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Fractal interpolation on the Koch Curve

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ABSTRACT

The Koch Curve can be obtained as an iterated function system construction. Self-similar interpolation is possible for any function on the sets that are defined recursively. We prove that the Koch Curve (KC) is an analogue of the fractal interpolation theorem of Barnsley. Also the classical harmonic functions are defined on the KC as the degree 1 polynomials for self-similar interpolation.

1. Introduction

The theory of fractal interpolation has become a powerful and useful tool in applied science and engineering since Barnsley introduced the concept of the fractal interpolation function [1]. Many researchers used the fractal interpolation function to analyze fractal sets and show that the functions have important applications to data interpolation [2]. Demir et al., computed derivatives of the restrictions of harmonic functions on the Sierpinski gasket to segments [3]. Kigami used harmonic functions on the segments of the Sierpinski gasket [4,5]. Needleman et al., defined calculus on the Sierpinski gasket [6]. Also Celik et al., proved the Sierpinski gasket (SG) to be an analogue of the fractal interpolation theorem of Barnsley [7].

An iterated function system construction is useful for sets that are defined recursively. The Cantor Set, Sierpinski Gasket and Koch Curve are three well known self-similar fractals. The Koch Curve (KC) is a continuous curve which is nowhere differentiable. The KC has infinite length and bounds a finite area. In the popular presentation of the theory of fractals, the Koch curve is defined heuristically in the following way: The segment $K_0 = [0, 1] \times \{0\} \subset R^2$ is called the initiator of the curve. The middle third of this segment is removed and replaced by two equal segments that would form an equilateral triangle with the removed piece. The resulting four sided zigzag K_1 is called the generator of the curve. The next step consists of subjecting each of the four segments of K_1 to the same process (removal of the middle third and replacement by two equal segments), as if each of them were a new initiator of length $1/3$. Apply the above procedure to all the four segments of K_1, we obtain a 16 sided zigzag K_2. This procedure is carried out ad infinitum and the Koch curve is seen as the figure obtained in the limit, which we shall denote by K_∞. This definition is easy in understanding intuitively the nature of the Koch curve. Due to the mathematical deficiency of the heuristic definition, we prefer to use different definition. Instead of dealing with the successive curves K_n, we will consider the finite sets V_n consisting of the vertices of K_n. A vertex, i.e. an element of V_n, is an endpoint of one of the constitutive segments of K_n. As an example we have, the KC starting with the line segment $((0, 0), (1, 0))$ as shown in Figs. 1 and 2.

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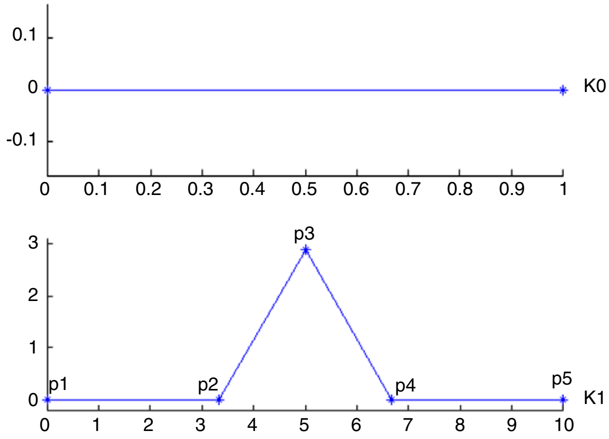

Fig. 1. Example of construction of Koch Curve K_0 and K_1.

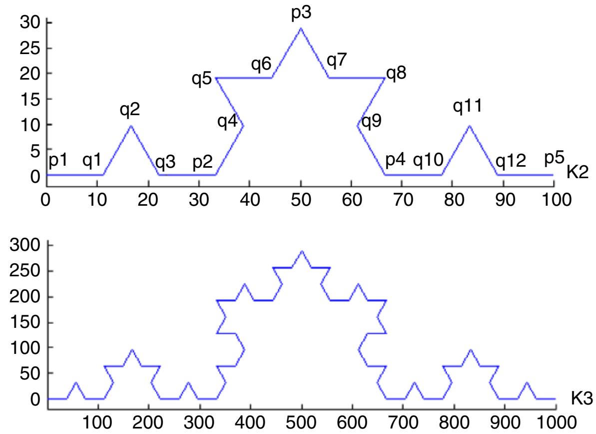

Fig. 2. Example of construction of Koch curve K_2 and K_3.

The four contractions f_1, f_2, f_3, f_4 are given below:

$$f_1(x,y) = \left(\frac{x}{3}, \frac{y}{3}\right)$$

$$f_2(x,y) = \left(-\frac{x}{6} - \frac{\sqrt{3}y}{6} + \frac{1}{2}, -\frac{\sqrt{3}x}{6} - \frac{y}{6} + \frac{\sqrt{3}}{6}\right)$$

$$f_3(x,y) = \left(\frac{x}{6} + \frac{\sqrt{3}y}{6} + \frac{1}{2}, -\frac{\sqrt{3}x}{6} - \frac{y}{6} + \frac{\sqrt{3}}{6}\right)$$

$$f_4(x,y) = \left(\frac{x}{3} + \frac{2}{3}, \frac{y}{3}\right).$$

V_1 contains the following set of vertices:

$$V_1 = \left\{(0,0), \left(\frac{1}{3}, 0\right), \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right), \left(\frac{2}{3}, 0\right), (1,0)\right\}$$

where (x, y) denotes a generic point of $\mathbb{R}^2$. We now consider the union

$$V_\infty = \bigcup_{i \in \mathbb{N}} V_i$$

KC is the countable set and it contains 4^k line segments of length 3^{-k}.

Definition 1.1. The Koch curve KC is the closure of V_∞ in $\mathbb{R}^2$, namely $KC = \overline{V}_\infty$.

Since V_∞ is a countable set, we can construct a bijection between V_∞, and a certain countable subset A of the closed unit interval $[0, 1] \subset \mathbb{R}$. Since the curve K_n is open and connected, its vertices can be consistently numbered from 0 to 4^n starting at the origin $(0, 0) \in \mathbb{R}^2$ and $(1, 0) \in \mathbb{R}^2$. Let V_n^i $(i = 0, 1, \dots, 4^n)$ denote the ith vertex of K_n.

Let $V_1 = \{p_1, p_2, p_3, p_4, p_5\}$ be the vertices of the KC, and

$$u_i(x) = \frac{1}{3}(x + p_i), \quad (i = 1, 2, 3, 4, 5)$$

the set of contractions of the plane, of which the KC is the attractor. Fix a number n and consider the iteration $u_w = u_{w_1} u_{w_2} \cdots u_{w_n}$ for any sequence $w = (w_1, w_2, \dots, w_n) \in \{0, 1, \dots, 4^n\}$. The union of the images of V_0 under these iterations is the set of nth stage vertices V_n of the KC.

Let $F : V_n \to \mathbb{R}$ be any function. Given any numbers α_w, $w \in \{0, 1, \dots, 4^n\}$ with $0 < |\alpha_w| < 1$, there exists a unique continuous extension $f : KC \to \mathbb{R}$ of F, such that

$$f(u_w(x)) = \alpha_w f(x) + h_w(x)$$

for $x \in KC$, where h_w are harmonic function on the KC for $w \in \{0, 1, \dots, 4^n\}$. Interpreting the harmonic functions as the "degree 1 polynomials" on the KC is thus a self similar interpolation obtained for any start function $F : V_n \to \mathbb{R}$.

We first recall a version of the fractal interpolation theorem of Barnsley [1].

Theorem 1.1. *Let $[x_0, x_N] \subset \mathbb{R}$ be an interval and*

$$x_0 < x_1 < \dots < x_{i-1} < x_i < \dots < x_N$$

a subdivision of this interval ($N \geq 2$). Let $F_i \in \mathbb{R}$ ($i = 0, 1, \dots, N$) be some arbitrary values attached to the points x_i and which are to be interpolated over the interval $[x_0, x_N]$ by a continuous function $f : [x_0, x_N] \to \mathbb{R}$ with $f(x_i) = F_i$ ($i = 0, 1, 2, \dots, N$).

Let $u_i : [x_0, x_N] \to [x_{i-1}, x_i]$ be the invertible maps

$$u_i(x) = \frac{x_i - x_{i-1}}{x_N - x_0} x + \frac{x_N x_{i-1} - x_0 x_i}{x_N - x_0} \quad (i = 1, \dots, N),$$

$\alpha_i \in \mathbb{R}$ ($i = 1, \dots, N$) be any given numbers (called the vertical scaling factors) with $0 < |\alpha_i| < 1$ and $h_i : [x_0, x_N] \to \mathbb{R}$ be the linear functions

$$h_i(x) = \left(\frac{F_i - F_{i-1}}{x_N - x_0} - \alpha_i \frac{F_N - F_0}{x_N - x_0} \right) x + \frac{x_N F_{i-1} - x_0 F_i}{x_N - x_0} - \alpha_i \frac{x_N F_0 - x_0 F_N}{x_N - x_0}$$

for $i = 1, \dots, N$.

Then, there exist a unique continuous function $f : [x_0, x_N] \to \mathbb{R}$ with $f(x_i) = F_i$ ($i = 0, 1, \dots, N$) such that the following condition holds:

$$f(u_i(x)) = \alpha_i f(x) + h_i(x) \quad \text{for } x \in [x_0, x_N] \text{ and } i = 1, 2, \dots, N.$$

M.F. Barnsley proved that the graph of an interpolation function in the above sense can be realized as the attractor of an iterated functions system on the plane and thus it represents a fractal generically. Here we consider the other aspect of this construction: This interpolation function is "self similar" in the sense that if we magnify its restriction to $[x_{i-1}, x_i]$ to the whole interval $[x_0, x_N]$ by means of u_i, then it becomes almost the same f again (up to a multiplication by a number and modulo addition of a polynomial of degree 1) [6].

Interpreting the polynomials of degree 1 as classical harmonic functions on an interval and replacing them on the Koch Curve (KC) by harmonic functions of fractal analysis, we will obtain an analogue of the Barnsley fractal interpolation theorem for the KC.

2. Fractal interpolation theorem for the Koch Curve

Let $V_1 = \{p_1, p_2, p_3, p_4, p_5\}$ be the set of vertices on the plane $\mathbb{R}^2$ and $u_i(x) = \frac{1}{3}(x + p_i)$, $(i = 1, 2, 3, 4, 5)$ with the set of contractions of the plane which constitute an iterated functions system [4,5]. The KC is the attractor of this system:

$$KC = u_1(KC) \cup u_2(KC) \cup u_3(KC) \cup u_4(KC).$$

Fix a number n and consider the iteration $u_w = u_{w_1} u_{w_2} \cdots u_{w_n}$ for any sequence $w = (w_1, w_2, \dots, w_n) \in \{0, 1, \dots, 4^n\}$. The union of the images of V_0 under these iterations constitutes the set of nth stage vertices V_n of the KC.

Given any function $f : V_n \to \mathbb{R}$, there is an operator H_n, defined by $H_n(f) : V_n \to \mathbb{R}$

$$H_n(f)(p) = \sum_{q \in N_{p,n}} (f(q) - f(p)),$$

where $N_{p,n}$ denotes the "neighbourhood" of p in V_n, the set of "next neighbours" of p in V_n, 2 in number for $p \in V_n \setminus V_1$ and 1 or 2 for $p \in V_1$. f is called harmonic on V_n if $H_n(f)(p) = 0$ for all $p \in V_n \setminus V_1$.

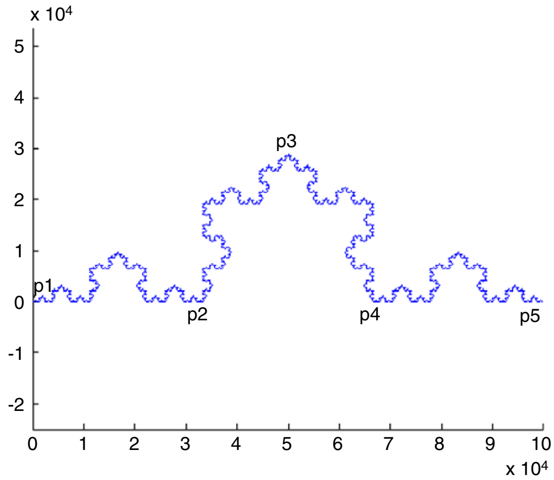

Fig. 3. Example of fractal interpolation.

A continuous function $f : KC \to \mathbb{R}$ is called harmonic, if its restriction to V_n is harmonic for all n.

Consider an initial set of interpolation values on the nth stage vertices of the KC, for the given function $F : V_n \to \mathbb{R}$ (n-harmonic function on the KC). By applying the above harmonic-extension theorem to all $u_w(V_1) \subset u_w(KC)$ locally, from this we obtained n-harmonic function on the KC. This n-harmonic extension is not self-similar in the sense of a relationship between local and global. Therefore we construct another extension, which is self-similar in a very nice way and which has a close resemblance with the fractal interpolation theorem of Barnsley. The harmonic functions are used as correction terms for correct matching after rescaling of the function.

Theorem 2.1. *Let $F : V_n \to R$ be any given function $(n \geq 1)$. For any given numbers α_w $(w \in \{0, 1, \dots, 4^n\}$ with $0 < |\alpha_w| < 1$, there exists a unique continuous function $f : KC \to R$, such that $f|_{V_n} = F$ and*

$$f(u_w(x)) = \alpha_w f(x) + h_w(x) \quad \text{for } x \in KC,$$

where h_w are harmonic functions on the KC for all $w \in \{0, 1, \dots, 4^n\}$.

Proof. Let

$$\mathscr{F} = \{g : KC \to R \text{ continuous with } g(p_1) = F(p_1), g(p_2) = F(p_2), g(p_3) = F(p_3), g(p_4) = F(p_4), g(p_5) = F(p_5)\}.$$

$\mathscr{F}$ is a complete metric space with the maximum metric. Define the operator $T : \mathscr{F} \to \mathscr{F}$ by

$$T(g)(y) = \alpha_w g(u_w^{-1}(y)) + h_w(u_w^{-1}(y)) \quad \text{for } y \in u_w(KC) \text{ and } w \in \left\{0, 1, \dots, 4^n\right\}$$

where h_w is the harmonic function on the KC with vertex values

$$h_w(p_i) = F(u_w(p_i)) - \alpha_w F(p_i) \quad \text{for } i = 1, 2, 3, 4, 5.$$

Then $T(g)$ is well defined, continuous and $T(g)(u_w(p_i)) = F(u_w(p_i))$, thus $T(g)(p_i) = F(p_i)$ and $T(g) \in \mathscr{F}$. T is a contraction on $\mathscr{F}$ with contractivity ratio $\max |\alpha_w|$ since

$$\max_{x \in KC} |T(g)(x) - T(\widetilde{g})(x)| \leq \max_{w \in \{0,1,\dots,4^n\}} |\alpha_w| \max x \in KC\, |g(x) - \widetilde{g}(x)|$$

for $g, \widetilde{g} \in \mathscr{F}$. The unique fixed point f of T satisfies

$$f(y) = \alpha_w f(u_w^{-1}(y)) + h_w(u_w^{-1}(y)) \quad \text{for } y \in u_w(KC) \text{ and } w \in \left\{0, 1, \dots, 4^n\right\},$$

which means

$$f(u_w(x)) = \alpha_w f(x) + h_w(x) \quad \text{for } x \in KC \text{ and } w \in \left\{0, 1, \dots, 4^n\right\}$$

as required. □

Remark. The application of the above theorem includes image decoding (approximate the points in the image segments).

Example 1. Let $V_1 = \{p_1, p_2, p_3, p_4, p_5\}$ and $F : V_1 \to R$ be a function with $F(p_1) = 0$, $F(p_2) = \frac{2}{9}$, $F(p_3) = \frac{1}{6}$, $F(p_4) = \frac{1}{9}$, $F(p_5) = \frac{1}{3}$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \frac{1}{3}$. Then the graph of the interpolation function on the KC which takes these values on V_1 is given in Fig. 3. The graph of the 1-harmonic function on the KC with vertex values $F(p_1) = 0$, $F(p_2) = \frac{2}{9}$, $F(p_3) = \frac{1}{6}$, $F(p_4) = \frac{1}{9}$, $F(p_5) = \frac{1}{3}$ in Fig. 4.

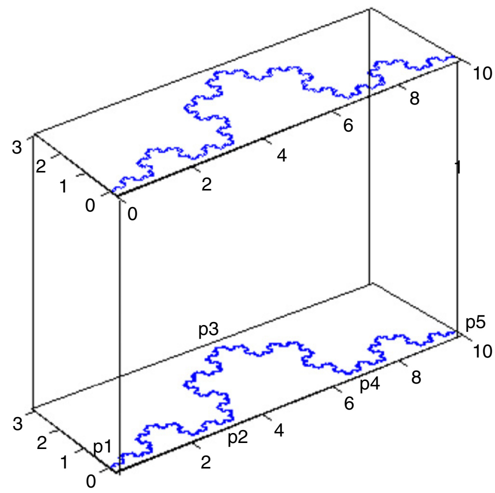

Fig. 4. Example of 1-harmonic function.

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