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The lattice of integer flows of a regular matroid

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ABSTRACT

For a finite multigraph G, let $\Lambda(G)$ denote the lattice of integer flows of G – this is a finitely generated free abelian group with an integer-valued positive definite bilinear form. Bacher, de la Harpe, and Nagnibeda show that if G and H are 2-isomorphic graphs then $\Lambda(G)$ and $\Lambda(H)$ are isometric, and remark that they were unable to find a pair of nonisomorphic 3-connected graphs for which the corresponding lattices are isometric. We explain this by examining the lattice $\Lambda(\mathcal{M})$ of integer flows of any regular matroid $\mathcal{M}$. Let $\mathcal{M}_\bullet$ be the minor of $\mathcal{M}$ obtained by contracting all co-loops. We show that $\Lambda(\mathcal{M})$ and $\Lambda(\mathcal{N})$ are isometric if and only if $\mathcal{M}_\bullet$ and $\mathcal{N}_\bullet$ are isomorphic.

1. Introduction

Let $G = (V, E)$ be a (finite undirected connected multi-) graph. Choose an arbitrary orientation for each edge of G, and let D be the corresponding signed incidence matrix: D is the V-by-E matrix with entries given by

$$D_{ve} = \begin{cases} +1 & \text{if } e \text{ points into } v \text{ but not out,} \\ -1 & \text{if } e \text{ points out of } v \text{ but not in,} \\ 0 & \text{otherwise.} \end{cases}$$

The matrix D defines a linear transformation $D : \mathbb{R}^E \to \mathbb{R}^V$. The *lattice of integer flows* of G is $\Lambda(G) = \ker(D) \cap \mathbb{Z}^E$. This is a finitely generated free abelian group with a positive definite integer-valued inner product $\langle \cdot, \cdot \rangle$ induced by the Euclidean dot product on $\mathbb{R}^E$. Of course, the set $\Lambda(G)$ depends on the choice of orientations defining the matrix D. Reversing the orientation of the edge $e \in E$

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results in changing the sign of the e-th coordinate of every element of $\Lambda(G)$. This changes neither the group structure nor the inner product structure of the lattice $(\Lambda(G), +, \langle\cdot,\cdot\rangle)$. Thus, the isometry class of this lattice is independent of the choice of orientations of the edges, and depends only on the isomorphism class of G. (An *isometry* of lattices Λ and Λ' is a bijection $\psi : \Lambda \to \Lambda'$ such that both ψ and ψ^{-1} are abelian group homomorphisms that preserve the bilinear forms on the lattices.) Bacher, de la Harpe, and Nagnibeda [1] and Biggs [2] thoroughly develop the theory of these lattices and their many interpretations, connections, and analogues.

A natural question of reconstruction arises: to what extent can properties of the graph G be determined from the isometry class of the lattice $\Lambda(G)$? Cut-edges of G contribute nothing to $\Lambda(G)$. Proposition 5 of Bacher, de la Harpe, and Nagnibeda [1] shows that if G and H are 2-isomorphic then $\Lambda(G)$ and $\Lambda(H)$ are isometric. They remark (on p. 197) that they were unable to find a pair of nonisomorphic 3-connected graphs with isometric lattices of integer flows. By Whitney's theorems [6] on 2-isomorphism of graphs, this suggests that $\Lambda(G)$ and $\Lambda(H)$ are isometric if and only if the graphic matroids $\mathcal{M}(G)$ and $\mathcal{M}(H)$ are isomorphic except for co-loops.

This is indeed the case, as follows from Theorem 1 below. For any matroid $\mathcal{M}$, let $\mathcal{M}_\bullet$ denote the minor of $\mathcal{M}$ obtained by contracting all co-loops of $\mathcal{M}$. Let $(\mathcal{M}, E)$ be a regular matroid of rank r on a ground-set E. Then $\mathcal{M}$ has a unique representation (over $\mathbb{R}$) as the column-matroid of a totally unimodular (TU) matrix M (modulo representation equivalence). The *lattice of integer flows* of $\mathcal{M}$ is $\Lambda(\mathcal{M}) = \ker(M) \cap \mathbb{Z}^E$. This generalizes the construction for graphs, in which case M is the signed incidence matrix of a connected graph with any row deleted. The isometry class of the lattice $\Lambda(\mathcal{M})$ is independent of the choice of representing matrix M, and depends only on the isomorphism class of $\mathcal{M}$. In his foundational work on representability of matroids, Tutte worked with a more general concept of "chain-groups" in which the coefficients are from any integral domain; see [5], for example. The chain-group of $\mathcal{M}$ with integer coefficients is, in our notation, $\Lambda(\mathcal{M}^*)$.

Theorem 1. *Let $\mathcal{M}$ and $\mathcal{N}$ be regular matroids. Then $\Lambda(\mathcal{M})$ and $\Lambda(\mathcal{N})$ are isometric if and only if $\mathcal{M}_\bullet$ and $\mathcal{N}_\bullet$ are isomorphic.*

Corollary 2. *Let G and H be 3-connected graphs. Then $\Lambda(G)$ and $\Lambda(H)$ are isometric if and only if G and H are isomorphic.*

Proof. Whitney [6] shows that 3-connected graphs G and H are isomorphic if and only if $\mathcal{M}(G)$ and $\mathcal{M}(H)$ are isomorphic. Also, since G has no cut-edges $\mathcal{M}(G)$ has no co-loops, so that $\mathcal{M}(G)_\bullet = \mathcal{M}(G)$, and similarly for $\mathcal{M}(H)$. The corollary now follows from Theorem 1. □

Our strategy for proving Theorem 1 is to identify metric properties of a basis $\mathcal{B}$ of an integral lattice Λ that correspond to Λ being the lattice $\Lambda(\mathcal{M})$ of integer flows of a regular matroid $\mathcal{M}$, and to $\mathcal{B}$ being a fundamental basis $\mathcal{B}(\mathcal{M}, B)$ of $\Lambda(\mathcal{M})$ consisting of signed circuits associated with a base B of $\mathcal{M}$. (Since we are dealing both with lattices and with matroids we use the word "basis" for a basis of a lattice, but "base" for what is usually called a basis of a matroid.)

The implementation of this strategy rests on two key ideas. The first key is a characterization of the signed circuits (or "simple flows") of $\mathcal{M}$ in terms of metric data of the lattice $\Lambda(\mathcal{M})$, without reference to their coordinates as vectors in $\mathbb{Z}^E$. The second key is to identify properties of a symmetric integer matrix A which correspond to the existence of a TU matrix U such that $U^\dagger U = A$: we find a necessary condition on A which we call "g-nonnegativity"; to any g-nonnegative matrix A we associate a certain $\{0,1\}$-matrix $X(A)$; finally, such a U exists if and only if $X(A)$ has a TU signing U such that $U^\dagger U = A$. An auxiliary result about TU matrices then enables us to complete the proof of Theorem 1.

In Section 2 we briefly review some preliminary facts concerning totally unimodular matrices, regular matroids, and integer flows and cuts. In Section 3 we develop some facts about signed circuits (or simple flows), culminating in their characterization by metric data. In Section 4 we introduce g-nonnegative, g-positive, and g-feasible matrices, and prove Theorem 1. In Section 5 we conclude with some subsidiary results and examples, and two conjectures.

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2. Preliminaries

2.1. Totally unimodular matrices

For a matrix M of real numbers, let $M^\sharp$ be the matrix of absolute values of the entries of M. A matrix U with entries in $\mathbb{Z}$ is *totally unimodular* (TU) if every square submatrix of U has determinant in the set $\{-1, 0, +1\}$. For a $\{0, 1\}$-matrix X, a *totally unimodular signing* of X is a TU matrix U such that $U^\sharp = X$. A matrix Q with entries in $\mathbb{Z}$ is *weakly unimodular* (WU) if every maximal square submatrix of Q has determinant in the set $\{-1, 0, +1\}$. Let I_s denote the s-by-s identity matrix. The proof of Lemma 3 is elementary, and is omitted.

Lemma 3. *If an m-by-s matrix U is WU and contains I_s as a submatrix, then U is TU.*

Lemma 4 *(Camion). (See Lemma 13.1.6 of [3].) Let Q and U be TU matrices such that $Q^\sharp = U^\sharp$. Then Q can be changed into U by multiplying some rows and columns by -1.*

Theorem 13.1.3 of [3] determines exactly which $\{0, 1\}$-matrices have TU signings, although we do not need this result until Example 20.

2.2. Regular matroids

A *regular* matroid $(\mathcal{M}, E)$ is the column-matroid of some r-by-m TU matrix M of rank r, represented over the real field $\mathbb{R}$. The columns of M are labelled by the set E. Two $\mathbb{F}$-representations M and M' of a matroid are *equivalent* if there is an r-by-r matrix F invertible over $\mathbb{F}$, an E-by-E $\mathbb{F}$-weighted permutation matrix P, and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ such that

$$M' = \sigma(FMP).$$

(The column labels E are also permuted according to P.) Regular matroids are *uniquely representable* over any field $\mathbb{F}$, meaning that any two $\mathbb{F}$-representations of a regular matroid are equivalent (Corollary 10.1.4 of [3]).

Let $\mathcal{M}$ be represented by a TU matrix M. If $B \subseteq E$ is a base of $\mathcal{M}$ then there is a signed permutation matrix P bringing the labels in B into the first r positions, and a matrix F, invertible over $\mathbb{Z}$, such that

$$FMP = [I_r\ L]$$

for some r-by-s matrix L, where $s = m - r$. This is a representation of $\mathcal{M}$ *coordinatized by B*. Since M is TU, F is invertible over $\mathbb{Z}$, and P is a signed permutation matrix, it follows that FMP is WU. From Lemma 3 (and transposition) it follows that $[I_r\ L]$ is also TU (see also Lemmas 2.2.20 and 2.2.21 of [3]).

2.3. Integer flows, duality, and integer cuts

Let $(\mathcal{M}, E)$ be a regular matroid represented by the r-by-m TU matrix M. The *lattice of integer flows of $\mathcal{M}$* is

$$\Lambda(\mathcal{M}) = \ker(M) \cap \mathbb{Z}^E,$$

defined up to isometry. If B is a base of $\mathcal{M}$ and $M = [I_r\ L]$ is a representation of $\mathcal{M}$ coordinatized by B, then the matrix

$$U = \begin{bmatrix} -L \\ I_s \end{bmatrix}$$

is such that $MU = O$. Since M is TU it follows that U is TU, and since U has rank $s = \dim\ker(M)$, the columns of U form an ordered basis $\mathcal{B}(\mathcal{M}, B) = \{\beta_1, \ldots, \beta_s\}$ of $\Lambda(\mathcal{M})$. This is a *fundamental basis of* $\Lambda(\mathcal{M})$ *coordinatized by* B.

If $\mathcal{M}$ is represented by $M = [I_r\ L]$ then the *dual matroid* $\mathcal{M}^*$ is represented by $U^\dagger = [-L^\dagger\ I_s]$. If M is TU then $U^\dagger$ is TU. The *lattice of integer cuts* of a regular matroid $\mathcal{M}$, represented by M, is

$$\Gamma(\mathcal{M}) = \mathrm{Row}(M) \cap \mathbb{Z}^E,$$

in which $\mathrm{Row}(M)$ denotes the row-space of M. As a set this depends on M, but it is well defined up to isometry. From the above, it is clear that $\Lambda(\mathcal{M})$ and $\Gamma(\mathcal{M}^*)$ are isometric. Since the definition of $\Lambda(\mathcal{M})$ implicitly involves matroid duality, some of our arguments could be simplified slightly by considering $\Gamma(\mathcal{M})$ instead. However, to keep things straight we will consider only $\Lambda(\mathcal{M})$, except in Section 5.1.

Lemma 5 is a familiar fact, but we prefer to phrase it just the way we want.

Lemma 5. *Let* $(\mathcal{M}, E)$ *be a regular matroid of rank* r *on a set* E *of size* m*, and let* $s = m - r$*. Let* $\mathcal{B}$ *be any basis for* $\Lambda(\mathcal{M})$*, and let* Q *be an* E*-by-*s *matrix with columns given by the elements of* $\mathcal{B}$*. Then* Q *is WU.*

Proof. Pick a base B of $\mathcal{M}$ and let $M = [I_r\ L]$ represent $\mathcal{M}$ coordinatized by B. Then $\mathcal{B}' = \mathcal{B}(\mathcal{M}, B)$ is another basis for $\Lambda(\mathcal{M})$, and any matrix U with these columns is TU. Since $\mathcal{B}'$ and $\mathcal{B}$ are both bases for the lattice $\Lambda(\mathcal{M})$, the change of basis matrix F such that $Q = UF$ has $\det F = \pm 1$. Since U is WU it follows that Q is WU. □

If $\mathcal{B} = \{\beta_1, \ldots, \beta_s\}$ is any ordered set of vectors in an inner-product space, then the *Gram matrix* $\mathrm{Gram}(\mathcal{B}) = A = (a_{ij})$ of $\mathcal{B}$ is the s-by-s matrix with entries $a_{ij} = \langle \beta_i, \beta_j \rangle$ for all $1 \leqslant i, j \leqslant s$. Two lattices Λ and Λ' are isometric if and only if they have ordered bases $\mathcal{B}$ and $\mathcal{B}'$, respectively, such that $\mathrm{Gram}(\mathcal{B}) = \mathrm{Gram}(\mathcal{B}')$.

3. Simple flows, or signed circuits

3.1. Basic facts

Let $(\mathcal{M}, E)$ be a regular matroid represented by a TU matrix M, and let $\Lambda(\mathcal{M}) = \ker(M) \cap \mathbb{Z}^E$ be its lattice of integer flows (relative to M). For a column vector $\beta \in \mathbb{Z}^E$, the *support* of β is the subset

$$\mathrm{supp}(\beta) = \{e \in E:\ \beta(e) \neq 0\}$$

of E. For $\beta \in \Lambda(\mathcal{M})$ we have $M\beta = \mathbf{0}$, so that if $\beta \neq \mathbf{0}$ then $\mathrm{supp}(\beta)$ is a dependent set in $\mathcal{M}$, and hence contains a circuit (*i.e.* a minimal dependent set) of $\mathcal{M}$.

We require the following familiar facts (and include supporting arguments as proof sketches).

Lemma 6. *For every* $\beta \in \Lambda(\mathcal{M})$*, if* $\mathrm{supp}(\beta)$ *is a circuit* C *then* β *spans the subspace of* $\ker(M)$ *consisting of vectors with support contained in* C*.*

Proof sketch. If there were another linearly independent vector in this subspace then we could produce a dependent set of $\mathcal{M}$ properly contained in C, a contradiction. □

Lemma 7. *For every* $\beta \in \Lambda(\mathcal{M})$*, if* $\mathrm{supp}(\beta)$ *is a circuit* C *then all nonzero coordinates of* β *have the same absolute value.*

Proof sketch. Let M_C be the submatrix of M supported on columns in C, and write one of the columns of M_C as a linear combination of the others. This system of linear equations may be redundant – reducing to an irredundant subsystem, it can then be solved by Cramer's Rule, and all the determinants involved are in $\{-1, 0, +1\}$ since M is TU. □

An element of $\Lambda(\mathcal{M})$ is a *simple flow* (or signed circuit) if it is nonzero, all of its coordinates are in the set $\{-1, 0, +1\}$, and its support is a circuit of $\mathcal{M}$. Let $\mathcal{S}(\mathcal{M})$ denote the set of all simple flows in $\Lambda(\mathcal{M})$.

Lemma 8. *For every circuit C of $\mathcal{M}$ there are exactly two simple flows $\pm\alpha_C$ with support equal to C.*

Proof sketch. Since C is dependent, there is a nonzero $\beta \in \ker(M)$ with $\mathrm{supp}(\beta) \subseteq C$. Since C is a circuit, $\mathrm{supp}(\beta) = C$. Now Lemma 8 follows from Lemmas 6 and 7. □

Lemma 9. *If B is a base of a regular matroid $\mathcal{M}$ then every element of $\mathcal{B}(\mathcal{M}, B)$ is a simple flow in $\Lambda(\mathcal{M})$, and hence $\mathcal{S}(\mathcal{M})$ spans $\Lambda(\mathcal{M})$.*

Proof sketch. Each element of $\mathcal{B}(\mathcal{M}, B)$ is supported on a circuit, as one easily verifies. □

Lemma 10. *If M' is another matrix that represents $\mathcal{M}$ then an element of $\Lambda(\mathcal{M})$ is a simple flow relative to M' if and only if it is a simple flow relative to M.*

Proof sketch. By uniqueness of representation for regular matroids, $M' = FMP$ as in Section 2.2. Note that $\beta \in \ker(M') \cap \mathbb{Z}^E$ corresponds to $P\beta \in \ker(M) \cap \mathbb{Z}^E$, and that P is a signed permutation matrix. □

3.2. Consistent decompositions

By Lemma 9, each flow in $\Lambda(\mathcal{M})$ can be expressed as a sum of simple flows. For $\beta \in \Lambda(\mathcal{M})$, a *consistent decomposition of β* is a multiset $\mathcal{A}$ of simple flows such that:

(i) $\beta = \sum_{\alpha \in \mathcal{A}} \alpha$;
(ii) for all $\alpha \in \mathcal{A}$, $\mathrm{supp}(\alpha) \subseteq \mathrm{supp}(\beta)$;
(iii) for all $\alpha \in \mathcal{A}$ and $e \in E$, $\alpha(e)\beta(e) \geqslant 0$.

Proposition 11 is due to Tutte (Theorem 6.2 of [4] or Theorem 5.43 of [5]). We reproduce his proof for completeness and the readers' convenience.

Proposition 11 *(Tutte). Let $(\mathcal{M}, E)$ be a regular matroid represented by a WU matrix M. Then every $\beta \in \Lambda(\mathcal{M})$ has a consistent decomposition $\mathcal{A}$.*

Proof. We begin by showing that if $\beta \neq \mathbf{0}$ then there exists a simple flow α that *conforms to* β in the sense that $\mathrm{supp}(\alpha) \subseteq \mathrm{supp}(\beta)$ and $\alpha(e)\beta(e) > 0$ for all $e \in \mathrm{supp}(\alpha)$. If there is a counterexample then there is such a counterexample β with $\mathrm{supp}(\beta)$ minimal. By Lemmas 6, 7, and 8, $\mathrm{supp}(\beta)$ is not a circuit. By Lemma 8, again, there is a simple flow α with $\mathrm{supp}(\alpha) \subseteq \mathrm{supp}(\beta)$. Let $e \in \mathrm{supp}(\alpha)$ be such that $|\beta(e)|$ is minimal. Replacing α by $-\alpha$ if necessary, we may assume that $\alpha(e)\beta(e) > 0$. Now $\beta' = \beta - \beta(e)\alpha$ has $\mathrm{supp}(\beta') \subset \mathrm{supp}(\beta)$. If $\beta' = \mathbf{0}$ then α conforms to β. Otherwise, since β was a minimal counterexample, there is a simple flow α' conforming to β'. From the choice of $e \in \mathrm{supp}(\alpha)$ it follows that α' conforms to β as well, a contradiction.

The proposition now follows from the base case $\beta = \mathbf{0}$ (which has the consistent decomposition $\mathcal{A} = \varnothing$) by an easy induction on $\|\beta\| = \sum_{e\in E} |\beta(e)|$. For the induction step, let $\|\beta\| > 0$ and let α be a simple flow conforming to β. Then $\beta' = \beta - \alpha$ has $\|\beta'\| < \|\beta\|$, so by induction it has a consistent decomposition $\mathcal{A}'$. Thus, $\mathcal{A} = \mathcal{A}' \cup \{\alpha\}$ is a consistent decomposition of β. □

3.3. Metric characterization

Proposition 12. *Let $(\mathcal{M}, E)$ be a regular matroid represented by a WU matrix M. For any nonzero $\alpha \in \Lambda(\mathcal{M})$, the following are equivalent:*

(a) *the element α is a simple flow of $\Lambda(\mathcal{M})$ (relative to M);*
(b) *for all nonzero $\beta, \gamma \in \Lambda(\mathcal{M})$ such that $\alpha = \beta + \gamma$, $\langle \beta, \gamma \rangle < 0$.*

Proof. First, assume that (a) holds, and let $\alpha = \beta + \gamma$ with nonzero $\beta, \gamma \in \Lambda(\mathcal{M})$. For every $e \in E$ we have $\alpha(e) = \beta(e) + \gamma(e)$, and since $\alpha(e) \in \{-1, 0, +1\}$ we must have $\beta(e)\gamma(e) \leqslant 0$. Since the support of α is a circuit of $\mathcal{M}$, the supports of β and γ cannot be disjoint (since each contains at least one circuit of $\mathcal{M}$). Therefore $\langle \beta, \gamma \rangle < 0$, so that (b) holds.

Conversely, assume that (a) fails to hold. By Proposition 11, α has a consistent decomposition $\mathcal{A}$. Since α is nonzero, $\mathcal{A}$ is nonempty. If $|\mathcal{A}| = 1$ then α is a simple flow. Thus, assume that $|\mathcal{A}| \geqslant 2$, and let $\beta \in \mathcal{A}$ and $\gamma = \alpha - \beta$. Now β and γ are nonzero, $\alpha = \beta + \gamma$, and $\beta(e)\gamma(e) \geqslant 0$ for all $e \in E$, from the definition of consistent decomposition. This shows that $\langle \beta, \gamma \rangle \geqslant 0$, so that (b) fails to hold. □

For an arbitrary lattice Λ we define the set of *simple elements* to be the set $\mathcal{S}(\Lambda)$ of nonzero elements $\alpha \in \Lambda$ satisfying condition (b) in Proposition 12. Lemma 13 is immediate.

Lemma 13. *Let $\psi : \Lambda \to \Lambda'$ be an isometry of integer lattices. Then ψ restricts to a (metric-preserving) bijection from $\mathcal{S}(\Lambda)$ to $\mathcal{S}(\Lambda')$.*

Lemma 13 already severely constrains the possibilities for an isometry $\psi : \Lambda(\mathcal{M}) \to \Lambda(\mathcal{N})$. How to get an isomorphism $\phi : \mathcal{M}_\bullet \to \mathcal{N}_\bullet$ from this is still not clear, however. This is resolved in the next section.

4. g-Feasible matrices, and proof of Theorem 1

Let $\mathcal{B}(\mathcal{M}, B) = \{\beta_1, \ldots, \beta_s\}$ be a fundamental basis of $\Lambda(\mathcal{M})$ (coordinatized by some base B and representing TU matrix M). Let U be the m-by-s matrix with $\{\beta_1, \ldots, \beta_s\}$ as columns. The Gram matrix $A = U^\dagger U$ determines the isometry class of $\Lambda(\mathcal{M})$. The main effort in the proof of Theorem 1 is to reconstruct (as far as possible) the matrix U from its Gram matrix A. This is accomplished by Camion's Lemma 4 and Corollary 15 below.

4.1. Inclusion/Exclusion

Let $\mathcal{C} = \{C_1, \ldots, C_s\}$ be a collection of subsets of a finite set E, and let $[s] = \{1, 2, \ldots, s\}$. For every $S \subseteq [s]$, define

$$\phi_{\mathcal{C}}(S) = \left|\bigcap_{i \in S} C_i\right| \quad \text{and} \quad \gamma_{\mathcal{C}}(S) = \left|\bigcap_{i \in S} C_i \setminus \bigcup_{j \in [s] \setminus S} C_j\right|.$$

Here, by convention, $\bigcap \varnothing = E$. One sees that for every $S \subseteq [s]$,

$$\phi_{\mathcal{C}}(S) = \sum_{S \subseteq S' \subseteq [s]} \gamma_{\mathcal{C}}(S').$$

By Inclusion/Exclusion, it follows that for every $S \subseteq [s]$,

$$\gamma_{\mathcal{C}}(S) = \sum_{S \subseteq S' \subseteq [s]} (-1)^{|S' \setminus S|} \phi_{\mathcal{C}}(S').$$

Note that $\gamma_{\mathcal{C}}(S) \geqslant 0$ for all $S \subseteq [s]$, from the definition.

4.2. g-Feasible matrices

Let $A = (a_{ij})$ be an s-by-s symmetric matrix of integers, with positive diagonal entries. The three-element subsets (or *triples*) $\{h, i, j\}$ of $[s]$ are divided into three types: $\{h, i, j\}$ is *positive*, *null*, or *negative* depending on whether

$$a_{hi} \cdot a_{ij} \cdot a_{jh}$$

is positive, zero, or negative. Let $\Delta(A)$ denote the set of negative triples of $[s]$. Define a function $f_A : 2^{[s]} \to \mathbb{N}$ as follows: for each $S \subseteq [s]$,

$$f_A(S) = \begin{cases} 0 & \text{if } S = \varnothing, \\ 0 & \text{if } Y \subseteq S \text{ for some } Y \in \Delta(A), \\ a_{ii} & \text{if } S = \{i\}, \\ \min\{|a_{ij}|\colon \{i, j\} \subseteq S\} & \text{otherwise.} \end{cases}$$

Define a second function $g_A : 2^{[s]} \to \mathbb{Z}$ by Inclusion/Exclusion: for each $S \subseteq [s]$,

$$g_A(S) = \sum_{S \subseteq S' \subseteq [s]} (-1)^{|S' \setminus S|} f_A\big(S'\big).$$

The matrix A is *g-nonnegative* provided that $g_A(S) \geqslant 0$ for all $\varnothing \neq S \subseteq [s]$, and is *g-positive* if it is g-nonnegative and such that $g_A(\{i\}) > 0$ for all $i \in [s]$. Notice that, since $f_A(\varnothing) = 0$, if A is g-positive and $[s] \neq \varnothing$ then

$$g_A(\varnothing) = -\sum_{\varnothing \neq S \subseteq [s]} g_A(S) \leqslant -s < 0.$$

Proposition 14. *Let $\mathcal{B} = \{\beta_1, \ldots, \beta_s\} \subseteq \{-1, 0, +1\}^E$ be a set of column vectors, let U be the E-by-s matrix with columns $\{\beta_1, \ldots, \beta_s\}$, and let $A = (a_{ij}) = U^\dagger U = \mathrm{Gram}(\mathcal{B})$. For each $i \in [s]$ let $C_i = \mathrm{supp}(\beta_i)$, and let $\mathcal{C} = \{C_1, \ldots, C_s\}$. If U is TU then for all $\varnothing \neq S \subseteq [s]$ we have $f_A(S) = \phi_{\mathcal{C}}(S)$ and $g_A(S) = \gamma_{\mathcal{C}}(S)$, so that $U^\dagger U$ is g-nonnegative.*

Proof. We use the notation $\beta_i(e) = U_{ei}$ for the entries of the matrix U. We claim that for all $\varnothing \neq S \subseteq [s]$,

$$f_A(S) = \phi_{\mathcal{C}}(S).$$

From this it follows by Inclusion/Exclusion that for all $\varnothing \neq S \subseteq [s]$,

$$g_A(S) = \gamma_{\mathcal{C}}(S).$$

The combinatorial meaning of $\gamma_{\mathcal{C}}$ then shows that A is g-nonnegative.

To prove the claim, consider any nonempty $S \subseteq [s]$.

If $S = \{i\}$ then

$$f_A\big(\{i\}\big) = a_{ii} = \langle \beta_i, \beta_i \rangle = |C_i| = \phi_{\mathcal{C}}\big(\{i\}\big).$$

If $S = \{i, j\}$ then consider any $\{e, f\} \subseteq C_i \cap C_j$. Since U is TU, the submatrix Z of U supported on rows e and f and columns i and j has $\det Z \in \{-1, 0, +1\}$. All four entries of Z are in $\{-1, +1\}$. Computing the determinants of all possibilities one finds that Z has an even number of -1's, that $\det Z = 0$, and that Z has rank one. That is,

$$\beta_i(e)\beta_j(e) = \beta_i(f)\beta_j(f).$$

It follows that the function $e \mapsto \beta_i(e)\beta_j(e)$ is constant on $C_i \cap C_j$, so that $|C_i \cap C_j| = |a_{ij}|$. Equivalently, for any $e \in C_i \cap C_j$,

$$a_{ij} \cdot \beta_i(e)\beta_j(e) > 0.$$

(This is true even if $a_{ij} = 0$, since then $C_i \cap C_j = \varnothing$.) Therefore,

$$f_A\big(\{i, j\}\big) = |a_{ij}| = |C_i \cap C_j| = \phi_{\mathcal{C}}\big(\{i, j\}\big).$$

It remains to consider the case that $|S| \geqslant 3$.

First, consider any $\{h, i, j\} \subseteq S$. If $e \in C_h \cap C_i \cap C_j$ then from the above it follows that

$$a_{hi}a_{ij}a_{jh} \cdot \beta_h(e)^2\beta_i(e)^2\beta_j(e)^2 > 0,$$

and hence that $\{h, i, j\}$ is a positive triple for A. Thus, if S contains a negative or a null triple $\{h, i, j\}$ then

$$f_A(S) = 0 = |C_h \cap C_i \cap C_j| = \left|\bigcap_{k\in S} C_k\right| = \phi_{\mathcal{C}}(S).$$

Finally, consider the case that every triple contained in S is positive. We show that $f_A(S) = \phi_{\mathcal{C}}(S)$ by contradiction, so suppose that there exists a set $S \subseteq [s]$ such that $f_A(S) \neq \phi_{\mathcal{C}}(S)$. Then there is such a set for which S is minimal according to set inclusion; by the above observations, $|S| = t \geqslant 3$. Replacing β_i by $-\beta_i$ as necessary, we can assume that $a_{ij} > 0$ for all $\{i, j\} \subseteq S$. (This is proved by induction on t; the base case $|t| = 3$ and the induction step both rely on the fact that every triple contained in S is positive.) Then, multiplying rows of U by -1 as necessary, we can assume that $\beta_i(e) = 1$ for all $i \in S$ and $e \in C_i$. Let $\{i, j\} \subset S$ be such that a_{ij} is minimal. Note that since $A = \mathrm{Gram}(\mathcal{B})$ and each $\beta_i \in \{-1, 0, +1\}^E$, we have $a_{ij} \leqslant \min\{a_{ii}, a_{jj}\}$ for all $\{i, j\} \subseteq [s]$. Also note that for every $S' \subseteq S$ with $|S'| \geqslant 2$, we have $f_A(S') \geqslant f_A(S) = a_{ij}$. Now

$$\phi_{\mathcal{C}}(S) = \left|\bigcap_{\ell\in S} C_\ell\right| \leqslant |C_i \cap C_j| = a_{ij} = f_A(S).$$

Since S is a minimal set for which $f_A(S) \neq \phi_{\mathcal{C}}(S)$, it follows that $\phi_{\mathcal{C}}(S) < f_A(S)$, and that for every $h \in S$,

$$\phi_{\mathcal{C}}\big(S \setminus \{h\}\big) = f_A\big(S \setminus \{h\}\big) \geqslant f_A(S) > \phi_{\mathcal{C}}(S).$$

Therefore, for every $h \in S$ there is an element

$$e_h \in \left(\bigcap_{\ell\in S\setminus\{h\}} C_\ell\right) \setminus C_h.$$

These elements are pairwise distinct. Let Z be the submatrix of U supported on columns $\{\beta_i\colon\ i \in S\}$ and rows $\{e_h\colon\ h \in S\}$. By permuting rows and columns of Z we can bring this into the form $J_t - I_t$, in which J_t is the t-by-t all-ones matrix. This is the adjacency matrix of the complete graph K_t, which has eigenvalues $t - 1$ of multiplicity 1 and -1 of multiplicity $t - 1$. Therefore, since $t \geqslant 3$, we see that

$$\det Z = \pm \det(J_t - I_t) = \pm(t - 1) \notin \{-1, 0, +1\}.$$

This contradicts the hypothesis that U is TU, showing that the defective set $S \subseteq [s]$ is impossible. This completes the proof. $\square$

Let $A = (a_{ij})$ be an s-by-s g-nonnegative matrix, and let $k = -g_A(\varnothing) \geqslant 0$. Define a k-by-s $\{0, 1\}$-matrix $X(A)$ by saying that for each $\varnothing \neq S \subseteq [s]$, exactly $g_A(S)$ rows of $X(A)$ are equal to the indicator row-vector of the subset $S \subseteq [s]$. (Note that $X(A)$ has no zero rows.) The matrix $X(A)$ is defined only up to arbitrary permutation of the rows. When A is g-positive we usually permute the rows of $X(A)$ so that the bottom s rows form an identity submatrix I_s.

Corollary 15. *Let U be a TU matrix, and let $A = U^\dagger U$ (which is g-nonnegative). Then the rows of $X(A)$ can be permuted so that they are exactly the nonzero rows of $U^\sharp$.*

Proof. Since U is TU, we have $g_A(S) = \gamma_{\mathcal{C}}(S)$ for all $\varnothing \neq S \subseteq [s]$, using the result and notation of Proposition 14. Thus, for all $\varnothing \neq S \subseteq [s]$, exactly $g_A(S)$ rows of U have support equal to the set S of columns. By definition, the same is true of $X(A)$. The matrix U may also have some zero rows. $\square$

A symmetric matrix A is *g-feasible* if there is a TU matrix U such that $U^\dagger U = A$. By Proposition 14, this implies that A is g-nonnegative. Corollary 15 and Camion's Lemma 4 show that if such a matrix U exists then it is unique (modulo deleting zero rows, permuting the rows, and changing the signs of some rows and columns). This is the uniqueness result at the heart of our proof of Theorem 1.

4.3. *Proof of Theorem 1*

One last technical detail is required.

Lemma 16. *Let U be an m-by-s TU matrix containing I_s as a submatrix. Then every WU matrix Q such that $Q^\dagger Q = U^\dagger U$ is TU.*

Proof. We proceed by induction on s. The basis of induction, $s = 1$, is trivial since in this case if Q is WU then Q is TU.

For the induction step we begin by showing that all $(s-1)$-by-$(s-1)$ minors of Q are in $\{-1, 0, +1\}$. Let Z' be a nonsingular $(s-1)$-by-$(s-1)$ submatrix of Q. Let Z be a nonsingular s-by-s submatrix of Q that contains Z'. Then $\det(Z) = \pm 1$, since Q is WU, so that $F = Z^{-1}$ also has $\det(F) = \pm 1$. Now QF is WU and contains I_s as a submatrix, so QF is TU by Lemma 3. Permuting this I_s submatrix of QF to the bottom s rows, the columns of QF are a fundamental basis of a lattice $\Lambda(\mathcal{N})$ for some regular matroid $\mathcal{N}$. Similarly, the columns of U are a fundamental basis of a lattice $\Lambda(\mathcal{M})$ for some regular matroid $\mathcal{M}$ (after permuting the I_s submatrix of U to the bottom s rows). Since $Q^\dagger Q = U^\dagger U$, it follows that $(QF)^\dagger QF = (UF)^\dagger UF$. Thus, the i-th column of UF is the image of the i-th column of QF (for each $i \in [s]$) by means of an isometry from $\Lambda(\mathcal{N})$ to $\Lambda(\mathcal{M})$. Since the columns of QF are simple flows in $\Lambda(\mathcal{N})$ (by Lemma 10), it follows from Lemma 13 that the columns of UF are simple flows in $\Lambda(\mathcal{M})$. Thus, by Proposition 12, the columns of UF are $\{-1, 0, +1\}$-valued. Since U contains I_s, UF contains $I_s F = F$ as a submatrix. Thus, the entries of $F = Z^{-1} = \mathrm{adj}(Z)/\det(Z)$ are all in the set $\{-1, 0, +1\}$. Therefore $\det(Z') = \pm 1$, as required.

Now, for any $i \in [s]$, let Q_i be the submatrix of Q obtained by deleting column i from Q, and define U_i similarly. Clearly U_i is TU, U_i contains I_{s-1} as a submatrix, and $Q_i^\dagger Q_i = U_i^\dagger U_i$. The previous paragraph shows that each Q_i is WU. Finally, the induction hypothesis shows that each Q_i is TU, and since Q is also WU it follows that Q is TU. This completes the induction step, and the proof. $\square$

Proof of Theorem 1. We begin by proving that if $\mathcal{M}$ and $\mathcal{N}$ are regular matroids for which $\mathcal{M}_\bullet$ and $\mathcal{N}_\bullet$ are isomorphic, then $\Lambda(\mathcal{M})$ and $\Lambda(\mathcal{N})$ are isometric. Let $\phi : E(\mathcal{M}_\bullet) \to E(\mathcal{N}_\bullet)$ be an isomorphism, let r be the rank of $\mathcal{M}_\bullet$ and let $k = |E(\mathcal{M}_\bullet)|$. Let B be any base of $\mathcal{M}_\bullet$ and let $\phi(B)$ be the corresponding base of $\mathcal{N}_\bullet$. Coordinatized by these bases, both $\mathcal{M}_\bullet$ and $\mathcal{N}_\bullet$ are represented by the same r-by-k matrix of the form $[I_r\ L]$ for some r-by-s TU matrix L (in which $s = k - r$). Since $\mathcal{M}_\bullet$ has no co-loops, L has no zero rows. Let $\mathcal{M}$ have p co-loops, and let $\mathcal{N}$ have q co-loops. Then $\mathcal{M}$ and $\mathcal{N}$ are represented by the matrices M and N, respectively, in which

$$M = \begin{bmatrix} I_p & O_{p\times r} & O_{p\times s} \\ O_{r\times p} & I_r & L \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} I_q & O_{q\times r} & O_{q\times s} \\ O_{r\times q} & I_r & L \end{bmatrix}.$$

Here, $O_{a\times b}$ denotes the a-by-b all-zero matrix.

As in Section 2.3, the lattices $\Lambda(\mathcal{M})$ and $\Lambda(\mathcal{N})$ have bases given by the columns of the matrices

$$Q_{\mathcal{M}} = \begin{bmatrix} O_{p\times s} \\ -L \\ I_s \end{bmatrix} \quad \text{and} \quad Q_{\mathcal{N}} = \begin{bmatrix} O_{q\times s} \\ -L \\ I_s \end{bmatrix},$$

respectively. One sees immediately that

$$Q_{\mathcal{M}}^{\dagger} Q_{\mathcal{M}} = Q_{\mathcal{N}}^{\dagger} Q_{\mathcal{N}},$$

and it follows that the lattices $\Lambda(\mathcal{M})$ and $\Lambda(\mathcal{N})$ are isometric.

Conversely, assume that $\mathcal{M}$ and $\mathcal{N}$ are regular matroids and let $\psi : \Lambda(\mathcal{M}) \to \Lambda(\mathcal{N})$ be an isometry. Let s be the rank of $\Lambda(\mathcal{M})$ and $\Lambda(\mathcal{N})$. Let $|E(\mathcal{M})| = m$ and $|E(\mathcal{N})| = n$.

Let $\mathcal{B} = \mathcal{B}(\mathcal{M}, B)$ be a fundamental basis of $\Lambda(\mathcal{M})$ coordinatized by a base B of $\mathcal{M}$. Let U be an m-by-s matrix with the elements $\beta_i \in \mathcal{B}$ for $i \in [s]$ as columns. Fix a TU matrix N representing $\mathcal{N}$ over $\mathbb{R}$, such that $\Lambda(\mathcal{N}) = \ker(N) \cap \mathbb{Z}^n$. Let Q be the n-by-s matrix with the elements $\psi(\beta_i)$ for $i \in [s]$ as columns.

Now U is an m-by-s TU matrix that contains I_s as a submatrix, and since ψ is an isometry it follows that $Q^{\dagger}Q = U^{\dagger}U$. Since ψ is an isometry and $\mathcal{B}$ is a basis for $\Lambda(\mathcal{M})$, the columns of Q form a basis for $\Lambda(\mathcal{N})$. From Lemma 5 it follows that Q is WU, and then from Lemma 16 it follows that Q is TU.

Now both U and Q are TU matrices such that $A = U^{\dagger}U = Q^{\dagger}Q$. By Corollary 15, the rows of U and of Q can be permuted so that the nonzero rows of $U^{\sharp}$ and of $Q^{\sharp}$ both agree with $X = X(A)$. Let $k = -g_A(\varnothing)$ be the number of rows of X, let $r = k - s$, let $p = m - k$, and let $q = n - k$. We may assume that the last s rows of X support an I_s submatrix, so that $X = [K^{\dagger}\ I_s]^{\dagger}$ for some r-by-s matrix K with no zero rows. Thus, the matrices $U^{\sharp}$ and $Q^{\sharp}$ have the forms

$$U^{\sharp} = \begin{bmatrix} O_{p\times s} \\ K \\ I_s \end{bmatrix} \quad \text{and} \quad Q^{\sharp} = \begin{bmatrix} O_{q\times s} \\ K \\ I_s \end{bmatrix}.$$

By Camion's Lemma 4, there are diagonal matrices H and F, invertible over $\mathbb{Z}$, such that the submatrix in the last k rows of $Q' = HQF$ equals the submatrix in the last k rows of U. The columns of Q' form a basis for $\Lambda(\mathcal{N})$, and the matrices U and Q' have the forms

$$U = \begin{bmatrix} O_{p\times s} \\ -L \\ I_s \end{bmatrix} \quad \text{and} \quad Q' = \begin{bmatrix} O_{q\times s} \\ -L \\ I_s \end{bmatrix}$$

for some r-by-s TU matrix L with no zero rows. Thus the regular matroids $\mathcal{M}$ and $\mathcal{N}$ are represented (over $\mathbb{R}$) by the matrices

$$M = \begin{bmatrix} I_p & O_{p\times r} & O_{p\times s} \\ O_{r\times p} & I_r & L \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} I_q & O_{q\times r} & O_{q\times s} \\ O_{r\times q} & I_r & L \end{bmatrix},$$

respectively. From the forms of these representing matrices one sees that $\mathcal{M}_\bullet$ and $\mathcal{N}_\bullet$ are both represented by $[I_r\ L]$, and thus are isomorphic. □

5. Concluding observations

5.1. The lattice of integer cuts

Recall the lattice $\Gamma(\mathcal{M})$ of integer cuts of a regular matroid $\mathcal{M}$, defined in Section 2.3. Theorem 1 is equivalent to each of the following two statements. (We omit the trivial proofs by duality.) For any matroid $\mathcal{M}$, let $\mathcal{M}^\circ$ denote the minor of $\mathcal{M}$ obtained by deleting all loops of $\mathcal{M}$.

Corollary 17. *Let $\mathcal{M}$ and $\mathcal{N}$ be regular matroids. Then $\Gamma(\mathcal{M})$ and $\Gamma(\mathcal{N})$ are isometric if and only if $\mathcal{M}^\circ$ and $\mathcal{N}^\circ$ are isomorphic.*

Corollary 18. *Let $\mathcal{M}$ and $\mathcal{N}$ be regular matroids. Then $\Lambda(\mathcal{M})$ and $\Gamma(\mathcal{N})$ are isometric if and only if $\mathcal{M}_\bullet$ and $(\mathcal{N}^\circ)^* = (\mathcal{N}^*)_\bullet$ are isomorphic.*

5.2. Lattices in general

For convenience, a basis $\mathcal{B}$ of a lattice Λ is said to be g-nonnegative, g-positive, or g-feasible depending on whether $\mathrm{Gram}(\mathcal{B})$ has that property.

Proposition 19. *Let Λ be an integral lattice. The following are equivalent*:

(a) *Λ has a g-feasible basis.*
(b) *Λ has a g-feasible and g-positive basis.*
(c) *Λ is isometric with $\Lambda(\mathcal{M})$ for some regular matroid $\mathcal{M}$.*

Proof. For (a) implies (b): let $\mathcal{B}$ be a g-feasible basis for Λ. Let $A = \mathrm{Gram}(\mathcal{B})$, let $X = X(A)$, and let U be a TU signing of X such that $U^\dagger U = A$. Say that U is an m-by-s matrix. Since A has rank s, there is an invertible s-by-s submatrix Z of U. Since U is TU, $\det(Z) = \pm 1$, so that $F = Z^{-1}$ is an integer matrix and $\det(F) = \pm 1$ as well. Now, $Q = UF$ is an m-by-s WU matrix that contains I_s as a submatrix, so by Lemma 3, Q is TU. The columns of Q form a basis for Λ (since F is invertible over $\mathbb{Z}$), and $Q^\dagger Q$ is g-positive (by Proposition 14, and since Q contains I_s). Since $Q^\dagger Q$ is clearly g-feasible, this proves (b).

For (b) implies (c): if $\mathcal{B}$ is a g-feasible and g-positive basis of Λ then $A = \mathrm{Gram}(\mathcal{B})$ is g-positive and $X = X(A)$ has a TU signing U such that $U^\dagger U = A$. The columns of U form a basis $\mathcal{B}'$ of the lattice $\Lambda(\mathcal{M})$ of some regular matroid. Now

$$\mathrm{Gram}\big(\mathcal{B}'\big) = Q^\dagger Q = A = \mathrm{Gram}(\mathcal{B}),$$

so that Λ and $\Lambda(\mathcal{M})$ are isometric.

Trivially (b) implies (a). For (c) implies (b): assume that $\psi : \Lambda(\mathcal{M}) \to \Lambda$ is an isometry, and let B be any base of $\Lambda(\mathcal{M})$. Then $\mathcal{B} = \mathcal{B}(\mathcal{M}, B)$ is a g-feasible and g-positive basis of $\Lambda(\mathcal{M})$, so that $\psi(\mathcal{B})$ is a g-feasible and g-positive basis of Λ. $\square$

5.3. Some examples

Example 20. *A g-positive matrix that is not g-feasible*. The matrix A shown below is g-positive, with $X = X(A)$ as shown:

$$A = \begin{bmatrix} 3 & 1 & 1 & 2 \\ 1 & 3 & 1 & 2 \\ 1 & 1 & 3 & 2 \\ 2 & 2 & 2 & 5 \end{bmatrix}, \qquad X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Theorem 13.1.3 of [3] shows that X does not have a TU signing (by pivoting on the top-right entry). Thus, A is not g-feasible.

Example 21. *A g-nonnegative matrix A such that $X(A)$ has a TU signing, but A is not g-feasible*. The matrix A shown below is g-nonnegative, with $X = X(A)$ as shown:

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix}, \qquad X = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

One checks that X itself is TU. By Camion's Lemma 4, any TU signing U of X is obtained from X by multiplying some rows and columns of X by -1. For any such matrix U, the Gram matrix $U^\dagger U$ is obtained from $X^\dagger X$ by multiplying some rows and the same columns by -1. But $X^\dagger X = A^\sharp$, and A cannot be obtained from $A^\sharp$ by means of this operation. Thus, there is no TU signing Q of X such that $Q^\dagger Q = A$. (A g-positive example of this is $A + I_4$.)

Example 22. *A matrix Q such that $Q^\dagger Q$ is g-positive and g-feasible, but Q is not WU.* This example relates to the hypotheses of Lemma 16. Clearly $Q = [2]$ is not WU. The matrix $A = Q^\dagger Q = [4]$ is g-positive with $X = X(A) = [1\ 1\ 1\ 1]^\dagger$. Clearly X is TU with $X^\dagger X = A$, so A is g-feasible.

Example 23. *The body-centered cubic lattice is $\Lambda(K_4) \simeq \Gamma(K_4)$.* To see this, let the cycles of length four in K_4 be C_1, C_2, and C_3, and let α_i be a simple flow supported on C_i for each $i \in \{1, 2, 3\}$. Now $\{\alpha_1, \alpha_2, \alpha_3\}$ spans a sublattice Π of $\Lambda(K_4)$, and has Gram matrix $4I_3$. Thus, Π is a cubical lattice with minimum length 2. Now, $\alpha_1 + \alpha_2 + \alpha_3 = 2\beta$ for some simple flow $\beta \in \Lambda(K_4)$ supported on a three-cycle. In fact $\Lambda(K_4)$ is the disjoint union of Π and $\Pi + \beta$, proving the claim. The lattices $\Lambda(K_n)$ are discussed on pp. 194–196 of [1].

Example 24. *The root lattice A_n is $\Lambda(\mathcal{U}_{1,n+1}) \simeq \Gamma(\mathcal{U}_{n,n+1})$. (The face-centered cubic lattice is A_3.)* To see this, for each $i \in [n+1]$ let $\mathbf{e}_i$ be the coordinate column vector of length $n+1$ with all entries 0 except for a 1 in row i. The root lattice A_n has as a basis the vectors $s_i = \mathbf{e}_{i+1} - \mathbf{e}_i$ for all $i \in [n]$. Since $\mathcal{U}_{1,n+1}$ is represented by the all-ones matrix with one row and $n+1$ columns, it is easy to see that $\{s_1, \ldots, s_n\}$ is a basis for $\Lambda(\mathcal{U}_{1,n+1})$ as well. The argument on p. 194 of [1] shows that these are the only root lattices of the form $\Lambda(\mathcal{M})$ for some regular matroid.

5.4. Sixth-root-of-unity matroids

Let $\omega = e^{i\pi/3}$ be a primitive sixth-root of unity, and let

$$\mathbb{E} = \{z \in \mathbb{C}:\ z = a + b\omega \text{ for some } a, b \in \mathbb{Z}\}$$

be the ring (in fact a PID) of *Eisenstein integers*. A *sixth-root-of-unity matrix* ($\sqrt[6]{1}$ matrix, for short) is a matrix with entries in $\mathbb{C}$ such that every square submatrix has determinant d such that either $d = 0$ or $d^6 = 1$. A *sixth-root-of-unity matroid* ($\sqrt[6]{1}$ matroid, for short) is one which can be represented over $\mathbb{C}$ by a $\sqrt[6]{1}$ matrix. Clearly, regular matroids are $\sqrt[6]{1}$. Lemma 5.8 of [7] gives sufficient conditions for a $\sqrt[6]{1}$ matroid to be uniquely representable over $\mathbb{C}$ (by a $\sqrt[6]{1}$ matrix).

For example, $\mathcal{U}_{2,4}$ is not a binary matroid (hence not regular) but it is represented over $\mathbb{C}$ by the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \omega \end{bmatrix}.$$

The vector space of real flows of $\mathcal{U}_{2,4}$ (relative to this representation) is the real span of the column vector $[-1\ \ -1\ 1\ 0]^\dagger$. The vector space of complex flows of $\mathcal{U}_{2,4}$ is the complex span of both $[-1\ \ -1\ 1\ 0]^\dagger$ and $[-1\ \ -\overline{\omega}\ 0\ 1]^\dagger$. The analogue of the lattice of integer flows for a matroid $(\mathcal{M}, E)$ represented over $\mathbb{C}$ by a $\sqrt[6]{1}$ matrix M is the *lattice of Eisenstein flows*

$$\Lambda_{\mathbb{E}}(\mathcal{M}) = \ker_{\mathbb{C}}(M) \cap \mathbb{E}^E.$$

The inner product on $\Lambda_{\mathbb{E}}(\mathcal{M})$ is induced by the Hermitian inner product on $\mathbb{C}^E$.

Such a lattice is not just an abelian group, but even an $\mathbb{E}$-module. This allows a stronger version of isometry: $\psi : \Lambda \to \Lambda'$ is an $\mathbb{E}$-*isometry* if it is a bijection such that both ψ and ψ^{-1} are $\mathbb{E}$-module homomorphisms that preserve the inner products on the lattices. Clearly an $\mathbb{E}$-isometry is an isometry in the usual sense.

If M and M' are $\sqrt[6]{1}$ matrices representing the same matroid $\mathcal{M}$, then $\ker_{\mathbb{C}}(M) \cap \mathbb{E}^E$ and $\ker_{\mathbb{C}}(M') \cap \mathbb{E}^E$ are $\mathbb{E}$-isometric if and only if M and M' are equivalent representations of $\mathcal{M}$. (This follows easily from the definition of representation equivalence.) Thus, $\mathcal{M}$ is uniquely representable by a $\sqrt[6]{1}$ matrix if and only if the $\mathbb{E}$-isometry class of $\Lambda_{\mathbb{E}}(\mathcal{M})$ is independent of the representing matrix M. It is not too difficult to see that for a regular matroid $\mathcal{M}$,

$$\Lambda_{\mathbb{E}}(\mathcal{M}) = \Lambda(\mathcal{M}) \otimes \mathbb{E},$$

but, as the example of $\mathcal{U}_{2,4}$ shows, this does not hold for all $\sqrt[6]{1}$ matroids. (In fact, this equality holds if and only if $\mathcal{M}$ is regular, since if it holds then $\Lambda_{\mathbb{E}}(\mathcal{M})$ has an $\mathbb{E}$-basis $\mathcal{B}$ that is also a $\mathbb{Z}$-basis of $\Lambda(\mathcal{M})$. Thus, the matrix Q formed from the column vectors in $\mathcal{B}$ is $\sqrt[6]{1}$ and real, hence TU. Therefore $\mathcal{M}^*$ and hence $\mathcal{M}$ are regular.)

The two-sum $\mathcal{U}_{2,4} \oplus_2 \mathcal{U}_{2,4}$ has two inequivalent representations by $\sqrt[6]{1}$ matrices, and these yield Eisenstein flow lattices that have bases with Gram matrices

$$\begin{bmatrix} 3 & 1+\omega & 1+\omega \\ 1+\overline{\omega} & 4 & 4-\sigma \\ 1+\overline{\omega} & 4-\overline{\sigma} & 4 \end{bmatrix}$$

in which $\omega, \sigma \in \mathbb{C}$ are primitive sixth-roots of unity. The two cases $\sigma = \omega$ and $\sigma = \overline{\omega}$ yield lattices which are not $\mathbb{E}$-isometric, but seem to be isometric.

Conjecture 25. *If M and M' are sixth-root-of-unity matrices representing the same matroid $\mathcal{M}$, then $\ker_{\mathbb{C}}(M) \cap \mathbb{E}^E$ and $\ker_{\mathbb{C}}(M') \cap \mathbb{E}^E$ are isometric.*

Conjecture 26. *Let $\mathcal{M}$ and $\mathcal{N}$ be sixth-root-of-unity matroids. Then $\Lambda_{\mathbb{E}}(\mathcal{M})$ and $\Lambda_{\mathbb{E}}(\mathcal{N})$ are isometric if and only if $\mathcal{M}_\bullet$ and $\mathcal{N}_\bullet$ are isomorphic.*

One could perhaps adopt a strategy similar to the one we used to prove Theorem 1 for regular matroids. The Gram matrix of a basis of $\Lambda_{\mathbb{E}}(\mathcal{M})$ is in general complex Hermitian with entries in $\mathbb{E}$. If one can identify a metric characterization of simple flows, and an appropriate generalization of g-feasible matrices, then much of our argument could carry over.

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