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Tree Constructions of Free Continuous Algebras

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Continuous algebras are algebras endowed with a partial order which is complete with respect to specified joins and such that the operations preserve these specified joins. We prove the existence of free continuous algebras by actually giving a concrete description of them in terms of trees, for any type of algebras and any choice of the "specified" joins.

Contents.

0. INTRODUCTION

Ordered algebraic structures have received a great deal of interest lately in connection with the theory of abstract data types and mathematical semantics of programming languages. The idea of using ordered algebras in this connection originated with Dana Scott, who, for example, in [24], showed how to model flow diagrams as elements of a complete lattice, where the operations (composition, conditionals) were to be not only order-preserving but also continuous, which meant preserving directed joins. Elements of infinite height in these lattices interpreted the "unfoldments" of loops, and thereby provided solutions to certain recursive equations.

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It has been argued by some (see, for example, Markowsky and Rosen [16] and Reynolds [21, 22]) that complete lattices are not the appropriate order-theoretic notion; directed-complete or chain-complete p.o. sets, and other modifications, too, have been proposed instead. Goguen, Thatcher, Wagner and Wright (ADJ [2–4]) have shown how to follow Scott's approach to abstract semantics by using "initial continuous algebras"; there, "continuous" referred to the existence and preservation of certain specified joins, which varied from joins of ω-chains to joins of all directed sets, or all PC (=pairwise compatible) sets. Later, ADJ [5] introduced the notion of "subset system" $\mathbf{Z}$ which encompassed all variations of join-completeness which had previously been investigated. The question of existence of initial $\mathbf{Z}$-continuous algebras was open.

Now, an initial continuous algebra of some type is just the free continuous algebra of that type over the empty set of generators, and conversely the free continuous algebra of a given type over a set X of generators is just the initial algebra of the expanded type which has the elements of X added as constants (nullary operations). Our paper is devoted to the construction of free continuous algebras.

The approach of ADJ [3, 4] of constructing initial continuous algebras was inadequate to handle binary joins, and also did not include infinitary operations. An alternative approach for proving the existence of free $\mathbf{Z}$-continuous algebras is to show that $\mathbf{Z}$-continuous algebras have "bounded generation," i.e., that for each set X there is a cardinal number $\mathfrak{m}$ such that any $\mathbf{Z}$-continuous algebra "generated" by X has cardinality at most $\mathfrak{m}$; this could then be used to verify the solution set condition in the Adjoint Functor Theorem and thereby infer the existence of free continuous algebras. This approach was followed in Nelson [20] for finitary algebras; however, we do not know how to prove this property for infinitary operations and arbitrary $\mathbf{Z}$, and conjecture that it is not true. Completely different methods were used by Adámek [1] to construct free $\mathbf{Z}$-continuous algebras in the presence of ω-joins.

In this paper we will describe a unified approach which, for *all* types of algebras and *all* subset systems $\mathbf{Z}$, yields an explicit construction of free Z-continuous algebras as algebras of labelled trees. Moreover, at the same time, we strengthen ADJ's results in a different direction: we actually construct, for an arbitrary $\mathbf{Z}$-complete p.o. set P, the free $\mathbf{Z}$-continuous algebra over P; in the case that P is the free $\mathbf{Z}$-completion (see Section 1) of a discrete set X this reduces to the type of free continuous algebra considered by ADJ.

In Section 1 we set down the definitions of subset systems $\mathbf{Z}$, of $\mathbf{Z}$-completeness, $\mathbf{Z}$-continuity and $\mathbf{Z}$-continuous algebras, and prove some new results about $\mathbf{Z}$-complete p.o. sets which will be used in the subsequent sections. In Section 2 we describe the trees that will constitute the free algebras, and some of their basic properties. Sections 3 and 4 contain the constructions of free $\mathbf{Z}$-continuous algebras for all $\mathbf{Z}$, first for those $\mathbf{Z}$ which include the empty join (in Section 3), and then for those that do not (in Section 4). Finally, in Section 5 we describe free strict $\mathbf{Z}$-continuous algebras, where "strict" refers to the fact that the operations now preserve the empty join, something which is not mandatory for $\mathbf{Z}$-continuous algebras.

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1. Definitions and Preliminaries

In this section, we collect all the relevant definitions concerning **Z**-complete partially ordered sets (p.o. sets) and **Z**-continuous algebras, and prove various results about **Z**-complete p.o. sets which will be used later on.

First, two small items of terminology: A map $f: P \to Q$ of p.o. sets P and Q is called an *embedding* provided that $p \leqslant q$ iff $f(p) \leqslant f(q)$. A subset $Y \subseteq P$ is called a *down set* (in P) when $p \leqslant q$, $q \in Y$ implies $p \in Y$. Joins (least upper bounds) are denoted $x \vee y$ or $\bigvee Y$ for elements x and y or a subset Y of a given p.o. set, and the least element (if it exists) is denoted by $\perp (= \bigvee \varnothing)$.

1.1. *Subset Systems*

Recall from ADJ [5] that a *subset system* is a map **Z** which

(i) assigns to each p.o. set P a collection $\mathbf{Z}(P)$ of its subsets and

(ii) for each order-preserving map $f: P \to Q$ is such that if $X \in \mathbf{Z}(P)$ then $f(X) \in \mathbf{Z}(Q)$, where $f(X) = \{f(x) \mid x \in X\}$.

A p.o. set P is called **Z**-complete if every set in $\mathbf{Z}(P)$ has a join in P. An order-preserving map $f: P \to Q$ between p.o. sets P and Q is called **Z**-continuous if it preserves all existing **Z**-joins, i.e., whenever $X \in \mathbf{Z}(P)$ has a join $\bigvee X$ in P then $f(X)$ has a join in Q and $f(\bigvee X) = \bigvee f(X)$. Note that if $\mathbf{Z}(P)$ contains all two-element chains in P then a map preserving **Z**-joins is automatically order-preserving.

Below are some examples of subset systems **Z**:

1. $\perp(P) = \{\varnothing\}$; $\perp$-complete p.o. sets are precisely the *strict* p.o. sets, i.e., those p.o. sets P with a smallest element $\perp = \bigvee \varnothing$, and $\perp$-continuous maps are precisely the *strict* order-preserving maps, i.e., those preserving $\perp$ whenever it exists.

2. $\boldsymbol{\omega}(P)$ is the set of all finite non-empty chains and all ω-chains in P; $\boldsymbol{\omega}$-complete p.o. sets are precisely those in which each ω-chain has a join. More generally:

3. $\boldsymbol{\alpha}(P) = \{Y \subseteq P \mid$ there exists an order-preserving map from α onto $Y\}$ for each ordinal α.

4. $\mathbf{P}(P) =$ power set of P, i.e., all subsets of P.

5. $\mathbf{P}_{\mathfrak{m}}(P) =$ all subsets of P of cardinality $< \mathfrak{m}$, for each cardinal number $\mathfrak{m} \neq 0$.

6. $\mathbf{S}(P) =$ all non-empty subsets of P.

7. $\mathbf{S}_{\mathfrak{m}}(P) =$ all non-empty subsets of P of cardinality $< \mathfrak{m}$.

8. $\Delta(P) =$ all directed subsets of P.

9. $\mathbf{C}(P)$ = all chains in P.

10. $\mathbf{PC}(P)$ = all pairwise compatible subsets of P, i.e., all $Y \subseteq P$ such that for all $p, q \in Y$ there exists $r \in P$ with $p \leqslant r$ and $q \leqslant r$.

Convention. For each subset system $\mathbf{Z}$, $\mathbf{Z}_\perp(P) = \mathbf{Z}(P) \cup \{\varnothing\}$ provides another subset system. For example, $(S_\mathfrak{m})_\perp = \mathbf{P}_\mathfrak{m}$. Further, for each p.o. set P, $P_\perp$ is the p.o. set obtained by adjoining one additional element $\perp$ to P which is strictly smaller than all the elements of P.

1.2. *The Norm of a Subset System*

For two subset systems $\mathbf{Z}_1$ and $\mathbf{Z}_2$ we define $\mathbf{Z}_1 \leqslant \mathbf{Z}_2$ if every $\mathbf{Z}_2$-complete p.o. set is also $\mathbf{Z}_1$-complete and every $\mathbf{Z}_2$-continuous map $f\colon P \to Q$, where P and Q are $\mathbf{Z}_2$-complete is also $\mathbf{Z}_1$-continuous.

Note that if for each p.o. set P, $\mathbf{Z}_1(P) \subseteq \mathbf{Z}_2(P)$ then evidently $\mathbf{Z}_1 \leqslant \mathbf{Z}_2$. But the converse is not true; for example, $\mathbf{S}_{\aleph_0} \leqslant \mathbf{S}_3$ and $\mathbf{P}_{\aleph_0} \leqslant \mathbf{P}_3$. A more interesting example of this phenomenon, which follows from Iwamura's Lemma, is that $\Delta \leqslant \mathbf{C}$ (see Markowsky [14]). Since joins of singleton sets always exist, we have $\mathbf{S}_2 \leqslant \mathbf{Z}$ for all $\mathbf{Z}$.

We define the *norm* of $\mathbf{Z}$, $\|\mathbf{Z}\|$, to be the largest cardinal number $\mathfrak{m}$ such that $\mathbf{S}_\mathfrak{m} \leqslant \mathbf{Z}$ if it exists, otherwise $\|\mathbf{Z}\| = \infty$. We extend the usual order on cardinals so that $\mathfrak{m} < \infty$ for all cardinals $\mathfrak{m}$; then $\mathbf{S}_\mathfrak{m} \leqslant \mathbf{Z}$ for all $\mathfrak{m} \leqslant \|\mathbf{Z}\|$. For example, $\|\boldsymbol{\omega}\| = 2$, $\|S_\mathfrak{m}\| = \mathfrak{m}$, and $\|S\| = \infty$. Note that for any $\mathbf{Z}$, either $\|\mathbf{Z}\|$ is infinite or $\|\mathbf{Z}\| = 2$.

Remark. $\|\mathbf{Z}\| = \aleph_0$ iff

(i) each $\mathbf{Z}$-complete p.o. set is a join semilattice;

(ii) each $\mathbf{Z}$-continuous map of $\mathbf{Z}$-complete p.o. sets preserves finite non-empty joins; and

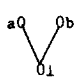
(iii) all sets in $\mathbf{Z}(\omega)$ are finite.

Proof. Parts (i) and (ii) are evidently equivalent with $\|\mathbf{Z}\| \geqslant \aleph_0$, and (iii) implies $\|\mathbf{Z}\| \leqslant \aleph_0$. Conversely, if $X \in \mathbf{Z}(\omega)$ for some infinite subset $X \subseteq \omega$ then there is an order-preserving map $f\colon \omega \to \omega$ with $f(X) = \omega$ and thus $\omega \in \mathbf{Z}(\omega)$. This, together with $\|\mathbf{Z}\| \geqslant \aleph_0$, implies that $\mathbf{Z}$-complete p.o. sets have *all* countable joins and that $\mathbf{Z}$-continuous maps preserve them, which means $\|Z\| > \aleph_0$.

The following result is also proved in Adámek [1].

PROPOSITION 1. *If* $\perp \leqslant \mathbf{Z}$, *then* $\|\mathbf{Z}\| = 2$ *iff* $\mathbf{Z}(P) \subseteq \mathbf{PC}(P)$ *for each p.o. set* P.

Proof. The p.o. set

Q =


is $\mathbf{PC}$-complete, and hence $\|\mathbf{PC}\| = 2$. Thus $\mathbf{Z}(Q) \subseteq \mathbf{PC}(Q)$ implies $\|\mathbf{Z}\| = 2$.

Conversely, if there exists a p.o. set P and $Y \in \mathbf{Z}(P)$ with $Y \notin \mathbf{PC}(P)$, then there are elements $p, q \in Y$ with no common upper bound in P. Define $f\colon P \to Q$ by:

$$\begin{aligned} f(x) &= a && \text{if } p \leqslant x \\ &= b && \text{if } q \leqslant x \\ &= \bot && \text{otherwise.} \end{aligned}$$

Then f is order-preserving and so $f(Y) \in \mathbf{Z}(Q)$. Now both a and b belong to $f(Y)$ and thus since every $\mathbf{Z}$-complete p.o. set has a smallest element, it follows that every pair (and hence every finite set) of elements in a $\mathbf{Z}$-complete p.o. set have a join, and that $\mathbf{Z}$-continuous maps preserve these joins, and consequently $\|\mathbf{Z}\| \geqslant \aleph_0$. ∎

PROPOSITION 2. *If $\|\mathbf{Z}\|$ is infinite then for each $\mathbf{Z}$-complete p.o. set P, $P_\bot$ is also $\mathbf{Z}$-complete. Moreover, $\|\mathbf{Z}\| = \|\mathbf{Z}_\bot\|$.*

Proof. Suppose P is $\mathbf{Z}$-complete, and $Y \in \mathbf{Z}(P_\bot)$. If $Y = \varnothing$ then $\bigvee Y = \bot$. If $Y \neq \varnothing$ then choose $p \in Y$ and consider the map $P_\bot \to P$ given by $q \rightsquigarrow p \vee q$; this map is order preserving and hence the set $Y' = \{p \vee q \mid q \in Y\}$ is in $\mathbf{Z}(P)$ and so $\bigvee Y'$ exists in P and evidently $\bigvee Y' = \bigvee Y$ in $P_\bot$.

Since $\mathbf{Z} \leqslant \mathbf{Z}_\bot$ we always have $\|\mathbf{Z}\| \leqslant \|\mathbf{Z}_\bot\|$ and the reverse inequality now follows from the preceeding paragraph. ∎

Note that Proposition 2 does not hold for $\|\mathbf{Z}\|$ finite; for example, if $\mathbf{Z}(P)$ consists of all subsets of P with a smallest element (for each p.o. set P) then the two element discrete set is $\mathbf{Z}$-complete, but $\mathbf{Z}$-completeness is destroyed when $\bot$ is adjoined.

1.3. *Free Z-completions*

For every subset system $\mathbf{Z}$, every p.o. set P has a free $\mathbf{Z}$-completion $\bar{P}$, (see Banaschewski and Nelson [8] or Meseguer [17]), that is, a $\mathbf{Z}$-complete extension $\bar{P} \supseteq P$ such that every order-preserving map from P into a Z-complete p.o. set Q has a unique extension to a $\mathbf{Z}$-continuous map on $\bar{P}$. For discrete sets X we have the following description of this free completion.

PROPOSITION 3. *For any subset system $\mathbf{Z}$ with $\|\mathbf{Z}\| = \mathfrak{m}$ and any discrete p.o. set X, the free $\mathbf{Z}$-completion of X is $\mathbf{P}_\mathfrak{m}(X)$ if $\bot \leqslant \mathbf{Z}$ and $S_\mathfrak{m}(X)$ if $\bot \nleqslant \mathbf{Z}$, ordered by inclusion.*

Proof. Assume $\bot \leqslant \mathbf{Z}$; the proof if $\bot \nleqslant \mathbf{Z}$ is analogous. First, note that $\mathfrak{m} \nleqslant \mathbf{Z}$: since $\|\mathbf{Z}\| = \mathfrak{m}$ implies $\mathbf{S}_\mathfrak{m} \leqslant \mathbf{Z}$, every $\mathbf{Z}$-complete p.o. set has joins of all subsets of cardinality $<\mathfrak{m}$. If in addition $\mathfrak{m} \leqslant \mathbf{Z}$ then each Z-complete p.o. set would also have joins of all $\mathfrak{m}$-chains, and hence joins of all sets of cardinality $\leqslant\mathfrak{m}$. Also, each $\mathbf{Z}$-continuous map on a $\mathbf{Z}$-complete p.o. set would preserve all joins of cardinality $\leqslant\mathfrak{m}$, and hence $\|\mathbf{Z}\| > \mathfrak{m}$, a contradiction.

Next, we show that for each $A \in Z(\mathbf{P}_\mathfrak{m}(X))$, the union, $\bigcup A$, of the sets in A, is in

$\mathbf{P}_{\mathfrak{m}}(X)$: If $\text{card}(\bigcup A) \geqslant \mathfrak{m}$ then (viewing $\mathfrak{m}$ as the set of ordinals $<\mathfrak{m}$) there is a one-one *set* map $h\colon \mathfrak{m} \to \bigcup A$. Define $h^*\colon \mathbf{P}_{\mathfrak{m}}(X) \to \mathfrak{m}$ by

$$h^*(T) = \bigvee \{\alpha < \mathfrak{m} \mid h(\alpha) \in T\}.$$

This join exists because $\text{card}(T) \leqslant \mathfrak{m}$ implies that $\{\alpha < \mathfrak{m} \mid h(\alpha) \in T\}$ is bounded in $\mathfrak{m}$. Moreover, h^* is evidently order-preserving, and hence $h^*(A) \in \mathbf{Z}(\mathbf{m})$. Further, for each $\alpha \in \mathfrak{m}$, since $h(\alpha) \in \bigcup A$ there exists $T \in A$ with $h(\alpha) \in T$ and then $h^*(T) \geqslant \alpha$; this shows that $h^*(A)$ is cofinal in $\mathfrak{m}$. However, the existence of a set in $\mathbf{Z}(\mathfrak{m})$ which is cofinal in $\mathfrak{m}$ clearly implies $\mathbf{m} \leqslant \mathbf{Z}$, contradicting the preceeding paragraph.

This establishes the fact that $\mathbf{P}_{\mathfrak{m}}(X)$ is $\mathbf{Z}$-complete, and that $\mathbf{Z}$-joins are unions. Clearly the map $x \rightsquigarrow \{x\}$ is an order-embedding of X into $\mathbf{P}_{\mathfrak{m}}(X)$. To see that $\mathbf{P}_{\mathfrak{m}}(X)$ is the desired completion, consider a map $f\colon X \to Y$ where Y is $\mathbf{Z}$-complete. Define an extension $\bar{f}\colon \mathbf{P}_{\mathfrak{m}}(X) \to Y$ by $\bar{f}(T) = \bigvee f(T)$ for each $T \in \mathbf{P}_{\mathfrak{m}}(X)$ (since card $(T) < \mathfrak{m}$ it follows that card $f(T) < \mathfrak{m}$ and hence the set $f(T)$ has a join in Y). Moreover, $\bar{f}$ is $\mathbf{Z}$-continuous: if $A \in \mathbf{Z}(\mathbf{P}_{\mathfrak{m}}(X))$ then by the preceeding paragraph we have $\bigvee A = \bigcup A$ and hence

$$\bar{f}\left(\bigvee A\right) = \bar{f}\left(\bigcup A\right) = \bigvee f\left(\bigcup A\right) = \bigvee_{T \in A} \bigvee_{t \in T} f(t) = \bigvee_{T \in A} \bar{f}(T) = \bigvee \bar{f}(A)$$

as required. Since $\bar{f}$ is clearly uniquely determined by f, this completes the proof. ∎

1.4. *Products and* **Z**-*Coproducts*

For any family $(P_i)_{i \in I}$ of p.o. sets, the cartesian product $\prod_{i \in I} P_i = \{f\colon I \to \bigcup P_i \mid f(i) \in P_i \text{ for each } i \in I\}$ is again a p.o. set with the "pointwise" or "componentwise" order defined by

$$f \leqslant g \quad \text{iff} \quad f(i) \leqslant g(i) \quad \text{for each } i \in I.$$

Moreover, for each $j \in I$, the projection map $\text{pr}_j\colon \prod_{i \in I} P_i \to P_j$ defined by $\text{pr}_j(f) = f(j)$ is evidently order-preserving.

The proofs of the following easy observations are left to the reader.

Remarks. 1. For any p.o. sets P_i $(i \in I)$, all joins which exist in $\prod P_i$ are componentwise, i.e., a set $Y \subseteq \prod P_i$ has a join iff so do the sets $\text{pr}_i(Y)$ for each $i \in J$ and then $\text{pr}_i(\bigvee Y) = \bigvee \text{pr}_i(Y)$.

2. If each P_i is $\mathbf{Z}$-complete so is $\prod P_i$.

3. If $f_i\colon P_i \to Q_i$ are $\mathbf{Z}$-continuous maps then so is $\prod f_i\colon \prod P_i \to \prod Q_i$ defined by $(\prod f_i)((x_i)_{i \in I}) = (f_i(x_i))_{i \in I}$.

The $\mathbf{Z}$-*coproduct* of two $\mathbf{Z}$-complete p.o. sets P and Q is a $\mathbf{Z}$-complete p.o. set $P \amalg Q$ together with $\mathbf{Z}$-continuous maps $i\colon P \to P \amalg Q$ and $j\colon Q \to P \amalg Q$ such that for any $\mathbf{Z}$-continuous $f\colon P \to R$, $g\colon Q \to R$ where R is $\mathbf{Z}$-complete, there is a unique $\mathbf{Z}$-continuous map $(f \sqcup g)\colon P \amalg Q \to R$ with $(f \sqcup g)i = f$, $(f \sqcup g)j = g$. The existence of $\mathbf{Z}$-

coproducts, for any $\mathbf{Z}$, is established in Banaschewski and Nelson [8]. However, we now give an explicit description of $\mathbf{Z}$-coproducts, which will be useful later on.

PROPOSITION 4. *For any* $\mathbf{Z}$-*complete p.o. sets* P *and* Q, *the* $\mathbf{Z}$-*coproduct is described as follows*:

(a) *If* $\|\mathbf{Z}\| \geqslant \aleph_0$ *and* $\perp \leqslant \mathbf{Z}$ *then* $P \amalg Q = P \times Q$.

(b) *If* $\|\mathbf{Z}\| \geqslant \aleph_0$ *and* $\perp \nleqslant \mathbf{Z}$ *then* $P \amalg Q = P_\perp \times Q_\perp - \{(\perp, \perp)\}$.

(c) *If* $\|\mathbf{Z}\| = 2$ *and* $\perp \leqslant \mathbf{Z}$ *then* $P \amalg Q = \{(p, q) \in P \times Q \mid p = \perp$ *or* $q = \perp\}$.

(d) *If* $\|\mathbf{Z}\| = 2$ *and* $\perp \nleqslant \mathbf{Z}$ *then* $P \amalg Q = \{(p, q) \in P_\perp \times Q_\perp \mid p = \perp$ *or* $q = \perp\} - \{(\perp, \perp)\}$.

In all these cases, the coproduct maps i *and* j *are defined by* $i(p) = (p, \perp)$ *for* $p \in P$, $j(q) = (\perp, q)$ *for* $q \in Q$.

Proof. (a) The $\mathbf{Z}$-completeness of $P \times Q$ and the $\mathbf{Z}$-continuity of i and j follow from Remark 1 above. Given a $\mathbf{Z}$-complete p.o. set R and $\mathbf{Z}$-continuous maps $f\colon P \to R$, $g\colon Q \to R$, define $f \sqcup g\colon P \times Q \to R$ by

$$(f \sqcup g)(p, q) = f(p) \vee g(q).$$

This join exists because R is $\mathbf{Z}$-complete and $\|\mathbf{Z}\|$ is infinite. Also, since $(p, q) = (p, \perp) \vee (\perp, q)$, and binary joins are preserved by $\mathbf{Z}$-continuous maps, $f \sqcup g$ is uniquely determined by f and g, and all that remains is to show that it is $\mathbf{Z}$-continuous. Suppose $X \in Z(P \times Q)$ with $\bigvee X = (p, q)$; then by Remark 1 above we have $\bigvee \mathrm{pr}_1(X) = p$ and $\bigvee \mathrm{pr}_2(X) = q$. Thus

$$\begin{aligned}(f \sqcup g)(\textstyle\bigvee X) &= f(p) \vee g(q)\\ &= f(\textstyle\bigvee \mathrm{pr}_1(X)) \vee g(\textstyle\bigvee \mathrm{pr}_2(X))\\ &= \textstyle\bigvee f(\mathrm{pr}_1(X)) \vee \textstyle\bigvee g(\mathrm{pr}_2(X))\\ &= \textstyle\bigvee (f \sqcup g)(X),\end{aligned}$$

as required.

(b) Since $\|\mathbf{Z}\|$ is infinite, $P_\perp$ and $Q_\perp$ are $\mathbf{Z}$-complete by Proposition 2 and hence so is $P_\perp \times Q_\perp$. If $X \in \mathbf{Z}(P_\perp \times Q_\perp - \{(\perp, \perp)\})$ then because the natural embedding $P_\perp \times Q_\perp - \{(\perp\ \perp)\} \to P_\perp \times Q_\perp$ is order-preserving, X is in $\mathbf{Z}(P_\perp \times Q_\perp)$. Also, $X \neq \varnothing$ and hence $\bigvee X \neq (\perp, \perp)$ and so $\bigvee X$ is also the join in $P_\perp \times Q_\perp - \{(\perp\ \perp)\}$. This shows that the indicated set is $\mathbf{Z}$-complete, and the rest of the proof is analogous to case (a).

(c) Let $S = \{(p, q) \in P \times Q \mid p = \perp$ or $q = \perp\}$. By Proposition 1, $\mathbf{Z}(S) \subseteq \mathbf{PC}(S)$ and hence for each $X \in \mathbf{Z}(S)$, either $X \subseteq P \times \{\perp\}$ or $X \subseteq \{\perp\} \times Q$; in both cases $\bigvee X$ exists in S and hence S is $\mathbf{Z}$-complete.

Given a **Z**-complete p.o. set R and Z-continuous maps $f: P \to R$ and $g: Q \to R$, define $f \sqcup g: S \to R$ by

$$(f \sqcup g)(p, \perp) = f(p) \quad \text{and} \quad (f \sqcup g)(\perp, q) = g(q) \quad \text{for} \quad p \in P, q \varepsilon Q.$$

Then $f \sqcup g$ is the unique Z-continuous map extending f and g.

(d) Let $S = \{(p, q) \in P_\perp \times Q_\perp \mid p = \perp \text{ or } q = \perp \text{ but not both}\}$. Define $h: S \to \{a, b\}$, where $\{a, b\}$ is discretely ordered, by

$$h(p, \perp) = a, \qquad h(\perp, q) = b \qquad \text{for} \quad p \in P, q \in Q.$$

Since $\|\mathbf{Z}\| = 2$ implies $\{a, b\} \notin \mathbf{Z}(\{a, b\})$, it follows that for each $X \in \mathbf{Z}(S)$, either $h(X) = \{a\}$ or $h(X) = \{b\}$; the former implies $X \subseteq P \times \{\perp\}$ and the latter $X \subseteq \{\perp\} \times Q$. If $X \subseteq P \times \{\perp\}$, choose any $(p_0, \perp) \in X$ and consider the map $\bar{h}: S \to P$ given by $\bar{h}(p, \perp) = p$, $\bar{h}(\perp, q) = p_0$. This map is order-preserving and so $\bar{h}(X) \in \mathbf{Z}(P)$ and $(\bigvee \bar{h}(X), \perp) = \bigvee X$. Similarly, if $X \subseteq \{\perp\} \times Q$ it has a join in S, and this proves that S is **Z**-complete. Maps are extended as above and **Z**-continuity is argued as in the preceeding sentence. ∎

Note that the coproduct in case (d) *is essentially the disjoint union of P and Q, and in case* (c) *is the disjoint union with* $\perp$ *elements identified, sometimes called strict disjoint union.*

Note also that in all cases the maps $i: P \to P \sqcup Q$ *and* $j: Q \to P \sqcup Q$ *are embeddings, and the images of P and Q are down-sets in* $P \sqcup Q$.

1.5. *(Universal) Algebras*

Here, we list some basic definitions and well-known facts about (universal) algebras which will be needed later.

A *type* of algebras is a set Σ together with a map $\sigma \leadsto |\sigma|$ assigning to each $\sigma \in \Sigma$ a cardinal number $|\sigma|$ called the *arity* of σ. If $|\sigma|$ is finite for all $\sigma \in \Sigma$ then the type is called *finitary*. An *algebra* of type Σ consists of a set A together with, for each $\sigma \in \Sigma$, an $|\sigma|$-ary operation $\sigma_A: A^{|\sigma|} \to A$. We will follow the usual custom of writing simply "σ" for "σ_A". A *homomorphism* from an algebra A to an algebra B is a map $f: A \to B$ such that for all $\sigma \in \Sigma$ and all $q_i \in A$ $(i < |\sigma|)$,

$$\sigma(f(a_i)_{i<|\sigma|}) = f(\sigma((a_i)_{i<|\sigma|})).$$

Recall the familiar fact that for each Σ and each set X there exists an *absolutely free* algebra $\mathfrak{F}X$ over X, characterized by the following properties: $X \subseteq \mathfrak{F}X$, and each set map of X into an algebra A has a unique extension to a homomorphism from $\mathfrak{F}X$ into A. An alternative characterization of $\mathfrak{F}X$ is that it is generated by X, and satisfies the "Peano properties," i.e.,

(P1) $\sigma((a_i)_{i \leqslant |\sigma|}) \notin X$ for any $a_i \in \mathfrak{F}X$, $\sigma \in \Sigma$.

(P2) $\sigma((a_i)_{i<|\sigma|}) = \lambda((b_j)_{j<|\lambda|})$ implies $\sigma = \lambda$ and $a_i = b_i$, all $i < |\sigma|$.

Moreover, $\mathfrak{F}X$ is built inductively from the elements of X as follows: For each ordinal α define a set $V_\alpha \subseteq \mathfrak{F}X$ (the elements of "complexity $\leqslant \alpha$"),

$$V_0 = X,$$
$$V_{\alpha+1} = X \cup \{\sigma((a_i)_{i<|\sigma|}) \mid \sigma \in \Sigma, a_i \in V_\alpha\},$$
$$V_\beta = \bigcup_{\alpha<\beta} V_\alpha \qquad \text{for a limit ordinal } \beta.$$

Then $V_\alpha \subseteq V_\beta$ whenever $\alpha < \beta$, and $\mathfrak{F}X = V_\mathfrak{m}$ whenever $\mathfrak{m}$ is an infinite regular cardinal $> |\sigma|$ for all $\sigma \in \Sigma$.

1.6. ***Z**-continuous Algebras*

Now we define the continuous algebras which are the main concern of this paper.

For a subset system $\mathbf{Z}$ and a type Σ of algebras, a ***Z**-continuous algebra* is an algebra A, which is endowed with a $\mathbf{Z}$-complete partial order, such that all operations $\sigma \in \Sigma$ preserve all non-empty $\mathbf{Z}$-joins, i.e., if $Y \in \mathbf{Z}(A^{|\sigma|})$ and $Y \neq \varnothing$, then, writing $y = (y_i)_{i<|\sigma|}$ for each $y \in Y$

$$\sigma\left(\left(\bigvee_{y\in Y} y_i\right)_{i<|\sigma|}\right) = \bigvee_{y \in Y} \sigma((y_i)_{i<|\sigma|}).$$

If $\perp \leqslant \mathbf{Z}$ and $\sigma(\perp, \perp, \perp, ...) = \perp$ for each $\sigma \in \Sigma$ then A is called a *strict Z-continuous algebra.*

For any $\mathbf{Z}$-complete p.o. set P, a $\mathbf{Z}$-continuous algebra V together with a $\mathbf{Z}$-continuous map $f: P \to V$ is called the *free Z-continuous algebra over P* iff for every $\mathbf{Z}$-continuous map g of P into a $\mathbf{Z}$-continuous algebra A there exists a unique $\mathbf{Z}$-continuous algebra homomorphism $\bar{g}: V \to P$ with $\bar{g}f = g$.

The usual arguments show that V is unique up to isomorphism over P. In Sections 3 and 4 we construct, for all $\mathbf{Z}$ and all $\mathbf{Z}$-complete p.o. sets P, the free $\mathbf{Z}$-continuous algebra over P; f will always be an order-embedding and $\bar{g}$ will be the unique extension of g to a $\mathbf{Z}$-continuous homomorphism.

Analogously, for $\perp \leqslant \mathbf{Z}$, a strict $\mathbf{Z}$-continuous algebra V together with a $\mathbf{Z}$-continuous map $f: P \to V$ is called the *free strict **Z**-continuous algebra over P* iff for every $\mathbf{Z}$-continuous map g from P into a strict $\mathbf{Z}$-continuous algebra A there exists a unique $\mathbf{Z}$-continuous homomorphism $\bar{g}$ such that $\bar{g}f = g$; free strict $\mathbf{Z}$-continuous algebras will be described in Section 5.

Remarks. 1. For an unordered set X of generators, the free $\mathbf{Z}$-continuous algebra V over the free $\mathbf{Z}$-completion P of X has the following property:

> Every *set* map $g: X \to A$, with A a $\mathbf{Z}$-continuous algebra, has a unique extension to a $\mathbf{Z}$-continuous homomorphism $\bar{g}: V \to A$.

Indeed, g has a unique $\mathbf{Z}$-continuous extension $\tilde{g}: P \to A$, and then $\tilde{g}$ has a unique extension $\bar{g}$. Algebras with this property were the type of free continuous algebras constructed by ADJ [4].

2. The concept of free **Z**-continuous algebra can be phrased in the terminology of category theory as follows: Let **ZAlg** be the category of **Z**-continuous algebras (of a given type) and **Z**-continuous homomorphisms. Let **ZPOS** be the category of **Z**-complete p.o. sets and **Z**-continuous maps. The construction of free **Z**-continuous algebras yields a left adjoint to the natural forgetful functor U: **ZAlg** $\to$ **ZPOS**. Further, the existence of **Z**-completions of discrete sets provides a left-adjoint to the forgetful functor U_0: **ZPOS** $\to$ **Set** (the category of sets) and so the composite of these adjoints is a left-adjoint to the forgetful functor $U_0 U$: **ZAlg** $\to$ **Set**.

Finally, free strict **Z**-continuous algebras yield a left adjoint to the restriction of U to the full subcategory **ZAlg**^{s} of strict algebras.

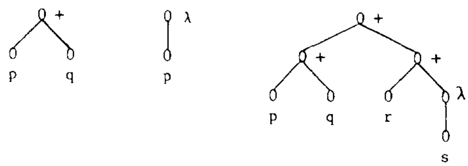
However, since we will not need any categorical results in this paper, we will not use categorical terminology either.

2. Arboriculture

In this section we describe the trees which will constitute all the free algebras considered later.

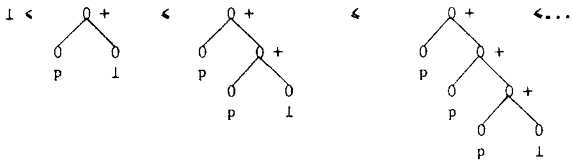
2.1. *Historical Remarks about Trees*

First, we recall that the elements of the absolutely free algebra of type Σ over a set X (see Section 1.5) can be represented by rooted trees, labelled with elements from $X \amalg \Sigma$, the disjoint union of X and Σ. Any node labelled with an operation $\sigma \in \Sigma$ has exactly $|\sigma|$ immediate successors, and a node is a leaf (i.e., has no successors) iff its label is an element of X or a nullary operation. For example, if Σ contains a binary operation $+$ and a unary operation λ then the following trees represent $p+q$, $\lambda(p)$ and $(p+q)+(r+\lambda(s))$:

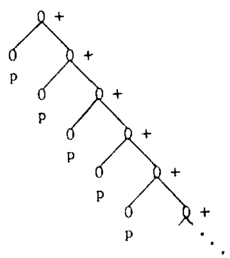


Note that we follow the usual custom of drawing trees upside down, so that the root is at the top and the branches reach down. This description can be used for infinitary operations as well as finitary ones; in this case, although a tree might be infinite (in width) each path down the tree from the root is necessarily finite.

The modification of such trees, introduced by Scott [24] to model abstract data types and flow diagrams, was to add an "undetermined" label $\perp$ which was to be smaller (carry less information) than everything else, and then to take the operations order-preserving. This yielded such inequalities as $\perp \leqslant p + \perp \leqslant p + q$, for example. The main advantage of this approach was the appearance of countable chains such as

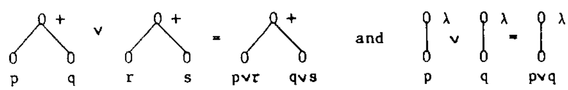


The joins of such ω-chains modelled the unfoldments of loops in flow charts, thereby providing solutions to certain recursive equations. For example, the join t of the chain pictured above satisfies $t = p + t$. This join can be depicted by the following tree, no longer having only finite paths:



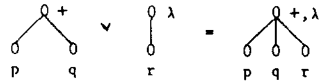
Ordered, ω-complete algebras, with ω-continuous operations, were subsequently studied by ADJ [3, 4], who constructed an algebra of trees which was, among other things, the free ω-continuous algebra over a set (see Remark 1 of Section 1.6).

The approach as outlined above, and pursued by ADJ, is not adequate, however, to handle binary joins. If the generating set X itself has binary joins, and if the operations are to preserve them, then we would necessarily have new equalities of the following kind:

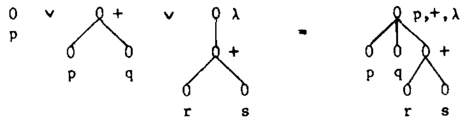


Some new technique is needed to deal with joins like $(p+q) \vee \lambda(r+s)$, or $p \vee (q+r)$.

The method we have developed to handle this situation allows a node to be labelled with more than one operation symbol, simultaneously with one of the generating elements, and then to have the appropriate number of immediate successors. The generating set will be a partially ordered set P, with $\perp$, and with appropriate completeness. Thus, for example, if Σ consists of one binary operation $+$ and one unary operation λ, then



and



For the sake of homogeneity and to simplify later arguments, all trees will be taken as subtrees of one master tree (see Diagram 1) in which each node a has, for each $\sigma \in \Sigma$, $|\sigma|$ immediate successors; these will be labelled $\perp$ in case σ is not part of the label of a. To provide each node with the right number of successors, the nodes are "enumerated" by finite sequences of pairs (σ, i), where $i < |\sigma|$. The root node is the empty sequence $\varnothing$ and the immediate successors of $a = (\sigma_1, i_1)(\sigma_2, i_2) \cdots (\sigma_k, i_k)$ are all sequences $a(\sigma, i) = (\sigma_1, i_1)(\sigma_2, i_2) \cdots (\sigma_k, i_k)(\sigma, i)$ (see Diagram 1).

2.2. *Formal Definition of* $T(P)$

Now we state the formal definition of the trees with which we shall work. To simplify notation we write σi instead of (σ, i) and just $\perp$ for the smallest element $(\perp, \varnothing)$ of $P \times \mathbf{P}(\Sigma)$.

For a type Σ, let $\tilde{\Sigma} = \{\sigma i \mid \sigma \in \Sigma, i < |\sigma|\}$ and let $\tilde{\Sigma}^*$ be the set of all finite sequences of elements from $\tilde{\Sigma}$.

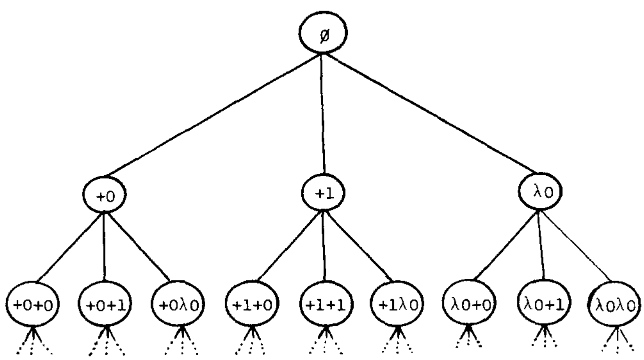


DIAGRAM I. Master tree for $\Sigma = \{+, \lambda\}$.

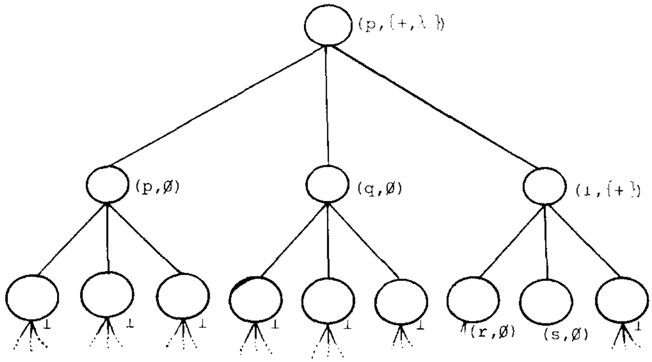


DIAGRAM II. Labelled tree for $p \vee (p+q) \vee \lambda(r+s)$.

For each partially ordered set P with $\perp$, a P-labelled Σ-tree is a map $t\colon \tilde{\Sigma}^* \to P \times \mathbf{P}(\Sigma)$ such that for all $a \in \tilde{\Sigma}^*$ and $\sigma i \in \tilde{\Sigma}$,

$$\text{If } t(a) = (p, k) \text{ and } t(a\sigma i) \neq \perp \text{ then } \sigma \in k. \qquad (*)$$

Let $T(P)$ be the set of all P-labelled Σ-trees, and define for each cardinal number $\mathfrak{m} > 1$,

$$T_{\mathfrak{m}}(P) = \{t \in T(P) \mid t(a) = (p, k) \text{ implies card } k < \mathfrak{m}\}.$$

Thus $T_{\mathfrak{m}}(P)$ consists of all trees in $T(P)$ with image in $P \times \mathbf{P}_{\mathfrak{m}}(\Sigma)$. Note that $T_{\mathfrak{m}}(P) = T(P)$ for any $\mathfrak{m} > \text{card}\, \Sigma$.

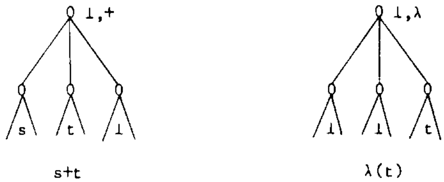
There is a natural embedding of P into $T(P)$ given by $p \rightsquigarrow \hat{p}$, where $\hat{p}(\emptyset) = (p, \emptyset)$ and $\hat{p}(a) = \perp$ for all $a \neq \emptyset$. Of course, $\hat{p} \in T_{\mathfrak{m}}(P)$ for all $\mathfrak{m}$.

We make $T(P)$ into a Σ-algebra as follows: For $\sigma \in \Sigma$ and $t_i \in T(P)$ for $i < |\sigma|$, the tree $t = \sigma((t_i)_{i<|\sigma|})$ is defined by:

$$\begin{aligned} &t(\emptyset) = (\perp, \{\sigma\}), \\ &t(\sigma j a) = t_j(a) \qquad \text{for all } j < |\sigma|, \text{ all } a \in \tilde{\Sigma}^*, \\ &t(\lambda j a) = \perp \qquad \text{for all } \lambda \neq \sigma, \text{ all } j < |\lambda|, \text{ all } a \in \tilde{\Sigma}^*. \end{aligned}$$

The resulting function t evidently belongs to $T(P)$. Further for each $\mathfrak{m}$, $T_{\mathfrak{m}}(P)$ is closed under all these operations.

EXAMPLE. For Σ consisting of one binary operation $+$ and one unary operation λ we illustrate these operations below.



2.3. *Properties of* $T(P)$

The set $T(P)$, as a subset of $(P \times \mathbf{P}(\Sigma))^{\tilde{\Sigma}^*}$, has the natural pointwise ordering on it; that is, $s \leqslant t$ iff for each $a \in \tilde{\Sigma}^*$, if $s(a) = (p, k)$ and $t(a) = (q, h)$ then $p \leqslant q$ and $k \subseteq h$.

PROPOSITION 5. *$T(P)$ is closed under pointwise joins in $(P \times \mathbf{P}(\Sigma))^{\tilde{\Sigma}^*}$. All joins that exists in $T(P)$ are formed pointwise and the operations on $T(P)$ preserve all existing non-empty pointwise joins. Further, the natural embedding of P into $T(P)$ preserves all existing joins.*

Proof. The first statement is obvious. For the second, suppose $t = \bigvee Y$ in $T(P)$. Let $a \in \tilde{\Sigma}^*$, $t(a) = (p, k)$, $s(a) = (p_s, k_s)$ for $s \in Y$. If $(p, k) \neq \bigvee_{s \in Y} (p_s, k_s)$ then either there is an upper bound p' of $\{p_s \mid s \in Y\}$ in P with $p \nleqslant p'$, or there is $\sigma_0 \in k$ with $\sigma_0 \notin \bigcup_{s \in Y} k_s$. In the former case, let $t' \in T(P)$ be the tree obtained from t by redefining t at a by $t'(a) = (p', k)$; then t' is an upper bound for Y with $t \nleqslant t'$, a contradiction. The latter case is analogous, we redefine t by $t'(a) = (p, k - \{\sigma_0\})$, $t'(a\sigma_0 ib) = \perp$ for all $i < |\sigma|$ and all $b \in \tilde{\Sigma}^*$. For the third statement, let $\sigma \in \Sigma$ and suppose $Y \subseteq T(P)^{|\sigma|}$ is a non-empty set with join $\bigvee Y = u \in T(P)^{|\sigma|}$. This means that for each $i < |\sigma|$, $\bigvee_{y \in Y} y_i = u_i$. We must prove that for each $a \in \tilde{\Sigma}^*$,

$$\left(\bigvee_{y \in Y} \sigma((y_i)_{i < |\sigma|}) \right)(a) = \sigma((u_i)_{i < |\sigma|})(a).$$

If $a = \varnothing$ then by the definition of the operation σ in $T(P)$, both sides equal $(\perp, \{\sigma\})$. If $a = \lambda jb$ for some $\lambda \neq \sigma$ then both sides equal $\perp$. Finally, if $a = \sigma jb$ then

$$\begin{aligned} \left(\bigvee_{y \in Y} \sigma((y_i)_{i < |\sigma|}) \right)(a) &= \bigvee_{y \in Y} \sigma((y_i)_{i < |\sigma|})(a) \\ &= \bigvee_{y \in Y} y_j(b) \\ &= u_j(b) \\ &= \sigma((u_i)_{i < |\sigma|})(a) \end{aligned}$$

as required. ∎

The last statement is obvious.

COROLLARY. *For every subset system* $\mathbf{Z}$, *if* P *is* $\mathbf{Z}$*-complete then* $T(P)$ *is a* $\mathbf{Z}$*-continuous algebra with* $\mathbf{Z}$*-joins formed pointwise.*

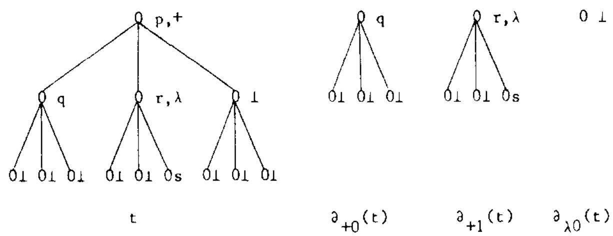
Note that the operations in $T(P)$ do not preserve the join of the empty set; for example, $\bot + \bot$ has its root labelled by $(\bot, \{+\})$, whereas the smallest element of $T(P)$ is the constant function with value $\bot$.

We conclude this section with two further remarks about the internal structure of $T(P)$.

First, for all $\sigma \in \Sigma$ and $i < |\sigma|$, define $\partial_{\sigma i}: T(P) \to T(P)$ by

$$\partial_{\sigma i}(t)(a) = t(\sigma i a).$$

The tree $\partial_{\sigma i}(t)$ is the maximal branch of t with root σi; some illustrative examples follow, for Σ consisting of one binary operation $+$ and one unary operation λ.



Note that for each $t \in T(P)$, with $t(\varnothing) = (p, k)$, we have

$$t = \hat{p} \vee \bigvee_{\sigma \in k} \sigma(\partial_{\sigma i}(t)_{i<|\sigma|}).$$

Moreover, the maps $\partial_{\sigma i}$ preserve all pointwise joins.

Finally, we say that a tree t is *path-finite* iff for every countable sequence $\sigma_1 i_1, \sigma_2 i_2 \cdots$ of elements of $\tilde{\Sigma}$, there exists a natural number n with $t(\sigma_1 i_1 \sigma_2 i_2 \cdots \sigma_n i_n) = \bot$ (and hence $t(\sigma_1 i_1 \cdots \sigma_m i_m) = \bot$ for all $m \geqslant n$). Define

$$F(P) = \{t \in T(P) \mid t \text{ is path-finite}\}$$

and for each $m > 1$,

$$F_m(P) = F(P) \cap T_m(P).$$

Clearly $F(P)$, and all the $F_m(P)$, are closed under all the operations in $T(P)$.

3. FREE **Z**-CONTINUOUS ALGEBRAS WHEN $\bot \leqslant \mathbf{Z}$

In this section we will describe, for an arbitrary **Z**-complete p.o. set P and an arbitrary type Σ of algebras (not necessarily finitary) the free **Z**-continuous algebras over P (as defined in Section 1.6).

In the following, $\bar{\Sigma}$ will be the free **Z**-completion of the discretely ordered set Σ, so that by Proposition 3 (Section 1.3) $\bar{\Sigma} = \{k \subseteq \Sigma \mid \text{card } k < \|Z\|\}$.

3.1. *Case* 1 $\|\mathbf{Z}\| > \aleph_0$

In this case, each **Z**-complete p.o. set is countably complete, and **Z**-continuous maps on **Z**-complete p.o. sets preserve countable joins. We will prove that for each **Z**-complete p.o. set P, the free **Z**-continuous algebra over P is $T_{\mathfrak{m}}(P)$, where $\mathfrak{m} = \|\mathbf{Z}\|$.

As mentioned in Section 2.1, $T_{\mathfrak{m}}(P)$ is closed under the operations in $T(P)$. Further, since $P \times \bar{\Sigma}$ is **Z**-complete and the natural embedding of $P \times \bar{\Sigma}$ into $P \times \mathbf{P}(\Sigma)$ preserves **Z**-joins (see Proposition 3), it follows that for every $Y \in \mathbf{Z}(T_{\mathfrak{m}}(P))$, the pointwise join t of Y exists, and takes values in $P \times \bar{\Sigma}$; further, t is in $T(P)$ by Proposition 5 and so t belongs to $T_{\mathfrak{m}}(P)$. Consequently $T_{\mathfrak{m}}(P)$ is **Z**-complete, and the operations on $T_{\mathfrak{m}}(P)$ are **Z**-continuous by Proposition 5. This shows that $T_{\mathfrak{m}}(P)$ is a **Z**-continuous algebra.

Before proceeding with the main proof, we will outline a technique of tree-cutting. For each tree $t \in T(P)$, define a countable sequence of trees $t \mid 0 \leqslant t \mid 1 \leqslant t \mid 2 \leqslant \cdots \leqslant t$ as follows:

$$\begin{aligned} (t \mid n)(a) &= t(a) && \text{if length } (a) < n \\ &= \bot && \text{otherwise.} \end{aligned}$$

It is clear that for each $t \in T(P)$, $t = \bigvee t \mid n$ $(n \in \omega)$. A tree is said to have *finite height* if $t = t \mid n$ for some n, and the smallest such n is called the *height of t.*

THEOREM 1. *If* **Z** *is a subset system with* $\bot \leqslant \mathbf{Z}$ *and* $\|\mathbf{Z}\| = \mathfrak{m} > \aleph_0$ *then the free* **Z**-*continuous algebra over a* **Z**-*complete p.o. set P is* $T_{\mathfrak{m}}(P)$.

Proof. We have seen just above that $T_{\mathfrak{m}}(P)$ is a **Z**-continuous algebra. Suppose $g: P \to A$ is a **Z**-continuous map into a **Z**-continuous algebra A. We first extend g to the trees in $T_{\mathfrak{m}}(P)$ of finite height by induction on height. Since $t \in T_{\mathfrak{m}}(P)$ implies $t \mid n \in T_{\mathfrak{m}}(P)$ for each n, and $t = \bigvee t \mid n$, we then use the fact that A is ω-complete to extend g to arbitrary trees.

The only tree of height 0 is $\bot$; define $\bar{g}(\bot) = \bot$. Proceeding inductively, assume that $\bar{g}$ has been defined on all trees of height $\leqslant n$, that t has height $n+1$ and that $t(\varnothing) = (p, k)$. Since by definition (see Section 2.3) $\partial_{\sigma i}(t)$ has height $\leqslant n$ for each $\sigma i \in \tilde{\Sigma}$, we can define

$$\bar{g}(t) = g(p) \vee \bigvee_{\sigma \in k} \sigma(\bar{g}\partial_{\sigma i}(t)_{i < |\sigma|}).$$

Note that $\bar{g}(p)=g(p)$ for each $p \in P$ and so $\bar{g}$ does extend g. A simple induction on height shows that for all s, t of finite height, $\bar{g}(s) \leqslant \bar{g}(t)$ whenever $s \leqslant t$, and hence $\bar{g}(t)=\bigvee_{n \in \omega} \bar{g}(t \mid n)$ since the latter join is essentially trivial.

Now for arbitrary $t \in T_{\mathfrak{m}}(P)$, define

$$\bar{g}(t)=\bigvee_{n \in \omega} \bar{g}(t \mid n).$$

This join exists because the $\bar{g}(t \mid n)$ form an ω-chain and A is ω-complete.

Then $\bar{g}$ is evidently order preserving; it remains to show that $\bar{g}$ preserves all operations and all $\mathbf{Z}$-joins.

If $t=\sigma((t_i)_{i<|\sigma|})$ has finite height then it follows directly from the definition of $\bar{g}$ that $\bar{g}(t)=\sigma(\bar{g}(t_i)_{i<|\sigma|})$. Moreover, for any $\sigma \in \Sigma$ and $t_i \in T_{\mathfrak{m}}(P)$ $(i<|\sigma|)$, clearly,

$$\sigma((t_i)_{i<|\sigma|}) \mid n=\sigma((t_i \mid n-1)_{i<|\sigma|}) \qquad \text{for each} \quad n \geqslant 1.$$

Thus for any $t_i \in T_{\mathfrak{m}}(P)$ we have:

$$\begin{aligned}
\bar{g}(\sigma(t_i)_{i<|\sigma|}) &= \bigvee_{n \in \omega} \bar{g}(\sigma((t_i)_{i<|\sigma|}) \mid n) \\
&= \bigvee_{n \geqslant 1} \bar{g}(\sigma((t_i \mid n-1)_{i<|\sigma|})) \\
&= \bigvee_{n \geqslant 1} \sigma(\bar{g}(t_i \mid n-1)_{i<|\sigma|}) \\
&= \sigma\left(\left(\bigvee_{n \geqslant 1} \bar{g}(t_i \mid n-1)\right)_{i<|\sigma|}\right) \qquad \text{by the } \omega\text{-continuity of } A \\
&= \sigma(\bar{g}(t_i)_{i<|\sigma|}).
\end{aligned}$$

Concerning the preservation of $\mathbf{Z}$-joins, we first prove by induction on n that for any tree t of height n, if $t=\bigvee Y$ for $Y \in \mathbf{Z}(T_{\mathfrak{m}}(P)$ then $\bar{g}(t)=\bigvee_{s \in Y} \bar{g}(s)$.

If $n=0$ then $t=\perp$ and $s=\perp$ for each $s \in Y$ and so this equality holds trivially. If t has height $n+1$ and the claim is true for all joins of height $\leqslant n$, let $t(\varnothing)=(p, k)$ and $s(\varnothing)=(p_s, k_s)$ for each $s \in Y$. Then

$$\bar{g}(t)=g(p) \vee \bigvee_{\sigma \in k} \sigma(\bar{g} \partial_{\sigma i}(t)_{i<|\sigma|})$$

and

$$\bar{g}(s)=g(p_s) \vee \bigvee_{\sigma \in k_s} \sigma(\bar{g} \partial_{\sigma i}(s)_{i<|\sigma|})$$

for each $s \in Y$.

Since the maps $\partial_{\sigma i}$ preserve all joins, we have for each $\sigma \in k$ and $i < |\sigma|$, $\{\partial_{\sigma i}(s) \mid s \in Y\} \in Z(T_{\mathfrak{m}}(P))$ and $\partial_{\sigma i}(t) = \bigvee_{s \in Y} \partial_{\sigma i}(s)$ and so we may apply the induction hypothesis to the latter join (being of height n) to obtain

$$\bar{g}\partial_{\sigma i}(t) = \bigvee_{s \in Y} \bar{g}\partial_{\sigma i}(s).$$

Now, the map $T_{\mathfrak{m}}(P) \to A^{|\sigma|}$ given by $u \rightsquigarrow (\bar{g}\partial_{\sigma i}(u))_{i<|\sigma|}$ is order-preserving, and hence $\{(\bar{g}\partial_{\sigma i}(s))_{i<|\sigma|} \mid s \in Y\} \in \mathbf{Z}(A^{|\sigma|})$ so the $\mathbf{Z}$-continuity of the operation σ on A yields

$$\sigma\left(\bigvee_{s \in Y} \bar{g}\partial_{\sigma i}(s))_{i<|\sigma|}\right) = \bigvee_{s \in Y} \sigma(\bar{g}\partial_{\sigma i}(s)_{i<|\sigma|}).$$

Further, since the map $T_{\mathfrak{m}}(P) \to P$ which assigns to each tree u with $u(\varnothing) = (q, h)$ the element q is order-preserving, we have $\{p_s \mid s \in Y\} \in \mathbf{Z}(P)$ and hence because all joins in $T_{\mathfrak{m}}(P)$ are formed pointwise and because g is $\mathbf{Z}$-continuous it follows that $g(p) = g(\bigvee_{s \in Y} p_s) = \bigvee_{s \in Y} g(p_s)$. All together, this yields

$$\bar{g}(t) = \bigvee_{s \in Y} g(p_s) \vee \bigvee_{\sigma \in k} \bigvee_{s \in Y} \sigma(\bar{g}\partial_{\sigma i}(s)_{i<|\sigma|}).$$

Now, $\bar{g}$ is order-preserving, and so it is enough to show that $\bar{g}(t) \leqslant \bigvee_{s \in Y} \bar{g}(s)$. For each $s \in Y$, $g(p_s) \leqslant \bar{g}(s)$ and so $\bigvee_{s \in Y} g(p_s) \leqslant \bigvee_{s \in Y} \bar{g}(s)$. Also, for each $\sigma \in k$ and $u \in Y$, if $\sigma \in k_u$ then $\sigma(\bar{g}\partial_{\sigma i}(u)_{i<|\sigma|}) \leqslant \bar{g}(u)$. If $\sigma \notin k_u$ then $\partial_{\sigma i}(u) = \bot$ for all $i < |\sigma|$; since all $\mathbf{Z}$-joins in $T_{\mathfrak{m}}(P)$ are formed pointwise (by the second paragraph of 3.1), there exists some $s \in Y$ with $\sigma \in k_s$ and then

$$\sigma(\bar{g}\partial_{\sigma i}(u)_{i<|u|}) = \sigma(\bar{g}(\bot)_{i<|\sigma|}) \leqslant \sigma(\bar{g}\partial_{\sigma i}(s)_{i<|\sigma|}) \leqslant \bar{g}(s)$$

and hence $\sigma(\bar{g}\partial_{\sigma i}(u)_{i<|\sigma|}) \leqslant \bigvee_{s \in Y} \bar{g}(s)$. Thus $\bar{g}(t) \leqslant \bigvee_{s \in Y} \bar{g}(s)$ as required, completing the induction step.

Now for an arbitrary $t \in T_{\mathfrak{m}}(P)$, if $t = \bigvee Y$ for some $Y \in \mathbf{Z}(T_{\mathfrak{m}}(P))$ then

$$\bar{g}(t) = \bigvee_{n \in \omega} \bar{g}(t \mid n) = \bigvee_{n \in \omega} \bar{g}\left(\bigvee_{s \in Y} s \mid n\right)$$

$$= \bigvee_{n \in \omega} \bigvee_{s \in Y} \bar{g}(s \mid n) \qquad \text{because } \bar{g} \text{ preserves } \mathbf{Z}\text{-joins of finite height}$$

$$= \bigvee_{s \in Y} \bigvee_{n \in \omega} \bar{g}(s \mid n) = \bigvee_{s \in Y} \bar{g}(s), \qquad \text{as required.} \quad \blacksquare$$

Remark. There is one curious aspect of this free continuous algebra: Even if the operations are infinitary, the generation process of $T_{\mathfrak{m}}(P)$ from P has a strong hint of finitariness, which will not be shared by the next case: Let $P_0 = \{\hat{p} \mid p \in P\}$ and for each natural number n,

$$P_{n+1} = P_n \cup \{\sigma((t_i)_{i<|\sigma|}) \mid \sigma \in \Sigma, t_i \in P_n\}.$$

Then every member of $T_{\mathfrak{m}}(P)$ is an ω-join of members of $\bigcup_{n\in\omega} P_n$.

3.2. *Case* 2 $\|\mathbf{Z}\| = \aleph_0$.

Here, $\mathbf{Z}$-continuous p.o. sets have binary joins, $\mathbf{Z}$-continuous maps on $\mathbf{Z}$-complete p.o. sets preserve binary joins, and $\mathbf{Z}(\omega)$ contains only finite sets (see Remark in 1.2).

In the preceeding case, certain arguments worked very smoothly because every tree was an ω-join of trees of finite height. Here, ω-joins are not available, and the free $\mathbf{Z}$-continuous algebra will be constructed by transfinite iteration, similar to the case of the usual (unordered) free algebras (see Section 1.5). Since $\|\mathbf{Z}\| = \aleph_0$, binary joins are available, and the free $\mathbf{Z}$-completion $\bar{\Sigma}$ of Σ is just all finite subsets of Σ. We will see that in this case the desired free continuous algebra is $F_{\aleph_0}(P)$, the set of all path-finite trees in $T_{\aleph_0}(P)$, as defined at the end of Section 2.3.

PROPOSITION 6. *$F_{\aleph_0}(P)$ is a $\mathbf{Z}$-continuous Algebra.*

Proof. $F_{\aleph_0}(P)$ is clearly closed under all the operations in $T(P)$.

We will show that if $Y \in \mathbf{Z}(F_{\aleph_0}(P))$ and $t = \bigvee Y$ (the join in $T(P)$, see Proposition 5 and its corollary), then $t \in F_{\aleph_0}(P)$. This will imply both that $F_{\aleph_0}(P)$ is $\mathbf{Z}$-complete, and that the operations restricted to $F_{\aleph_0}(P)$ are $\mathbf{Z}$-continuous.

Given such Y and $\sigma_1 i_1, \sigma_2 i_2, \sigma_3 i_3 \cdots \in \tilde{\Sigma}$, define a map $f\colon F_{\aleph_0}(P) \to \omega$ such that $f(s)$ is the smallest n with $s(\sigma_1 i_1 \cdots \sigma_n i_n) = \perp$. Then f is order-preserving, which implies $f(Y) \in \mathbf{Z}(\omega)$, and hence $f(Y)$ must be finite. Thus there exists n with $s(\sigma_1 i_1 \cdots \sigma_n i_n) = \perp$ for all $s \in Y$, and consequently $t(\sigma_1 i_1 \cdots \sigma_n i_n) = \bigvee_{s\in Y} s(\sigma_1 i_1 \cdots \sigma_n i_n) = \perp$. This shows that t is path-finite. Using the same proof as in the second paragraph of 3.1, we see that $t \in T_{\aleph_0}(P)$. Thus $t \in F_{\aleph_0}(P)$, as required.

Before proceeding with the proof that $F_{\aleph_0}(P)$ is the desired continuous algebra, we develop some facts about the way in which $F_{\aleph_0}(P)$ is generated from the elements of P. For each ordinal α, define $W_\alpha \subseteq F_{\aleph_0}(P)$ as follows:

$$W_0 = \{\hat{p} \mid p \in P\} \qquad \text{(c.f. 2.2)},$$
$$W_{\alpha+1} = \{t \in F_{\aleph_0}(P) \mid \partial_{\sigma i}(t) \in W_\alpha \text{ for all } (\sigma, i) \in \tilde{\Sigma}\},$$
$$W_\beta = \bigcup_{\alpha<\beta} W_\alpha \qquad \text{for each limit ordinal } \beta,$$

LEMMA 1. *$W_\alpha \subseteq W_\beta$ whenever $\alpha \leqslant \beta$, each W_α is a down set in $F_{\aleph_0}(P)$ and $F_{\aleph_0}(P) = W_{\mathfrak{n}}$, where $\mathfrak{n} = 2^{\operatorname{card} \tilde{\Sigma}^*}$.*

Proof. In view of the definition of W_β for limit ordinals β, it is enough, for the first statement, to prove that $W_\alpha \subseteq W_{\alpha+1}$ for all α, and for this we proceed by induction on α.

Since for all $p \in P$ and all $(\sigma, i) \in \tilde{\Sigma}$, $\partial_{\sigma i}(p) = \bot \in W_0$ it follows that $W_0 \subseteq W_1$. Also, if $W_\alpha \subseteq W_{\alpha+1}$ then it follows directly from the above definition that $W_{\alpha+1} \subseteq W_{(\alpha+1)+1}$. If α is a limit ordinal and the claim is true for all $\beta < \alpha$, then for each $t \in W_\alpha$ there exists $\beta < \alpha$ with $t \in W_{\beta+1}$; by the above definition this implies that for each $(\sigma, i) \in \tilde{\Sigma}$, $\partial_{\sigma i}(t) \in W_\beta \subseteq W_\alpha$ and consequently $t \in W_{\alpha+1}$. This proves the first part of the lemma.

W_0 is clearly a down set in $F_{\aleph_0}(P)$, and the fact that each W_α is a down set is a straightforward induction.

For the last part of the lemma, we define sets $S_\alpha(t) \subseteq \tilde{\Sigma}^*$ for each ordinal α and each $t \in F_{\aleph_0}(P)$ as follows:

$$S_0(t) = \{a \in \tilde{\Sigma}^* \mid t(a) = \bot\},$$

$$S_{\alpha+1}(t) = \{a \in \tilde{\Sigma}^* \mid a\sigma i \in S_\alpha(t) \text{ for all } (\sigma, i) \in \tilde{\Sigma}\},$$

$$S_\beta(t) = \bigcup_{\alpha<\beta} S_\alpha(t) \qquad \text{for each limit ordinal } \beta.$$

First we prove three technical statements.

I. $\sigma i a \in S_\alpha(t)$ iff $a \in S_\alpha(\partial_{\sigma i}(t))$.

Proof. We proceed by induction on α. The case $\alpha = 0$ is clear, and further

$$\begin{aligned} \sigma i a \in S_{\alpha+1}(t) &\Leftrightarrow \sigma i a \lambda j \in S_\alpha(t) && \text{for all } (\lambda, j) \in \tilde{\Sigma} \\ &\Leftrightarrow a\lambda j \in S_\alpha(\partial_{\sigma i}(t)) && \text{by inductive hypothesis} \\ &\Leftrightarrow a \in S_{\alpha+1}(\partial_{\sigma i}(t)) && \text{by definition of } S_{\alpha+1}. \end{aligned}$$

The case α is a limit ordinal is clear, and this proves the claim.

II. $\varnothing \in S_\alpha(t)$ implies $t \in W_\alpha$ for all $t \in F_{\aleph_0}(P)$.

Proof. If $\alpha = 0$ then $\varnothing \in S_0(t)$ implies $t = \bot \in W_0$. For a limit ordinal α the argument is clear. If the claim is true for α, and $\varnothing \in S_{\alpha+1}(t)$ then $\sigma i \in S_\alpha(t)$ for all $(\sigma, i) \in \tilde{\Sigma}$ and hence by *I*, $\varnothing \in S_\alpha(\partial_{\sigma i}(t))$. By the inductive hypothesis, this implies $\partial_{\sigma i}(t) \in W_\alpha$ for all σi and so $t \in W_{\alpha+1}$ as required.

III. $S_\alpha(t) = S_{\alpha+1}(t)$ implies $\varnothing \in S_\alpha(t)$.

Proof. If $\varnothing \notin S_\alpha(t)$ then $\varnothing \notin S_{\alpha+1}(t)$ and hence there exists $\sigma_1 i_1 \in \tilde{\Sigma}$ with $\sigma_1 i_1 \notin S_\alpha(t)$. Again, $\sigma_1 i_1 \notin S_{\alpha+1}(t)$ and hence there exists $\sigma_2 i_2$ with $\sigma_1 i_1 \sigma_2 i_2 \notin S_\alpha(t)$. Continuing, we find an infinite sequence $\sigma_n i_n \in \tilde{\Sigma}$ with $\sigma_1 i_1 \cdots \sigma_k i_k \notin S_\alpha(t)$ for any k. Since $a \notin S_\alpha(t)$ implies $t(a) \neq \bot$, this contradicts the path-finiteness of t.

Finally, we prove that $F_{\aleph_0}(P) = W_{\mathfrak{n}}$ for $\mathfrak{n} = 2^{\operatorname{card} \tilde{\Sigma}^*}$. For each $t \in F_{\aleph_0}(P)$, clearly $\alpha \leqslant \beta$ implies $S_\alpha(t) \subseteq S_\beta(t) \subseteq \tilde{\Sigma}^*$, and moreover, if $S_\alpha(t) = S_{\alpha+1}(t)$ then $S_\alpha(t) = S_\beta(t)$ for all $\beta \geqslant \alpha$. Thus for each t, the sequence of $S_\alpha(t)$'s must be constant from $\mathfrak{n} = 2^{\operatorname{card} \tilde{\Sigma}^*}$ on, and consequently, $S_{\mathfrak{n}}(t) = S_{\mathfrak{n}+1}(t)$ which by III implies $t \in W_{\mathfrak{n}}$, as required. ∎

THEOREM 2. *Let $\mathbf{Z}$ be a subset system with $\|\mathbf{Z}\| = \aleph_0$ and $\bot \leqslant \mathbf{Z}$. Then the free $\mathbf{Z}$-continuous algebra over a $\mathbf{Z}$-complete p.o. set P is $F_{\aleph_0}(P)$.*

Proof. By Proposition 6, $F_{\aleph_0}(P)$ is a $\mathbf{Z}$-continuous algebra. Suppose $g: P \to A$ is a $\mathbf{Z}$-continuous map into a $\mathbf{Z}$-continuous algebra A. We extend g to $F_{\aleph_0}(P)$ by induction on the W_α as follows: For $t = \hat{p} \in W_0$, define $\bar{g}(t) = g(p)$, so $\bar{g}$ does extend g. For each $t \in W_{\alpha+1} - W_\alpha$ and each $\sigma i \in \tilde{\Sigma}$ we have $\partial_{\sigma i}(t) \in W_\alpha$. We define

$$\bar{g}(t) = g(p) \vee \bigvee_{\sigma \in k} \sigma(\bar{g}\partial_{\sigma i}(t)_{i<|\sigma|}) \qquad \text{where} \quad (p, k) = t(\varnothing).$$

This join exists in A because k is finite and A has all finite joins.

Moreover, since $\bar{g}$ is to be a $\mathbf{Z}$-continuous homomorphism, $\bar{g}$ is clearly uniquely determined by g.

Since $F_{\aleph_0}(P) = \bigcup W_\alpha$, this defines a map $\bar{g}: F_{\aleph_0}(P) \to A$ and a straightforward induction shows that $\bar{g}$ is order preserving. Moreover, $\bar{g}$ is also a homomorphism: If $t = \sigma((t_i)_{i<|\sigma|})$ then $t(\varnothing) = (\bot, \{\sigma\})$ and $\partial_{\sigma i}(t) = t_i$ so that $\bar{g}(t) = g(\bot) \vee \sigma(\bar{g}(t_i)_{i<|\sigma|}) = \sigma(\bar{g}(t_i)_{i<|\sigma|})$. It remains to show that $\bar{g}$ is $\mathbf{Z}$-continuous.

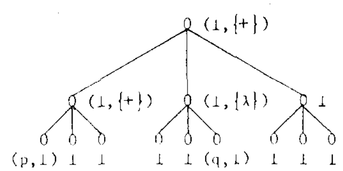
We prove by induction on α that for all $t \in W_\alpha$, if $t = \bigvee Y$ for some $Y \in \mathbf{Z}(F_{\aleph_0}(P))$ then $\bar{g}(t) = \bigvee_{s \in Y} \bar{g}(s)$. If $t \in W_0$ then each $s \in Y$ also belongs to W_0. The map from $F_{\aleph_0}(P)$ to P, sending each tree u with $u(\varnothing) = (p, k)$ to p, is $\mathbf{Z}$-continuous. Hence the set $Y' = \{p \in P \mid \hat{p} \in Y\} \in \mathbf{Z}(P)$. Since g is $\mathbf{Z}$-continuous we have $\bigvee \bar{g}(Y) = \bigvee g(Y') = g(\bigvee Y') = \bar{g}(\bigvee Y)$. If the claim is true for α, and $t \in W_{\alpha+1} - W_\alpha$ with $t(\varnothing) = (p, k)$ and $s(\varnothing) = (p_s, k_s)$ for each $s \in Y$ then the proof is the same as the induction step in the proof of Theorem 1. ∎

3.3. *Case* 3A $\|\mathbf{Z}\| = 2$ *and* $\omega \leqslant \mathbf{Z}$

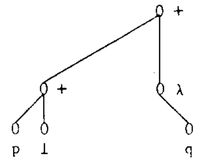
We shall prove that the free $\mathbf{Z}$-continuous algebra is the following subalgebra of $T(P)$:

$$T^0(P) = \{t \in T(P) \mid \text{for all } a \in \tilde{\Sigma}^*, \text{ either } t(a) = (p, \varnothing) \text{ with } p \in P \text{ or }$$
$$t(a) = (\bot, \{\sigma\}) \text{ with } \sigma \in \Sigma\}.$$

These trees are equivalent to the ordinary (infinite) trees considered by Scott [24, 25], ADJ [4], and Taylor and Mycielski [19]. The equivalence is obtained by "pruning extraneous branches," and replacing $(p, \bot)$ by p and $(\bot, \{\sigma\})$ by σ. Extraneous branches in t are just those with root $a\sigma i$, where σ is not in $t(a)$. For example, if Σ consists of one binary operation $+$ and one unary operation λ, then in our trees, every node has exactly three immediate successors, and the first three levels of the tree representing $(p + \bot) + \lambda(q)$ are



and on all subsequent levels all labels are $\perp$. The result of pruning is



THEOREM 3. *Let* $\mathbf{Z}$ *be a subset system with* $\|\mathbf{Z}\| = 2$, $\perp \nleqslant \mathbf{Z}$ *and* $\boldsymbol{\omega} \leqslant \mathbf{Z}$. *The free* $\mathbf{Z}$*-continuous algebra over a* $\mathbf{Z}$*-complete p.o. set* P *is* $T^0(P)$.

Proof. $T^0(P)$ is evidently closed under all operations of $T(P)$. Also, since $\|\mathbf{Z}\| = 2$, it follows that $\{(p, \varnothing) \mid p \in P\} \cup \{(\perp, \{\sigma\}) \mid \sigma \in \Sigma\}$ is closed under formation of $\mathbf{Z}$-joins in $P \times \mathfrak{P}(\Sigma)$ and hence $T^0(P)$ is a $\mathbf{Z}$-continuous algebra, which evidently contains $\hat{p}$ for each $p \in P$. The proof of the extension property for maps is the same as Theorem 1 (Section 3.1): every tree in $T^0(P)$ is an ω-join of trees in $T^0(P)$ of finite height, and in the inductive definition of $\bar{g}$ for trees of height $n+1$ the joins have at most one member distinct from $\perp$, and hence exist trivially. ∎

For these $\mathbf{Z}$, discrete P, and finitary Σ, these free continuous algebras were described in ADJ [3, 4], and for arbitrary P and Σ in Nelson [20].

3.4. *Case* 3B. $\|\mathbf{Z}\| = 2$ *and* $\boldsymbol{\omega} \nleqslant \mathbf{Z}$

Let $\mathfrak{F}(P)$ be the absolutely free (unordered) algebra generated by the elements of P (see Section 1.5). Take the least partial order $\leqslant$ on $\mathfrak{F}(P)$ which

(i) extends the order on P;

(ii) has $\perp$ ($\in P$) as its smallest element;

(iii) is preserved by the operations of $\mathfrak{F}(P)$.

We shall prove that $\mathfrak{F}(P)$, with this order, is actually the desired free continuous algebra, by finding an isomorphic algebra $F^0(P)$ of trees and proving it free.

First, define $F^0(P) = F(P) \cap T^0(P)$. Then $F^0(P)$ is clearly closed under all the operations in $T(P)$, and the same argument as in the proof of Proposition 6 (Section 3.2) shows that $F^0(P)$ is closed under formation of all $\mathbf{Z}$-joins in $T(P)$ and hence is a $\mathbf{Z}$-continuous algebra.

THEOREM 4. *Let* $\mathbf{Z}$ *be a subset system with* $\perp \leqslant \mathbf{Z}$, $\|\mathbf{Z}\| = 2$ *and* $\boldsymbol{\omega} \nleqslant \mathbf{Z}$. *The free* $\mathbf{Z}$*-continuous algebra over a* $\mathbf{Z}$*-complete p.o. set* P *is* $F^0(P)$.

Proof. The proof is analogous to the proof of Theorem 2 (Section 3.2); the role of the W_α is now played by the $W_\alpha \cap T^0(P)$ and the joins appearing in the inductive definition of $\bar{g}$ have at most one member distinct from $\perp$, and hence exist trivially.

THEOREM 5. *Let* $\mathbf{Z}$ *and* P *be as in Theorem* 4; *then* $F(P)$ *is isomorphic, as an ordered algebra, to* $F^0(P)$.

Proof. Let $n = 2^{\text{card}\,\tilde{\Sigma}^*}$ and put $W_\alpha^0 = F^0(P) \cap W_\alpha$ for each ordinal α; then $F^0(P) = W_n^0 = \bigcup W_\alpha^0$ by Lemma 1. Further, $\mathfrak{F}(P) = \bigcup_{\alpha < n} V_\alpha$, where the V_α are the sets defined in Section 1.5.

Define maps $\psi_\alpha\colon V_\alpha \to W_\alpha^0$, so that $\psi_\alpha \subseteq \psi_\beta$ for $\alpha \leqslant \beta$, inductively as follows: $\psi_0(p) = \hat{p}$, $\psi_\beta = \bigcup_{\alpha<\beta} \psi_\alpha$ for a limit ordinal β, and

$$\psi_{\alpha+1}(\sigma((y_i)_{i<|\sigma|})) = \sigma(\psi_\alpha(y_i)_{i<|\sigma|});$$

the latter is a legitimate definition by the Peano property (P2) in Section 1.5. Then a straightforward induction shows that each ψ_α is one-to-one and onto; since $\psi_n = \psi_{n+1}$, ψ_n is an algebraic isomorphism from $\mathfrak{F}(P)$ onto $F^0(P)$.

Finally, the order of $F^0(P)$ has the properties corresponding to (i) to (iii) above: It is the least partial order extending that of P having $\perp$ $(\in P)$ as its smallest element and such that all operations are order-preserving. Hence the order on $\mathfrak{F}(P)$ defined by $x \leqslant y$ iff $\psi_n(x) \leqslant \psi_n(y)$ is the least order on $\mathfrak{F}(P)$ subject to (i) to (iii) above, and this implies that ψ_n is an order-isomorphism.

Remark. We have described above the free **Z**-continuous algebra for all subset systems **Z** with $\perp \leqslant \mathbf{Z}$. The various cases have the following common features: The free **Z**-continuous algebra is a subalgebra of the algebra $T(P)$, and is also a down set in $T(P)$ which is closed under (pointwise) formation of **Z**-joins. Moreover, these subalgebras have a more unified description: The free **Z**-continuous algebra over P consists of those trees $t \in T(P)$ which map $\tilde{\Sigma}^*$ into the **Z**-coproduct $P \amalg \tilde{\Sigma}$ of P and the free **Z**-completion of Σ and which, in case $\omega \not\leqslant \mathbf{Z}$, are path-finite.

4. Free Z-Continuous Algebras when $\perp \not\leqslant \mathbf{Z}$

In this section we assume throughout that all **Z**-sets are non-empty, so that **Z**-continuous algebras need not have bottom elements, and even if they do, homomorphisms need not preserve them. As before, P will be an arbitrary **Z**-complete p.o. set and Σ an arbitrary type of algebras; we will construct the free **Z**-continuous algebra over P.

4.1. *Preliminaries for $\|\mathbf{Z}\|$ Infinite*

We again make case distinctions as in Section 3, but first develop some preliminaries which will be used in both of the first two cases, namely $\|\mathbf{Z}\| > \aleph_0$ and $\|\mathbf{Z}\| = \aleph_0$. Note that in these cases, for D any two-element discrete set, D is not **Z**-complete and hence $D \in \mathbf{Z}(D)$, which implies that all two-element subsets of any p.o. set are **Z**-sets. The appropriate trees will belong to $T(P_\perp)$ and will satisfy:

$$t(\varnothing) \neq \perp,$$
$$t(a) = (p, k) \text{ implies } t(a\sigma i) \neq \perp \qquad \text{for all } \sigma \in k \text{ and } i < |\sigma|. \qquad (\neq)$$

We define, for each cardinal number $\mathfrak{m} \neq 0$,

$$\mathfrak{F}^{\#}_{\mathfrak{m}}(P) = \{t \in \mathfrak{F}_{\mathfrak{m}}(P_{\perp}) \mid t \text{ satisfies } (\#)\}$$

and

$$T^{\#}_{\mathfrak{m}}(P) = \{t \in T_{\mathfrak{m}}(P_{\perp}) \mid t(a) \neq \perp \text{ implies there exists } s \leqslant t \text{ with } s \in \mathfrak{F}^{\#}_{\mathfrak{m}}(P) \text{ and } s(a) \neq \perp\}.$$

We will show that, for $\|\mathbf{Z}\| > \aleph_0$, $T^{\#}_{\|\mathbf{Z}\|}(P)$ is the desired free algebra. To do so, we will first develop some properties of $\mathfrak{F}^{\#}_{\mathfrak{m}}(P)$ for $\mathfrak{m}$ any infinite cardinal (even $\aleph_0$); we shall prove that $F^{\#}_{\aleph_0}(P)$ is itself the free algebra if $\|\mathbf{Z}\| = \aleph_0$.

Although $F^{\#}_{\mathfrak{m}}(P)$ need not be $\mathbf{Z}$-complete, we will prove that it has the homomorphism extension property expected of the free algebra.

To do so, we introduce the analogues of the sets W_α defined in Section 3.2. For each ordinal α, define $W^{\#}_\alpha$ inductively, as follows:

$$W^{\#}_0 = \{\hat{p} \mid p \in P\},$$

$$W^{\#}_{\alpha+1} = \{t \in F^{\#}_{\mathfrak{m}}(P) \mid \partial_{\sigma i}(t) \in W^{\#}_\alpha \text{ for all } (\sigma, i) \in \tilde{\Sigma} \text{ with } (\perp, \{\sigma\}) \leqslant t(\varnothing)\},$$

$$W^{\#}_\beta = \bigcup_{\alpha<\beta} W^{\#}_\alpha \text{ for a limit ordinal } \beta.$$

LEMMA 2. *$W^{\#}_\alpha \subseteq W^{\#}_\beta$ whenever $\alpha \leqslant \beta$, each $W^{\#}_\alpha$ is a down set in $\mathfrak{F}^{\#}_{\mathfrak{m}}(P)$, and $\mathfrak{F}^{\#}_{\mathfrak{m}}(P) = W^{\#}_n$, where $n = 2^{\operatorname{card} \tilde{\Sigma}^*}$.*

Proof. The proof is the same as the proof of Lemma 1 (see Section 3.2). ∎

PROPOSITION 7. *If $\|\mathbf{Z}\| = \mathfrak{m} \geqslant \aleph_0$, then every $\mathbf{Z}$-continuous map of P into a $\mathbf{Z}$-continuous algebra has a unique extension to a $\mathbf{Z}$-continuous homomorphism on $\mathfrak{F}^{\#}_{\mathfrak{m}}(P)$.*

Proof. The proof is similar to that of Theorem 2 (Section 3.2); nevertheless, we exhibit all the details here.

Suppose $g: P \to A$ is $\mathbf{Z}$-continuous and that A is a $\mathbf{Z}$-continuous algebra. We extend g to $\tilde{g}: F^{\#}_{\mathfrak{m}}(P) \to A$ by induction on the $W^{\#}_\alpha$. Define $\tilde{g}(\hat{p}) = g(p)$ for each $p \in P$. For $t \in W^{\#}_{\alpha+1} - W^{\#}_\alpha$ with $t(\varnothing) = (p, k)$, define

$$\tilde{g}(t) = g(p) \vee \bigvee_{\sigma \in k} \sigma(\tilde{g}\partial_{\sigma i}(t)_{i<|\sigma|})$$

if $p \neq \perp$. If $p = \perp$ then drop $g(p)$ from this expression. This join exists in A because card $k < \|\mathbf{Z}\|$ and A is $\mathbf{Z}$-complete.

Since $F^{\#}_{\mathfrak{m}}(P) = \bigcup W^{\#}_\alpha$, this gives a homomorphism $\tilde{g}: F^{\#}_{\mathfrak{m}}(P) \to A$, and a straightforward induction shows that $\tilde{g}$ is order-preserving.

We prove by induction on α that for all $t \in W^{\#}_{\alpha}$, if $t = \bigvee Y$ for some $Y \in \mathbf{Z}(F^{\#}_{\mathrm{m}}(P))$ then $\tilde{g}(t) = \bigvee_{s \in Y} \tilde{g}(s)$.

If $t = \hat{p} \in W^{\#}_0$ then $Y \subseteq W^{\#}_0$; take any $\hat{p}_0 \in Y$. The map from $F^{\#}_{\mathrm{m}}(P)$ into P taking a tree u with $u(\varnothing) = (q, h)$ to the element $q \vee p_0$ is $\mathbf{Z}$-continuous, and hence the set $Y' = \{q \vee p_0 \mid \hat{q} \in Y\}$ is in $\mathbf{Z}(P)$. Since g is $\mathbf{Z}$-continuous, it preserves the join $\bigvee Y' = p$ and thus

$$\tilde{g}(t) = g(p) = \bigvee_{\hat{q} \in Y} g(q) \vee g(p_0) = \bigvee_{q \in Y} g(q) = \bigvee \tilde{g}(Y).$$

It $t \in W^{\#}_{\alpha+1} - W^{\#}_{\alpha}$ and the claim is proved for all joins belonging to $W^{\#}_{\alpha}$ then let $t(\varnothing) = (p, k)$ and $s(\varnothing) = (p_s, k_s)$ for each $s \in Y$. Since joins in $F^{\#}_{\mathrm{m}}(P)$ are formed pointwise it follows that $p = \bigvee_{s \in Y} p_s$, and the argument in the preceeding paragraphs shows that if $p \neq \bot$ then $g(p) = \bigvee g(p_s)$ $(s \in Y, p_s \neq \bot)$. If $k = \varnothing$ then $k_s = \varnothing$ for all $s \in Y$ and the preceeding equality is all that is needed; if $k \neq \varnothing$ then, again because joins in $F^{\#}_{\mathrm{m}}(P)$ are formed pointwise, for each $\sigma \in k$ there exists $s_\sigma \in Y$ with $\sigma \in k_{s_\sigma}$; let $Y_\sigma = \{s \vee s_\sigma \mid s \in Y\}$. Since the map $s \rightsquigarrow s \vee s_\sigma$ is order-preserving, $Y \in \mathbf{Z}(F^{\#}_{\mathrm{m}}(P))$ implies $Y_\sigma \in \mathbf{Z}(F^{\#}_{\mathrm{m}}(P))$, and clearly $\bigvee Y = \bigvee Y_\sigma$ for each $\sigma \in k$. Now, for each $u \in Y_\sigma$, $\partial_{\sigma i}(u)$ satisfies (#); since $\partial_{\sigma i}$ preserve all joins, $\partial_{\sigma i}(t) = \bigvee_{u \in Y_\sigma} \partial_{\sigma i}(u)$ and we may apply the inductive hypothesis to the latter join to obtain

$$\tilde{g}\partial_{\sigma i}(t) = \bigvee_{u \in Y_\sigma} \tilde{g}\partial_{\sigma i}(u).$$

Further, the map $\mathfrak{F}^{\#}_{\mathrm{m}}(P) \to A^{|\sigma|}$ given by $u \rightsquigarrow (\tilde{g}\partial_{\sigma i}(u))_{i < |\sigma|}$ is order-preserving, and hence $\{(\tilde{g}\partial_{\sigma i}(u))_{i < |\sigma|} \mid u \in Y_\sigma\} \in \mathbf{Z}(A^{|\sigma|})$ so the $\mathbf{Z}$-continuity of the operation σ on A implies:

$$\sigma(\tilde{g}\partial_{\sigma i}(t)_{i < |\sigma|}) = \sigma\left(\left(\bigvee_{u \in Y_\sigma} \tilde{g}\partial_{\sigma i}(u)\right)_{i < |\sigma|}\right) = \bigvee_{u \in Y_\sigma} \sigma(\tilde{g}\partial_{\sigma i}(u)_{i < |\sigma|}).$$

All together this yields:

$$\begin{aligned}\tilde{g}(t) &= \bigvee_{\substack{s \in Y \\ p_s \neq \bot}} g(p_s) \vee \bigvee_{\sigma \in k} \bigvee_{u \in Y_\sigma} \sigma(\tilde{g}\partial_{\sigma i}(u)_{i < |\sigma|}) \\ &= \bigvee_{\substack{s \in Y \\ p_s \neq \bot}} \tilde{g}(\hat{p}_s) \vee \bigvee_{\sigma \in k} \bigvee_{s \in Y} \sigma(\tilde{g}\partial_{\sigma i}(s \vee s_\sigma)_{i < |\sigma|}).\end{aligned}$$

Since $\tilde{g}$ is order-preserving it is enough to prove $\tilde{g}(t) \leqslant \bigvee_{s \in Y} \tilde{g}(s)$ and for this it is enough, by the preceeding equality, to prove for each $\sigma \in k$ and each $u \in Y$ that $\sigma(\tilde{g}\partial_{\sigma i}(u \vee s_\sigma)_{i < |\sigma|}) \leqslant \bigvee_{s \in Y} \tilde{g}(s)$.

If $\sigma \in k_u$ then (since binary joins are $\mathbf{Z}$-joins) we can use the inductive hypothesis on the join $\partial_{\sigma i}(u) \vee \partial_{\sigma i}(s_\sigma) = \partial_{\sigma i}(u \vee s_\sigma)$: $W^{\#}_{\alpha}$ is a down-set in $\mathfrak{F}^{\#}_{\mathrm{m}}(P)$ by Lemma 2, and $\partial_{\sigma i}(u) \vee \partial_{\sigma i}(s_\sigma) \leqslant \partial_{\sigma i}(t)$, so $\partial_{\sigma i}(u) \vee \partial_{\sigma i}(s_\sigma) \in W^{\#}_{\alpha}$ and by inductive hypothesis,

$\tilde{g}(\partial_{\sigma i}(u) \vee \partial_{\sigma i}(s_\sigma)) = \tilde{g}\partial_{\sigma i}(u) \vee \tilde{g}\partial_{\sigma i}(s_\sigma)$ for each $i < |\sigma|$. Thus, since σ preserves binary joins we obtain

$$\sigma(\tilde{g}\partial_{\sigma i}(u \vee s_\sigma)_{i<|\sigma|} = \sigma(\tilde{g}\partial_{\sigma i}(u)_{i<|\sigma|}) \vee \sigma(\tilde{g}\partial_{\sigma i}(s_\sigma)_{i<|\sigma|} \leqslant \bigvee_{s \in Y} \tilde{g}(s).$$

If $\sigma \notin k_u$ then $\partial_{\sigma i}(u) = \bot$ for all $i < |u|$ and $\partial_{\sigma i}(u \vee s_\sigma) = \partial_{\sigma i}(s_\sigma)$ and hence we again have the desired inequality. ∎

4.2. *Case* 1 $\|\mathbf{Z}\| > \aleph_0$

Before proving that the desired free continuous algebra is $T^{\#}_{\mathfrak{m}}(\mathbf{Z})$, where $\mathfrak{m} = \|\mathbf{Z}\|$, we need two additional results.

PROPOSITION 8. *For each infinite cardinal* $\mathfrak{m}$, $T^{\#}_{\mathfrak{m}}(P)$ *is closed under the formation of all non-empty pointwise joins in* $T_{\mathfrak{m}}(P_\bot)$, *and each element of* $T^{\#}_{\mathfrak{m}}(P)$ *is a join of an* ω*-chain in* $F^{\#}_{\mathfrak{m}}(P)$.

Proof. The first statement is clear from the definition of (#). For the second claim, let $t \in T^{\#}_{\mathfrak{m}}(P)$. For each $a \in \tilde{\Sigma}^*$ with $t(a) \neq \bot$ choose $s_a \leqslant t$ with $s_a(a) \neq \bot$ and $s_a \in F^{\#}_{\mathfrak{m}}(P)$. Define, for each natural number n a tree t_n as follows:

$$\begin{aligned} t_n(c) &= t(c) && \text{if length } (c) < n \\ &= s_a(c) && \text{if } c = ab \text{ where length } (a) = n \text{ and } t(a) \neq \bot \\ &= \bot && \text{otherwise.} \end{aligned}$$

It is straightforward to check that t_n is path-finite, and clearly $t = \bigvee t_n$. Thus, $t_0 \leqslant t_0 \vee t_1 \leqslant t_0 \vee t_1 \vee t_2 \leqslant \cdots$ provides an ω-chain of path-finite trees with join t.

LEMMA 3. *Let* $\{t_n \mid n \in \omega\}$ *be an* ω*-chain* in $F^{\#}_{\mathfrak{m}}(P)$ *and suppose* $s \in F^{\#}_{\mathfrak{m}}(P)$ *satisfies* $s \leqslant \bigvee_{n\in\omega} t_n$ *(join in* $T^{\#}_{\mathfrak{m}}(P)$*). Then there exists an* ω*-chain* $\{\bar{t}_n \mid n \in \omega\}$ *in* $F^{\#}_{\mathfrak{m}}(P)$ *with* $\bar{t}_n \leqslant t_n$ *for each* $n \in \omega$ *such that* $\bigvee_{n\in\omega} \bar{t}_n \in F^{\#}_{\mathfrak{m}}(P)$ *and* $s \leqslant \bigvee_{n\in\omega} \bar{t}_n$.

Proof. The proof proceeds by induction on α, where $s \in W_\alpha$. Let $s(\varnothing) = (p, k)$ and $t_n(\varnothing) = (p_n, k_n)$ for each n.

If $\alpha = 0$ then $p \neq \bot$ (and $k = \varnothing$) and hence there exists m with $p_m \neq \bot$. For each n, let $\bar{t}_n = t_n$ if $n < m$, and $\bar{t}_n = t_m \vee \hat{p}_n$ if $n \geqslant m$; then $s \leqslant \bigvee \bar{t}_n$ and this join is clearly path-finite.

Suppose $\alpha > 0$. For each $\sigma \in k$ there exists $m_\sigma \in \omega$ with $\sigma \in k_{m_\sigma}$, and hence $\sigma \in k_n$ for all $n \geqslant m_\sigma$. Since $\partial_{\sigma i} s \leqslant \bigvee_{n \geqslant m_\sigma} \partial_{\sigma i} t_n$, for each $i < |\sigma|$, by the inductive hypothesis there is an ω-chain $\bar{t}^{\sigma i}_n \leqslant \partial_{\sigma i} t_n$ for $n \geqslant m_\sigma$ with $\partial_{\sigma i} s \leqslant \bigvee_{n > m_\sigma} \bar{t}^{\sigma i}_n$ and the latter join path-finite. Define $\bar{t}_n$ as follows:

$$\begin{aligned} \bar{t}_n(\varnothing) &= (p_n, k_n) \\ \partial_{\sigma i} \bar{t}_n &= \partial_{\sigma i} t_n && \text{if } n < m_\sigma \\ &= \bar{t}^{\sigma i}_n && \text{if } n \geqslant m_\sigma \text{ for each } \sigma \in k_n, i < |\sigma|. \end{aligned}$$

Clearly $\bar{t}_n \leqslant t_n$, and $s \leqslant \bigvee \bar{t}_n$. Moreover, since $\partial_{\sigma i}(\bigvee \bar{t}_n) = \bigvee \bar{t}_n^{\sigma i}$ is path-finite for every σi, it follows that $\bigvee \bar{t}_n$ is path-finite, as required. ∎

THEOREM 6. *Let* $\mathbf{Z}$ *be a subset system with* $\|\mathbf{Z}\| = \mathfrak{m} > \aleph_0$ *and* $\perp \nleqslant \mathbf{Z}$. *The free* $\mathbf{Z}$*-continuous algebra over a* $\mathbf{Z}$*-complete p.o. set* P *is* $T^{\#}_{\mathfrak{m}}(P)$.

Proof. By Proposition 2, $P_\perp$ is $\mathbf{Z}_\perp$-complete. It follows (see Section 3.1) that $T_{\mathfrak{m}}(P_\perp)$ is a $\mathbf{Z}$-continuous algebra, and then so is $T^{\#}_{\mathfrak{m}}(P)$ by Proposition 8.

Suppose $g: P \to A$ is a $\mathbf{Z}$-continuous map into a $\mathbf{Z}$-continuous algebra A. By Proposition 7 (Section 4.1), there is a $\mathbf{Z}$-continuous homomorphism $\tilde{g}: \mathfrak{F}^{\#}_{\mathfrak{m}}(P) \to A$ extending g. We extend $\tilde{g}$ to a $\mathbf{Z}$-continuous homomorphism $\bar{g}: T^{\#}_{\mathfrak{m}}(P) \to A$ as follows: For each $t \in T^{\#}_{\mathfrak{m}}(P)$, choose an ω-chain $t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots$ in $\mathfrak{F}^{\#}_{\mathfrak{m}}(P)$ with $t = \bigvee_{n\in\omega} t_n$ (by Proposition 8), and define $\bar{g}(t) = \bigvee \tilde{g}(t_n)$. This definition is independent of the choice of the ω-chain $\{t_n\}$; in fact, $\bar{g}(t) = \bigvee \tilde{g}(s)$, the join over all $s \in F^{\#}_{\mathfrak{m}}(P)$ with $s \leqslant t$. This is seen as follows: since $s \leqslant t$, by Lemma 3 there is an ω-chain $\{\bar{t}_n\}$ in $F^{\#}_{\mathfrak{m}}(P)$ with $s \leqslant \bigvee \bar{t}_n \in F^{\#}_{\mathfrak{m}}(P)$ and $\bar{t}_n \leqslant t_n$ for each n, so that, by the $\mathbf{Z}$-continuity of $\tilde{g}$, $\tilde{g}(s) \leqslant \tilde{g}(\bigvee \bar{t}_n) = \bigvee \tilde{g}(\bar{t}_n) \leqslant \bigvee \tilde{g}(t_n) = \bar{g}(t)$.

First, we prove that $\bar{g}$ is a homomorphism: If $t = \sigma((t_i)_{i<|\sigma|})$ then for each $i < |\sigma|$ choose an ω-chain $t_{i0} \leqslant t_{i1} \leqslant \cdots$ of path-finite trees with $\bigvee_{n\in\omega} t_{in} = t_i$. Since σ is ω-continuous, we have $t = \bigvee_{n\in\omega} \sigma((t_{in})_{i<|\sigma|})$, and since the latter is a join of path-finite trees we have:

$$\begin{aligned}
\bar{g}(t) &= \bigvee_{n\in\omega} \tilde{g}(\sigma((t_{in})_{i<|\sigma|})) && \text{by the definition of } \bar{g}\\
&= \bigvee_{n\in\omega} \sigma(\tilde{g}(t_{in})_{i<|\sigma|}) && \text{because } \tilde{g} \text{ is a homomorphism}\\
&= \sigma\left(\left(\bigvee_{n\in\omega} \tilde{g}(t_{in})\right)_{i<|\sigma|}\right) && \text{by the continuity of } \sigma \text{ in } A\\
&= \sigma\left(\bar{g}\left(\bigvee_{n\in\omega} t_{in}\right)_{i<|\sigma|}\right) && \text{by the definition of } \bar{g}\\
&= \sigma(\bar{g}(t_i)_{i<|\sigma|})
\end{aligned}$$

as required.

Next we show that $\bar{g}$ preserves binary joins. Given $s, t \in T^{\#}_{\mathfrak{m}}(P)$, choose ω-chains $\{s_n\}$ and $\{t_n\}$ in $F^{\#}_{\mathfrak{m}}(P)$ with $s = \bigvee s_n$, $t = \bigvee t_n$. Then $\{s_n \vee t_n\}$ is an ω-chain in $F^{\#}_{\mathfrak{m}}(P)$ with join $s \vee t$, hence

$$\begin{aligned}
\bar{g}(s \vee t) &= \bigvee_{n\in\omega} \tilde{g}(s_n \vee t_n)\\
&= \bigvee_{n\in\omega} \tilde{g}(s_n) \vee \tilde{g}(t_n) && \text{because } \tilde{g} \text{ preserves binary joins}\\
&= \bigvee_{n\in\omega} \tilde{g}(s_n) \vee \bigvee_{n\in\omega} \tilde{g}(t_n) = \bar{g}(s) \vee \bar{g}(t).
\end{aligned}$$

Finally, we show that $\bar{g}$ preserves all $\mathbf{Z}$-joins. For this it is enough to prove for all $Y \in \mathbf{Z}(T^{\#}_{\mathfrak{m}}(P))$; and for each path-finite tree $t \in F^{\#}_{\mathfrak{m}}(P)$, that

$$t \leqslant \bigvee Y \text{ implies } \bar{g}(t) \leqslant \bigvee \bar{g}(Y).$$

This then implies that $\bar{g}(\bigvee Y) \leqslant \bigvee \bar{g}(Y)$ and the reverse inequality is trivial. We proceed by induction on α where $t \in W^{\#}_{\alpha}$. For each $s \in Y$, put $s(\varnothing) = (p_s, k_s)$.

If $t \in W^{\#}_0$ then $t = \hat{p}$ for some $p \in P$, and since $\perp \neq \hat{p} \leqslant \bigvee Y$ there exists $s_0 \in Y$ with $p_{s_0} \neq \perp$. The map from $T^{\#}_{\mathfrak{m}}(P)$ to P, sending each u with $u(\varnothing) = (q, h)$ to $q \vee s_0$, is order-preserving, and thus the set $Y' = \{p_s \vee p_{s_0} \mid s \in Y\}$ is in $\mathbf{Z}(P)$. Since g is $\mathbf{Z}$-continuous and $\bigvee Y' = p$ it follows that

$$\bar{g}(t) = g(p) = \bigvee g(Y') = \bigvee_{p_s \neq \perp} g(p_s) \vee g(p_{s_0}) = \bigvee_{p_s \neq \perp} g(p_s) \leqslant \bigvee_{s \in Y} \bar{g}(s).$$

Next, let $t \in W^{\#}_{\alpha+1} - W^{\#}_{\alpha}$ and suppose the claim is true for $W^{\#}_{\alpha}$. We first deal with the case $t = \sigma((t_i)_{i<|\sigma|})$ for some $t_i \in W^{\#}_{\alpha}$ and $\sigma \in \Sigma$. There exists $s_0 \in Y$ with $\sigma \in k_{s_0}$ and then $\partial_{\sigma i}(s_0) \neq \perp$ for all $i < |\sigma|$. Thus for each $i < |\sigma|$ we have an order-preserving map $T^{\#}_{\mathfrak{m}}(P) \to T^{\#}_{\mathfrak{m}}(P)$ given by $u \rightsquigarrow \partial_{\sigma i}(u \vee s_0)$ and we may use the induction hypothesis on the sets $\{\partial_{\sigma i}(s \vee s_0) \mid s \in Y\} \in \mathbf{Z}(T^{\#}_{\mathfrak{m}}(P))$ with joins $\bigvee \{\partial_{\sigma i}(s \vee s_0) \mid s \in Y\} \geqslant t_i$. Thus

$$\begin{aligned}
\bar{g}(t) &= \sigma(\bar{g}(t_i)_{i<|\sigma|}) && \text{because } \bar{g} \text{ is a homomorphism} \\
&\leqslant \sigma\Big(\bigvee_{s \in Y} \bar{g}(\partial_{\sigma i}(s \vee s_0))_{i<|\sigma|}\Big) && \text{by inductive hypothesis} \\
&= \bigvee_{s \in Y} \sigma(\bar{g}\partial_{\sigma i}(s \vee s_0)_{i<|\sigma|}) && \text{by the continuity of } \sigma \text{ on } A \\
&= \bigvee_{s \in Y} \bar{g}(\sigma(\partial_{\sigma i}(s \vee s_0)_{i<|\sigma|})) && \text{because } \bar{g} \text{ is a homomorphism} \\
&\leqslant \bigvee_{s \in Y} \bar{g}(s \vee s_0) && \text{since } u \leqslant \sigma(\partial_{\sigma i}(u)_{i<|\sigma|}) \text{ for any tree } u \text{ with } u(\varnothing) = (\perp, \{\sigma\}) \\
&= \bigvee_{s \in Y} \bar{g}(s) \vee \bar{g}(s_0) && \text{because } \bar{g} \text{ preserves binary joins} \\
&= \bigvee \bar{g}(Y).
\end{aligned}$$

Now for an arbitrary $t \in W^{\#}_{\alpha+1}$ with $t(\varnothing) = (p, k)$, we have, by the definition of $\bar{g}$ (see the proof of Proposition 7), that

$$\bar{g}(t) = \tilde{g}(t) = g(p) \vee \bigvee_{\sigma \in k} \sigma(\tilde{g}\partial_{\sigma i}(t)_{i<|\sigma|}),$$

with possibly $g(p)$ missing. But the discussion for the case $W_0^{\#}$ shows that $g(p) \leqslant \bigvee \bar{g}(Y)$ and the preceeding paragraph shows that $\sigma(\tilde{g}\partial_{\sigma i}(t)_{i<|\sigma|}) = \tilde{g}(\sigma(\partial_{\sigma i}(t)_{i<|\sigma|})) \leqslant \bigvee \bar{g}(Y)$ and altogether this implies $\bar{g}(t) \leqslant \bigvee \bar{g}(Y)$, as required.

Finally, we verify that $\bar{g}$ is unique: its restriction $\tilde{g}$ to $F_{\mathfrak{m}}^{\#}(P)$ is unique by Proposition 7, and since $\bar{g}$, being a $\mathbf{Z}$-continuous map on a $\mathbf{Z}$-complete p.o. set, preserves all joins of cardinality $< \|\mathbf{Z}\|$ and hence in particular all countable joins, and since each element of $T_{\mathfrak{m}}^{\#}(P)$ is a countable join of elements of $F_{\mathfrak{m}}^{\#}(P)$, this uniquely determines $\bar{g}$ as a $\mathbf{Z}$-continuous extension of $\tilde{g}$. ∎

4.3. *Case* 2. $\|\mathbf{Z}\| = \aleph_0$.

THEOREM 7. *Let* $\mathbf{Z}$ *be a subset system with* $\perp \nleqslant \mathbf{Z}$ *and* $\|\mathbf{Z}\| = \aleph_0$. *Then the free* $\mathbf{Z}$*-continuous algebra over a* $\mathbf{Z}$*-complete p.o. set* P *is* $F_{\aleph_0}^{\#}(P)$.

Proof. In view of Proposition 7, it is only necessary to prove that $F_{\aleph_0}^{\#}(P)$ is $\mathbf{Z}$-complete. Since $\|\mathbf{Z}\|$ is infinite, and P is $\mathbf{Z}$-complete, it follows from Proposition 2 (Section 1.2) that $P_\perp$ is $\mathbf{Z}$-complete. Hence $T_{\aleph_0}(P_\perp)$ is $\mathbf{Z}$-complete, and so it is enough to show that $F_{\aleph_0}^{\#}(P)$ is closed under $\mathbf{Z}$-joins in $T_{\aleph_0}(P_\perp)$. The proof that a $\mathbf{Z}$-join of path-finite trees is again path-finite is the same as the proof of the analogous fact in Proposition 6 (Section 3.2), and the preservation of property (#) under formation of joins is clear. ∎

4.4. *Case* 3 $\|\mathbf{Z}\| = 2$

We will see in this case, as in Section 3.4, that the free continuous algebra is the absolutely free algebra $\mathfrak{F}(P)$ (see Section 1.5) generated by the elements of P, with an appropriate order. The technique of proof is different from all other cases; because $\|\mathbf{Z}\|$ is finite, $P_\perp$ need not be $\mathbf{Z}$-complete when P is, and so we cannot use the trees in $T(P_\perp)$ as we did in the first two cases of this section. In fact, we will not use trees at all here.

The order on $\mathfrak{F}(P)$ is defined inductively, as follows:

$$s \leqslant t \text{ iff either } s, t \in P \text{ and } s \leqslant t \text{ in } P, \text{ or there is } \sigma \in \Sigma \text{ with } s = \sigma((s_i)_{i<|\sigma|}) \text{ and } t = \sigma((t_i)_{i<|\sigma|}) \text{ and } s_i \leqslant t_i \text{ for all } i < |\sigma|. \quad (*)$$

Note that if P is discrete then so is $\mathfrak{F}(P)$.

THEOREM 8. *Let* $\mathbf{Z}$ *be a subset system with* $\|\mathbf{Z}\| = 2$ *and* $\perp \nleqslant \mathbf{Z}$. *The free* $\mathbf{Z}$*-continuous algebra over the* $\mathbf{Z}$*-complete p.o. set is* $\mathfrak{F}(P)$.

Proof. Let $\mathfrak{F}(1)$ be the absolutely free algebra over the singleton set $1 = \{0\}$, and let $h: \mathfrak{F}(P) \to \mathfrak{F}(1)$ be the unique homomorphism extending the constant map $P \to 1$. Using the inductive definition (*) of the order on $\mathfrak{F}(P)$ it is easy to see that h is order preserving, which means that

$$s \leqslant t \text{ implies } h(s) = h(t). \quad (**)$$

For each $d \in \mathfrak{F}(1)$, let $U_d = h^{-1}(d)$, as a sub p.o. set of $\mathfrak{F}(P)$; then by (**) $F(P)$ is the disjoint union of the p.o. sets U_d for $d \in \mathfrak{F}(1)$.

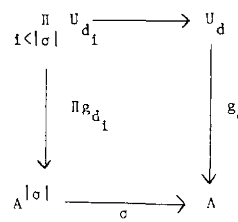
Since h is a homomorphism, it follows that for each $d = \sigma((d_i)_{i<|\sigma|}) \in \mathfrak{F}(1)$, the restriction of the operation σ in $\mathfrak{F}(P)$ provides an order-isomorphism $\prod_{i<|\sigma|} U_{d_i} \to U_d$.

We define, by induction on the complexity of d, a cardinal number $|d|$ and an order-isomorphism $k_d: P^{|d|} \to U_d$ as follows: If $d = 0$ $(\in V_0)$, put $|d| = 1$ and let $k_d: P \to P$ be the identity map. If $d = \sigma((d_i)_{i<|\sigma|})$ then $|d| = \sum_{i<|\sigma|} |d_i|$ (cardinal sum) and $k_d: P^{|d|} \to P$ is the composite of the natural isomorphism $P^{|d|} \to \prod_{i<|\sigma|} P^{|d_i|}$, the map $\prod_{i<|\sigma|} k_{d_i}: \prod_{i<|\sigma|} P^{|d_i|} \to \prod_{i<|\sigma|} U_{d_i}$, and the order-isomorphism $\prod_{i<|\sigma|} U_{d_i} \to U_d$ given by the restriction of σ.

Each $P^{|d|}$ is **Z**-complete (see Remark 2 in 1.4) and so is U_d. Moreover, since $\|\mathbf{Z}\| = 2$ and $\bot \not\leqslant \mathbf{Z}$, it follows as in the proof of Proposition 5(d) (Section 1.4) that any disjoint union of **Z**-complete p.o. sets is **Z**-complete, and from this it follows that $\mathfrak{F}(P)$ is **Z**-complete.

To prove that the operations are **Z**-continuous, it suffices to verify the continuity of the restriction of each $\sigma \in \Sigma$ to $\prod_{i<|\sigma|} U_{d_i}$ for $d_i \in F(1)$, $i < |\sigma|$. But here, σ is an order-isomorphism onto U_d where $d = \sigma((d_i)_{i<|\sigma|}$.

Finally, given a Z-continuous algebra A and a **Z**-continuous $g: P \to A$, there is a unique homomorphism $\bar{g}: F(P) \to A$ extending g, and we must show that it is **Z**-continuous. For this, it suffices to verify the **Z**-continuity of its restriction $g_d: U_d \to A$ for each $d \in F1$, and this is accomplished by induction on the complexity of d. If $d = 0$, $g_d = g$. If $d = \sigma((d_i)_{i<|\sigma|})$ then consider the following commutative diagram:



Here, the top arrow is the restriction of σ to $\prod U_{d_i}$ and hence is an isomorphism. Since $\prod g_{d_i}$ is **Z**-continuous by inductive hypothesis and the operation σ on A is **Z**-continuous, it follows that g_d is also **Z**-continuous, as required. ∎

5. Free Strict Z-Continuous Algebras

Throughout this section we assume $\bot \leqslant \mathbf{Z}$.

Recall that a **Z**-continuous algebra is *strict* iff the operations preserve the join of the empty set as well as the other **Z**-joins, so that $\sigma(\bot, \bot, \bot, \ldots) = \bot$ for each $\sigma \in \sum$. In this section we will describe, for each subset system **Z** with $\bot \leqslant \mathbf{Z}$ the free strict **Z**-

continuous algebra over a **Z**-complete p.o. set P (see Section 1.6). The free algebra will again consist of trees t from $T(P)$, now satisfying the additional restriction:

(s) if $t(a) = (p, k)$ then for each $\sigma \in k$ there exists $i < |\sigma|$ and $b \in \tilde{\Sigma}^*$ with $t(a\sigma ib) = (q, h)$ where $q \neq \bot$.

Thus, we define

$$T^s(P) = \{t \in T(P) \mid t \text{ satisfies } (s)\}.$$

The operations in $T^s(P)$ are defined exactly as they are in $T(P)$, except that for each $\sigma \in \Sigma$, if $t_i = \bot$ for all $i < |\sigma|$ then $\sigma((t_i)_{i<|\sigma|}) = \bot$.

Recall that in Section 3, the free **Z**-continuous algebra was in each case a subalgebra of $T(P)$; here the free strict **Z**-continuous algebra will be the analogous subalgebra of $T^s(P)$. Note that if $V \subseteq T(P)$ is closed under the operations in $T(P)$ then $V \cap T^s(P)$ is closed under the operations in $T^s(P)$.

THEOREM 9. *Let* **Z** *be a subset system with* $\bot \leqslant \mathbf{Z}$, *and let* $V \subseteq T(P)$ *be the free Z-continuous algebra over P described in Chapter* 3. *Then* $V \cap T^s(P)$ *is the free strict Z-continuous algebra over P.*

Proof. Since V is closed under the formation of **Z**-joins in $T(P)$, (see the Remark at the end of Section 3) it follows that $V \cap T^s(P)$ is closed under formation of **Z**-joins in $T^s(P)$. Moreover, the natural embedding $e\colon V \cap T^s(P) \to V$ is **Z**-continuous and hence $V \cap T^s(P)$ is a **Z**-continuous algebra.

Let $g\colon P \to A$ be a **Z**-continuous map into a strict **Z**-continuous algebra A. Then A is a **Z**-continuous algebra and hence there exists a **Z**-continuous homomorphism $\bar{g}\colon V \to A$ extending g. Let $\bar{\bar{g}} = \bar{g}e\colon V \cap T^s(P) \to A$ be the restriction of $\bar{g}$ to $V \cap T^s(P)$. Since the natural embedding e is **Z**-continuous, $\bar{\bar{g}}$ is also **Z**-continuous. To prove that $\bar{\bar{g}}$ is a homomorphism it is sufficient to show that $\bar{\bar{g}}(\sigma(\bot, \bot, \bot,...)) = \sigma(\bar{\bar{g}}(\bot), \bar{\bar{g}}(\bot),...)$, but since the operations in both $V \cap T^s(P)$ and A are strict, this follows from the fact that $\bar{\bar{g}}(\bot) = \bot$.

Finally, we prove the uniqueness of g. Define a map $f\colon V \to V \cap T^s(P)$ as follows: given a tree $t \in V$ and $a \in \tilde{\Sigma}^*$ with $t(a) = (p, k)$, put $f(t)(a) = (p, \bar{k})$, where

$$\bar{k} = \{\sigma \in k \mid \text{for some } a\sigma ib \in \tilde{\Sigma}^*,\ t(a\sigma ib) = (q, h) \text{ with } q \neq \bot\}.$$

It is easy to see that $f(t)$ is really a tree in $V \cap T^s(P)$ and that $f(t) = t$ for each $t \in V \cap T^s(P)$; moreover f is clearly a **Z**-continuous homomorphism.

Let $\bar{\bar{g}}'\colon V \cap T^s(P) \to A$ be a **Z**-continuous homomorphism extending g. Then $\bar{\bar{g}}f$ and $\bar{\bar{g}}'f$ are both **Z**-continuous homomorphisms from V to A extending g, and hence are equal. Since $f(t) = t$ for $t \in V \cap T^s(P)$ it follows that $\bar{\bar{g}} = \bar{\bar{g}}'$, as required. ∎

REFERENCES

1. J. ADÁMEK, Construction of free ordered algebras, to appear.
2. ADJ (= J. A. GOGUEN, J. W. THATCHER, E. G. WAGNER, AND J. B. WRIGHT), Some fundamentals of order-algebraic semantics, *in* "Proceedings, Symp. Math. Found. of Comp. Sci. Gdansk, Poland, Sept. 1976," Lecture Notes in Computer Science No. 45, pp. 153–168, Springer-Verlag, Berlin/New York, 1976.
3. ADJ, "Initial algebra semantics and continuous theories, *J. Assoc. Comput. Mach.* **24** (1977), 68–95.
4. ADJ, "Free Continuous Theories," IBM Res. Report 6909, Yorktown Heights, 1977.
5. ADJ. A uniform approach to inductive posets and inductive closure, *in* "Math Foundations of Computer Science" (J. Gruska, Ed.), Lecture Notes in Computer Science No. 53, Springer-Verlag, Berlin/New York, 1977; also appeared in *Theoret. Comput. Sci.* **7** (1978), 57–77.
6. M. A. ARBIB AND E. MANES, "Partially Additive Categories and Computer Semantics," Rep. 78-12, Univ. of Mass., Amherst, 1978.
7. A. ARNOLD AND M. NIVAT, Metric interpretations of recursive program schemes, *in* "Proceedings, 2nd Workshop on Categorical and Algebraic Methods in Computer Science, pp. 11–22, Dortmund, 1978.
8. B. BANASCHEWSKI AND E. NELSON, Completions of Partially Ordered Sets, *SIAM J. Comput.*, to appear.
9. S. L. BLOOM, Varieties of ordered algebras, *J. Comput. System Sci.* **13** (1976), 200–212.
10. I. GUESSARIAN, On continuous completions, *in* "Proceedings 4th GI," Lecture Notes in Computer Science No. 67, pp. 142–152, Springer-Verlag, Berlin/New York, 1979.
11. G. JARZEMBSKI, Algebras in the category of p.o. sets, Preprint 1, Copernicus Univ., Torun, 1978.
12. D. LEHMAN, On the algebra of order, manuscript, Mathematics Dept., Hebrew U; Jerusalem, 1979.
13. D. J. LEHMANN AND M. B. SMYTH, Data types, *in* "Proceedings, 18th Ann. Symp. F.O.C.S.," pp. 7–12, I.E.E.E., New York, 1977.
14. G. MARKOWSKY, Chain-complete p.o. sets and directed sets with applications, *Algebra Universalis* **6** (1976), 53–68.
15. G. MARKOWSKY, Categories of chain-complete p.o. sets, *Theoret. Comput. Sci.* **4** (1977), 125–135.
16. G. MARKOWSKY AND B. ROSEN, Bases for chain complete p.o. sets. *IBM J. Res. Develop.* **20** (1976), 138–147.
17. J. MESEGUER. Ideal monads and Z-p.o. sets, Manuscript, Math. Dept., Berkeley.
18. J. MESEGUER, On order-complete universal algebra and enriched functorial semantics, *in* "Fundamentals of Computation Theory," Lecture Notes in Computer Science No., **56**, pp. 294–301, Springer-Verlag, Berlin/New York, 1977.
19. J. MYCIELSKI AND W. TAYLOR, A compactificiation of the algebra of terms, *Algebra Universalis* **6** (1976), 159–163.
20. E. NELSON, Free Z-continuous algebras, *in* "Proceedings, Workshop on Continuous Lattices," Lecture Notes in Mathematics No. **871**, pp. 315–334, Springer-Verlag, Berlin/New York, 1981.
21. J. REYNOLDS, Notes on a lattice-theoretic approach to the theory of computation, manuscript, Syracuse University, 1972.
22. J. REYNOLDS, Semantics of the domain of flow diagrams, *J. Assoc. Comput. Mach.* **24** (1977), 484–503.
23. D. SCOTT, Outline of a math. theory of computation, *in* "Proceedings 4th Ann. Princeton Conf. on Inf. Sciences and Systems, 1969," pp. 169–176.
24. D. SCOTT, The lattice of flow diagrams, *in* "Semantics of Algorithmic Languages," Lecture Notes in Mathematics No. 188, pp. 311–366, Springer-Verlag, Berlin/New York, 1971.
25. D. SCOTT, Data types as lattices, *SIAM J. Comput.* **5** (1976), 522–587.
26. D. SCOTT, Lattice theory, data types and semantics, *in* "Formal Semantics of Programming Languages," Courant Comp. Sci. Symp. 2, Sept. 1970 (R. Rustin, Ed.), Prentice–Hall, Engelwood Cliffs, N. J., 1972.

27. M. B. SMYTH AND G. D. PLOTKIN, The category-theoretic solution of recursive domain equations, *in* "Proceedings 18th Ann. Symp. F.O.C.S.," pp. 13–17 I.E.E.E., New York, 1977.
28. J. TIURYN, Fixed points and algebras with infinitely long expressions, *Fund. Inform*, in press.
29. M. WAND, Fixed-point constructions in order-enriched categories, *Theoret. Comput. Sci.* **8** (1979), 13–30.