On Automorphisms and Derivations of
Cayley–Dickson Algebras

PAUL EAKIN AND AVINASH SATHAYE

Department of Mathematics, University of Kentucky,
Lexington, Kentucky 40506

Communicated by Melvin Hochster
Received September 4, 1987

In [D], Dickson gave a construction which derived the Cayley algebra
from the quaternions in a manner analogous to the construction of the
complex numbers from the reals. Albert [A] adapted this to derive, from
any base field F, sequences of algebras with involution, $A_0 = F < A_1 < \ldots$, such that

(i) $A_{i+1} = A_i + A_i u_i$,

(ii) $(a + bu_i) = \bar{a} - bu_i$,

(iii) $(a + bu_i)(c + du_i) = (ac + \gamma_i u_i \bar{d}b) + (bc + da)u_i$,

where $(\gamma_i)$ is a sequence of nonzero elements of F called the sequence of structure constants. Albert called these Cayley–Dickson algebras. The construction, extended to general initial algebras $A_0$, is known as the Cayley–Dickson process [S1, p. 451 and the algebras $A_n$ that it generates are known as generalized Cayley–Dickson algebras of order n.

In the “classical case”, $A_0$ is the real numbers and $\gamma_i = -1$ for each i. Then $A_1$ is the complex numbers, $A_2$ the quaternions, and $A_3$ the Cayley numbers. We are concerned here with the case when $A_0$ is a field, F, of characteristic other than 2 or 3, its involution is the identity function, and $(\gamma_i)$ is an arbitrary sequence of nonzero elements in F. For $n \geq 3$ these Cayley–Dickson algebras are not associative but they do satisfy the flexible law $(x(yx) = (xy)x)$ for each pair $x, y \in A_n$ [S2]. This can be stated as saying that the associator $\langle x, y, z \rangle$ vanishes identically. The strong non-associativity conjecture of P. Yiu [YIU] asserts that the flexible law is the only identity of this type. It says that for $x, z \in A_n$, the associator $\langle x, y, z \rangle$ vanishes identically if and only if 1, $x$, and $z$ are linearly dependent over F. We prove a special case of this conjecture (Proposition 1.5) which we then
apply to give a proof of a theorem of Schafer [S2] on the derivations of Cayley–Dickson algebras (Theorem 1.6). This result is the principal tool in Section 2, which is devoted to the calculation of the automorphism groups of Cayley–Dickson algebras. We prove (Corollary 2.5) that if $F$ is a field of characteristic other than 2 or 3 and $n \geq 3$ then $\text{Aut}_F(A_{n+1}) \cong \text{Aut}_F(A_n) \oplus G$, where $G \cong \mathbb{Z}/2$ if $\sqrt{-3\gamma_{n+1}^{-1}} \notin F$ and $G \cong S_3$ otherwise. Jacobson [J] has calculated the automorphisms of the algebras $A_3$. Thus this gives a complete description of $\text{Aut}_F(A_n)$, which had been conjectured by Brown [B]. (He proved it for $n=4$). Finally, we give an observation (Corollary 2.6) concerning when two Cayley–Dickson algebras $A_n$ and $A_n'$ of order $n$ (with possibly different structure constants $(\gamma_i)$ and $(\gamma_i')$) are isomorphic as $F$-algebras. The conclusion is that this holds precisely when $A_1$ and $A_1'$ are isomorphic as $F$-algebras and $\gamma_i' \gamma_i^{-1}$ is a square in $F$ for $i \geq 4$.

Since the isomorphisms of Cayley–Dickson algebras of order at most 3 are already well-known [S1, p. 701], this gives a complete classification of (finite dimensional) Cayley–Dickson algebras, which had also been conjectured by Brown [B].

0. Preliminaries

Let $F$ be a base field of characteristic other than 2 and let $\Gamma = (\gamma_i)$ be a sequence in $F$ called the sequence of structure constants. It is assumed that $\gamma_i \neq 0$ for any $i$. Consider $F$ as an $F$-algebra $A_0^\Gamma$ with involution $(-)$ given by the identity map. The Cayley Dickson process [S1] then inductively generates a sequence $A_i^\Gamma$ of algebras with involution defined as

$$A_{i+1}^\Gamma = A_i^\Gamma \times A_i^\Gamma$$

with its inherited $F$-linear structure, its involution defined by

$$(a, b) = (\bar{a}, -b),$$

and its multiplication defined by

$$(a, b)(c, d) = (ac + \gamma_i \gamma_{i+1} \bar{d}b, \bar{b}c + da).$$

The algebra $A_n^\Gamma$ is a $2^n$ dimensional vector space over $F$ with a standard basis inductively defined as

$$e_0^\Gamma = 1.$$

If $\{e_i^\Gamma\}_{i=0}^{2^n-1}$ is the basis defined for $A_{n-1}^\Gamma$, then for $2^{n-1} \leq q < 2^n - 1$,

$$e_i^\Gamma = (0, e_{a^\Gamma}^\Gamma),$$

where $\{e_i^\Gamma\}_{i=0}^{2^n-1}$ are a set of algebra generators for $A_n^\Gamma$ over $F$. In
CAYLEY–DICKSON ALGEBRAS

265

general if \( A \subset A^\Gamma = \bigcup_{i=0}^\infty A_i^\Gamma \) and \( B \) is a subalgebra of \( A^\Gamma \), \( B \llcorner A \rrcorner \) denotes the subalgebra of \( A^\Gamma \) generated over \( B \) by \( A \). Thus \( A_n^\Gamma = F\llcorner \{e_i^\Gamma\}_{i=0}^{n+1} \rrcorner \) and \( A_{n+1}^\Gamma = A_n^\Gamma \llcorner e_{n+1}^\Gamma \rrcorner \).

If \( A = (\gamma_{\phi(i)}) \) is a subsequence of \( \Gamma \) then there is a natural injection

\[ A^A \to A^\Gamma \]

given by

\[ e_{2^\phi(i)}^A \mapsto e_{2^\phi(i)}^\Gamma \]

which is an algebra isomorphism of \( A^A \) onto the algebra \( F\llcorner \{e_{2^\phi(i)}\} \rrcorner \). In the particular case where \( A = (\gamma_d, \gamma_{d+1}, \ldots) \) for a fixed integer \( d \), we denote this algebra by \( dA^\Gamma \).

In the sequel we assume that the field \( F \) and the sequence \( \Gamma \) are fixed and that the identifications described above are understood. Hence we drop the symbol “\( \Gamma \)” from our notation. We will generally be working in \( A_n \) for some fixed \( n \geq 1 \). When this is understood we set \( A_n = A \) and \( A_{n-1} = B \). Then \( \{e_i, i = 0, \ldots, 2^n - 1\} \) is a standard basis for \( B \) and the \( F \)-linear mapping \( (-): B \to B \) satisfies:

\begin{enumerate}
  \item \( \overline{e_0} = e_0 \)
  \item \( \overline{e_i} = -e_i \) for \( i > 0 \)
  \item \( e_i e_j = e_j e_i \).
\end{enumerate}

\( A \) is the \( F \)-vector space \( B + Be \) where for each pair \( x, y \) in \( B \), the new element \( e = e_{2^n-1} \) satisfies

\begin{enumerate}
  \item \( (xe)(ye) = \gamma(\bar{y}x) \), where \( \gamma = \gamma_n = ee \in F \)
  \item \( (xe) y = (x\bar{y})e \)
  \item \( x(ye) = (yx)e \) for each pair \( x, y \) in \( B \).
\end{enumerate}

If \( n > 1 \) then \( e_{2^n-2} \) will be denoted by \( f \).

For \( a \in A \) the (unique) representation \( a = b_1 + b_2 e \) with \( b_i \in B \) is called the standard decomposition of \( a \). The standard basis for \( B \) is extended to that for \( A \) by

\[ e_i \gamma_{n-1} = e_i e \quad \text{for} \quad 0 < i < 2^n - 1 \]

and conjugation extended so that (i)–(iii) continue to hold.

We will refer to the elements \( e_i \) as the basic elements of \( A \). An element \( a = \sum_{i=0}^\infty \lambda_i e_i \) is pure if \( \lambda_0 = 0 \) (equivalently, if \( a + \bar{a} = 0 \)). We denote the set of pure elements of \( A \) by \( \overline{A} \). For \( 1 \leq d < n \), \( A_d \) is the subalgebra \( F\llcorner e_{\gamma_0}, \ldots, e_{\gamma_{d-1}} \rrcorner \) and \( dA \) is the subalgebra \( F\llcorner e_{2d}, e_{2d-1}, \ldots \rrcorner \). Moreover, for any basic \( e_i \in A \) there are unique basic elements \( e_j \in dA \) and \( e_k \in A_d \) such
that $e_i = e_k e_j$. (In fact, $k$ is simply the residue of $i$ mod $2^d$.) Thus each
element of $A$ has a unique representation $\sum \lambda_{kj} x_k \beta_j$, where $x_k$ and $\beta_j$
are basic elements of $A_d$ and $aA$, respectively, and $\lambda_{kj} \in F$. We call this the level
d expansion of $A$.

There is a natural dot product defined on $A$ by

$$
e_i \cdot e_j = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j = 0 \\
\prod_{r=1}^{l} \left( -\gamma_{i_r} + 1 \right) & \text{if } 0 \neq i = j = 2^{h_1} + \cdots + 2^{h_l}.
\end{cases}
$$

Equivalently,

$$\quad a \cdot b = \frac{1}{2}(ab + ba).$$

In the classical case this is the usual Euclidean inner product.

We use the following standard notation:

(1) The commutator: $[x, y] = xy - yx$

(2) The associator: $\langle x, y, z \rangle = (xy)z - x(yz)$

(3) The norm: $|x|^2 = x\bar{x} = 2(x \cdot 1)x - x^2$

(4) $x \cdot 1 = \frac{1}{2}(x + \bar{x})$.

The following are standard identities which we will use repeatedly:

(5) $ab \cdot z = a \cdot zb = b \cdot \bar{a}z$

(6) $|a + b|^2 = |a|^2 + |b|^2 + 2(a \cdot b)$

(7) $ab + ba = 2((b \cdot 1)a + (a \cdot 1)b - a \cdot b)$. 

Equation (5) is a statement of the fact that left (resp. right) multiplication
by $\bar{x}$ is the adjoint of left (right) multiplication by $x$ [S2]. To derive (6)
set $x = a + b$ in (3) to get $|a + b|^2 = |a|^2 + |b|^2 + ab + \bar{a}b = |a|^2 + |b|^2 + 2(a \cdot b)$. The last equality follows from the above description of the dot
product. Identity (7) is similarly derived by substituting $a + b$ for $x$ in the
second equality of (3) and applying (6) and (3).

In general, the norm is not multiplicative for $n > 3$. However, there are
pairs of subspaces on which it is. The following gives some trivial, though
very useful, examples.

If $u = \sum \alpha_i e_i$ is a standard decomposition, then by $\text{Supp}(u)$ we denote
$\{i \mid \alpha_i \neq 0\}$. Let $S$ and $T$ be two sets of nonnegative integers. We call $S$
and $T$ extremely disjoint if $e_{s_1} e_{t_1}$ and $e_{s_2} e_{t_2}$ are linearly independent over
$F$ for distinct $(s_1, t_1), (s_2, t_2) \in S \times T$. The following is verified by a
straightforward expansion:

(8) If $\text{Supp}(u)$ and $\text{Supp}(v)$ are extremely disjoint then $|uv|^2 = |vu|^2 = |u|^2 \cdot |v|^2$. In particular, this is the case if $u \in A_m$ and $v \in mA$. 

1. Nonassociativity

The algebras $A_i$ are not commutative when $i \geq 2$ ($[e_1, e_2] = 2e_1e_2 = 2e_3$) and not associative for $i \geq 3$ ($\langle e_1, e_2, e_4 \rangle = [e_1, e_2]e_4 = 2e_7$). However, as Schafer noted [S2], they are flexible (i.e., $\langle x, y, x \rangle = 0$ always holds). This can be seen, for instance by observing that $\langle x, y, x \rangle$ is pure ($\langle x, y, x \rangle \cdot 1 = (xy)x \cdot 1 - x(yx) \cdot 1 = -yx \cdot x + xy \cdot x = y \cdot (\bar{x})^2 - y \cdot (\bar{x})^2$) and observing $-\langle x, y, x \rangle = \langle x, y, x \rangle = -\langle \bar{x}, \bar{y}, \bar{x} \rangle = \langle x, y, x \rangle$.

Yiu [YIU] has conjectured that if $n > 3$ then $\langle x, -z, z \rangle$ can vanish identically if and only if $1, x, \text{ and } z$ are linearly dependent over $F$. In this section we will prove a special case of the conjecture and apply it to prove a theorem of Schafer [S2] concerning the derivations of the algebras $A_i$.

**Lemma 1.1.** Let $y \in A$ such that $\langle x_1, y, x_2 \rangle = 0$ for all $x_1, x_2 \in A$. If $B$ is noncommutative (i.e., $n \geq 3$), then $y \in F$.

**Proof.** Write the standard decomposition $y = y_1 + y_2e$ and note the identities:

1. $\langle x_1, y_1 + y_2e, e \rangle = \gamma[x_1, y_2] - \gamma[y_1, x_1]e$ if $x_1 \in B$
2. $\langle x_1, y_1 + y_2e, x_2 \rangle = \langle x_1, y_1, x_2 \rangle - ((y_2x_2)x_1 - (y_2x_1)x_2)e$ if $x_1, x_2 \in B$.

From (1) it follows that $y_1, y_2 \in F$, the center of $B$. Now (2) implies that $y_2[x_2, x_1]e = 0$ for all $x_1, x_2 \in B$. Since $B$ is not commutative, $y_2 = 0$. ■

**Lemma 1.2.** Let $y = y_1 + y_2e$ with $y_1, y_2 \in B$. In general, if $y_1, y_2 \in F$ then $\langle x, y \rangle = 0$ for all $x \in A$. Conversely, if $B$ is nonassociative ($n \geq 4$) and $\langle x, y \rangle = 0$ for all $x \in A$ then $y_1, y_2 \in F$. If $n \leq 3$ then $\langle x, x, y \rangle = 0$ for every $y \in A$.

**Proof.** If $x = x_1 + x_2e$ with $x_1, x_2 \in B$ then we have the identity

1. $\langle x, y \rangle = \gamma(x_1, x_1, y_1) + \gamma(x_2, x_2, y_1) - \gamma(x_1, y_2, x_2) + (\gamma(x_1, y_2, y_2) - \gamma(x_1, x_2, y_2) - \gamma(x_1, y_1, x_2))e$.

The first statement is now obvious. If $B$ is associative (i.e., $n \leq 3$) then again this implies $\langle x, y \rangle = 0$ for all $x, y$. Now we proceed by induction on $n$. If $n = 4$, then by the above, (1) reduces to

2. $\langle x, y \rangle = -\langle x_1, y_2, x_2 \rangle - \langle x_1, y_1, x_2 \rangle e$.

It follows that $y_1, y_2$ satisfy the conditions of Lemma 1.1 and hence $y_1, y_2 \in F$ as required.

Now let $n > 4$. Taking $x = x_1 \in B$ it follows that $y_1, y_2 \in B$ satisfy

$\langle x, x, y_1 \rangle = \langle x, x, y_2 \rangle = 0$ for all $x \in B$. 


By the induction hypothesis on $B$, we get that $y_1, y_2$ are both in the vector space $F1 + Ff$ and again (1) reduces to (2).

As before we get $y_1, y_2 \in F$ by Lemma 1.1.

**Lemma 1.3.** Let $D : A \to A$ be a derivation. Then we have:

(i) $Dx \cdot 1 = Dx \cdot x = 0$ for all $x \in A$.

(ii) $Dx \cdot y + x \cdot Dy = 0$ for all $x, y \in A$.

(iii) $De = 0$ if $n \geq 4$.

**Proof.** Recall that

(1) $x^2 - 2(x \cdot 1)x + |x|^2 = 0$ for all $x \in A$.

Taking derivatives

(2) $xDx + (Dx)x - 2(x \cdot 1) Dx = 0$.

The identity (7) in Section 0 yields

(3) $2((x \cdot 1) Dx + (Dx \cdot 1)x - Dx \cdot x) - 2(x \cdot 1) Dx = 0$.

Thus

(4) $((Dx \cdot 1)x - Dx \cdot x) = 0$.

If $x \in F$ then (i) is evident; while if $x \notin F$ then 1, $x$ are linearly independent over $F$ and (i) follows from (4).

We easily deduce (ii) by linearizing the equality $Dx \cdot x = 0$.

Consider the identity

(5) $D\langle x, x, y \rangle = \langle Dx, x, y \rangle + \langle x, Dx, y \rangle + \langle x, x, Dy \rangle = \langle x + Dx, x + Dx, y \rangle - \langle x, x, y \rangle - \langle Dx, Dx, y \rangle + \langle x, x, Dy \rangle$.

Taking $y = e$ and using Lemma 1.2, reduce (5) to

$0 = \langle x, x, De \rangle$ for all $x \in A$.

Thus $De$ satisfies the conditions of Lemma 1.2 and $De = a + be$, with $a, b \in F$. Now (i) implies $De \cdot 1 = a = 0$, $De \cdot e = -by = 0$. Hence $De = 0$.

**Lemma 1.4.** Let $x, y \in A$ such that $[x, y] = xy - yx = 0$. Then the vector space $V$ spanned by $\{1, x, y\}$ is a commutative algebra and any two elements $a, b \in V$ satisfy

$ab = (a \cdot 1)b + (b \cdot 1)a - a \cdot b$.

**Proof.** Using $xy = yx$ in identity (7) of Section 0, we get

$xy = yx = (x \cdot 1)y + (y \cdot 1)x - x \cdot y$
This proves that \( V \) is an algebra; it is obviously commutative. The conclusion for \( a, b \) follows as for \( x, y \).

**Proposition 1.5** (Special case of Yiu's strong nonassociativity conjecture). Let \( n \geq 3 \) and \( x, z \in A \) such that

\[
\langle x, y, z \rangle = 0 \quad \text{for all } y \in A.
\]

Write the standard decomposition \( z = z_1 + z_2 e, x = x_1 + x_2 e \) with \( z_1, z_2, x_1, x_2 \in B \). Let \( \tilde{z}_1 = z_1 - (z_1 \cdot 1) \) and \( \tilde{z}_2 = z_2 - (z_2 \cdot 1) \) be the "purifications." Moreover, assume that one of the following holds:

\[
(**) \quad z_2 = 0 \quad \text{and either } z_1 \in F \text{ or } |\tilde{z}_1|^2 \neq 0
\]

\[
(**') \quad z_1 = 0 \quad \text{and either } z_2 \in F \text{ or } |\tilde{z}_2|^2 \neq 0.
\]

Then the vector space spanned by \( \{1, x, z\} \) is a commutative algebra of dimension \( \leq 2 \).

**Proof.** Note that \( n \geq 3 \) implies that the algebra \( B \) is noncommutative and has trivial center. Consider first the case \( z_2 = 0 \). We may assume that \( z = z_1 \notin F \), since otherwise there is nothing to prove. We may assume \( z_1 \) to be pure (i.e., replace \( z_1 \) by \( \tilde{z}_1 \)), since this affects neither the conclusion nor the hypothesis. Taking \( y = p \) or \( pe \) with \( p \in B \) and calculating the expression in \((*)\) we derive

\[
(1) \quad x_1 (pz_1) - (x_1 p) z_1 = 0
\]

\[
(2) \quad x_2 (z_1 \tilde{p}) - (x_2 \tilde{p}) z_1 = 0
\]

\[
(3) \quad (z_1 \tilde{p}) x_2 - (\tilde{p} x_2) z_1 = 0
\]

\[
(4) \quad (pz_1) x_1 - (px_1) z_1 = 0.
\]

Setting \( p = 1 \), we get from \((3)\) and \((4)\)

\[
(5) \quad z_1 x_2 = x_2 z_1, \quad z_1 x_1 = x_1 z_1.
\]

Taking \( p = z_1 \) (and remembering \( \tilde{z}_1 = -z_1 \)) we get from \((1)\), \((2)\)

\[
(6) \quad -x_1 |z_1|^2 - (x_1 z_1) z_1 = 0
\]

\[
(7) \quad x_2 |z_1|^2 + (x_2 z_1) z_1 = 0.
\]

Applying Lemma 1.4 to pairs \( z_1, x_2 \) and \( z_1, x_1 \) and then expanding \((6)\), \((7)\), we get

\[
(8) \quad x_1 |z_1|^2 - (x_1 \cdot z_1) z_1 - (x_1 \cdot 1) |z_1|^2 = 0
\]

\[
(9) \quad x_2 |z_1|^2 - (x_2 \cdot z_1) z_1 - (x_2 \cdot 1) |z_1|^2 = 0.
\]

Since \( |z_1|^2 \neq 0 \), \((9)\) implies that \( \{1, x_2, z_1\} \) are linearly dependent. By successive reductions we will prove that \( x_2 \) has to be zero. This will make \( x = x_1 \) and \( z = z_1 \) dependent upon \( 1 \) by virtue of \((8)\) and complete the case \( z_2 = 0 \).
Case 1. \( x_2 \in F \).
In this case (2) forces \( z_1 \in F \), the center of \( B \), a contradiction since \( z_1 \) is pure and nonzero.

Case 2. \( x_2 \notin F \) and \((x_2 \cdot 1) \neq 0\).
Write \( z_1 = \lambda x_2 + \mu \) with \( \lambda, \mu \in F \) and \( \lambda \) necessarily nonzero. Substituting in (2), (3) and replacing \( p \) by \( \bar{p} \), we get

\[
(10) \quad x_2(x_2p) = (x_2p)x_2 = (px_2)x_2 \quad \text{for all } p \in B.
\]

Using the flexible law on the middle term we deduce that \( x_2 \) commutes with \( x_2p \) as well as \( px_2 \) and hence with \( x_2p + px_2 = 2(p \cdot 1)x_2 + 2(x_2 \cdot 1)p - 2(x_2 \cdot p) \).

Thus, if \((x_2 \cdot 1) \neq 0\) then \( x_2 \) commutes with all \( p \in B \) and \( x_2 \in F \), a contradiction. Thus it is enough to derive a contradiction in

Case 3. \( x_2 \) is pure and nonzero.
Note that since \( x_2 \) is pure, \( |x_2|^2 \neq 0 \), as we now have \( z_1 = \lambda x_2 \), and \( |\lambda|^2 \neq 0 \) by hypothesis. Since \( x_2p \) commutes with \( x_2 \) we have, by Lemma 1.4,

\[
(11) \quad x_2(x_2p) = (x_2p)x_2 + (x_2p \cdot 1)x_2 - x_2 \cdot x_2p.
\]

If we let \( p = \bar{f} \) in (11) then since \( x_2 \) and \( \bar{f} \) are pure, we have \( |x_2|^2 \bar{f} = -(x_2 \bar{f} \cdot 1)x_2 \). Therefore, \( x_2 \) is a scalar multiple of \( \bar{f} \) and if, in (10), we take \( p \) to be any pure, basic element other than \( \bar{f} \) we get a contradiction. This completes the case \( z_2 = 0 \).

Now let \( z_1 = 0 \). The argument is analogous to the above and we indicate related equations and cases by appending the symbol "I". Thus the corresponding equations are:

\[
(1') \quad (\bar{p} \bar{z}_2)x_2 - \bar{z}_2(x_2 \bar{p}) = 0
\]
\[
(2') \quad (z_2 \bar{p})x_1 - z_2(x_1 \bar{p}) = 0
\]
\[
(3') \quad x_1(\bar{z}_2 \bar{p}) - \bar{z}_2(p x_1) = 0
\]
\[
(4') \quad x_2(\bar{p}z_2) - z_2(\bar{p}x_2) = 0.
\]

Just as before (3'), (4') with \( p = 1 \) yield

\[
(5') \quad [x_1, \bar{z}_2] = [x_2, z_2] = 0.
\]

Again \( p = \bar{z}_2 \) in (1'), (2') yields

\[
(6') \quad x_2 |\bar{z}_2|^2 - \bar{z}_2(x_2 \bar{z}_2) = 0
\]
\[
(7') \quad |z_2|^2 x_1 - z_2(x_1 \bar{z}_2) = 0.
\]

If \( z_2 = 0 \), then there is nothing to prove. If \( \bar{z}_2 = 0 \) (i.e., \( z_2 \in F \) but \( z_2 \neq 0 \)) then (1'), (2') imply that \( x_1, x_2 \in F \), the center of \( B \).

Note also that \( x_1 \) can be modified by any scalar without affecting the
hypothesis or the conclusion. Hence we assume that \( x_1 \) is pure. Thus if \( x_1 \in F \), then \( x_1 = 0 \).

**Case 1'.** \( z_2 \in F \).
Then the above discussion shows that we are reduced to \( x = x_2 e \), \( z = z_2 e \), and these are dependent.
Now we apply Lemma 1.4 as before and the coefficients of \( x_1 \), \( x_2 \) in (6'), (7') are easily seen to be \(|z_2|^2 - (z_2 \cdot 1)^2 = |z_2|^2\).
As before, this implies that \( x_1 \), \( x_2 \) are linearly dependent upon \( \{1, z_2\} \) over \( F \) and it remains to show \( x_1 = 0 \) (under the assumption of purity). We now imitate the previous proof with \( (x_1, z_2) \) playing the roles of \( (z_1, x_2) \).

**Case 2'.** \( z_2 \notin F \) and \( (z_2 \cdot 1) \neq 0 \).
Write \( x_1 = \lambda z_2 + \mu \), with \( \lambda, \mu \in F \). We may assume \( \lambda \neq 0 \) since otherwise we are done. We will deduce a contradiction.
Substitution in (2'), (3') leads to
\[
(10') \quad (z_2 p)z_2 = z_2(z_2 p), \quad z_2(\overline{z_2} p) = \overline{z_2}(pz_2) \quad \text{for all } p \in B.
\]
Using the flexible law on the second equation we see that \( z_2 \) commutes with \( z_2 p \) and \( \overline{z_2} p \) for all \( p \) in \( B \). Hence it commutes with \( z_2 p + \overline{z_2} p = (z_2 \cdot 1) p \) for all \( p \in B \). Since \( z_2 \cdot 1 \neq 0 \), \( z_2 \in F \), a contradiction. Thus we may assume \( z_2 \) is pure and as above we come to

**Case 3'.** \( z_2 \) is pure and nonzero.
Now, we get a contradiction exactly as in Case 3.  

**Theorem 1.6 (Schafer [S2]).** Suppose the characteristic of \( F \) is neither 2 nor 3 and \( D: A \rightarrow A \) is a derivation. For \( x \in B \) write the decomposition \( Dx = d(x) + g(x)e \) with \( d(x), g(x) \in B \). If \( n \geq 4 \) then \( g(x) = 0 \) for all \( x \) and \( D \) is an extended derivation so that \( D(x_1 + x_2 e) = d(x_1) + d(x_2)e \), where \( d: B \rightarrow B \) is a derivation of \( B \).

**Proof.** Computation of \( D(xy) \) for \( x, y \in B \) yields
\[
(1) \quad d(xy) = d(x) y + x d(y)
(2) \quad g(xy) = g(x) \tilde{y} + g(y) x.
\]
Since \( De = 0 \) by Lemma 1.3, the whole proof is finished by showing \( g(x) = 0 \) for all \( x \).
Applying (2) repeatedly we get
\[
(3) \quad g(x(xy)) = g(x)(\tilde{y} \tilde{x}) + (g(x) \tilde{y} + g(y) x)x.
\]
If \( x, y \) are pure then this reduces to
\[
(4) \quad g(x(xy)) - (g(y) x)x = -\langle g(x), y, x \rangle.
\]
If $x$ is taken to be a pure basic element then the LHS of (4) is zero. This is seen by checking $x(xz) = (zx)x = x^2z$ with $x^2 \in F$ for any basic $x$ and any $z \in A$. Thus when $x$ is a pure basic element

(5) $\langle g(x), y, x \rangle = 0$ for all pure $y$ (and hence for all $y$).

From Lemma 1.3(ii), (iii) we deduce that $Dx \cdot e = 0 = g(x)e \cdot e = -g(x) \cdot 1$ and hence $g(x)$ is pure. Now if $x$ is a basic element of $B$ then certainly one component of its standard decomposition is zero and $|x|^2 \neq 0$. Thus Proposition 1.5 implies

(6) $g(x) = \lambda_x x$ with $\lambda_x \in F$.

Using this in (2) we get

(7) $\lambda_{xy} = -\lambda_x y + \lambda_y x$ for all pure, basic $x \neq y$.

Therefore

(8) $\lambda_{xy} + \lambda_x + \lambda_y = 0$ for all pure, basic $x \neq y$.

We wish to prove that $\lambda_x = 0$ for all pure, basic $x$.

First let $n = 4$. The algebra $B$ is alternative (i.e. $\langle x, x, y \rangle = 0$) [S1, p. 46] (or Lemma 1.2), so the discussion following (4) holds for any pure $x \in B$ and hence (6) holds for any $x \in B$ in which the standard decomposition $x = x_1 + x_2$ has:

\[
\begin{cases} 
  x_1 = 0, \ x_2 \text{ pure, and } |x_2| \neq 0 \\
  \text{or} \\
  x_2 = 0, \ x_1 \text{ pure, and } |x_1| \neq 0.
\end{cases}
\]

(\*)

Define an order on the basic elements of $A$ by

$e_i < e_j$ if $i < j$.

Now let $p_1$ and $p_2$ be two pure elements of $B$, each of nonzero norm. Since $\text{char}(F) \geq 5$, there is a linear combination of these two having nonzero norm. (One of $p_1 \pm p_2$, $p_1 \pm 2p_2$ works. This is easily checked by computing the norms using (5) in Section 0.) Now if $p_1$, $p_2$, and the appropriate combination are all eigenvectors of $g$, then it follows that they belong to the same eigenvalue. Apply this to any two pure, basic elements, $e_i$, $e_j > f$ (respectively $< f$). They and their combinations will satisfy (\*) above. Thus $\lambda_{e_i} = \lambda_{e_j}$ for $i > f$ (respectively $i < f$). Set these common values equal to $v_1$ and $v_2$, respectively. Using (8) with distinct, pure, basic $x$, $y$ less than $f$ we get

\[
\lambda_{xy} + \lambda_x + \lambda_y = 3v_t = 0.
\]
Similarly, for basic, pure \( x, y \) greater than \( f \) we get
\[
\hat{\lambda}_{xy} + \hat{\lambda}_x + \hat{\lambda}_y = v_x + 2v_y = 0.
\]
Thus \( v_x = v_y = 0 \) (since \( \text{char} (F) \neq 2, 3 \)) and it follows that \( g(x) = 0 \) for all \( x \).

Now for \( n > 4 \) we proceed by induction. Let \( D_0 \) be the derivation
\[
D_0(x_1 + x_2 e) = D(x_1 + x_2 e) - d(x_1) - d(x_2)e.
\]
This restricts to a derivation of \( A = F \langle e_2, \ldots \rangle \) and the corresponding function \( g(x) \) is the restriction of the one from \( A \). Thus by induction we get that \( \hat{\lambda}_x = 0 \) for all pure, basic \( x = e_{2i} \in 1B \). It remains to show that \( \hat{\lambda}_{e_{1+z}} = 0 \) for any \( r \). But from (8) \( \hat{\lambda}_{e_{1+z}} + \hat{\lambda}_{e_{1+z}} = 0 \) for any distinct \( r, s \) and hence \( \hat{\lambda}_{e_{1+z}} = 0 \) for all \( r \) since it is possible to choose at least three distinct values of \( r \) (namely \( r = 0, 1, 2 \)).

**Remark 1.7.** The above argument follows Schafer’s original scheme. Our use of the “nonassociativity conjecture” avoids a detailed analysis of certain tuples and allows us to essentially ignore the structure constants.

We will need the following elementary lemma in the next section.

**Lemma 1.8.** If \( x \in A_i \), for \( i \in \{2, 3\} \), is a pure, basic element, then there is \( D \in \text{Der}_F(A_i) \) such that \( \text{kernel}(D) = F + Fx \), a two dimensional, commutative algebra.

**Proof.** Let \( y \in A_i \) such that \( y \cdot 1 = y \cdot x = 0 \neq |y|^2 \) (e.g., \( y \) is some other pure, basic element). Then \( [xy, y] = 2xy^2 = -2x |y|^2 \). It is straightforward to show that the operator \( [L_{xy}, L_x] + [R_{xy}, R_x] + [L_{xy}, R_x] \) is a derivation on \( A_i \) \([S1, p. 77]\) (Here \( L \) and \( R \) respectively denote left and right multiplications). A direct calculation shows that the kernel is \( F1 + Fx \) and the images of the other \( 2^i - 2 \) pure, basic elements are linearly independent.

**Corollary 1.9.** If \( A \) is a Cayley–Dickson algebra, then
\[
\bigcap_{D \in \text{Der}_F(A)} \text{Kernel}(D) = \begin{cases} 
3A & \text{if } n \geq 4 \\
A & \text{if } n = 0, 1 \\
F & \text{if } n = 2, 3.
\end{cases}
\]

**Proof.** If \( n \leq 1 \) this is obvious as there are no nontrivial \( F \)-derivations of a separable algebraic extension of \( F \). If \( n \in \{2, 3\} \) it follows immediately from Lemma 1.8. For \( n \geq 4 \) the RHS is contained in the LHS by Schafer’s theorem since, in this case, \( D(e_{2i}) = 0 \) for all \( i \). Conversely, suppose \( x \in A \). Then there is a unique representation
\[
x = \sum x_i(x) e_{2i}
\]
with \( \alpha_i(x) \in A_3 \). If \( x \notin A \) then for some \( i, \alpha_i(x) \notin F \). It follows then, from the case \( n = 3 \), that there is a derivation \( d \) of \( A_3 \) such that \( d(\alpha_i(x)) \neq 0 \). This extends to a derivation \( D \) of \( A \) such that \( D(x) \neq 0 \).

### 2. Automorphisms

For \( n \geq 1 \), the automorphism group of \( B, \text{Aut}_F(B) \) is naturally isomorphic to a subgroup of \( \text{Aut}_F(A) \) under the correspondence

\[
\sigma \mapsto \hat{\sigma},
\]

where \( \hat{\sigma}(a + be) = \sigma(a) + \sigma(b)e \) for \( a, b \in B \).

We will identify \( \sigma \) with \( \hat{\sigma} \) and view \( \text{Aut}_F(B) \) as a subgroup of \( \text{Aut}_F(A) \) under this identification. It is natural then to ask how the two are related.

Schafer's Theorem says that if \( n \geq 4 \) then every derivation of \( A \) is the trivial extension of one from \( B \). In [B] Brown studied the automorphisms of the Cayley-Dickson algebras and noted that there is always a nontrivial, discrete subgroup \( G \subseteq \text{Aut}_F(A) \) which is isomorphic to either \( \mathbb{Z}/2 \) or \( S_3 \). He gave explicit generators for this group and conjectured that \( \text{Aut}_F(A) = \text{Aut}_F(B) \oplus G \). In this section we will prove Brown's conjecture.

Observe that for any \( \sigma \in \text{Aut}_F(A) \) and \( a \in \tilde{A}, \sigma(a) \in \tilde{A} \). This is evident for \( a \in F \). For \( a \notin F \), one has

\[
a \in \tilde{A} \iff a \cdot 1 = 0 \iff a^2 \in F \iff (\sigma(a))^2 \in F.
\]

The second implication above follows from (3) of Section 0. Since conjugation is multiplication by \(-1\) on \( \tilde{A} \) and identity on \( F \), it follows that \( \sigma(\tilde{x}) = \overline{\sigma(x)} \) for all \( x \).

Therefore we get

\[
\sigma(a \cdot b) = \sigma(\frac{1}{2}(ab + b\tilde{a})) = \frac{1}{2}(\sigma(a) \overline{\sigma(b)} + \sigma(b) \overline{\sigma(a)}) = \sigma(a) \cdot \sigma(b).
\]

Thus \( \sigma \) preserves scalar products and norms.

**Lemma 2.1.** Let \( \sigma \in \text{Aut}_F(A) \), then

(i) \( \sigma(e) = \pm e \) if \( n \geq 4 \)

(ii) \( \sigma(\tilde{A}) = \tilde{A} \).

**Proof.** It follows immediately from Lemma 1.2 that \( \sigma \) leaves \( n-1 \cdot A = F1 + Fe \) invariant. Thus \( \sigma(e) = x + \beta e \). By the above observations \( \sigma(e) \) is orthogonal to \( 1 \) and of norm \(-\gamma \). Therefore \( x = 0 \) and \( \beta = \pm 1 \). Part (ii) follows from Schafer's theorem and Lemma 1.9 since these imply that \( \tilde{A} \) is precisely the intersection of all kernels of all derivations of \( A \). This
representation implies that it is mapped onto itself by every automorphism of $A$.

Recall that if $A$ is a Cayley–Dickson algebra then $\mathcal{A}$ denotes the set of pure elements of $A$.

**Lemma 2.2 (Brown [B]).** Suppose that $\text{char}(F) \neq 2, 3$. Let $2 \leq m \leq n$ and

$$\zeta_m = -\frac{1}{2} - \frac{1}{2} \sqrt{-3} \gamma^{-1} e_{2^{m-1}},$$

for $x \in A$ define

$$\begin{align*}
\mu_m(x) &= \begin{cases} 
x & \text{if } x \in m^{-1}A \\
x_{m-1}^* & \text{if } x \in \mathcal{A}_{m-1}
\end{cases} \\
\tau_m(x) &= \begin{cases} 
x & \text{if } x \in mA \\
a - be_{2^{m-1}} & \text{if } x = a + be_{2^{m-1}} \text{ with } a, b \in \mathcal{A}_{m-1}.
\end{cases}
\end{align*}$$

Then, in view of the level $m$ expansion of $A$, $\tau_m$ extends uniquely to an $F$-automorphism of $A$; if $\sqrt{-3} \gamma^{-1} \in F$, then the same is true of $\mu_m$. Moreover $\tau_n$ and $\mu_n$ commute with $\text{Aut}_F(B)$ and the only relations between them are:

(i) $\mu_n^2 = \tau_n^2 = \text{identity}$

(ii) $\mu_n \tau_n = \tau_n \mu_n^2$.

**Proof.** Both $\tau_m$ and $\mu_m$ are obviously $F$-linear monomorphisms. It is straightforward (using the equations (a)–(c) in Section 0) to check that they preserve multiplication and commute with the conjugation operator ($\overline{}$). The remaining assertions are straightforward verifications.

Let $G$ be the subgroup of $\text{Aut}_F(A)$ generated by $\tau_n$ and $\mu_n$ if $\mu_n$ is defined over $F$, and simply the subgroup generated by $\tau_n$ otherwise. Then the above relations imply that $G$ is isomorphic either to $\mathbb{Z}/2$ or $S_3$. Since it is obvious from the definitions that $G \cap \text{Aut}_F(B)$ is the identity, one has

$$\text{Aut}_F(B) \oplus G \subset \text{Aut}_F(A).$$

**Lemma 2.9.** Let $F$ be a field of characteristic not equal to 2 or 3. Suppose $A$ is a Cayley–Dickson algebra and $\sigma \in \text{Aut}_F(A)$. Then there is $\theta \in \mathcal{A}$ such that $|\theta| = 1$ and

$$\sigma(A_3) = F + \mathcal{A}_3 \theta.$$

**Proof.** Let $x \in \mathcal{A}_3$ be any nonzero scalar multiple of a basic element of
$A_3$ and $\sigma(x) = \sum \alpha_i(x)e_{8i}$, with $\alpha_i(x) \in A_3$. Since $A_3 \cong \sigma(A_3)$, by Lemma 1.8 there is a derivation $d_\sigma \in \text{Der}_F(\sigma(A_3))$ such that kernel $d_\sigma = F(1 + F\sigma(x))$. This extends to a derivation $D$ of $A$ which has nullspace of dimension $2^n - 2$.

By Schafer's theorem this is, in turn, the extension to $A$ of some $d \in \text{Der}_F(A_3)$ which must therefore have nullspace of dimension 2. Thus the kernel of $d$ is a commutative subalgebra of $A_3$.

Since

$$0 = D(\sigma(x)) = \sum d(\alpha_i(x))e_{8i},$$

it follows that $\{\alpha_i(x)\} \subseteq \text{kernel } d$. Since $\sigma(x) \notin \mathfrak{3}A$, the elements $\{\alpha_i(x)\}$ generate a 2-dimensional commutative subalgebra of $A_3$. Thus there is a pure $v_x$ such that

$$\alpha_i(x) = a_i + b_i v_x \quad \text{with } a_i, b_i \in F \text{ for each } i.$$

Thus $\sigma(x) = a + b v_x$ with $a, b \in \mathfrak{3}A$. However, since $x$ is orthogonal to $\mathfrak{3}A$, $\sigma(x)$ is orthogonal to $\sigma(\mathfrak{3}A) = \mathfrak{3}A$. Thus $a = 0$ and

$$\sigma(x) = v_x \theta_x \quad \text{with } \theta_x \in \mathfrak{3}A, v_x \in \mathfrak{A}_3.$$

Now if $x$ and $y$ are nonzero multiples of distinct, pure, basic elements of $A_3$ then the corresponding derivations $d$ must have distinct kernels and it follows that $v_x$ and $v_y$ are linearly independent. Moreover if $x$ and $y$ are taken so that $x+y$ is a nonzerodivisor of $A_3$ then the latter is, under an automorphism of $A_3$, itself a multiple of a basic element [J]. Thus there is an equation

$$v_x \theta_x + v_y \theta_y = v_{x+y} \theta_{x+y}.$$

By the independence of $v_x$ and $v_y$ this can happen only if $\theta_x$ and $\theta_y$ are proportional. That is, $\theta_x = r \theta_y$ for some $r \in F$. Thus $\sigma(A_3) = F + \sum F \sigma(e_i) = F + \sum F v_{e_i} \theta_{e_i} = F + \sum F v_{e_i} \theta_{e_i} = F + \mathfrak{A}_3 \theta$.

It is well known that $|uv|^2 = |u|^2 |v|^2$ if $u, v \in A_3$ [S1, p. 73]. Since $\sigma$ is an automorphism, it follows that $|\sigma(uv)|^2 = |\sigma(u)|^2 |\sigma(v)|^2$ for $u, v \in A_3$. So since $0 \neq |\sigma(e_i)|^2 = |v_{e_i} \theta|^2 = |v_{e_i}|^2 |\theta|^2$, we get that $|\theta|^2$ is not zero.

Pick $x_1, x_2 \in A_3$ such that $\sigma(x_1) = e_1 \theta$, $\sigma(x_2) = e_2 \theta$. Then $\sigma(x_1 x_2) = (e_1 \theta)(e_2 \theta) = \delta(e_1 e_2) \theta$ for some $\delta \in F$.

The last equality is checked as follows: expand $\theta = \sum \alpha_{8i} e_{8i}$, with $\alpha_{8i} \in A_3$, then calculate $(e_1 \theta)(e_2 \theta)$. This is quickly seen to have the form $(e_1)(e_2) \sum \beta_i e_{8i}$. Since this is also in $\mathfrak{A} \theta$ the equality follows by comparing coefficients.

Taking norms of both sides of the second equality and using (8) from Section 0, we get $|e_1|^2 |e_2|^2 |\theta|^4 = \delta^2 |e_1|^2 |e_2|^2 |\theta|^2$ and thus $|\theta|^2 = \delta^2$. Replacing $\theta$ by $\delta^{-1} \theta$ we get the desired conclusion.
Lemma 2.4. Let $F$ be a field of characteristic not equal to 2 or 3. If $\sigma \in \text{Aut}_F(A)$ and $n \geq 4$ then there are integers $i, j$ such that $\tau_i \mu_n^j \sigma \in \text{Aut}_F(B)$.

Proof. By Lemma 2.1 $\sigma(e) = \pm e$. Thus replacing $\sigma$ by $\tau_n \sigma$, if necessary, we may assume $\sigma(e) = e$. By Lemma 2.3 there is $\theta \in \mathbb{A}_3$ of norm 1 such that for each $x \in \mathbb{A}_3$, $\sigma(x) = v_x \theta$.

Choose $x_1, x_2, x_3, x_4 \in \mathbb{A}_3$ such that $\sigma(x_i) = e_i \theta$. If, as above, one expands $\theta = \sum a_i x_i$ and calculates $(e_i \theta)(e_j \theta)$ one arrives at the form $\sum \beta_i e_i e_j e_k$ with $\beta_i \in F$. Thus from the form $\sigma(x, x_j) = (e_i \theta)(e_j \theta) = v_x \beta_i, \beta_j \theta$ and the fact that $\sigma$ is norm preserving, it follows that $v_x \beta_i, \beta_j \theta = v_x \beta_i, \beta_j$.

By (8) of Section 0, it is easy to see that $|e, \theta|^2 = |e|^2$. Since $\sigma$ is an isometry it follows that the $x_i$ are mutually orthogonal, that $|x_i|^2 = |e_i|^2$, and that $x_i x_j$ is orthogonal to $x_k$ for distinct $i, j, k$. Thus by [J] there is an automorphism $\sigma^* \in \text{Aut}_F(A_3)$ defined by $(e_1, e_2, e_4) \mapsto (x_1, x_2, x_4)$. There is no loss in replacing $\sigma$ by $\sigma \sigma^*$, which results in $\sigma(e_i) = e_i \theta$ for $i = 1, 2, 4$. It follows that $\sigma(e_i, e_j) = \psi(e_i e_j \theta)$ where $\psi = \pm 1$. Let $\sigma^{**} \in \text{Aut}_F(A_3)$ be defined by $(e_1, e_2, e_4) \mapsto (\psi e_1, \psi e_2, e_4)$. Replacing $\sigma$ by $\sigma \sigma^{**}$ and $\theta$ by $\psi \theta$ we may assume

$$(#) (e_1, e_2, e_1 e_2) \mapsto (e_1, \theta, e_2 \theta, (e_1 e_2) \theta).$$

If $n = 4$, set $\theta = \alpha + \beta e_8$. Then (#) yields

$$\alpha^2 + \gamma \beta^2 = \alpha$$

$$-2 \alpha \beta = \beta.$$

If $\beta = 0$ then $\sigma(A_4) \subseteq A_4$ and $\sigma(e) = e$ so $\sigma \in \text{Aut}_F(A_3)$. If $\beta \neq 0$ then $\theta = \zeta_4$ or $\zeta_4^3$ and respectively $\mu_3^2 \sigma$ or $\mu_2 \sigma \in \text{Aut}_F(B)$. This establishes the theorem for $n = 4$. We now proceed by induction on $n$.

Under the identifications given in Section 0, $\mathbb{A}$ is a Cayley–Dickson algebra of order $n-1$. Moreover, since $\sigma(e_i) = e_i \theta$ for $i = 2, 4$ and $\sigma(\mathbb{A}) = \mathbb{A}$, it follows that $\sigma(\mathbb{A}) \subseteq \mathbb{A}$. Thus $\sigma \sigma \in \text{Aut}_F(\mathbb{A})$. Let $\mu_n^i, \mathbb{A}_n$ and $\tau_n \mathbb{A}_n$ denote the associated elements of $\text{Aut}_F(\mathbb{A})$. By induction, there exist $i, j$ such that

$$1\tau_n^i, \mu_n^j \sigma \in \text{Aut}_F(1\mathbb{A}_n) = \text{Aut}_F(1B).$$

But $1\mu_{n-1} = \mu_n \mathbb{A}$ and $1\tau_{n-1} = \tau_n \mathbb{A}$.

Consider $\tau_n^i \mu_n^j \sigma = \sigma_B$. Clearly $\sigma_B(1\mathbb{A}_{n-1}) \subseteq 1\mathbb{A}_{n-1} \subseteq B$. Also $\sigma_B$ continues to satisfy (#). That is, we have

$$(#') (e_1, e_2, e_1 e_2) \mapsto \sigma_B(e_1 \theta', e_2 e_2 \theta', (e_1 e_2) \theta')$$

for some $\theta' \in \mathbb{A}_4$. Since $e_2 \in 1\mathbb{A}_{n-1}$, $\sigma_B(e_2) = e_2 \theta' \in B$. Hence $e_2(e_2 \theta') = \gamma_2 \theta' \in B$. Therefore $\theta' \in B$ and since $e_1 \in B$, $e_1 \theta' = \sigma_B(e_1) \in B$. So by (#'), $\sigma_B(e_1) \in B$. As $B$ is clearly generated by $e_1$ and $1\mathbb{A}_{n-1}$ we get $\sigma_B(B) \subseteq B$. 


COROLLARY 2.5. Let $F$ be a field of characteristic not equal to 2 or 3. If $n \geq 4$, then

$$\text{Aut}_F(A_n) \cong \text{Aut}_F(A_{n-1}) \oplus \begin{cases} \mathbb{Z}/2 & \text{if } \sqrt{-3\gamma_n^{-1}} \notin F \\ S_3 & \text{if } \sqrt{-3\gamma_n^{-1}} \in F. \end{cases}$$

Finally, straightforward adaptations of the proofs of Lemma 2.3 and Lemma 2.4 yield the following corollary. This was also conjectured by Brown [B]. We refrain from giving an explicit proof, since it only requires one to rewrite the proofs of these lemmas with slightly more complicated notation.

COROLLARY 2.6. Let $F$ be a field of characteristic not equal to 2 or 3. Suppose $n \geq 4$ and $\Gamma$ and $\Lambda$ are two sequences of structure constants. Then $A_n^\Gamma \cong A_n^\Lambda$ if and only if $\gamma_n \lambda_n^{-1}$ is a square in $F$ and $A_{n-1}^\Gamma \cong A_{n-1}^\Lambda$.

ACKNOWLEDGMENTS

The authors are grateful to Professors F. Cohen and M. Peim for numerous helpful conversations. We are particularly appreciative of Professor Cohen's pregnant questions which prompted this study. We are indebted to the referee for many helpful suggestions and simplifications which significantly improved this exposition.

Note added in proof. The authors have subsequently completed the proof of Yiu's Conjecture mentioned in the introduction. See Yiu's Conjecture on the Cayley-Dickson Algebras, by Paul Eakin and Avinash Sathaye, preprint 1989.

REFERENCES