \cong k^S$, we have $\mathrm{Det}(S)^*[-2|S|] \cong \mathrm{Det}(S)$. Note also that $\mathrm{Det}^2(S) \cong k[-2|S|]$.

For a simplex σ, the symbol $\mathrm{Det}(\sigma)$ will denote the determinant of the set of vertices of σ. When the ground field $k = \mathbb{R}$, a choice of a nonzero element in $\mathrm{Det}(\sigma)$ up to a positive real factor is equivalent to providing σ with an orientation in the usual sense.

1. Verdier duality for simplicial complexes

In this section we formulate and prove certain results on Verdier duality for sheaves on simplicial complexes. These results, in a slightly different situation of spaces stratified into cells, were stated in [7].

Definition 1.1. Let X be a finite simplicial complex. A sheaf of dg-vector spaces over a ground field k on X is called *constructible*, if its restriction to each open face of X is a constant sheaf whose stalk is a dg-vector space.

Remark 1.2. Ginzburg and Kapranov use the term "combinatorial sheaf." We follow the more conventional terminology adopted in, e.g. [10].

Any simplicial complex X admits an open covering U_σ where σ runs through the faces of X; namely U_σ is the open star of σ, the union of the interiors of those faces of X which contain σ. Any sheaf determines a contravariant functor from the poset $\{U_\sigma\}$ into the category of dg-vector spaces. Conversely, let $\mathscr{F}$ be a constructible sheaf on X. Let $x \in X$ and consider the face σ of smallest dimension containing x. Then the space of sections of $\mathscr{F}$ over any sufficiently small neighborhood of x will coincide with the space $\Gamma(U_\sigma, \mathscr{F})$ of sections of $\mathscr{F}$ over U_σ. Therefore, $\mathscr{F}$ is completely determined by the corresponding functor.

Consider the category whose objects are the simplices of X and the morphisms are inclusions of faces. We will call a *coefficient system* on X any covariant functor from this category to the category of dg-vector spaces.

Proposition 1.3. *There is a one-to-one correspondence between constructible sheaves and coefficient systems on a simplicial complex X.*

Proof. Indeed, it suffices to note that the category of faces of X is opposite to the category of open stars of X. □

Remark 1.4. The cohomology of a constructible sheaf could be computed using the Čech complex of the covering $\{U_\sigma\}$, as follows from Kashiwara and Schapira [10, Proposition 8.1.4]. Cohomology in this paper will always mean hypercohomology.

Proposition 1.5. *Let $\mathscr{F}$ be a constructible sheaf and $\{\mathscr{F}_\sigma\}$ be the corresponding coefficient system on X. Then the cohomology of a constructible sheaf on X coincides with the cohomology of the cochain complex*

$$\bigoplus_{\tau} \mathscr{F}_\tau \otimes \operatorname{Det}(\tau)[1] \tag{1.1}$$

on which the differential acts as the sum of the internal differential on $\mathscr{F}$ and a map

$$\mathscr{F}_\sigma \otimes \operatorname{Det}(\sigma)[1] \mapsto \bigoplus_{\substack{\tau \supset \sigma \\ \dim \tau = \dim \sigma + 1}} \mathscr{F}_\tau \otimes \operatorname{Det}(\tau)[1],$$

where the last map is induced by inclusions $\sigma \hookrightarrow \tau$.

Proof. According to Remark 1.4, the cohomology of $\mathscr{F}$ could be computed using the Čech (bi)complex of the covering of $\{U_\sigma\}$ of X. A simple inspection shows that this complex is isomorphic to the complex (1.1). □

We will now discuss Verdier duality in the simplicial context. Recall that for a sheaf $\mathscr{F}$, considered as an object of the derived category of sheaves on X, its *Verdier dual $D\mathscr{F}$* is defined by $R\Gamma(U, D\mathscr{F}) = [R\Gamma_c(U, \mathscr{F})]^*$ for each open set $U \subset X$, where $R\Gamma$ and $R\Gamma_c$ denote the derived functors of sections, all and with compact supports, respectively. It is easy to see that for a constructible sheaf $\mathscr{F}$, its Verdier dual complex $D\mathscr{F}$ will have constructible cohomology and therefore, by [10, Theorem 8.1.10], can be represented by a complex of constructible sheaves.

Proposition 1.6. *Let $\mathscr{F}$ be a constructible sheaf on X. Then its Verdier dual $D\mathscr{F}$ may be represented by a constructible complex $\sigma \mapsto D\mathscr{F}_\sigma$, where $D\mathscr{F}_\sigma$ is the following cochain complex:*

$$\bigoplus_{\tau \supset \sigma} \bigl(\mathscr{F}_\tau \otimes \operatorname{Det}(\tau)[1]\bigr)^*$$

whose differential is the dual to that in (1.1).

Note that under our grading convention for dual spaces, $\deg(\mathrm{Det}(\tau)[1])^* = -\dim\tau$.

Proof. Consider the open star $\mathrm{st}(\sigma)$ of the simplex σ. We will denote by $i : \mathrm{st}(\sigma) \to \overline{\mathrm{st}(\sigma)}$ the inclusion of $\mathrm{st}(\sigma)$ into its closure. Then the extension by zero $i_!\mathscr{F}|_{\mathrm{st}(\sigma)}$ is a constructible sheaf on the simplicial complex $\overline{\mathrm{st}(\sigma)}$. It follows that

$$R\Gamma_c\big(\mathrm{st}(\sigma), \mathscr{F}\big) = R\Gamma_c\big(\overline{\mathrm{st}(\sigma)}, i_!\mathscr{F}|_{\mathrm{st}(\sigma)}\big) = R\Gamma\big(\overline{\mathrm{st}(\sigma)}, i_!\mathscr{F}|_{\mathrm{st}(\sigma)}\big).$$

Note that $\overline{\mathrm{st}(\sigma)}$ is the union of all simplices containing σ. The sheaf $i_!(\mathscr{F}|_{\mathrm{st}(\sigma)})$ corresponds to the coefficient system on $\overline{\mathrm{st}(\sigma)}$ so that

$$(i_!\mathscr{F}|_{\mathrm{st}(\sigma)})_\tau = \begin{cases} \mathscr{F}_\tau & \text{if } \tau \supset \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Now Proposition 1.5 implies that $R\Gamma i_!(\mathscr{F}|_{\mathrm{st}(\sigma)})$ is represented by the complex

$$\bigoplus_{\tau\supset\sigma} \mathscr{F}_\tau \otimes \mathrm{Det}(\tau)[1],$$

and the desired statement follows. □

Remark 1.7. This result was formulated in the stratified setting in [7, Proposition 3.5.12(b)].

2. Equivariant Verdier duality

In this section we generalize our theory to the case of *orbi-simplicial* complexes. We will not discuss orbi-simplicial complexes in full generality, restricting ourselves to the case when there exists a global group action. For the rest of the paper, the ground field k will have characteristic 0. Let X be a topological space and G be a group acting properly discontinuously on X. That means that the stabilizer G_x of every point $x \in X$ is finite and every point $x \in X$ has a neighborhood U_x such that $gU_x \cap U_x = \emptyset$ if $g \notin G_x$. Let Y denote the space of orbits X/G and by $f : X \to Y$ the projection map. We now recall some standard definitions and facts about equivariant sheaves, cf. [8] or a more modern reference [1].

Definition 2.1. A *G-equivariant sheaf* $\mathscr{F}$ on X is a sheaf of k-vector spaces with a G-action. More precisely, for any $g \in G$ and any open set $U \subset X$ there is an isomorphism $g_U : \Gamma(U, \mathscr{F}) \to \Gamma(gU, \mathscr{F})$ which is compatible with the restriction maps in the sense that for any open subsets $V \subset U$ in X the following diagram is commutative:

$$\begin{tikzcd} \Gamma(U,\mathscr{F}) \arrow[r, "g_U"] \arrow[d] & \Gamma(gU,\mathscr{F}) \arrow[d] \\ \Gamma(V,\mathscr{F}) \arrow[r, "g_V"] & \Gamma(gV,\mathscr{F}) \end{tikzcd}$$

where the downward arrows are the restriction maps. In addition we require the following cocycle conditions:

- 1_U is the identity isomorphism for any open set U;
- $h_{gU} \circ g_U = (h \circ g)_U$ for any $h, g \in G$ and any open set $U \in X$.

Note that $\Gamma(X, \mathscr{F})$ has a G-action. We will denote by $\Gamma^G(X, \mathscr{F})$ the space of G-invariants: $\Gamma^G(X, \mathscr{F}) = [\Gamma(X, \mathscr{F})]^G$. A morphism $\mathscr{F}_1 \to \mathscr{F}_2$ between two equivariant sheaves is an element in $\Gamma^G(X, \mathscr{H}om(\mathscr{F}_1, \mathscr{F}_2))$. G-equivariant sheaves on X form an abelian category. For any sheaf $\mathscr{F}$ on Y the sheaf $f^{-1}\mathscr{F}$ is naturally a G-equivariant sheaf on X. The direct image sheaf $f_*\mathscr{F}$ is a G-equivariant sheaf on Y, where G is assumed to act trivially.

Definition 2.2. The *G-equivariant direct image* $f_*^G\mathscr{F}$ is the sheaf of G-invariants of $f_*\mathscr{F}$ so that for $V \in Y$ we have $\Gamma(V, f_*^G\mathscr{F}) = \Gamma(V, f_*\mathscr{F})^G$.

The functor f^{-1} embeds the category of sheaves on Y as a full subcategory into the category of G-equivariant sheaves on X. Moreover, $f_*^G \circ f^{-1}$ is isomorphic to the identity functor on the category of sheaves on Y. Since the functor f_*^G is exact these statements continue to hold on the level of derived categories, cf. [1, Theorem 8.6.1].

Now assume that X is a finite-dimensional simplicial complex and that G acts simplicially, i.e. for any simplex $\sigma \in X$ and $g \in G$ the image $g(\sigma)$ is another simplex of X and $g : \sigma \to g(\sigma)$ is an affine map. Our standing assumptions on the action imply that the stabilizer of each simplex is finite. As a topological space Y is glued from *orbi-simplices*, i.e. quotients of simplices by actions of finite groups. One has one n-dimensional orbi-simplex of Y for each orbit of the action of G on the set of n-simplices of X.

Definition 2.3. A sheaf $\mathscr{F}$ on Y is called *constructible*, if $f^{-1}\mathscr{F}$ is constructible on X.

In other words a constructible sheaf is constant when restricted onto each orbi-simplex. Just as in the non-equivariant situation, a constructible sheaf $\mathscr{F}$ on Y is equivalent to a coefficient system on Y, i.e. a functor $\sigma \mapsto \mathscr{F}_\sigma$ from the poset of orbi-simplices of Y into k-vector spaces. Then we have the following (almost verbatim) analogue of Proposition 1.5.

Proposition 2.4. *Let $\mathscr{F}$ be a constructible sheaf on Y and $\{\mathscr{F}_\sigma\}$ be the corresponding coefficient system on Y. Then the cohomology of a constructible sheaf on X coincides with the cohomology of the complex*

$$\bigoplus_\tau \mathscr{F}_\tau \otimes \mathrm{Det}(\tau)[1]. \tag{2.1}$$

Here the direct sum is over the orbi-simplices of Y, and the differential acts as in the non-equivariant situation.

Proof. According to the correspondence between equivariant sheaves on X and non-equivariant sheaves on Y we have an isomorphism $R\Gamma^G(X, f^{-1}\mathscr{F}) \cong R\Gamma(Y, \mathscr{F})$. Since the complex (2.1) is just the complex of G-invariants of the Čech complex of $f^{-1}\mathscr{F}$ and the latter does compute $R\Gamma(X, \mathscr{F})$ by Proposition 1.5 the statement of our proposition follows. □

Similar arguments can be used to prove the following analogue of Proposition 1.6.

Proposition 2.5. *Let $\mathscr{F}$ be a constructible sheaf on Y and $D\mathscr{F}$ be its Verdier dual. Then $D\mathscr{F}$ is represented by the constructible complex $\sigma \mapsto D\mathscr{F}_\sigma$ where $D\mathscr{F}_\sigma$ is the following complex:*

$$\bigoplus_{\tau \supset \sigma} \bigl(\mathscr{F}_\tau \otimes \mathrm{Det}(\tau)[1]\bigr)^*. \tag{2.2}$$

Here τ runs over the orbi-simplices of Y having σ as a face, the grading convention and the formula for the differential are the same as in the non-equivariant situation.

3. Graph complexes and spaces of metric graphs

3.1. Graph complexes

A graph is specified by a set of vertices, a set of half-edges, and (rather obvious) combinatorial relations between them, cf. for example, [6] for precise definitions. One may also think of a graph as an isomorphism class of a 1-dimensional CW complex. We will only consider connected, finite graphs whose vertices have valence three or higher, i.e., for each vertex the number of incident half-edges must be at least three. The sets of vertices and edges of a graph Γ will be denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. The set of half-edges incident to a vertex $v \in V(\Gamma)$ will be denoted by $H(v)$.

Let $\mathscr{O}$ be a cyclic operad in the category of chain complexes of k-vector spaces. For simplicity, we will assume that $\mathscr{O}(1) = k$ and $\mathscr{O}(n)$ is a finite-dimensional dg-vector space for each $n \geqslant 2$, as this is the case for the standard examples of $\mathscr{O} = \mathscr{C}omm$, $\mathscr{A}ss$, and $\mathscr{L}ie$. The more general case of an admissible operad, see [7, 3.1.5], can also be treated by taking tensor products over the associative algebra $K = \mathscr{O}(1)$, rather than the ground field k.

If S is a set of $n+1$ elements, $n > 0$, one can define $\mathscr{O}((S))$ by using the coinvariants trick:

$$\mathscr{O}((S)) := \bigl(\mathscr{O}(n) \times \mathrm{Iso}\bigl(S, [n]\bigr)\bigr)_{S_{n+1}},$$

where $\mathrm{Iso}(S, [n])$ is the set of bijections between S and $[n] := \{0, 1, \ldots, n\}$ and the symmetric group S_{n+1} acts diagonally. Recall the notion of an $\mathscr{O}$-graph complex [3].

Definition 3.1. An *$\mathscr{O}$-decorated graph* or simply an *$\mathscr{O}$-graph* is a graph Γ together with a decoration which associates to any vertex v of Γ an element in $\mathscr{O}((H(v)))$.

The *space of $\mathscr{O}$-decorations on Γ* is the chain complex

$$\Gamma^{\mathscr{O}} = \bigotimes_{v \in V(\Gamma)} \mathscr{O}((H(v))).$$

The *orientation space* of a graph Γ is the one-dimensional graded vector space $\mathrm{Or}(\Gamma) := \mathrm{Det}(E(\Gamma)) \otimes \mathrm{Det}^{-1} H_1(\Gamma)[\chi]$, concentrated in degree $e(\Gamma) - 1$, where $e(\Gamma) = |E(\Gamma)|$ and $\chi = \chi(\Gamma)$ is the Euler characteristic of the graph Γ as a CW complex. A *twisted orientation space* of a graph Γ is the vector space $\mathrm{Det}(E(\Gamma))[1]$. A *(twisted) orientation* on a graph Γ is a choice of a nonzero element $\mathtt{or}$ in $\mathrm{Or}(\Gamma)$ ($\mathrm{Det}(E(\Gamma)[1])$, respectively). We identify a graph Γ with (twisted) orientation σ with the negative to Γ with the opposite (twisted, respectively) orientation $-\sigma$.

Remark 3.2. When $k=\mathbb{R}$, an orientation on a graph (up to a positive real factor) is equivalent to an ordering of its vertices and directing its edges (up to even permutation), cf. [3,6,16].

The following cyclic operads are of particular importance:

(1) the commutative operad $\mathscr{C}omm(n)=k$ for $n>0$;
(2) the associative operad $\mathscr{A}ss(n)=k[S_n]$ for $n>0$;
(3) the Lie operad, whose nth space $\mathscr{L}ie(n)$ is the k-vector space spanned by all Lie monomials in n variables containing each variable exactly once.

The corresponding $\mathscr{O}$-graphs are called commutative, ribbon, and Lie graphs, respectively.

Definition 3.3. The *$\mathscr{O}$-graph complex* is the following complex of k-vector spaces:

$$C_\bullet^{\Gamma\mathscr{O}}=\bigoplus_{[\Gamma,\mathtt{or}]}\Gamma^{\mathscr{O}}\otimes \operatorname{Or}(\Gamma),$$

where the summation runs over the isomorphism classes $[\Gamma,\mathtt{or}]$ of oriented graphs; thus if Γ has an orientation-reversing automorphism, its contribution to the graph complex is zero. The grading comes from the internal grading on $\mathscr{O}$ and the grading on $\operatorname{Or}(\Gamma)$, which sits in degree $e(\Gamma)-1$, so that, provided that $\mathscr{O}$ is non-negatively graded, the graph complex in general would end in degree $-\chi(\Gamma)$, corresponding to graphs with one vertex. The differential is the sum of the internal differential coming from the operad $\mathscr{O}$ and the graph differential $d: C_n^{\Gamma\mathscr{O}}\to C_{n-1}^{\Gamma\mathscr{O}}$ which acts as follows:

$$d(\Gamma\otimes\mathtt{or})=\sum_e \Gamma_e\otimes\mathtt{or}_e, \tag{3.1}$$

where Γ is an $\mathscr{O}$-graph and $\mathtt{or}\in\operatorname{Or}(\Gamma)\setminus\{0\}$ is an orientation on Γ. Here Γ_e is the graph obtained from Γ by contracting an edge e and the summation is taken over all edges of Γ which are not loops.

The orientation $\mathtt{or}_e$ of Γ_e is induced from the orientation $\mathtt{or}$ of Γ in such a way that $\mathtt{or}=e\wedge\mathtt{or}_e$. The $\mathscr{O}$-decoration on Γ_e is defined as follows. Let v_1 and v_2 denote the two endpoints of the edge e in Γ and v_e the vertex into which the edge e contracts in Γ_e. Then the vertex obtained by coalescing v_1 and v_2 is decorated by the element $v_1(\mathscr{O})\circ v_2(\mathscr{O})\in\mathscr{O}((H(v_e)))$, where $\mathscr{O}((H(v_1)))\circ\mathscr{O}((H(v_2)))\to\mathscr{O}((H(v_e)))$ is the structure map of the cyclic operad $\mathscr{O}$ corresponding to grafting the half-edges making up the edge e.

Similarly, the *twisted* graph complex $\widetilde{C}_\bullet^{\Gamma\mathscr{O}}$ is formed by the isomorphism classes of $\mathscr{O}$-graphs with twisted orientation; the grading and the differential are defined like in the untwisted case, so that, for example, terms corresponding to graphs Γ with a single vertex decorated by an element of $\mathscr{O}$ of degree zero would sit in degree $-\chi(\Gamma)$.

The homology of the complexes $C_\bullet^{\Gamma\mathscr{O}}$ and $\widetilde{C}_\bullet^{\Gamma\mathscr{O}}$ are denoted by $H_\bullet^{\Gamma\mathscr{O}}$ and $\widetilde{H}_\bullet^{\Gamma\mathscr{O}}$ respectively and called *$\mathscr{O}$-graph homology*. The cohomology of the k-dual cochain complexes $C^\bullet_{\Gamma\mathscr{O}}=[C_\bullet^{\Gamma\mathscr{O}}]^\vee=\bigoplus_{[\Gamma,\mathtt{or}]}(\Gamma^{\mathscr{O}})^\vee\otimes\operatorname{Det}(E(\Gamma))\otimes\operatorname{Det}^{-1}(H_1(\Gamma)^*)[\chi]$ and $\widetilde{C}^\bullet_{\Gamma\mathscr{O}}=[\widetilde{C}_\bullet^{\Gamma\mathscr{O}}]^\vee=\bigoplus_{[\Gamma,\mathtt{or}]}(\Gamma^{\mathscr{O}})^\vee\otimes\operatorname{Det}(E(\Gamma))[1]$ are called *$\mathscr{O}$-graph cohomology* and *twisted $\mathscr{O}$-graph cohomology*, respectively.

Lemma 3.4. *A quasi-isomorphism between two dg-operads $\mathcal{O}_1$ and $\mathcal{O}_2$ induces a quasi-isomorphism between the complexes $C_\bullet^{\Gamma\mathcal{O}_1}$ and $C_\bullet^{\Gamma\mathcal{O}_2}$.*

Proof. It is clear that a given map between $\mathcal{O}_1$ and $\mathcal{O}_2$ induces a chain map on the corresponding graph complexes. Both could be considered as double complexes with the horizontal differential given by edge-contractions and the vertical one being induced by the differentials in $\mathcal{O}_1$ and $\mathcal{O}_2$. We see that the appropriate spectral sequences converging to $C_\bullet^{\Gamma\mathcal{O}_1}$ and $C_\bullet^{\Gamma\mathcal{O}_2}$ are isomorphic from the term E^1 onwards and so the statement of the lemma follows. $\square$

For an integer $n > 1$, we will consider graphs Γ with $H_1(\Gamma, \mathbb{Z})$ being a free abelian group of rank n. These graphs form a subcomplex $C_\bullet^{\Gamma\mathcal{O}}(n)$; clearly $C_\bullet^{\Gamma\mathcal{O}} \cong \bigoplus_n C_\bullet^{\Gamma\mathcal{O}}(n)$. Note that $C_\bullet^{\Gamma\mathcal{O}}(n)$ are complexes of finite-dimensional vector spaces and have finite lengths.

Furthermore in the case $\mathcal{O} = \mathcal{A}ss$ an $\mathcal{O}$-graph—a ribbon graph—has an additional invariant, the *number $b > 0$ of boundary components*, cf. for example, the survey [9]. It is convenient to introduce the *genus* $g \geqslant 0$ of a ribbon graph by the formula $g = 1/2(n + 1 - b)$; then graphs with fixed g and b form a subcomplex $C_\bullet^{\Gamma\mathcal{A}ss}(g, b)$ inside $C_\bullet^{\Gamma\mathcal{A}ss}$ and

$$C_\bullet^{\Gamma\mathcal{A}ss}(n) \cong \bigoplus_{\substack{g\geqslant 0,\, b\geqslant 1\\ 2-2g-b=1-n<0}} C_\bullet^{\Gamma\mathcal{A}ss}(g, b).$$

3.2. Metric graphs

We now introduce the moduli space of metric graphs, cf. [4]. A *metric graph* is a graph Γ together with a map $l : E(\Gamma) \to \mathbb{R}_+$ so that $\sum_{e\in E(\Gamma)} l(e) = 1$; the positive number $l(e)$ is called the *length* of the edge e.

The set of metric graphs has the structure of a topological space; moreover it could be identified with a subset of a certain orbi-simplicial complex. We will recall how this may be done, referring the reader to [2] for details. It is convenient to use the notion of a stable graph introduced in [6]. Recall that a *stable graph* is a graph each of whose vertices is labeled by a non-negative integer, its *genus*; all vertices of genus 0 are required to be at least trivalent, and all vertices of genus 1—at least univalent. The (total) *genus* of a stable graph is the sum of the genera of its vertices and its first Betti number. We will construct the moduli space of stable metric graphs of genus n as follows.

Let S be a finite set of cardinality not less than $6n - 6$. Note that $6n - 6$ is the maximal number of half-edges of a stable graph of genus n. Then an *S-labeled* (or simply, *labeled*) stable graph is a stable graph together with labelings of each of its half-edges by a distinct element of S. The set of labeled stable metric graphs of genus n clearly form a simplicial complex $\overline{X}_n$ with each labeled stable graph contributing a simplex; the face maps correspond to edge-contractions. Note that if a loop is contracted, then the genus of the corresponding vertex increases by one. Inside $\overline{X}_n$ lies a subset (not a simplicial subcomplex) X_n consisting of stable metric graphs with each vertex having genus zero; these are called *labeled metric graphs*.

Furthermore, the group $\mathrm{Aut}(S)$ of permutations of the set S acts on $\overline{X}_n$ by changing the labels of the half-edges of the labeled stable metric graphs. The resulting quotient does not depend on the choice of the labeling set S and will be denoted $\overline{Y}_n$. This is the *moduli space of stable metric graphs*, which is an orbi-simplicial complex by construction. Let Y_n denote the subset of (unstable) metric graphs and $i : Y_n \hookrightarrow \overline{Y}_n$ the corresponding inclusion.

We note that there is a different realization of Y_n as a quotient of a certain contractible space called the *Outer Space* by a properly discontinuous action of $\mathrm{Out}(F_n)$, the group of outer automorphisms of the free group on n generators, cf. [4], so that Y_n is rationally a classifying space of $\mathrm{Out}(F_n)$. This realization allows one to identify the rational homology of $\mathrm{Out}(F_n)$ with that of Y_n.

We will now introduce certain constructible sheaves on $\overline{Y}_n$.

Definition 3.5.

(1) For an orbi-simplex σ corresponding to a graph Γ, we set $\overline{\mathscr{H}}_\sigma = \mathrm{Det}^{-1}(H_1(\Gamma))[-n]$. For $\sigma \in \overline{Y}_n \setminus Y_n$, we set $\overline{\mathscr{H}}_\sigma = 0$. The resulting sheaf on $\overline{Y}_n$ will be denoted $\overline{\mathscr{H}}$. The sheaf $i^{-1}\overline{\mathscr{H}}$ on Y_n will be denoted $\mathscr{H}$.

(2) Associated to a cyclic chain operad $\mathscr{O}$ is a constructible complex $\overline{\mathscr{F}}^{\mathscr{O}}$ on $\overline{Y}_n$ defined as follows. For an oriented simplex $\sigma \in \overline{Y}_n$ corresponding to a graph Γ, we set $\overline{\mathscr{F}}^{\mathscr{O}}_\sigma$ to be the cochain complex k-dual to the chain complex $\Gamma^{\mathscr{O}}$ of $\mathscr{O}$-decorations on Γ:

$$\overline{\mathscr{F}}^{\mathscr{O}}_\sigma := \left(\Gamma^{\mathscr{O}}\right)^\vee.$$

For $\sigma \in \overline{Y}_n \setminus Y_n$, set $\overline{\mathscr{F}}^{\mathscr{O}}_\sigma = 0$. If $\sigma \subset \tau$ is a face of τ, the corresponding morphism $\overline{\mathscr{F}}^{\mathscr{O}}_\sigma \to \overline{\mathscr{F}}^{\mathscr{O}}_\tau$ is defined as the dual to the one obtained by the operad composition of decorations along the edges in the graph Γ_τ being contracted to obtain Γ_σ, where Γ_σ and Γ_τ are the graphs corresponding to the simplices σ and τ, respectively. The complex of sheaves $i^{-1}\overline{\mathscr{F}}^{\mathscr{O}}$ on Y_n will be denoted $\mathscr{F}^{\mathscr{O}}$.

Remark 3.6.

- Clearly $i_!\mathscr{F}^{\mathscr{O}} \cong \overline{\mathscr{F}}^{\mathscr{O}}$; similarly $i_!\mathscr{H} \cong \overline{\mathscr{H}}$. Furthermore $\mathscr{H}$ is a locally free sheaf on Y_n.
- For two quasi-isomorphic operads $\mathscr{O}$ and $\mathscr{O}'$, the complexes $\mathscr{F}_{\mathscr{O}}$ and $\mathscr{F}_{\mathscr{O}'}$ are quasi-isomorphic.

Theorem 3.7. *There are canonical isomorphisms of graded k-vector spaces:*

(1) $\widetilde{H}^\bullet_{\Gamma\mathscr{O}}(n) \cong H^\bullet_c(Y_n, \mathscr{F}^{\mathscr{O}})$ *for the twisted $\mathscr{O}$-graph cohomology;*
(2) $H^\bullet_{\Gamma\mathscr{O}}(n) \cong H^\bullet_c(Y_n, \mathscr{F}^{\mathscr{O}} \otimes \mathscr{H})$ *for the $\mathscr{O}$-graph cohomology.*

Proof. We shall only prove part (1), the argument for (2) being virtually identical, as the standard orientation on a graph Γ differs from the twisted one by $\mathrm{Det}^{-1}(H_1(\Gamma))[-n]$. It suffices to prove the isomorphism $\widetilde{H}^\bullet_{\Gamma\mathscr{O}}(n) \cong H^\bullet(\overline{Y}_n, \overline{\mathscr{F}}^{\mathscr{O}})$. Using Proposition 2.4, we see that the complex computing $H^\bullet(\overline{Y}_n, \overline{\mathscr{F}}^{\mathscr{O}})$ coincides with the complex $\widetilde{C}^\bullet_{\Gamma\mathscr{O}}(n)$, so the statement of the theorem follows. □

Given a cyclic operad $\mathscr{O}$ of chain complexes, let us recall the construction of its *dg-dual operad* $D\mathscr{O}$ from [5–7]. The nth component $D\mathscr{O}(n)$ of the dg-dual cyclic operad $D\mathscr{O}$ for each $n > 0$ is defined as the linear dual of the space of oriented, unrooted trees decorated by a certain degree shift $\mathfrak{s}\mathscr{O}[-1]$ with leaves labeled by $0, 1, \dots, n$. More precisely,

$$D\mathscr{O}(n) := \bigoplus_{\text{unrooted } n+1\text{-trees } T} \left(T^{\mathfrak{s}\mathscr{O}[-1]}\right)^*, \tag{3.2}$$

where the summation runs over the isomorphism classes T of (unrooted) trees with vertices of valence at least three and $n+1$ leaves thought of as half-edges with free ends and labeled by numbers $0, 1, \dots, n$. Here $\mathfrak{s}\mathscr{O}$ is the *cyclic-operad suspension* [5]:

$$\mathfrak{s}\mathscr{O}(n) := \operatorname{Det}^{-1}\left(k^{n+1}\right)[-2] \otimes \mathscr{O}(n),$$

which results in

$$\mathfrak{s}\mathscr{O}((S)) = \operatorname{Det}^{-1}(S)[-2] \otimes \mathscr{O}((S))$$

for a finite set S. Also, $T^{\mathfrak{s}\mathscr{O}[-1]}$ is the space of $\mathfrak{s}\mathscr{O}[-1]$-decorations on T. Thus, if $\mathscr{O}$ happens to be concentrated in degree zero, $D\mathscr{O}(n)$ will be a chain complex spanning degrees $n-2$ through 0. The differential on $D\mathscr{O}(n)$ is the sum of the internal differential coming from the complex of $\mathscr{O}$-decorations on a tree and the differential linear dual to the differential (3.1) restricted from graphs to trees.

Remark 3.8. If S is a set of $n+1$ elements, one can make precise sense out of $D\mathscr{O}(S)$ by considering trees whose leaves are labeled by the elements of S.

Theorem 3.9. *There is a canonical isomorphism in the derived category of sheaves on Y_n:*

$$D\mathscr{F}^{\mathscr{O}} \cong \mathscr{F}^{D\mathscr{O}} \otimes \mathscr{H}[4-3n],$$

where $D\mathscr{F}$ is the Verdier dual sheaf and $D\mathscr{O}$ the dg-dual operad.

Proof. It suffices to provide a canonical isomorphism

$$i_! i^{-1} D\overline{\mathscr{F}}^{\mathscr{O}} \cong \overline{\mathscr{F}}^{D\mathscr{O}} \otimes \overline{\mathscr{H}}[4-3n]. \tag{3.3}$$

To prove it, we will evaluate (3.3) on an orbi-simplex σ and establish an isomorphism, natural with respect to isomorphisms of the corresponding graphs Γ_σ.

By definition,

$$\mathscr{F}_\sigma^{D\mathscr{O}} = \left(\Gamma_\sigma^{D\mathscr{O}}\right)^\vee = \bigotimes_{v\in V(\Gamma_\sigma)} D\mathscr{O}\left(\left(H(v)\right)\right)^\vee = \bigotimes_{v\in V(\Gamma_\sigma)} \bigoplus_{\substack{\text{unrooted}\\ H(v)\text{-trees } T_v}} \left(T_v^{\mathfrak{s}\mathscr{O}[-1]}\right)^{*\vee}.$$

Note that a graph Γ_σ with each vertex v decorated by a tree T_v whose leaves are labeled by the set $H(v)$ of half-edges emanating from v is literally the same as a graph Γ_τ with a collection of subtrees, such that contracting each of these subtrees returns the graph Γ_σ. We will call such graph Γ_τ a *vertex expansion* of Γ_σ. Moreover, $\mathfrak{s}\mathscr{O}[-1]$-decorations on the trees T_v will obviously result in $\mathfrak{s}\mathscr{O}[-1]$-decorations on the graphs Γ_τ. Thus, we see that

$$\mathscr{F}_\sigma^{D\mathscr{O}} = \bigoplus_{\substack{\text{vertex expansions}\\ \Gamma_\tau \text{ of } \Gamma_\sigma}} \left(\Gamma_\tau^{\mathfrak{s}\mathscr{O}[-1]}\right)^{*\vee}.$$

Now let us identify $(\Gamma^{\mathfrak{s}\mathscr{O}[-1]})^{*\vee}$:

$$\left(\Gamma^{\mathfrak{s}\mathscr{O}[-1]}\right)^{*\vee} = \bigotimes_{v\in V(\Gamma)} \operatorname{Det}\bigl(H(v)\bigr) \otimes \mathscr{O}\bigl(\bigl(H(v)\bigr)\bigr)^{*\vee}[3] \cong \left(\Gamma^{\mathscr{O}}\right)^{*\vee} \otimes \bigotimes_{v\in V(\Gamma)} \operatorname{Det}\bigl(H(v)\bigr)[3].$$

Leaving the factor $(\Gamma^{\mathscr{O}})^{*\vee}$ out for the time being, let us deal with orientations. We have

$$\begin{aligned}\bigotimes_{v\in V(\Gamma)} \operatorname{Det}\bigl(H(v)\bigr)[3] &\cong \operatorname{Det}^{-3}\bigl(V(\Gamma)\bigr) \otimes \bigotimes_{v\in V(\Gamma)} \operatorname{Det}\bigl(H(v)\bigr)\\ &\cong \operatorname{Det}^{-1}\bigl(V(\Gamma)\bigr)\bigl[2v(\Gamma)\bigr] \otimes \bigotimes_{v\in V(\Gamma)} \operatorname{Det}\bigl(H(v)\bigr),\end{aligned}$$

where $v(\Gamma) = |V(\Gamma)|$. Note that the set $\coprod_{v\in V(\Gamma)} H(v)$ is naturally isomorphic to the set $\coprod_{e\in E(\Gamma)} H(e)$, where $H(e)$ is the set of (two) half-edges making up an edge e, as both sets count the set of half-edges $H(\Gamma)$ of the graph, the former by grouping the set of half-edges by vertices, the latter by edges. By passing to determinants, we obtain

$$\bigotimes_{v\in V(\Gamma)} \operatorname{Det}\bigl(H(v)\bigr)[3] \cong \operatorname{Det}^{-1}\bigl(V(\Gamma)\bigr)\bigl[2v(\Gamma)\bigr] \otimes \bigotimes_{e\in E(\Gamma)} \operatorname{Det}\bigl(H(e)\bigr). \tag{3.4}$$

Note that the exact sequence

$$0 \to H_1(\Gamma) \to C_1(\Gamma) \to C_0(\Gamma) \to H_0(\Gamma) \to 0$$

yields a canonical isomorphism

$$\operatorname{Det} H_0(\Gamma) \otimes \operatorname{Det}^{-1} H_1(\Gamma) \cong C_0(\Gamma) \otimes \operatorname{Det}^{-1} C_1(\Gamma).$$

Further, we have the following natural isomorphisms:

$$\begin{aligned}\operatorname{Det} C_0(\Gamma) &\cong \operatorname{Det} V(\Gamma),\\ \operatorname{Det} C_1(\Gamma) &\cong \bigotimes_{e\in E(\Gamma)} \operatorname{Det} H(e)[1]\\ &\cong \operatorname{Det}^{-1} E(\Gamma) \otimes \bigotimes_{e\in E(\Gamma)} \operatorname{Det} H(e),\\ \operatorname{Det}\bigl(H_0(\Gamma)\bigr) &\cong k[-1].\end{aligned}$$

We conclude that the last expression in (3.4) is isomorphic to

$$\begin{aligned}&\operatorname{Det}\bigl(E(\Gamma)\bigr) \otimes \operatorname{Det}\bigl(H_1(\Gamma)\bigr) \otimes \operatorname{Det}^{-1}\bigl(H_0(\Gamma)\bigr)\bigl[2v(\Gamma)\bigr]\\ &\quad\cong \operatorname{Det}^{-1}\bigl(E(\Gamma)\bigr) \otimes \operatorname{Det}\bigl(H_1(\Gamma)\bigr) \otimes \operatorname{Det}^{-1}\bigl(H_0(\Gamma)\bigr)\bigl[2\bigl(v(\Gamma) - e(\Gamma)\bigr)\bigr]\\ &\quad\cong \operatorname{Or}^{-1}(\Gamma)[4-3n],\end{aligned}$$

which implies

$$\mathscr{F}_\sigma^{D\mathscr{O}} \cong \bigoplus_{\text{vertex expansions } \Gamma_\tau \text{ of } \Gamma_\sigma} \big(\Gamma_\tau^{\mathscr{O}}\big)^{*\vee} \otimes \mathrm{Or}^{-1}(\Gamma_\tau)[4-3n].$$

The differential on the complex $\mathscr{F}_\sigma^{D\mathscr{O}}$ is the sum of the internal differential on $\mathscr{O}$ and the summation over all contractions of a given graph Γ_τ along the edges arising in the vertex expansions of Γ_σ of the corresponding operad compositions. This is similar to the differential in the graph complex $C_\bullet^{\Gamma\mathscr{O}}$, with the same effect on the orientation factor, except that the resulting grading is now cohomological.

Now let us turn to the left-hand side of (3.3). The complex $i_! i^{-1} D\overline{\mathscr{F}}_\sigma^{\mathscr{O}}$ vanishes on $\overline{Y}_n \setminus Y_n$ and coincides with $D\overline{\mathscr{F}}_\sigma^{\mathscr{O}}$ on Y_n. Therefore, according to Proposition 2.5, for $\sigma \in Y_n$, the complex $i_! i^{-1} D\overline{\mathscr{F}}_\sigma^{\mathscr{O}}$ is represented by the complex

$$D\overline{\mathscr{F}}_\sigma^{\mathscr{O}} \cong \bigoplus_{\tau \supset \sigma} \big(\mathscr{F}_\tau^{\mathscr{O}} \otimes \mathrm{Det}(\tau)[1]\big)^* = \bigoplus_{\text{vertex expansions } \Gamma_\tau \text{ of } \Gamma_\sigma} \big(\Gamma_\tau^{\mathscr{O}}\big)^{*\vee} \otimes \mathrm{Det}^{-1}\big(E(\Gamma_\tau)\big)[-1],$$

with the same differential as for the twisted graph complex $\widetilde{C}_\bullet^{\Gamma\mathscr{O}}$. This immediately implies (3.3). $\square$

Remark 3.10. Note that the identification of $\mathscr{F}^{D\mathscr{O}}$ in the beginning of the proof of Theorem 3.9 shows that the complex computing the cohomology of $\mathscr{F}^{D\mathscr{O}}$ is a *graph complex decorated by a decorated tree complex*, all complexes being *cochain* complexes. That complex can obviously be identified with a decorated (cochain) graph complex, and the rest of the proof of Theorem 3.9 expresses the resulting decoration through the $\mathscr{O}$-decoration.

Remark 3.11. We would like to stress that, despite the previous result, there is no simple relationship between the $\mathscr{O}$-decorated graph complex and the $D\mathscr{O}$-decorated graph complex. This is a reflection of the fact that there is no simple relationship between the cohomology of a sheaf and its cohomology with compact supports on a *non-compact topological space*.

For a Koszul operad $\mathscr{O}$, the Verdier dual sheaf of $\mathscr{F}^{\mathscr{O}}$ has an especially simple description, since for such an operad, the dg-dual $D\mathscr{O}$ of $\mathscr{O}$ is quasi-isomorphic its Koszul dual $\mathscr{O}^!$. As a special case, we obtain a description of the dualizing sheaf on Y_n, using the fact that the Koszul dual to the operad $\mathscr{C}omm$ is the operad $\mathscr{L}ie$. Note that this dualizing sheaf is concentrated in a single degree (although it is not one-dimensional, as it would have been, had Y_n been a topological manifold).

Corollary 3.12. *Let $\mathscr{O}$ be a Koszul operad and $\mathscr{O}^!$ be its Koszul dual. There is the following isomorphism in the derived category of sheaves on Y_n:*

$$D\mathscr{F}^{\mathscr{O}} \cong \mathscr{F}^{\mathscr{O}^!} \otimes \mathscr{H}[4-3n].$$

In particular, the dualizing sheaf on Y_n is isomorphic to $\mathscr{F}^{\mathscr{L}ie} \otimes \mathscr{H}[4-3n]$.

Let $\tilde{k}$ denote the one-dimensional $\mathrm{Out}(F_n)$-module corresponding to the local system $\mathscr{H}$, concentrated in degree zero; an element in $\mathrm{Out}(F_n)$ acts on $\tilde{k}$ as multiplication by 1 or -1, equal to the determinant of the linear map induced on $H_1(\Gamma)$. Then we have the following result which was formulated by Kontsevich in [12] and given a different proof in [3].

Corollary 3.13. *There are the following isomorphisms of graded k-vector spaces:*

(1) $H_\bullet(\mathrm{Out}(F_n), k) \cong H^{3n-4-\bullet}_{\Gamma\mathscr{L}ie}(n)$;
(2) $H_\bullet(\mathrm{Out}(F_n), \tilde{k}) \cong \widetilde{H}^{3n-4-\bullet}_{\Gamma\mathscr{L}ie}(n)$.

Proof. As usual, we limit ourselves with proving the first statement. We have

$$\begin{aligned} H_\bullet\bigl(\mathrm{Out}(F_n), k\bigr) &\cong H_\bullet(Y_n, k) \quad \text{by [4]} \\ &\cong \bigl[H^\bullet(Y_n, k)\bigr]^\vee \\ &\cong \bigl[H^\bullet\bigl(Y_n, \mathscr{F}^{\mathscr{C}omm}\bigr)\bigr]^\vee \\ &\cong H_c^\bullet\bigl(Y_n, D\mathscr{F}^{\mathscr{C}omm}\bigr)^{*\vee} \quad \text{by definition of } D\mathscr{F} \\ &\cong H_c^\bullet\bigl(Y_n, \mathscr{F}^{D\mathscr{C}omm} \otimes \mathscr{H}[4-3n]\bigr)^{*\vee} \quad \text{by Theorem 3.9} \\ &\cong H_c^{\bullet+4-3n}\bigl(Y_n, \mathscr{F}^{D\mathscr{C}omm} \otimes \mathscr{H}\bigr)^{*\vee} \\ &\cong H_c^{3n-4-\bullet}\bigl(Y_n, \mathscr{F}^{\mathscr{L}ie} \otimes \mathscr{H}\bigr) \\ &\cong H^{3n-4-\bullet}_{\Gamma\mathscr{L}ie} \quad \text{by Theorem 3.7,} \end{aligned}$$

as required. Note that we used the fact that $D = D^{-1}$ where D is the functor of taking the Verdier dual. □

Remark 3.14. Compare this to a more straightforward computation to get a relation between the cohomology of Y_n with compact supports and commutative graph cohomology:

$$H_c^\bullet(Y_n, k) = H_c^\bullet\bigl(Y_n, \mathscr{F}^{\mathscr{C}omm}\bigr) = \widetilde{H}^\bullet_{\Gamma\mathscr{C}omm}(n).$$

4. Ribbon graphs

The theory developed in the previous section has an analogue for non-Σ operads and ribbon graph complexes. Recall that a ribbon graph is an $\mathscr{A}ss$-decorated graph; this is equivalent to having a cyclic ordering on the set of half-edges around each vertex. Given a ribbon graph Γ, there is a canonical way of producing a compact, oriented surface with boundary $S(\Gamma)$ of which the graph Γ is a deformation retract. In this way one attaches to a ribbon graph two invariants: the genus $g \geqslant 0$ and the number $n \geqslant 1$ of boundary components of the corresponding surface, $2 - 2g - n < 0$. An isomorphism between two ribbon graphs is an isomorphism preserving the cyclic ordering around each vertex. We will not specify whether the boundary components should be fixed (not necessarily point-wise) under an isomorphism or allowed to be permuted; both versions admit completely parallel treatments.

Now let $\mathscr{O}$ be a cyclic (chain) k-operad with $\mathscr{O}(1) = k$ without the action of the symmetric group, a so-called non-Σ operad. We introduce the notions of a ribbon $\mathscr{O}$-graph complex $C_\bullet^{\mathrm{Rib}\,\mathscr{O}}$, its (cochain) dual $C^\bullet_{\mathrm{Rib}\,\mathscr{O}}$, as well as the twisted versions $\widetilde{C}_\bullet^{\mathrm{Rib}\,\mathscr{O}}$ and $\widetilde{C}^\bullet_{\mathrm{Rib}\,\mathscr{O}}$ in precisely the same way as in the previous section. For two quasi-isomorphic operads, the corresponding ribbon graph complexes will be quasi-isomorphic. The subcomplex in $C_\bullet^{\mathrm{Rib}\,\mathscr{O}}$ consisting of ribbon graphs with fixed g and n will be denoted by $C_\bullet^{\mathrm{Rib}\,\mathscr{O}}(g,n)$ ($C^\bullet_{\mathrm{Rib}\,\mathscr{O}}(g,n)$ for the cohomological version). It is easy to see that

$$C_\bullet^{\mathrm{Rib}\,\mathscr{O}} \cong \bigoplus_{\substack{g\geqslant 0,\, n\geqslant 1\\ 2-2g-n<0}} C_\bullet^{\mathrm{Rib}\,\mathscr{O}}(g,n).$$

The most important example of a ribbon $\mathscr{O}$-graph complex corresponds to the associative non-Σ operad $\mathscr{T}(m)=k$ in all degrees $m\geqslant 1$. In this case we have an isomorphism $C_\bullet^{\mathrm{Rib}\,\mathscr{T}}(g,n)\cong C_\bullet^{\Gamma\mathscr{A}ss}(g,n)$ and similarly for the twisted versions, cf. the discussion at the end of Section 3.1.

We will now introduce the space of metric ribbon graphs $\mathscr{M}_{g,n}$. A point Γ in $\mathscr{M}_{g,n}$ is an isomorphism class of ribbon graphs of genus g with n boundary components such that each edge $e\in E(\Gamma)$ is supplied with length $l(e)>0$; we require that $\sum_{e\in E(\Gamma)} l(e)=1$. The space $\mathscr{M}_{g,n}$ naturally compactifies to a simplicial orbi-complex $\overline{\mathscr{M}}_{g,n}$ introduced by Kontsevich [11] in connection with his proof of the Witten conjecture. In fact, there are several competing definitions of such a compactification corresponding to various modular closures of the cyclic associative operad; we refer to the paper [2] for a relevant discussion. A detailed construction of these compactifications as well as their connection with the Deligne–Mumford compactification could be found in the papers by Looijenga [13], Zvonkine [19], Mondello [14], and Zuniga [18]. For our purposes any one of these compactifications could be used.

The importance of the space $\mathscr{M}_{g,n}$ stems from the fact that it is homeomorphic to the Cartesian product of the open standard simplex Δ^{n-1} and the moduli space of Riemann surfaces of genus g with n labeled punctures or its quotient by the diagonal action of the symmetric group S_n, depending on whether we allow graph isomorphisms permuting the boundary components. The space $\mathscr{M}_{g,n}$ also serves as a rational classifying space of the corresponding mapping class group.

Definitions of the sheaves $\mathscr{F}^{\mathscr{O}}$, $\overline{\mathscr{F}}^{\mathscr{O}}$, and $\mathscr{H}$ on $\mathscr{M}_{g,n}$ and $\overline{\mathscr{M}}_{g,n}$ transfer verbatim from the corresponding definitions in the previous section. We can now formulate analogues of the main results from Section 3. They are proved in precisely the same way as Theorems 3.7 and 3.9.

Theorem 4.1. *There are canonical isomorphisms of k-vector spaces:*

(1) $\widetilde{H}^\bullet_{\mathrm{Rib}\,\mathscr{O}}(g,n)\cong H_c^\bullet(\mathscr{M}_{g,n},\mathscr{F}^{\mathscr{O}})$;
(2) $H^\bullet_{\mathrm{Rib}\,\mathscr{O}}(g,n)\cong H_c^\bullet(\mathscr{M}_{g,n},\mathscr{F}^{\mathscr{O}}\otimes\mathscr{H})$.

Note the following explicit relation between the cohomology of Y_r and $\mathscr{M}_{g,n}$.

Corollary 4.2.

$$H_c^\bullet\big(Y_r,\mathscr{F}^{\mathscr{A}ss}\big)\cong \bigoplus_{\substack{g\geqslant 0,\, n\geqslant 1\\ 2-2g-n=1-r}} H_c^\bullet(\mathscr{M}_{g,n},k)\quad \textit{for } r>1.$$

Theorem 4.3. *There is a canonical isomorphism in the derived category of sheaves on $\mathscr{M}_{g,n}$:*

$$D\mathscr{F}^{\mathscr{O}}\cong \mathscr{F}^{D\mathscr{O}}\otimes\mathscr{H}[7-6g-3n].$$

Here $D\mathcal{O}$ is the dg-dual non-Σ operad to $\mathcal{O}$, defined for non-Σ operads the same way as in (3.2)*, except that the trees must be planar.*

The analogue of Corollary 3.13 reads as follows.

Corollary 4.4. *There are the following isomorphisms of graded k-vector spaces:*

(1) $H_\bullet(\mathcal{M}_{g,n}, k) \cong H^{6g+3n-7-\bullet}_{\mathrm{Rib}\,\mathcal{T}}(g,n) \cong H^{6g+3n-7-\bullet}_{\Gamma\mathcal{A}ss}(g,n)$;
(2) $H_\bullet(\mathcal{M}_{g,n}, \tilde{k}) \cong \widetilde{H}^{6g+3n-7-\bullet}_{\mathrm{Rib}\,\mathcal{T}}(g,n) \cong \widetilde{H}^{6g+3n-7-\bullet}_{\Gamma\mathcal{A}ss}(g,n)$.

The following corollary describes the dualizing sheaf on $\mathcal{M}_{g,n}$.

Corollary 4.5. *The dualizing sheaf on $\mathcal{M}_{g,n}$ is isomorphic to $\mathcal{H}[7-6g-3n]$.*

Remark 4.6. Note that the dualizing sheaf is locally constant in agreement with the well-known fact that $\mathcal{M}_{g,n}$ is an orbifold. Therefore, Verdier duality on $\mathcal{M}_{g,n}$ turns into Poincaré–Lefschetz duality:

$$H_\bullet(\mathcal{M}_{g,n}, k) = H_c^{6g+3n-7-\bullet}(\mathcal{M}_{g,n}, \tilde{k}).$$

When $\mathcal{M}_{g,n}$ stands for the moduli space of ribbon graphs with *labeled* boundary components, it will be an orientable orbifold, in which case $\tilde{k} \cong k$, but taking its quotient by the symmetric group permuting the boundary components to get the other version of $\mathcal{M}_{g,n}$ will destroy orientability, and the dualizing sheaf will no longer be constant.

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