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Classification of finite simple Lie conformal superalgebras

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Dedicated to Claudio Procesi on the occasion of his 60th birthday

1. Introduction

In recent years, two-dimensional conformal field theory has attracted the attention of many mathematicians and physicists. Borcherds [Bo] introduced the notion of a vertex algebra, which encodes the operator product expansion (OPE) of chiral fields in this theory. The singular parts of the OPE (or, equivalently, the commutators of fields) are encoded by a Lie conformal superalgebra [K2,K3]. By means of this formalism, computations of OPE's are greatly simplified.

In the language of the λ-bracket, a Lie conformal superalgebra R is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map

$$R\otimes R\to\mathbb{C}[\lambda]\otimes R,\qquad a\otimes b\mapsto[a_\lambda b],$$

satisfying the following axioms [DK,K2] ($a,b,c\in R$):

$$[\partial a_\lambda b]=-\lambda[a_\lambda b],\qquad [a_\lambda\partial b]=(\partial+\lambda)[a_\lambda b],\qquad\text{(sesquilinearity)}$$

$$[b_\lambda a]=-(-1)^{p(a)p(b)}[a_{-\lambda-\partial}b],\qquad\text{(skew-commutativity)}$$

$$\big[a_\lambda[b_\mu c]\big]=\big[[a_\lambda b]_{\lambda+\mu}c\big]+(-1)^{p(a)p(b)}\big[b_\mu[a_\lambda c]\big].\qquad\text{(Jacobi identity)}$$

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Finite (i.e. finitely generated as a $\mathbb{C}[\partial]$-module) simple Lie conformal algebras were classified in [DK] and their representation theory was further developed in [CK1,BKV].

On the other hand, Lie conformal superalgebras are closely connected to the notion of a formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$, i.e. a Lie superalgebra $\mathfrak{g}$ spanned by the coefficients of a family $\mathcal{F}$ of mutually local formal distributions. Namely, to a Lie conformal superalgebra R one canonically associates the maximal formal distribution Lie superalgebra $\operatorname{Lie} R = R[t,t^{-1}]/\tilde{\partial} R[t,t^{-1}]$ (see Section 3), which establishes an equivalence between the category of Lie conformal superalgebras and the category of equivalence classes of formal distribution Lie superalgebras obtained as quotients of $\operatorname{Lie} R$ by irregular ideals, see [K2].

In the present paper, we give the classification of finite simple Lie conformal superalgebras. The main result is the following theorem (announced in [K2,K6]):

Theorem 1.1. *Any finite simple Lie conformal superalgebra R is isomorphic to one of the Lie conformal superalgebras of the following list* (*see Section* 3.2 *for their construction*):

(1) W_N $(N \geqslant 0)$;
(2) $S_{N,a}$ $(N \geqslant 2,\ a \in \mathbb{C})$;
(3) $\widetilde{S}_N$ $(N$ *even*, $N \geqslant 2)$;
(4) K_N $(N \geqslant 0,\ N \neq 4)$;
(5) K_4';
(6) CK_6;
(7) $\operatorname{Cur} \mathfrak{s}$, *where $\mathfrak{s}$ is a simple finite-dimensional Lie superalgebra.*

The general outline of the proof of this theorem is similar to that of [DK] in the non "super" case. First of all, we extend to Lie superalgebras the classical Cartan–Guillemin theorem [G1,B,BB] which asserts that any minimal non-abelian closed ideal in a linearly compact Lie algebra L is of the form $\mathbb{C}[\![t_1,\dots,t_r]\!] \widehat{\otimes} \mathfrak{s}$, where $\mathfrak{s}$ is a simple linearly compact Lie algebra (see Theorem 2.7 and Corollary 2.8).

Secondly, we deduce that the annihilation algebra $\mathcal{A}(R)$ is an irreducible central extension of $\mathbb{C}[\![t_1,\dots,t_r]\!] \widehat{\otimes} \mathfrak{s}$, where $\mathfrak{s}$ is simple linearly compact Lie superalgebra (Lemma 5.3). Recall that $\mathcal{A}(R)$ is the completion of the image of $R[t]$ in $\operatorname{Lie} R$. It is linearly compact if R is finite. The operator $-\partial/\partial t$ on $R[t,t^{-1}]$ induces a derivation T of $\mathcal{A}(R)$, and the semi-direct product $\mathbb{C}T \ltimes \mathcal{A}(R)$ is called the extended annihilation algebra.

Thirdly, we remark that the growth of $\mathcal{A}(R)$ is smaller or equal than one, so that we obtain that either $r = 1$ and $\mathfrak{s}$ is a simple finite-dimensional Lie superalgebra or $r = 0$ and $\mathfrak{s}$ is a simple linearly compact Lie superalgebra of growth 1, and we may use the classifications of the papers [K4] and [K5], respectively. This produces a list of all possible annihilation algebras (Proposition 5.5).

The fourth step is the classification (up to conjugacy) of all even surjective continuous derivations of these candidates for the annihilation algebras. This leads to the list of all possibilities for the extended annihilation algebras (Theorem 5.11).

Finally, we reconstruct R from the extended annihilation algebra to obtain the main result (Theorem 5.12). An immediate corollary of this result is a classification of all finite simple formal distribution Lie superalgebras (Corollary 5.13).

Along the way we show in Section 2.2 that the growth of a simple linearly compact Lie superalgebra is independent of its algebra filtration, and we classify in Section 4 all central extensions of finite simple Lie conformal superalgebras (which are important for the construction of simple vertex algebras).

2. Linearly compact Lie superalgebras

2.1. The Cartan–Guillemin theorem

Recall that a *vector superspace* is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space, $V = V_{\bar{0}} \oplus V_{\bar{1}}$. We denote by $p(a) = \alpha$ the *parity* of an homogeneous element a, $a \in V_\alpha$, $\alpha \in \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$. A subspace U of V is by definition $\mathbb{Z}/2\mathbb{Z}$-graded, i.e. $U = (U \cap V_{\bar{0}}) \oplus (U \cap V_{\bar{1}})$. All vector superspaces, linear maps and tensor products are over the field $\mathbb{C}$ of complex numbers. Exterior and symmetric powers of a vector superspace are to be understood in the super-sense (see [K4]).

A *superalgebra* A is a vector superspace endowed with an algebra structure such that $A_\alpha A_\beta \subset A_{\alpha+\beta}$, with $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$.

A *Lie superalgebra* is a superalgebra satisfying super-anticommutativity and the super-Jacobi identity (see [K4]).

We endow $\mathbb{C}$ with the discrete topology. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a Hausdorff topological vector superspace. We will say that V is a *linearly compact* vector superspace if every family of closed affine linear varieties of V has non-empty intersection whenever every finite subset of the family has non-empty intersection. A topological Lie superalgebra L is called linearly compact if the underlying topological space is linearly compact.

Let $V^\star$ be the topological dual of V. Let U be a linearly compact subspace of V. We denote by $U^\perp$ the set of all continuous linear functionals which annihilate U. We define a topology on $V^\star$ by taking the collection of all sets of the form $U^\perp$ to be a fundamental system of neighborhoods of the origin.

In the following we list some properties of linearly compact vector superspaces.

Proposition 2.1 (see [G1]).

(1) *If A is a linearly compact subspace in a linearly compact vector superspace V, then A is closed.*

(2) *Direct products and inverse limits of linearly compact vector superspaces are linearly compact.*
(3) *A subspace of V is open if and only if it is closed and has finite codimension.*
(4) *A discrete topological vector superspace is linearly compact if and only if it is finite-dimensional.*
(5) *If V is discrete* (*respectively linearly compact*), *then $V^\star$ is linearly compact* (*respectively discrete*).
(6) *If V is discrete or linearly compact, the canonical linear map $V \to V^{\star\star}$ is an isomorphism.*
(7) *V is linearly compact if and only if it is isomorphic to a* (*topological*) *product of finite-dimensional discrete spaces.*
(8) *If V is linearly compact, then it is complete.*
(9) *The image of a linearly compact space under a continuous linear map is linearly compact.*
(10) (*Chevalley's principle*) *Suppose $F_1 \supset F_2 \supset \cdots$ is a sequence of closed subspaces in a linearly compact vector superspace V, such that $\bigcap_i F_i = \{0\}$. If U is a neighborhood of 0 in V, then there exists an integer i_0 such that $F_{i_0} \subset U$.*

The basic examples of linearly compact spaces are finite-dimensional vector superspaces with the discrete topology (see Proposition 2.1(4)) and the space of formal power series $V[[t]]$, where V is a finite-dimensional vector superspace, with the *formal topology* defined by taking as a fundamental system of neighborhoods of the origin the set $\{t^j V[[t]]\}_{j\in\mathbb{Z}_+}$ (see Proposition 2.1(2)). A closely related important example is the associative linearly compact superalgebra $\mathbb{C}[[t_1,\dots,t_r]] \otimes \wedge(m)$, where $\wedge(m)$ denotes the Grassman algebra on m anticommuting indeterminates $\xi_1,\dots,\xi_m$ and $p(t_i)=\bar{0}$, $p(\xi_i)=\bar{1}$, with the *formal* topology defined by $\{(t_1,\dots,t_r)^j\}_{j\in\mathbb{Z}_+}$.

Let V, W be linearly compact vector superspaces. Let $V^\star, W^\star$ be their topological duals. We form the tensor product $V^\star \otimes W^\star$, endow it with the discrete topology, and define the *completed tensor product* of V and W to be the space $(V^\star \otimes W^\star)^\star$. It is denoted by $V \widehat{\otimes} W$. Note that, if $\dim V < \infty$, then $V \widehat{\otimes} W = V \otimes W$.

A linearly compact Lie superalgebra L is called *simple* if it contains no non-trivial closed graded ideals. The same proof as in [G1, Proposition 4.3] shows that then L has no non-trivial graded ideals (closed or not). Due to [Sc, Proposition 2.1] then L has no non-trivial left or right ideals (graded or not) as well.

Lemma 2.2 (Schur's Lemma). *For a topological Lie superalgebra L we set*:

$$\Delta_L = \Big\{\tau \in (\operatorname{End} L)_\alpha,\ \alpha \in \mathbb{Z}/2\mathbb{Z} \;\Big|\; \tau\big([x,y]\big) = (-1)^{p(x)\alpha}\big[x,\tau(y)\big] \text{ for any } x,y \in L\Big\}.$$

If L is simple and linearly compact, then $\Delta_L = \mathbb{C}$.

Proof. Let $\tau \in \Delta_L$. Note that $\ker\tau$ is an ideal of L. Since L is simple, by the above discussion either $\tau = 0$ or τ is invertible, i.e. Δ_L is a skew-field. Now one can argue as in [G1, Proposition 4.4]. □

Lemma 2.3. *Let H be a closed subalgebra of the linearly compact Lie superalgebra L. Let V be a linearly compact H-module. Endow the induced L-module $U(L) \otimes_{U(H)} V^\star$ with the discrete topology, so that $(U(L) \otimes_{U(H)} V^\star)^\star$ is a linearly compact space. Endow the L-module $\mathrm{Hom}_{U(H)}(U(L), V)$ with the finite-open topology. Then $\mathrm{Hom}_{U(H)}(U(L), V)$ is linearly compact and is homeomorphic to $(U(L) \otimes_{U(H)} V^\star)^\star$ as an $U(L)$-module.*

Proof. The proof is the same as in the "even" case, see [B, Proposition 1] and [BB, Lemma 1.1]. □

Let L be a linearly compact Lie superalgebra and let V be a linearly compact (respectively discrete) L-module. The space V is called *topologically* (respectively *algebraically*) irreducible if it contains no non-trivial closed submodules (respectively no non-trivial submodules). The module V is called *topologically* (respectively *algebraically*) absolutely irreducible if it is topologically (respectively algebraically) irreducible and the commuting ring of L in $\mathrm{Hom}_{\mathbb{C}}^{\mathrm{Cont}}(V, V)$ (respectively $\mathrm{Hom}_{\mathbb{C}}(V, V)$) consists only of scalar operators. Remark that V is topologically absolutely irreducible iff $V^\star$ is algebraically absolutely irreducible.

Let I be a closed ideal of the linearly compact Lie superalgebra L. Let V be a topological I-module. The *stabilizer* of V is defined as follows:

$$H = \left\{x \in L \;\middle|\; \exists s \in \mathrm{Hom}_{\mathbb{C}}^{\mathrm{Cont}}(V, V)\colon\ [x, z]v = [s, z]v \text{ for any } z \in I,\ v \in V\right\}.$$

Then H is a closed subalgebra of L containing I.

Theorem 2.4 (Blattner). *Let I be a closed ideal in a linearly compact Lie superalgebra L. Let V be an algebraically absolutely irreducible discrete I-module, and let H be its stabilizer. Let W be an algebraically absolutely irreducible discrete H-module such that, as an I-module, it is a direct sum of copies of V. Then $U(L) \otimes_{U(H)} W$ is an algebraically absolutely irreducible L-module.*

Proof. As in the "even" case, see [BB, Theorem 3(b)]. □

Proposition 2.5. *In the notation of Theorem* 2.4*, let V be a topologically absolutely irreducible I-module. Let W be a topologically irreducible linearly compact H-module. Suppose that, as an I-module, it is topologically module-isomorphic to a direct product of copies of V. Then the L-module $\mathrm{Hom}_{U(H)}(U(L), W)$ is topologically absolutely irreducible.*

Proof. As in the "even" case, see [B, Theorem 1.2]. □

Proposition 2.6. *Let L be a linearly compact Lie superalgebra. Suppose L admits a non-abelian minimal closed ideal I. Then I possesses a maximal proper ideal J, which is closed, $\bar{I} := I/J$ is a simple non-abelian linearly compact Lie superalgebra and $N := N_L(J)$ is open. Let φ be the canonical map of I onto $\bar{I}$. Then $\bar{I}$ is a N-module and φ is a N-homomorphism. Furthermore, we have a L-module homomorphism*

$$\Theta : I \to \mathrm{Hom}_{U(N)}\big(U(L), \bar{I}\big),$$

where $\Theta(x)(a) = (-1)^{p(a)p(x)}\varphi((\mathrm{ad}\, a)(x))$ $(x \in I,\ a \in U(L))$.

Proof. As in the "even" case, see [B, Lemma 2.2]. □

Theorem 2.7 (Cartan–Guillemin). *Let I be a non-abelian minimal closed ideal in a linearly compact Lie superalgebra L. Then I is homeomorphic via Θ to* $\mathrm{Hom}_{U(N)}(U(L), \bar{I})$ *both as a Lie superalgebra and as a L-module.*

Proof. We can apply the Schur Lemma 2.2 to $\bar{I}$ and Propositions 2.5 and 2.6 in order to use the same argument as in [B, Theorem 2.4]. □

Corollary 2.8. *Let* $\dim(L/N)_{\bar{0}} = r$ *and* $\dim(L/N)_{\bar{1}} = m$. *Then, in the notation of Theorem* 2.7,

$$I \xrightarrow{\sim} \big(\mathbb{C}[\![t_1, \dots, t_r]\!] \otimes \wedge(m)\big) \mathbin{\widehat{\otimes}} \bar{I}$$

as topological Lie superalgebras.

Proof. By [B, Corollary to Proposition 7], $\mathrm{Hom}_{U(N)}(U(L), \bar{I})$ is isomorphic to $\mathrm{Hom}_{\mathbb{C}}(S(L/N), \bar{I})$ as a topological Lie superalgebra. Since N is open, $\dim(L/N) < \infty$. The fact that a linearly compact space can be identified with its double dual implies that $\mathrm{Hom}_{\mathbb{C}}(S(L/N), \bar{I})$ is isomorphic to $(\mathbb{C}[\![t_1, \dots, t_r]\!] \otimes \wedge(m)) \mathbin{\widehat{\otimes}} \bar{I}$. □

2.2. Growth

A *filtration* of a vector superspace V is a decreasing filtration by subspaces of finite codimension ($j_0 \in \mathbb{Z}$):

$$V = V_{j_0} \supset V_{j_0+1} \supset V_{j_0+2} \supset \cdots$$

such that $\bigcap_j V_j = \{0\}$. The *growth* of this filtration is defined as follows:

$$\mathrm{gw}(V) = \limsup_{j \to \infty} \frac{\log \dim(V/V_j)}{\log j}.$$

Given a subspace U of V, one has an induced filtration on U and on V/U. Clearly, $\mathrm{gw}(U) \leqslant \mathrm{gw}(V)$ and $\mathrm{gw}(V/U) \leqslant \mathrm{gw}(V)$. Also, if the filtration $\{V_j\}$ of V is shifted (i.e. we take $V_{(j)} = V_{j+a}$) or rescaled (i.e. we take $V_{(j)} = V_{nj}$ for a fixed positive integer n) then $\mathrm{gw}(V)$ remains unchanged. The first claim is obvious, while the second one follows from the observation that on the one hand, the rescaling of the filtration may obviously only increase the growth, but, on the other hand, it may only decrease the growth:

$$\mathrm{gw}(V) \geqslant \limsup_{j\to\infty} \frac{\log \dim(V/V_{nj})}{\log nj} = \limsup_{j\to\infty} \frac{\log \dim(V/V_{nj})}{\log j}.$$

An *algebra filtration* of a linearly compact Lie superalgebra L is a filtration of L by open subspaces L_i such that $[L_i, L_j] \subseteq L_{i+j}$. A similar definition applies to associative algebras.

Recall that a *fundamental subalgebra* of a linearly compact Lie superalgebra L is a proper open subalgebra that contains no non-zero ideals of L. Given a fundamental subalgebra L_0 of L one constructs the *canonical* (algebra) filtration of L associated to L_0,

$$L = L_{-1} \supset L_0 \supset L_1 \supset \cdots,$$

by letting inductively for $j \geqslant 1$ (cf. [GS]):

$$L_j = \{a \in L_{j-1} \mid [a, L] \subset L_{j-1}\}.$$

One knows [G1] that any linearly compact Lie superalgebra L contains a proper open subalgebra L_0, which is of course fundamental if L is simple.

One defines $\mathrm{gw}(L, L_0)$ to be the growth of the canonical filtration of L associated to L_0.

Proposition 2.9 [BDK]. *The number* $\mathrm{gw}(L, L_0)$ *is independent of the choice of the fundamental subalgebra* L_0 *of* L.

Proof. If we choose the jth member L_j ($j \geqslant 1$) of the canonical filtration associated to L_0 as another fundamental subalgebra, say $\widetilde{L_0}$, then, by definition, the associated canonical filtration is $\widetilde{L_k} = L_{k+j}$. Hence this change of the fundamental subalgebra does not affect the growth.

Now, if M_0 is another fundamental subalgebra of L and $\{M_j\}$ is the associated canonical filtration, then, by Chevalley's principle, $L_k \subset M_0$ for sufficiently large k, hence $L_{k+j} \subset M_j$ for all j. Therefore, $\mathrm{gr}(L, L_0) \geqslant \mathrm{gr}(L, M_0)$. Exchanging the roles of L_0 and M_0 we obtain the opposite inequality. □

We denote by $\mathrm{gw}(L)$ and call the *growth* of L the number $\mathrm{gw}(L, L_0)$ defined above. If L is simple, one can prove the following stronger result.

Theorem 2.10. *Let L be a simple linearly compact Lie superalgebra. Then any algebra filtration of L by open subspaces has growth equal to* $\mathrm{gw}(L)$.

Proof. If L contains an open subalgebra M_1 such that all $M_j = M_1^j$ are open subspaces of L (then this is automatically an algebra filtration), let us denote by $\mathrm{gw}'(L, M_1)$ the growth of this filtration. Since the growth is invariant under rescaling, we see (using Chevalley's principle as in the proof of Proposition 2.9) that $\mathrm{gw}'(L, M_1) = \mathrm{gw}'(L)$ is independent of the choice of M_1. Furthermore, we clearly have:

$$\mathrm{gw}(L) \leqslant \mathrm{gw}(\text{any algebra filtration}) \leqslant \mathrm{gw}'(L), \tag{2.1}$$

if we choose M_1 to be a fundamental subalgebra of L.

Furthermore, it follows from the classification of simple linearly compact Lie superalgebras [K5] that L has a Weisfeiler filtration

$$L = L_{-d}^{\mathrm{W}} \supset L_{-d+1}^{\mathrm{W}} \supset \cdots \supset L_{-1}^{\mathrm{W}} \supset L_0^{\mathrm{W}} \supset L_1^{\mathrm{W}} \supset \cdots,$$

for which the associated graded Lie superalgebra $\mathrm{Gr}(L) = \bigoplus_{j \geqslant -d} \mathfrak{g}_j$ has the property that the space $\mathfrak{g}_1 + \mathfrak{g}_2$ generates $\bigoplus_{j \geqslant 1} \mathfrak{g}_j$ (in fact, in all cases except for $L = \mathrm{Der}(\mathbb{C}[\![t]\!])$, one has $\mathfrak{g}_j = \mathfrak{g}_1^j$, $j > 0$), hence

$$L_{2k}^{\mathrm{W}} \subset \left(L_1^{\mathrm{W}}\right)^k \subset L_k^{\mathrm{W}} \quad \text{for all } k. \tag{2.2}$$

On the other hand, for the canonical filtration $\{L_j\}$ associated to L_0^{W} one has

$$L_j \subset L_{jd}^{\mathrm{W}} \quad \text{for all } j. \tag{2.3}$$

It follows from (2.2) and (2.3) that $\mathrm{gw}(L) \geqslant \mathrm{gw}'(L)$. Therefore, by (2.1), the growth of any algebra filtration of L is equal to $\mathrm{gw}(L)$. □

The following theorem follows from the classification of simple linearly compact Lie superalgebras [K5].

Theorem 2.11. *Any simple linearly compact Lie superalgebra of growth at most* 1 *is either finite-dimensional or is isomorphic to one of the following Lie superalgebras* (*see Section* 3.2 *for their description*): $\overline{W}(1, N)$, $N \geqslant 0$; $\overline{S}(1, N)'$, $N \geqslant 2$; $\overline{K}(1, N)$, $N \geqslant 0$; $\overline{E}(1, 6)$.

2.3. *On derivations of linearly compact Lie superalgebras*

In this section we prove three propositions which will be essential in the sequel. We shall denote by $\mathrm{Der}(L)$ the Lie superalgebra of all continuous derivations of a topological superalgebra L.

Proposition 2.12. *Let A be a commutative, associative, unital linearly compact superalgebra and let L be a simple linearly compact Lie superalgebra. Then*

$$\mathrm{Der}(A \mathbin{\widehat{\otimes}} L) = \mathrm{Der}(A) \otimes 1 + A \mathbin{\widehat{\otimes}} \mathrm{Der}(L).$$

Proof. Using Schur Lemma (see Proposition 2.2), one can apply the same argument as in [BDK, Proposition 6.12]. □

Proposition 2.13. *Let L be a linearly compact Lie superalgebra and let*

$$L = L_{-d} \supset L_{-d+1} \supset \cdots \supset L_0 \supset L_1 \supset \cdots$$

be an algebra filtration of L of depth $d > 0$. Let D be an even element of L such that

$$[D, L_k] = L_{k-d} \quad \text{for all } k \geqslant d. \tag{2.4}$$

Let V be a finite-dimensional Lie algebra acting by derivations on L such that

$$V(L_k) \subseteq L_k \quad \text{for all } k \geqslant 0, \tag{2.5}$$

and let $\widetilde{L} = V \ltimes L$. Then any even element of the form $D + v + g_0 \in \widetilde{L}$, where $v \in V$ and $g_0 \in L_0$, can be conjugated via a continuous inner automorphism of L to $D + v$.

Proof. Let m be the maximal integer such that $g_0 \in L_m \backslash L_{m+1}$. Then there exists $l_{m+d} \in L_{m+d}$ such that $[D, l_{m+d}] = g_0$. The automorphism $\exp(\mathrm{ad}(l_{m+d}))$ is well-defined and converges uniformly on $\widetilde{L}$, and we have:

$$\begin{aligned}\exp\bigl(\mathrm{ad}(l_{m+d})\bigr)(D + v + g_0) &= D + v + \bigl(g_0 + [l_{m+d}, D]\bigr) \\ &\quad + \bigl([l_{m+d}, g_0] - v(l_{m+d})\bigr) + \cdots \\ &= D + v + \text{terms in } L_{m+d}.\end{aligned}$$

By repeating this argument, we obtain $D + v$ in the limit. □

Proposition 2.14. *Any non-solvable finite-dimensional Lie superalgebra $\mathfrak{g}$ has no even surjective derivations.*

Proof. Let D be an even surjective derivation of $\mathfrak{g}$. It clearly transforms $\mathfrak{g}_{\bar{0}}$ surjectively into itself. Moreover, D leaves the radical $\mathfrak{r}$ of $\mathfrak{g}_{\bar{0}}$ invariant, hence it induces a derivation of $\mathfrak{g}_{\bar{0}}/\mathfrak{r}$ which is not inner because it is surjective. On the other hand, $\mathfrak{g}_{\bar{0}}/\mathfrak{r}$ is a semisimple Lie algebra, so that every derivation is inner. Consequently, $\mathfrak{g}_{\bar{0}} = \mathfrak{r}$ is solvable, but this in turn implies (see [K4, Proposition 1.3.3]) that $\mathfrak{g}$ is solvable. □

3. Formal distribution Lie superalgebras and Lie conformal superalgebras

3.1. Basic definitions

Let V be a vector superspace. A V-valued *formal distribution* in one indeterminate z is a formal power series

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \quad a_n \in V.$$

The vector superspace of all formal distributions in one indeterminate will be denoted by $V[\![z, z^{-1}]\!]$. It has a natural structure of a $\mathbb{C}[\partial_z]$-module. We define

$$\operatorname{Res}_z a(z) = a_0.$$

Similarly, one can define a formal distribution in two indeterminates:

$$a(z, w) = \sum_{m,n \in \mathbb{Z}} a_{m,n} z^{-m-1} w^{-n-1}.$$

It is called *local* if

$$(z - w)^N a(z, w) = 0 \quad \text{for } N \gg 0.$$

Let $\mathfrak{g}$ be a Lie superalgebra, and let $a(z), b(z)$ be two $\mathfrak{g}$-valued formal distributions. They are called local if $[a(z), b(w)]$ is local, i.e.

$$(z - w)^N \big[a(z), b(w)\big] = 0 \quad \text{for } N \gg 0.$$

Let $\mathfrak{g}$ be a Lie superalgebra, and let $\mathcal{F}$ be a family of $\mathfrak{g}$-valued mutually local formal distributions. The pair $(\mathfrak{g}, \mathcal{F})$ is called a *formal distribution Lie superalgebra* if $\mathfrak{g}$ is spanned by the coefficients of all formal distributions from $\mathcal{F}$.

Proposition 3.1 [K2]. *Two $\mathfrak{g}$-valued formal distributions $a(z), b(z)$ are local iff*

$$\big[a(z), b(w)\big] = \sum_{j \in \mathbb{Z}_+} c^j(w) \partial_w^j \delta(z - w)/j!, \tag{3.1}$$

where the sum is finite. Here $\delta(z - w) = \sum_{n \in \mathbb{Z}} z^n w^{-n-1}$ is the formal delta-function, and the OPE coefficients $c^j(w) \in \mathfrak{g}[\![w, w^{-1}]\!]$ can be computed as follows:

$$c^j(w) = \operatorname{Res}_z (z - w)^j \big[a(z), b(w)\big]. \tag{3.2}$$

The algebraic analogue of the Fourier transform, see [K2], provides an effective way to study the OPE.

The *formal Fourier transform* of a formal distribution in two indeterminates is defined as follows:

$$F_{z,w}^{\lambda}\big(a(z, w)\big) = \operatorname{Res}_z e^{\lambda(z-w)} a(z, w) \in \mathbb{C}[\![w, w^{-1}]\!][\![\lambda]\!].$$

One has:

$$F^{\lambda}_{z,w}\big(\partial^{j}_{w}\delta(z-w)\big)=\lambda^{j},$$

therefore the formal Fourier transform is the generating series of the OPE coefficients of $[a(z),b(w)]$. The λ-*bracket* of $a(w)$ and $b(w)$ is defined as

$$\big[a(w)_{\lambda}b(w)\big]=F^{\lambda}_{z,w}\big(\big[a(z),b(w)\big]\big).$$

The coefficient of $\lambda^n/n!$ in the λ-bracket is a $\mathfrak{g}$-valued formal distribution, called the nth product of $a(w)$ and $b(w)$ (it is computed by formula (3.2)).

The properties of the λ-bracket lead to the following basic definition (see [DK, K2]).

A *Lie conformal superalgebra* R is a left $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map, called the λ-*bracket*,

$$R\otimes R\to\mathbb{C}[\lambda]\otimes R,\quad a\otimes b\mapsto[a_{\lambda}b],$$

satisfying the following axioms ($a,b,c\in R$):

$$[\partial a_{\lambda}b]=-\lambda[a_{\lambda}b],\qquad [a_{\lambda}\partial b]=(\partial+\lambda)[a_{\lambda}b], \qquad \text{(sesquilinearity)}$$

$$[b_{\lambda}a]=-(-1)^{p(a)p(b)}[a_{-\lambda-\partial}b], \qquad \text{(skew-commutativity)}$$

$$\big[a_{\lambda}[b_{\mu}c]\big]=\big[[a_{\lambda}b]_{\lambda+\mu}c\big]+(-1)^{p(a)p(b)}\big[b_{\mu}[a_{\lambda}c]\big]. \qquad \text{(Jacobi identity)}$$

We write

$$[a_{\lambda}b]=\sum_{n\in\mathbb{Z}_+}\frac{\lambda^n}{n!}(a_{(n)}b). \tag{3.3}$$

The coefficient $a_{(n)}b$ is called the nth *product* of a and b. A *subalgebra* S of R is a $\mathbb{C}[\partial]$-submodule of R such that $S_{(n)}S\subseteq S$ for any $n\in\mathbb{Z}_+$. An *ideal* I of R is a $\mathbb{C}[\partial]$-submodule of R such that $R_{(n)}I\subseteq I$ for any $n\in\mathbb{Z}_+$. A Lie conformal superalgebra R is *simple* if it has no non-trivial ideals and the λ-bracket is not identically zero. A Lie conformal superalgebra R is *finite* if it is finitely generated as a $\mathbb{C}[\partial]$-module. We denote by R' the derived subalgebra of R, i.e. the $\mathbb{C}$-span of all nth products.

One knows that any torsion element of R (viewed as a $\mathbb{C}[\partial]$-module) has zero λ-bracket with R [DK], hence a finite simple Lie conformal superalgebra is free as a $\mathbb{C}[\partial]$-module.

One can associate to a formal distribution Lie superalgebra $(\mathfrak{g},\mathcal{F})$ a Lie conformal superalgebra as follows. Let $\overline{\mathcal{F}}$ be the minimal $\mathbb{C}[\partial_z]$-submodule of $\mathfrak{g}[\![z,z^{-1}]\!]$ that contains $\mathcal{F}$ and is closed under all nth products, $n\in\mathbb{Z}_+$. Then the λ-bracket defines a Lie conformal superalgebra structure on $\overline{\mathcal{F}}$.

Vice versa, given a Lie conformal superalgebra R, we can construct a formal distribution Lie superalgebra Lie R using the following definition.

The *affinization* of a Lie conformal superalgebra R is the Lie conformal superalgebra

$$\widetilde{R}=R\otimes\mathbb{C}\big[t,t^{-1}\big],$$

with $p(t) = \bar{0}$, the ∂-action defined by $\tilde{\partial} = \partial \otimes 1 + 1 \otimes \partial_t$ and the nth products defined by

$$(a \otimes f)_{(n)}(b \otimes g) = \sum_{j \geqslant 0} (a_{(n+j)}b) \otimes \big((\partial_t^j f)g\big)/j!,$$

where $a, b \in R$, $f, g \in \mathbb{C}[t, t^{-1}]$ and $n \in \mathbb{Z}_+$.

We let $\operatorname{Lie} R = \widetilde{R}/\tilde{\partial}\widetilde{R}$ and denote by a_n the image of $a \otimes t^n$ in $\operatorname{Lie} R$. Then $\operatorname{Lie} R$ is a Lie superalgebra with respect to the 0th product induced from $\widetilde{R}$. Explicitly, the bracket is

$$[a_m, b_n] = \sum_{j \geqslant 0} \binom{m}{j} (a_{(j)}b)_{m+n-j}, \quad a, b \in R,\ m, n \in \mathbb{Z}. \tag{3.4}$$

Also,

$$(\partial a)_m = -m a_{m-1}, \quad a \in R,\ m \in \mathbb{Z}. \tag{3.5}$$

The Lie superalgebra $\operatorname{Lie} R$ admits an even derivation T defined as $T a_n = -n a_{n-1}$.

Let

$$\mathcal{F}(R) = \left\{ a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \;\middle|\; a \in R \right\}.$$

Then (3.4) is equivalent to (3.1), where $c^j(w) = (a_{(j)}b)(w)$, hence all formal distributions in $\mathcal{F}(R)$ are pairwise local. The pair $(\operatorname{Lie} R, \mathcal{F}(R))$ is called *the maximal formal distribution Lie superalgebra* associated to the Lie conformal superalgebra R.

Let $(\mathfrak{g}, \mathcal{F})$ be a formal distribution Lie superalgebra. An ideal in $\mathfrak{g}$ is called *irregular* if it does not contain all the coefficients of a non-zero element of $\overline{\mathcal{F}}$. Then any formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$ such that $\overline{\mathcal{F}} \simeq R$ is a quotient of $\operatorname{Lie} R$ by an irregular ideal [K2].

An ideal I in $(\mathfrak{g}, \mathcal{F})$ is called *regular* if it is of the form $I = \{a_n \mid a \in J, n \in \mathbb{Z}_+\}$, where J is an ideal of the Lie conformal superalgebra $\overline{\mathcal{F}}$; I is clearly T-stable.

Let $(\operatorname{Lie} R, \mathcal{F}(R))$ be the maximal formal distribution Lie superalgebra associated to the Lie conformal superalgebra R. We let

$$(\operatorname{Lie} R)_- = \langle a_n \mid a \in R,\ n \in \mathbb{Z}_+ \rangle, \qquad (\operatorname{Lie} R)_+ = \langle a_n \mid a \in R,\ n < 0 \rangle.$$

Formula (3.4) implies that these are both T-invariant subalgebras of $\operatorname{Lie} R$.

Let R be a finite Lie conformal superalgebra, and let $\{a^j\}_{j \in J}$ be a finite set of generators of R as a $\mathbb{C}[\partial]$-module. Let $\mathcal{L}_m$ be the $\mathbb{C}$-span of $\{a_i^j \mid i \geqslant m,\ j \in J\}$.

It is easy to see using (3.4) that the subspaces $\mathcal{L}_m$ form a *quasi-filtration* of $(\operatorname{Lie} R)_-$ (see [DK]),

$$(\operatorname{Lie} R)_- = \mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2 \supset \cdots, \tag{3.6}$$

by subspaces of finite codimension, i.e. $\bigcap_i \mathcal{L}_i = \{0\}$ and there exists an integer $d \in \mathbb{Z}_+$ such that

$$[\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_{i+j-d}, \quad i, j \in \mathbb{Z}_+. \tag{3.7}$$

It also follows from the construction that

$$\dim \mathcal{L}_i/\mathcal{L}_{i+1} \leqslant |J|, \quad i \in \mathbb{Z}_+. \tag{3.8}$$

Furthermore,

$$[T, \mathcal{L}_i] = \mathcal{L}_{i-1}. \tag{3.9}$$

Due to Chevalley's principle, the completion $\mathcal{A}(R)$ of $(\operatorname{Lie} R)_-$ with respect to the topology induced by this filtration is independent of the choice of the $\mathbb{C}[\partial]$-generators of R. The Lie superalgebra $\mathcal{A}(R)$ is a linearly compact Lie superalgebra, called the *annihilation algebra* of R. Note that formula (3.7) (respectively (3.9)) implies that the bracket (respectively T) is continuous on $\mathcal{A}(R)$. The map T is surjective on $(\operatorname{Lie} R)_-$ by its very definition, hence it extends to an even continuous surjective derivation of $\mathcal{A}(R)$ (because $\mathcal{A}(R)$ is a Hausdorff topological space). The (linearly compact) Lie superalgebra

$$\mathcal{A}(R)^e = \mathbb{C}T \ltimes \mathcal{A}(R)$$

is called the *extended annihilation algebra* of R.

Proposition 3.2. *Let $(\mathfrak{g}, \mathcal{F})$ be a formal distribution Lie superalgebra. Suppose that $\overline{\mathcal{F}} = \mathbb{C}[\partial]\mathcal{F}$ and that all the coefficients of the formal distributions in $\mathcal{F}$ form a $\mathbb{C}$-basis of $\mathfrak{g}$. Then $(\mathfrak{g}, \mathcal{F})$ is the maximal formal distribution Lie superalgebra associated to the Lie conformal superalgebra $\overline{\mathcal{F}}$.*

Proof. Let $\mathcal{F} = \{a^i(z) \mid i \in \mathcal{F}\}$. Then $\operatorname{Lie} \overline{\mathcal{F}}$ is spanned by the set $\{a^i_n \mid i \in \mathcal{F}, n \in \mathbb{Z}\}$. We have a canonical surjective map $\operatorname{Lie} \overline{\mathcal{F}} \to \mathfrak{g}$, and since the images of the a^i_n's are linearly independent in $\mathfrak{g}$, the kernel of this map is zero. □

3.2. Some examples

Example 3.3. Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra. The *loop algebra* associated to $\mathfrak{g}$ is the Lie superalgebra

$$\tilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}], \qquad p(at^k) = p(a) \quad \text{for } a \in \mathfrak{g},\ k \in \mathbb{Z},$$

with bracket

$$[a \otimes t^n, b \otimes t^m] = [a, b] \otimes t^{n+m} \quad (a, b \in \mathfrak{g},\ m, n \in \mathbb{Z}).$$

We introduce the family $\mathcal{F}_\mathfrak{g}$ of formal distributions (known as currents)

$$a(z) = \sum_{n \in \mathbb{Z}} (a \otimes t^n) z^{-n-1}, \quad a \in \mathfrak{g}.$$

It is easily verified that

$$\big[a(z), b(w)\big] = [a, b](w)\delta(z - w),$$

hence $(\tilde{\mathfrak{g}}, \mathcal{F}_{\mathfrak{g}})$ is a formal distribution Lie superalgebra. The associated Lie conformal superalgebra is $\mathbb{C}[\partial] \otimes \mathfrak{g}$, with λ-bracket (we identify $1 \otimes \mathfrak{g}$ with $\mathfrak{g}$)

$$[a_\lambda b] = [a, b], \quad a, b \in \mathfrak{g};$$

it is called the *current conformal algebra* associated to $\mathfrak{g}$, and it is denoted by $\operatorname{Cur}\mathfrak{g}$. The Lie conformal superalgebra $\operatorname{Cur}\mathfrak{g}$ is simple iff $\mathfrak{g}$ is a simple Lie superalgebra. Indeed, an ideal I of $\mathfrak{g}$ generates the ideal $\mathbb{C}[\partial] \otimes I$ of $\operatorname{Cur}\mathfrak{g}$. Conversely, let $J = \mathbb{C}[\partial] \otimes V$ be an ideal of $\operatorname{Cur}\mathfrak{g}$. Take a non-zero $a = \sum_i \partial^i a_i \in J$, where $a_i \in \mathfrak{g}$. Then for any $b \in \mathfrak{g}$ we have:

$$[a_\lambda b] = \sum_i \frac{(-\lambda)^i}{i!}(a_{(i)}b),$$

hence $a_{(i)}b \in V$ for all $i \geqslant 0$, and V is an ideal in $\mathfrak{g}$. Due to Proposition 3.2, $(\tilde{\mathfrak{g}}, \overline{\mathcal{F}_{\mathfrak{g}}})$ is the maximal formal distribution Lie superalgebra $(\operatorname{Lie}\operatorname{Cur}\mathfrak{g}, \mathcal{F}(\operatorname{Cur}\mathfrak{g}))$ associated to $\operatorname{Cur}\mathfrak{g}$; also, it is clear that $T = -\partial/\partial t$. Hence the annihilation algebra and the extended annihilation algebra are respectively $\mathfrak{g}[\![t]\!]$ and $\mathbb{C}\frac{\partial}{\partial t} \ltimes \mathfrak{g}[\![t]\!]$.

Example 3.4. Let us denote by $\overline{\wedge}(1, N)$ (respectively $\wedge(\!(1, N)\!)$) the tensor product of $\mathbb{C}[\![x]\!]$ (respectively $\mathbb{C}[x, x^{-1}]$) and the exterior algebra $\wedge(N)$ in the indeterminates $\xi_1, \ldots, \xi_N$. They are associative, commutative superalgebras if we set $p(x) = \bar{0}$, $p(\xi_i) = \bar{1}$, $i = 1, \ldots, N$, and $\overline{\wedge}(1, N)$ is a linearly compact algebra in the formal topology. Let $\overline{W}(1, N)$ (respectively $W(\!(1, N)\!)$) be the Lie superalgebra of all continuous derivations of $\overline{\wedge}(1, N)$ (respectively all derivations of $\wedge(\!(1, N)\!)$). Then $\overline{W}(1, N)$ is a simple linearly compact Lie superalgebra [K5]. Occasionally we will be dealing also with the Lie superalgebra $W(1, N)$ of all derivations of $\wedge(1, N) = \mathbb{C}[\partial] \otimes \wedge(N)$ and its subalgebras, but the main role will be played by their completions in the formal topology. Every element of $\overline{W}(1, N)$ (respectively $W(\!(1, N)\!)$) can be written in the form

$$D = \sum_{i=0}^{N} P_i \partial_i, \tag{3.10}$$

where $P_i \in \overline{\wedge}(1, N)$ (respectively $\wedge(\!(1, N)\!)$); $\partial_0 := \partial/\partial x$ is an even derivation and $\partial_i := \partial/\partial\xi_i$, $i = 1, \ldots, N$, are odd derivations.

For each element $A \in \wedge(N)$, and for any $j = 0, 1, \ldots, N$ we define a $W(\!(1, N)\!)$-valued formal distribution

$$A^j(z) = \sum_{n\in\mathbb{Z}} (x^n A)\partial_j\, z^{-n-1}.$$

The commutation relations are ($A, B \in \wedge(N)$ and $i, j = 1, \ldots, N$):

$$\begin{aligned}
\big[A^i(z), B^j(w)\big] &= \big((A\partial_i B)^j(w) + (-1)^{p(A)}\big((\partial_j A)B\big)^i(w)\big)\delta(z-w);\\
\big[A^i(z), B^0(w)\big] &= (A\partial_i B)^0(w)\delta(z-w)\\
&\quad - (-1)^{p(B)}(AB)^i(w)\partial_w\delta(z-w);\\
\big[A^0(z), B^0(w)\big] &= -\partial_w(AB)^0(w)\delta(z-w) - 2(AB)^0(w)\partial_w\delta(z-w).
\end{aligned}$$

Hence the family $\mathcal{F}_W = \{A^j(z)\}_{A\in\wedge(N),\, j=0,\dots,N}$ consists of mutually local formal distributions and $(W(\!(1,N)\!), \mathcal{F}_W)$ is a formal distribution Lie superalgebra.

The associated Lie conformal superalgebra is

$$W_N = \mathbb{C}[\partial] \otimes \big(W(N) \oplus \wedge(N)\big),$$

where $W(N)$ denotes the Lie superalgebra of all derivations of $\wedge(N)$. The λ-bracket ($a, b \in W(N)$, $f, g \in \wedge(N)$) is as follows:

$$\begin{aligned}
&[a_\lambda b] = [a,b], \qquad [a_\lambda f] = a(f) - \lambda(-1)^{p(a)p(f)} f a,\\
&[f_\lambda g] = -\partial(fg) - 2\lambda fg.
\end{aligned}\tag{3.11}$$

The Lie conformal superalgebra W_N is simple for $N \geqslant 0$. Indeed, it is easy to see that W_0 and W_1 are simple. Let I be an ideal of W_N. Taking $[1_\lambda I]$ we conclude from (3.11) that I equals to the sum of its intersections with $\operatorname{Cur} W(N)$ and $\mathbb{C}[\partial] \otimes \wedge(N)$. If $N \geqslant 2$, then $\operatorname{Cur} W(N)$ is simple, hence either $I \subset \mathbb{C}[\partial] \otimes \wedge(N)$ or $I \subset \operatorname{Cur} W(N)$. Formula (3.11) implies that the first case is impossible and that in the second case $I = W_N$. Furthermore, W_N is a free $\mathbb{C}[\partial]$-module of rank $(N+1)2^N$.

We shall need the following representation of W_N on $\mathbb{C}[\partial] \otimes \wedge(N)$ (see [CK1]):

$$a_\lambda g = a(g), \quad f_\lambda g = -(\partial + \lambda) fg, \quad a \in W(N),\ f, g \in \wedge(N). \tag{3.12}$$

By Proposition 3.2,

$$\mathcal{A}(W_N) = \overline{W}(1,N) \quad \text{and} \quad \mathcal{A}(W_N)^e = \mathbb{C}\,\mathrm{ad}(\partial_0) \ltimes \mathcal{A}(W_N).$$

Example 3.5. Recall that the *divergence* of a differential operator (3.10) is defined by the formula

$$\operatorname{div} D = \partial_0 P_0 + \sum_{i=1}^{N} (-1)^{p(P_i)} \partial_i P_i;$$

its main property is

$$\operatorname{div}[D_1, D_2] = D_1(\operatorname{div} D_2) - (-1)^{p(D_1)p(D_2)} D_2(\operatorname{div} D_1).$$

It follows that

$$\begin{aligned}
&S(\!(1,N)\!) = \big\{D \in W(\!(1,N)\!) \bigm| \operatorname{div} D = 0\big\}\\
&\overline{S}(1,N) = \big\{D \in \overline{W}(1,N) \bigm| \operatorname{div} D = 0\big\}
\end{aligned}\tag{3.13}$$

are subalgebras of the Lie superalgebras $W(\!(1,N)\!)$ and $\overline{W}(1,N)$, respectively. We have

$$S(\!(1,N)\!) = S(\!(1,N)\!)' \oplus \mathbb{C}\xi_1\ldots\xi_N\partial_0,$$
$$\overline{S}(1,N) = \overline{S}(1,N)' \oplus \mathbb{C}\xi_1\ldots\xi_N\partial_0,$$

where $S(\!(1,N)\!)'$ and $\overline{S}(1,N)'$ denote the respective derived subalgebras. The Lie superalgebra $\overline{S}(1,N)'$ is a simple linearly compact Lie superalgebra for $N \geqslant 2$ [K5].

Let $\{A_i \mid i = 1,\ldots,(N-1)2^N+1\}$ be a basis of $S(N)$, the 0-divergence subalgebra of $W(N)$, and let $\{B_j \mid j = 1,\ldots,2^N-1\}$ be a set of homogeneous, linearly independent monomials in $\wedge(N)$, whose degree is strictly less than N. It is easy to see that the following family $\mathcal{F}_S$ of mutually local formal distributions is linearly independent over $\mathbb{C}[\partial_z]$ and that all their coefficients form a $\mathbb{C}$-basis of $S(\!(1,N)\!)'$:

$$A_i(z) = \sum_{n\in\mathbb{Z}} (x^n A_i) z^{-n-1},$$
$$B_j(z) = (\deg B_j - N)B_j^0(z) + \partial_z \sum_{i=1}^{N} (B_j\xi_i)^i(z). \tag{3.14}$$

Hence $(S(\!(1,N)\!)', \mathcal{F}_S)$ is a formal distribution Lie superalgebra, and the corresponding Lie conformal superalgebra $\overline{\mathcal{F}_S}$ has rank $N2^N$ over $\mathbb{C}[\partial_z]$.

Let us describe this Lie conformal superalgebra more explicitly. For an element $D = \sum_{i=1}^N P_i(\partial,\xi)\partial_i + f(\partial,\xi) \in W_N$, we define the corresponding notion of divergence:

$$\operatorname{div} D = \sum_{i=1}^{N} (-1)^{p(P_i)} \partial_i P_i - \partial f \in \mathbb{C}[\partial] \otimes \wedge(N).$$

The following identity holds in $\mathbb{C}[\partial] \otimes \wedge(N)$, where $D_1, D_2 \in W_N$ (cf. (3.12)):

$$\operatorname{div}[D_{1\lambda} D_2] = (D_1)_\lambda(\operatorname{div} D_2) - (-1)^{p(D_1)p(D_2)}(D_2)_{-\lambda-\partial}(\operatorname{div} D_1). \tag{3.15}$$

Therefore,

$$S_N = \{D \in W_N \mid \operatorname{div} D = 0\}$$

is a subalgebra of W_N. The Lie conformal superalgebra S_N is simple for $N \geqslant 2$; one can check this by using the same argument as for W_N. Also, it is a free $\mathbb{C}[\partial]$-module of rank $N2^N$. Furthermore, $(S(\!(1,N)\!)', \mathcal{F}_S)$, where $\mathcal{F}_S$ is defined by (3.14), is the maximal formal distribution Lie superalgebra associated to S_N. This follows from Proposition 3.2. The above discussion implies that

$$\mathcal{A}(S_N) = \overline{S}(1,N)' \quad \text{and} \quad \mathcal{A}(S_N)^e = \mathbb{C}\,\mathrm{ad}(\partial_0) \ltimes \overline{S}(1,N)'.$$

Example 3.6. For any $a \in \mathbb{C}$, we set

$$S(\!|1, N, a|\!) = \{D \in W(\!|1, N|\!) \mid \operatorname{div} e^{ax} D = 0\}.$$

This is a subalgebra of $W(\!|1, N|\!)$, which is spanned by the coefficients of the following family $\mathcal{F}_{S,a}$ of mutually local formal distributions (cf. (3.14)):

$$A_i(z), \quad B_{j,a}(z) = (l - N) B_j^0(z) + (\partial_z - a) \sum_{i=1}^{N} (B_j \xi_i)^i(z). \tag{3.16}$$

The associated Lie conformal superalgebra is constructed explicitly as follows. Let $D = \sum_{i=1}^{N} P_i(\partial, \xi)\partial_i + f(\partial, \xi)$ be an element of W_N. We define the deformed divergence to be

$$\operatorname{div}_a D = \operatorname{div} D + af.$$

It still satisfies formula (3.15), hence

$$S_{N,a} = \{D \in W_N \mid \operatorname{div}_a D = 0\}$$

is a subalgebra of W_N, which is simple for $N \geqslant 2$ and has rank $N2^N$.

As for the annihilation algebra, for $a \neq 0$ the automorphism of $\overline{\wedge}(1, N)$ sending x to $(e^{ax} - 1)/a$ and leaving the ξ_i's invariant induces an automorphism in the space of the differential forms, which transforms the standard volume form $dx \wedge v$ into $e^{ax}\, dx \wedge v$. Hence

$$\mathcal{A}(S_{N,a}) \simeq \overline{S}(1, N)'.$$

Moreover, using Lemmas 5.8 and 5.9, one can see that the induced automorphism in $\operatorname{Der}(\overline{S}(1, N)')/\operatorname{ad}(\overline{S}(1, N)')$ sends $\operatorname{ad}(\partial_0)$ to $\operatorname{ad}(\partial_0 - a\sum_{i=1}^{N} \xi_i \partial_i)$. Consequently,

$$\mathcal{A}(S_{N,a})^e = \mathbb{C} \operatorname{ad}\left(\partial_0 - a \sum_{i=1}^{N} \xi_i \partial_i\right) \ltimes \overline{S}(1, N)'.$$

Example 3.7. Let $N \in \mathbb{Z}_+$ be an even integer. We set

$$\widetilde{S}(\!|1, N|\!) = \{D \in W(\!|1, N|\!) \mid \operatorname{div}((1 + \xi_1 \dots \xi_N) D) = 0\}.$$

This is a subalgebra of $W(\!|1, N|\!)$, which is spanned by the coefficients of the following family $\mathcal{F}_{\widetilde{S}}$ of mutually local formal distributions (cf. (3.14)):

$$\widetilde{A}_i(z) = (1 - \xi_1 \dots \xi_N) A_i(z), \quad B_j(z). \tag{3.17}$$

The associated Lie conformal superalgebra $\widetilde{S}_N$ is constructed explicitly as follows:

$$\widetilde{S}_N = \{D \in W_N \mid \operatorname{div}((1 + \xi_1 \dots \xi_N) D) = 0\} \quad \big(= (1 - \xi_1 \dots \xi_N) S_N\big).$$

The Lie conformal superalgebra $\widetilde{S}_N$ is simple for $N \geqslant 2$ and has rank $N2^N$.

As for the annihilation algebra, we remark that the automorphism of $\overline{\wedge}(1, N)$ sending x to $(1 + \xi_1 \dots \xi_N)x$ and leaving the ξ_i's unchanged induces an automorphism in the space of differential forms which transforms the standard volume form $\mathrm{d}x \wedge v$ into $(1 + \xi_1 \dots \xi_N)\,\mathrm{d}x \wedge v$. Hence,

$$\mathcal{A}\big(\widetilde{S}_N\big) = \overline{S}(1, N)'.$$

Moreover, $\mathrm{ad}(\partial_0)$ is sent to $\mathrm{ad}(\partial_0 - \xi_1 \dots \xi_N \partial_0)$, so that

$$\mathcal{A}\big(\widetilde{S}_N\big)^e = \mathbb{C}\,\mathrm{ad}(\partial_0 - \xi_1 \dots \xi_N \partial_0) \ltimes \overline{S}(1, N)'.$$

Example 3.8. Given the differential form

$$\omega = \mathrm{d}x - \sum_{i=1}^{N} \xi_i \,\mathrm{d}\xi_i,$$

we define the following subalgebras of $W(\!|1, N|\!)$ and $\overline{W}(1, N)$, respectively:

$$K(\!|1, N|\!) = \big\{D \in W(\!|1, N|\!) \;\big|\; D\omega = P\omega \;\exists P \in \wedge(\!|1, N|\!)\big\},$$
$$\overline{K}(1, N) = \big\{D \in \overline{W}(1, N) \;\big|\; D\omega = P\omega \;\exists P \in \overline{\wedge}(1, N)\big\},$$

see [K5]. They consist of linear operators of the form

$$D^f = f\partial_0 + \frac{1}{2}(-1)^{p(f)} \sum_{i=1}^{N} (\xi_i \partial_0 + \partial_i)(f)(\xi_i \partial_0 + \partial_i)$$

for $f \in \wedge(\!|1, N|\!)$ and $\overline{\wedge}(1, N)$, respectively.

The Lie superalgebra $\overline{K}(1, N)$ is linearly compact and simple for all $N \in \mathbb{Z}_+$ [K5]. The space $\wedge(\!|1, N|\!)$ (respectively $\overline{\wedge}(1, N)$) can be identified with the Lie superalgebra $K(\!|1, N|\!)$ (respectively $\overline{K}(1, N)$) via the map $f \to D^f$, in which case the bracket becomes, for $f, g \in \wedge(\!|1, N|\!)$ (respectively $\overline{\wedge}(1, N)$):

$$[f, g] = \left(f - \frac{1}{2}\sum_{i=1}^{N} \xi_i \partial_i f\right)\partial_0 g - \partial_0 f\left(g - \frac{1}{2}\sum_{i=1}^{N} \xi_i \partial_i g\right) + (-1)^{p(f)} \frac{1}{2} \sum_{i=1}^{N} \partial_i f \partial_i g.$$

For $A \in \wedge(N)$, we define the $\wedge(\!|1, N|\!)$-valued formal distribution

$$A(z) = \sum_{n \in \mathbb{Z}} (x^n A) z^{-n-1}.$$

If $A = \xi_{i_1} \dots \xi_{i_r}$, $B = \xi_{j_1} \dots \xi_{j_s}$, we have

$$\big[A(z), B(w)\big] = \left(\left(\frac{r}{2} - 1\right)\partial_w AB(w) + (-1)^r \frac{1}{2}\sum_{i=1}^{N}(\partial_i A \partial_i B)(w)\right)\delta(z-w) + \left(\frac{r+s}{2} - 2\right)AB(w)\partial_w\delta(z-w),$$

hence the formal distributions in $\mathcal{F}_K = \{A(z)\}_{A\in\wedge(N)}$ are mutually local and $(K(\!|1, N|\!), \mathcal{F}_K)$ is a formal distribution Lie superalgebra. The associated Lie conformal superalgebra is

$$K_N = \mathbb{C}[\partial] \otimes \wedge(N)$$

with λ-bracket

$$[A_\lambda B] = \left(\left(\frac{r}{2} - 1\right)\partial(AB) + (-1)^r \frac{1}{2}\sum_{i=1}^{N}\partial_i A \partial_i B\right) + \lambda\left(\frac{r+s}{2} - 2\right)AB.$$

Using the same argument as for W_N, one can see that the Lie conformal superalgebra K_N is simple for all $N \in \mathbb{Z}_+$, $N \neq 4$, and is a free $\mathbb{C}[\partial]$-module of rank 2^N. By Proposition 3.2, we have

$$\mathcal{A}(K_N) = \overline{K}(1, N) \quad \text{and} \quad \mathcal{A}(K_N)^e = \mathbb{C}\,\mathrm{ad}(\partial_0) \ltimes \overline{K}(1, N).$$

Example 3.9. The Lie superalgebra $K(\!|1, 4|\!)$ is not simple:

$$K(\!|1, 4|\!) = K(\!|1, 4|\!)' \oplus \left(\mathbb{C}x^{-1}\xi_1 \ldots \xi_4\right),$$

but $K(\!|1, 4|\!)' = [K(\!|1, 4|\!), K(\!|1, 4|\!)]$ is simple. Also, $K(\!|1, 4|\!)'$ is a formal distribution Lie superalgebra, spanned by the coefficients of the family of mutually local formal distributions

$$\mathcal{F}_{K'} = \big\{A(z),\ \text{where } A \text{ is a monomial in } \wedge(4),\ A \neq \xi_1 \ldots \xi_4;\ \partial_z \xi_1 \ldots \xi_4(z)\big\}.$$

Its associated Lie conformal superalgebra is K_4', the derived subalgebra of $K_4 = K_4' \oplus \mathbb{C}\xi_1 \ldots \xi_4$. By formula (3.4), $(\partial\xi_1 \ldots \xi_4)_0$ is a central element of the formal distribution Lie superalgebra $(\mathrm{Lie}\, K_4', \mathcal{F}(K_4'))$. Recall that we have a surjective homomorphism $\mathrm{Lie}\, K_4' \to K(\!|1, 4|\!)'$. It follows from Proposition 3.2 that $\mathrm{Lie}\, K_4'$ is a central extension of $K(\!|1, 4|\!)'$ by a 1-dimensional center. We denote it by $CK(\!|1, 4|\!)'$. The corresponding cocycle is given by [KL, formula (4.22)] for $d = 0$. It follows that the annihilation algebra of K_4' is a central extension $C\overline{K}(1, 4)$ of $\overline{K}(1, 4)$ obtained by restricting the above central extension of $K(\!|1, 4|\!)'$ to the subalgebra $K(1, 4)$ and then going to the completion $\overline{K}(1, 4)$. The non-zero entries of the corresponding cocycle are given by:

$$\psi(1, \xi_1 \ldots \xi_4) = 1, \qquad \psi(\xi_i, \partial_i \xi_1 \ldots \xi_4) = \frac{1}{2}, \quad i = 1, 2, 3, 4. \tag{3.18}$$

Also, $\mathcal{A}(K_4')^e = \mathbb{C}\,\mathrm{ad}(\partial_0) \ltimes C\overline{K}(1, 4)$, where ∂_0 acts trivially on the center.

Example 3.10. The formal distribution Lie superalgebra $(K(\!(1,6)\!), \mathcal{F}_K)$ has a simple subalgebra, denoted by $(CK(\!(1,6)\!), \mathcal{F}_{CK})$. The associated Lie conformal superalgebra is CK_6. It is a simple rank 32 subalgebra of K_6, whose even part is $W_0 \ltimes \mathrm{Cur}\, \mathfrak{so}_6$ and whose odd part is spanned by six primary fields of conformal weight $3/2$ and ten primary fields of conformal weight $1/2$. For the explicit form of the commutation relations, as well as for more detailed information on CK_6, see [CK2].

The annihilation algebra of CK_6 is the exceptional simple linearly compact Lie superalgebra $\overline{E}(1,6)$ (see [CK2,K5]), which is a subalgebra of $\overline{K}(1,6)$. The extended annihilation algebra is $\mathcal{A}(CK_6)^e = \mathbb{C}\, \mathrm{ad}(\partial_0) \ltimes \overline{E}(1,6)$.

Remark 3.11. The Virasoro conformal algebra is the only non-abelian rank 1 Lie conformal algebra. One has $W_0 = \mathbb{C}[\partial]L$, where $[L_\lambda L] = (\partial + 2\lambda)L$. An even element L of a Lie conformal superalgebra satisfying this λ-bracket is called a Virasoro element.

Remark 3.12. The following are the most important Lie conformal superalgebras. $W_0 \simeq K_0$ is the Virasoro conformal algebra, K_1 is the Neveu–Schwarz Lie conformal superalgebra, $K_2 \simeq W_1$ is the $N = 2$ Lie conformal superalgebra, K_3 is the $N = 3$ Lie conformal superalgebra, $S_{2,0}$ is the $N = 4$ Lie conformal superalgebra, and K_4' is the big $N = 4$ Lie conformal superalgebra. The corresponding formal distribution Lie superalgebras (or, rather, their central extensions) are well known and play an important role in physics.

4. Central extensions

4.1. Central extensions of Lie superalgebras

We shall be dealing with the cohomology $H^2(\mathfrak{g})$ of a Lie superalgebra $\mathfrak{g}$ with coefficients in the trivial $\mathfrak{g}$-module $\mathbb{C}$, which parameterizes the central extensions of $\mathfrak{g}$. The following lemma is well known (cf. [K1, Exercise 7.6]).

Lemma 4.1. *Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra, which has an invariant, supersymmetric non-degenerate bilinear form* (*even or odd*)*. Then the* 2*-cocycles on $\mathfrak{g}$ with coefficients in $\mathbb{C}$ are in* 1–1 *correspondence with the derivations of $\mathfrak{g}$ that are skew-symmetric with respect to the form, the cocycle being trivial iff the corresponding derivation is inner. This correspondence is given by $\alpha_D(x,y) = (Dx,y)$, $D \in \mathrm{Der}(\mathfrak{g})$, $x, y \in \mathfrak{g}$.*

Proof. If α is a 2-cocycle on $\mathfrak{g}$, it can be written in the form $\alpha_D(x,y) = (Dx,y)$, where D is an endomorphism of the space $\mathfrak{g}$. One can easily check that the skew-symmetry of α is equivalent to the skew-symmetry of D, that the cocycle equation is equivalent to D being a derivation and that α is a trivial cocycle iff D is an inner derivation. □

In the remaining part of this subsection, we compute the central extensions of all simple finite-dimensional Lie superalgebras $\mathfrak{s}$ and their $\mathbb{C}[\![t]\!]$-current algebras $\mathfrak{s}[\![t]\!]$.

The main result in [K4] is the following theorem.

Theorem 4.2. *A simple finite-dimensional Lie superalgebra is isomorphic either to one of the simple Lie algebras or to one of the Lie superalgebras* $A(m,n)$ $(0 \leqslant m < n)$, $A(n,n)$ $(n > 0)$, $B(m,n)$ $(m \geqslant 0,\ n > 0)$, $C(n)$ $(n \geqslant 3)$, $D(m,n)$ $(m \geqslant 2,\ n > 0)$, $D(2,1,\alpha)$, $F(4)$, $G(3)$, $P(n)$ $(n \geqslant 2)$, $Q(n)$ $(n \geqslant 2)$, $W(n)$ $(n \geqslant 3)$, $S(n)$ $(n \geqslant 4)$, $\widetilde{S}(n)$ $(n \geqslant 4$, *n even*), *or* $H(n)$ $(n \geqslant 5)$.

Proposition 4.3. *A simple finite-dimensional Lie superalgebra* $\mathfrak{s}$ *has an even* (*respectively odd*) *supersymmetric, invariant bilinear form iff* $\mathfrak{s}$ *is isomorphic to* $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1,\alpha)$, $G(3)$, $F(4)$, $H(n)$ *with n even* (*respectively* $\mathfrak{s} \simeq H(n)$ *with n odd or* $Q(n)$). *Such a form is unique up to a constant factor.*

Proof. The Lie superalgebras $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1,\alpha)$, $G(3)$, and $F(4)$ are contragredient with a symmetrizable Cartan matrix, hence they have an even invariant form [K4].

The obvious pairing between the even and the odd part of $Q(n)$ is an odd invariant form.

Define on $\wedge(n)$ the Poisson bracket

$$\{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_i}.$$

This Lie superalgebra has a bilinear form $(f,g) = \int fg\,\mathrm{d}\xi_1\ldots\mathrm{d}\xi_n$ (where $\int$ denotes the Berezin integral, cf. [Be]), which is invariant on the derived subalgebra $\wedge(n)'$ i.e. the span of all monomials except the top one. The kernel of the restriction of the form to $\wedge(n)'$ is $\mathbb{C}$. Since $H(n) \simeq \wedge(n)'/\mathbb{C}$, the proposition is proved in this case as well.

In order to show that in the remaining cases there is no invariant form on $\mathfrak{s}$, we take a maximal reductive Lie subalgebra $\mathfrak{r}$ of $\mathfrak{s}$ and show that the $\mathfrak{r}$-modules $\mathfrak{s}$ and $\mathfrak{s}^\star$ are not isomorphic. □

Proposition 4.4. *A complete list of non-trivial* $H^2(\mathfrak{s})$ *for all simple finite-dimensional Lie superalgebras* $\mathfrak{s}$ *is as follows*:

$$H^2\big(A(1,1)\big) = H^2_{\mathrm{even}}\big(A(1,1)\big) = \mathbb{C}^3;$$

$$H^2\big(A(n,n)\big) = H^2_{\text{even}}\big(A(n,n)\big) = \mathbb{C}, \quad n > 1;$$
$$H^2\big(P(3)\big) = H^2_{\text{even}}\big(P(3)\big) = \mathbb{C};$$
$$H^2\big(Q(n)\big) = H^2_{\text{even}}\big(Q(n)\big) = \mathbb{C}, \quad n \geqslant 2;$$
$$H^2\big(H(n)\big) = H^2_{\text{even}}\big(H(n)\big) = \mathbb{C}, \quad n \geqslant 5.$$

Proof. In the cases when $\mathfrak{s}$ carries an invariant, supersymmetric, non-degenerate bilinear form, we apply Lemma 4.1 and the known description of derivations (see [K4, Proposition 5.1.2]). Notice that in all cases except $H(n)$ all derivations are skew-symmetric, whereas for $H(n)$, $n \geqslant 5$, only one of the two outer derivations, namely $D = \sum_i (\partial_i \xi_1 \ldots \xi_n)\partial_i$, has this property. In the remaining cases, i.e. $\mathfrak{s} = P(n)$, $S(n)$, $\widetilde{S}(n)$, and $W(n)$, we apply Lemma 2.1 of [KL] to compute $H^2(\mathfrak{s})$. □

Remark 4.5. The central extensions corresponding to the non-trivial cocycles listed above are as follows:

(1) $A(n,n)$: the canonical homomorphism $sl(n+1,n+1) \to A(n,n)$;
(2) $Q(n)$: the canonical homomorphism $\widetilde{Q}(n) \to Q(n)$;
(3) if $\mathfrak{s} = \bigoplus_{j \geqslant -1} \mathfrak{s}_j$ (respectively $\mathfrak{s} = \bigoplus_{j \leqslant 1} \mathfrak{s}_j$) is a consistent $\mathbb{Z}$-gradation of $\mathfrak{s}$ such that the $\mathfrak{s}_0$-module $\mathfrak{s}_{-1}$ carries a non-zero symmetric invariant bilinear form $(.\,,.)$, then we have the following 2-cocycle on $\mathfrak{s}$: $\alpha(x,y) = (x,y)$, if $x, y \in \mathfrak{s}_{-1}$ (respectively $x, y \in \mathfrak{s}_{+1}$), and $\alpha(x,y) = 0$ if $x \in \mathfrak{s}_i$, $y \in \mathfrak{s}_j$, and i or j is not -1 (respectively i or j is not $+1$); the central extensions of $H(n)$, $P(3)$, and the remaining two central extensions of $A(1,1)$ are obtained in this way.

Proposition 4.6. *Let $\mathfrak{s}$ be a simple finite-dimensional Lie superalgebra. Then any irreducible central extension of $\mathfrak{s}[\![t]\!]$ is isomorphic to $\hat{\mathfrak{s}}[\![t]\!]$, where $\hat{\mathfrak{s}}$ is an irreducible central extension of $\mathfrak{s}$.*

Proof. Note that all Kähler differentials of $\mathbb{C}[\![t]\!]$ are exact. Hence one can use the same argument as in [Sa]. □

4.2. Central extensions of Lie conformal superalgebras

Basic and reduced cohomology of Lie conformal algebras are defined in [BKV]. They are denoted respectively by $\widetilde{H}^\star$ and $H^\star$. The same definitions (with appropriate signs) apply to the case of Lie conformal superalgebras.

A central extension of a Lie conformal superalgebra R by a Lie conformal superalgebra K is a $\mathbb{C}[\partial]$-split (i.e. $S = K \oplus R$ as $\mathbb{C}[\partial]$-modules) short exact sequence of Lie conformal superalgebras

$$0 \to K \to S \to R \to 0,$$

in which K is central in S and the action of ∂ is trivial on K. Equivalence classes of 1-dimensional central extensions are parameterized by elements of the second reduced cohomology group $H^2(R)$ (cf. [BKV]).

Lemma 4.7. *Let R be a Lie conformal superalgebra. Suppose in R there is an element L such that $L_{(0)}a = \partial a$ for any $a \in R$. Then*

$$H^n(R) = H^n\big(\mathcal{A}(R)\big) \oplus H^{n+1}\big(\mathcal{A}(R)\big), \quad n \geqslant 0.$$

Proof. This follows from [BDK, Proposition 15.6]. □

Lemma 4.8. *Let R be a Lie conformal superalgebra such that $R = R'$. Then $H^1(R) = 0$.*

Proof. We denote by $\tilde{\mathrm{d}}^n$ the nth differential of the basic complex and by d^n the corresponding differential of the reduced complex. $\mathbb{C}$ is a trivial $\mathbb{C}[\partial]$-module, hence $\tilde{\mathrm{d}}^0 = \mathrm{d}^0 = 0$, and $H^1(R) = \ker \mathrm{d}^1$.

We remark that $\gamma \in \partial \widetilde{C}^1(R)$ if, and only if, for any $a \in R$, $\gamma_\lambda(a)$ has no constant term as polynomial in λ. Indeed, if $\gamma = \partial\omega$, $\gamma_\lambda(a) = (\partial\omega)_\lambda(a) = \lambda\omega_\lambda(a)$. Also, if $\gamma_\lambda(a) = \lambda P_a(\lambda)$ for any $a \in R$, we can define $\omega \in \widetilde{C}^1(R)$ such that $\omega_\lambda(a) = P_a(\lambda)$ and clearly $\gamma = \partial\omega$.

Recall that

$$\mathrm{d}^1 : \frac{\widetilde{C}^1(R)}{\partial\widetilde{C}^1(R)} \to \frac{\widetilde{C}^2(R)}{\partial\widetilde{C}^2(R)}.$$

Suppose $[\gamma] = \gamma + \partial\widetilde{C}^1(R) \in \ker \mathrm{d}^1$. Then $\mathrm{d}^1([\gamma]) = 0$, i.e. $\tilde{\mathrm{d}}^1\gamma \in \partial\widetilde{C}^2(R)$. It follows that, for any $a_1, a_2 \in R$,

$$\big(\tilde{\mathrm{d}}^1\gamma\big)_{\lambda_1,\lambda_2}(a_1, a_2) = (\partial\beta)_{\lambda_1,\lambda_2}(a_1, a_2) = (\lambda_1 + \lambda_2)\beta_{\lambda_1,\lambda_2}(a_1, a_2).$$

On the other hand, $(\tilde{\mathrm{d}}^1\gamma)_{\lambda_1,\lambda_2}(a_1, a_2) = -\gamma_{\lambda_1+\lambda_2}([a_{1\lambda_1}a_2])$. Therefore, the polynomial $\gamma_\lambda(a)$ has no constant term for any $a \in R'$. But $R = R'$, so this actually holds for any $a \in R$. It follows that $\gamma \in \partial\widetilde{C}^1(R)$, i.e. $[\gamma] = 0$. □

Lemma 4.9. *Let R be a Lie conformal superalgebra, such that $R = R'$. Suppose R has an element L such that $L_{(0)}a = \partial a$ for any $a \in R$. Then $H^2(\mathcal{A}(R)) = 0$. In particular, this holds for the annihilation algebras of the simple Lie conformal superalgebras W_N ($N \geqslant 0$), $S_{N,a}$ ($N \geqslant 2$), $\widetilde{S}_N$ (N even, $N \geqslant 2$), K_N ($N \geqslant 0$, $N \neq 4$), K_4', CK_6.*

Proof. This follows from Lemmas 4.7 and 4.8, since $0 = H^1(R) = H^1(\mathcal{A}(R)) \oplus H^2(\mathcal{A}(R))$. □

Lemma 4.10. *Let R be a finite Lie conformal superalgebra, which is free as a $\mathbb{C}[\partial]$-module. Let $\mathfrak{s}$ be a reductive Lie algebra, and let $\mathfrak{s} \to R$ be an injective homomorphism with respect to the* 0*th product on R. Assume that $R = \bigoplus_{i\in I} \mathfrak{g}_i$ is a decomposition of R into a direct sum of finite-dimensional, irreducible $\mathfrak{s}$-modules with respect to the* 0*th product. Then every reduced* 2*-cocycle ψ on R is equivalent to a cocycle α_λ such that*

(1) $\alpha_\lambda(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ *if $\mathfrak{g}_i, \mathfrak{g}_j$ are not contragredient $\mathfrak{s}$-modules*;
(2) $\alpha_\lambda(x, y) = P_{ij}\langle x, y\rangle$ *for all $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_j$, and some $P_{ij} \in \mathbb{C}[\lambda]$ if $\mathfrak{g}_i, \mathfrak{g}_j$ are contragredient $\mathfrak{s}$-modules* ($\langle\, ,\rangle$ *denotes the pairing between them*).

Proof. The Lie algebra $\mathfrak{s}$ acts completely reducibly on the space of reduced 2-cocycles, hence there exists an $\mathfrak{s}$-invariant subspace V complementary to the space of trivial 2-cocycles. Since R acts trivially on $H^2(R)$, we conclude that $\mathfrak{s}$ acts trivially on V. □

In the remaining part of this section, we compute the central extensions of all simple Lie conformal superalgebras listed in Section 3.2. The proofs are all based on Lemma 4.10. We give all the details of the computations only in the most involved case (cf. Proposition 4.16) and omit them in all other cases. We give only the non-zero entries of the non-trivial cocycles.

Proposition 4.11. *Let $\mathfrak{s}$ be a simple finite-dimensional Lie superalgebra. Then all central extensions of* $\mathrm{Cur}\,\mathfrak{s}$ *are given by the following* 2*-cocycles*

$$\alpha_\lambda(a, b) = \alpha_0(a, b) + (a, b)\lambda, \quad a, b \in \mathfrak{s},$$

where $\alpha_0(a, b)$ is a 2*-cocycle on $\mathfrak{s}$ and $(.\,,.)$ is a supersymmetric, invariant bilinear form on $\mathfrak{s}$* (*cf. Propositions* 4.3 *and* 4.4)*. Two such cocycles are equivalent iff the* 2*-cocycles α_0 are equivalent.*

Proof. See [K2, Section 2.7]. □

In what follows we shall often denote an element $1 \otimes (\sum_{i=1}^N f_i\partial_i + f) \in W_N = \mathbb{C}[\partial] \otimes (W(N) \oplus \wedge(N))$ by $\sum_i (f_i)^i + f$.

Proposition 4.12. *The Lie conformal superalgebras W_0, W_1, and W_2 have a unique, up to isomorphism, central extension. The corresponding* 2*-cocycles are as follows*:

$$W_0\colon\quad \alpha_3(1, 1) = \frac{1}{2};$$

$$W_1\colon\quad \alpha_1\big((\xi_1)^1, (\xi_1)^1\big) = \frac{1}{3}, \qquad \alpha_2\big((\xi_1)^1, 1\big) = \frac{1}{3},$$

$$\alpha_2\big((1)^1, \xi_1\big) = -\frac{1}{3};$$

$$W_2\colon\quad \alpha_1\big((\xi_1)^2,(\xi_2)^1\big)=\frac{1}{6},\qquad \alpha_1\big((\xi_1)^1,(\xi_2)^2\big)=-\frac{1}{6},$$
$$\alpha_1\big((\xi_1\xi_2)^2,(1)^1\big)=-\frac{1}{6},\qquad \alpha_1\big((\xi_1\xi_2)^1,(1)^2\big)=\frac{1}{6}.$$

Proposition 4.13. *The Lie conformal superalgebra W_N has no non-trivial central extensions if $N \geqslant 3$.*

Using [AF] and Lemma 4.7 one actually computes the whole basic and reduced cohomology of W_N.

Proposition 4.14. *The Lie conformal superalgebra $S_{2,a}$ has a unique, up to isomorphism, non-trivial central extension. The corresponding 2-cocycle is as follows*:

$$\alpha_3(L_a,L_a)=\frac{1}{2},\quad \textit{where } L_a=-1+\frac{1}{2}(\partial-a)\big((\xi_1)^1+(\xi_2)^2\big),$$
$$\alpha_2\big((1)^1,\xi_1-(\partial-a)(\xi_1\xi_2)^2\big)=-\frac{1}{3},$$
$$\alpha_1\big((1)^1,\xi_1-(\partial-a)(\xi_1\xi_2)^2\big)=\frac{a}{6},$$
$$\alpha_0\big((1)^1,\xi_1-(\partial-a)(\xi_1\xi_2)^2\big)=-\frac{a^2}{24},$$
$$\alpha_2\big((1)^2,\xi_2+(\partial-a)(\xi_1\xi_2)^1\big)=-\frac{1}{3},$$
$$\alpha_1\big((1)^2,\xi_2+(\partial-a)(\xi_1\xi_2)^1\big)=\frac{a}{6},$$
$$\alpha_0\big((1)^2,\xi_2+(\partial-a)(\xi_1\xi_2)^1\big)=-\frac{a^2}{24},$$
$$\alpha_1\big((\xi_1)^2,(\xi_2)^1\big)=\frac{1}{6},$$
$$\alpha_1\big((\xi_1)^1-(\xi_2)^2,(\xi_1)^1-(\xi_2)^2\big)=\frac{1}{3}.$$

Proposition 4.15. *The Lie conformal superalgebra $\widetilde{S}_2$ has a unique, up to isomorphism, non-trivial central extension. The corresponding 2-cocycle is as follows*:

$$\alpha_3(\widetilde{L},\widetilde{L})=\frac{1}{2},\quad \textit{where } \widetilde{L}=-(1-\xi_1\xi_2)+\frac{1}{2}\partial\big((\xi_1)^1+(\xi_2)^2\big),$$
$$\alpha_1\big((1-\xi_1\xi_2)^1,(1-\xi_1\xi_2)^2\big)=-\frac{1}{3},$$
$$\alpha_2\big((1-\xi_1\xi_2)^2,\xi_2+\partial(\xi_1\xi_2)^1\big)=-\frac{1}{3},$$

$$\alpha_2\big((1-\xi_1\xi_2)^1, \xi_1 - \partial(\xi_1\xi_2)^2\big) = -\frac{1}{3}, \qquad \alpha_1\big((\xi_1)^2, (\xi_2)^1\big) = \frac{1}{6},$$
$$\alpha_1\big((\xi_1)^1 - (\xi_2)^2, (\xi_1)^1 - (\xi_2)^2\big) = \frac{1}{3}.$$

Proposition 4.16. *The Lie conformal superalgebras* $S_{N,a}$ $(a \in \mathbb{C})$ *and* $\widetilde{S}_N$ (N *even*) *have no non-trivial central extensions if* $N > 2$.

Proof. We remark that $S_N \simeq S_{N,a} \simeq \widetilde{S}_N \simeq W(N)$ as $\mathfrak{sl}_N$-modules. By Lemma 4.10, we need to compute the cocycles corresponding to the $\mathfrak{sl}_N$-invariants of $W(N) \otimes W(N)$, which occur in the following cases $(1 \leqslant k \leqslant N-1)$:

$$R(\pi_1 + \pi_{N-1}) \otimes R(\pi_1 + \pi_{N-1}), \qquad R(0) \otimes R(0),$$
$$R(\pi_{N-1}) \otimes R(\pi_1), \qquad R(\pi_k) \otimes R(\pi_{N-k}).$$

Let

$$L_a = -1 + \frac{1}{N}(\partial - a)H \quad \text{and} \quad \widetilde{L} = -(1 - \xi_1 \dots \xi_N) + \frac{1}{N}\partial H$$

be Virasoro elements for $S_{N,a}$ and $\widetilde{S}_N$, respectively, where $H = \sum_{i=1}^N \xi_i \partial_i$ denotes the Euler operator. Let ∂_N (respectively $(1 - \xi_1 \dots \xi_N)\partial_N$) be the highest weight vector of $R(\pi_{N-1}) \subseteq W(N)_{-1}$ in $S_{N,a}$ (respectively $\widetilde{S}_N$). Let v_k^a (respectively $\widetilde{v_k}$) be the highest weight vector of $R(\pi_k) \subseteq W(N)_k$ in $S_{N,a}$ (respectively $\widetilde{S}_N$), $1 \leqslant k \leqslant N-1$. Then

$$v_k^a = (k - N)\xi_1 \dots \xi_k + (\partial - a)\xi_1 \dots \xi_k H,$$
$$\widetilde{v_k} = (k - N)\xi_1 \dots \xi_k + \partial \xi_1 \dots \xi_k H.$$

Let w_k^a (respectively $\widetilde{w_k}$) be the lowest weight vector of $R(\pi_{N-k}) = R(\pi_k)^\star$, which we view as lowest weight module $R(-\pi_k)^{\text{low}} \subseteq W(N)_{N-k}$ in $S_{N,a}$ (respectively $\widetilde{S}_N$), $1 \leqslant k \leqslant N-1$. Then

$$w_k^a = -k\xi_{k+1} \dots \xi_N + (\partial - a)\xi_{k+1} \dots \xi_N H,$$
$$\widetilde{w_k} = -k\xi_{k+1} \dots \xi_N + \partial \xi_{k+1} \dots \xi_N H.$$

The action of L_a and $\widetilde{L}$ is given by the following formulas $(1 \leqslant k \leqslant N-1)$:

$$[L_{a\lambda} \partial_N] = \frac{1}{N}\big(N\partial + (N+1)\lambda + a\big)\partial_N,$$
$$\big[\widetilde{L}_\lambda\big((1 - \xi_1 \dots \xi_N)\partial_N\big)\big] = \frac{1}{N}\big(N\partial + (N+1)\lambda\big)(1 - \xi_1 \dots \xi_N)\partial_N - v_{N-1}^0,$$
$$\big[L_{a\lambda} v_k^a\big] = \frac{1}{N}\big(N\partial + (2N - k)\lambda - ak\big)v_k^a,$$
$$\big[\widetilde{L}_\lambda \widetilde{v_k}\big] = \frac{1}{N}\big(N\partial + (2N - k)\lambda\big)\widetilde{v_k},$$

$$\left[L_{a\lambda} w_k^a\right] = \frac{1}{N}\bigl(N\partial + (N+k)\lambda - a(N-k)\bigr)w_k^a,$$
$$\left[\widetilde{L}_\lambda \widetilde{w_k}\right] = \frac{1}{N}\bigl(N\partial + (N+k)\lambda\bigr)\widetilde{w_k}.$$

We remark that $R(\pi_1 + \pi_{N-1}) \subseteq \operatorname{Cur} S(N)$ for $S_{N,a}$ and $R(\pi_1 + \pi_{N-1}) \subseteq \operatorname{Cur} \widetilde{S}(N)$ for $\widetilde{S}_N$. By Proposition 4.11, the corresponding cocycle is trivial.

In all other cases, by Lemma 4.10 we need to show that the cocycle pairing the highest weight vector and the lowest weight vector is trivial. The cocycle equation for the triples $(v_k^a, \xi_k\partial_k - \xi_{k+1}\partial_{k+1}, w_k^a)$, (L_a, v_k^a, w_k^a), and $(\widetilde{v_k}, \xi_k\partial_k - \xi_{k+1}\partial_{k+1}, \widetilde{w_k})$, $(\widetilde{L}, \widetilde{v_k}, \widetilde{w_k})$ shows that $\alpha_\lambda(v_k^a, w_k^a) = \alpha_\lambda(\widetilde{v_k}, \widetilde{w_k}) = 0$.

Finally, the cocycle equation for the triple (L_a, ∂_N, w_1^a) shows that $\alpha_\lambda(L_a, L_a)$ and $\alpha_\lambda(\partial_N, w_1^a)$ are trivial. Similarly, one can see that the cocycle equation for the triple $(\widetilde{L}, (1 - \xi_1 \ldots \xi_N)\partial_N, \widetilde{w_1})$ implies that $\alpha_\lambda(\widetilde{L}, \widetilde{L})$ and $\alpha_\lambda((1 - \xi_1 \ldots \xi_N)\partial_N, \widetilde{w_1})$ are also trivial. Therefore, $H^2(S_{N,a}) = H^2(\widetilde{S}_N) = 0$. □

Proposition 4.17. *The Lie conformal superalgebras K_0, K_1, K_2, and K_3 have a unique, up to isomorphism, central extension. The corresponding 2-cocycles are as follows*:

$$\alpha_3(1,1) = \frac{1}{2}, \qquad \alpha_2(\xi_i,\xi_i) = \frac{1}{6} \quad (i = 1,2), \qquad \alpha_1(\xi_1\xi_2, \xi_1\xi_2) = -\frac{1}{12},$$
$$\alpha_1(\xi_i\xi_j, \xi_i\xi_j) = -\frac{1}{12} \quad (i \neq j),$$
$$\alpha_0(\xi_i\xi_j\xi_k, \xi_i\xi_j\xi_k) = -\frac{1}{12} \quad (i \neq j \neq k).$$

Proposition 4.18. *The Lie conformal superalgebra K_4' has two, up to isomorphism, linearly independent central extensions. The corresponding 2-cocycles α and β are as follows*:

$$\alpha_3(1,1) = \frac{1}{2}, \qquad \alpha_2(\xi_i,\xi_i) = \frac{1}{6},$$
$$\alpha_1(\xi_i\xi_j, \xi_i\xi_j) = -\frac{1}{12} \quad (i \neq j),$$
$$\alpha_0(\xi_i\xi_j\xi_k, \xi_i\xi_j\xi_k) = -\frac{1}{12} \quad (i \neq j \neq k),$$
$$\alpha_1(\partial\xi_1\xi_2\xi_3\xi_4, \partial\xi_1\xi_2\xi_3\xi_4) = -\frac{1}{12};$$
$$\beta_2(1, \partial\xi_1\xi_2\xi_3\xi_4) = 2, \qquad \beta_1\bigl(\xi_i, \partial_i(\xi_1\xi_2\xi_3\xi_4)\bigr) = 1,$$
$$\beta_1\bigl(\xi_i\xi_j, \partial_i\partial_j(\xi_1\xi_2\xi_3\xi_4)\bigr) = -1.$$

Proposition 4.19. *The Lie conformal superalgebras K_N $(N \geqslant 5)$ and CK_6 have no non-trivial central extensions.*

5. Finite simple Lie conformal superalgebras

5.1. The annihilation algebra

In this section we study the annihilation algebra $\mathcal{A}(R)$ of a finite simple Lie conformal superalgebra R. We shall use the following two propositions, whose proof is the same as in the non-super case, see [DK, Lemma 4.3 and Proposition 5.1].

Proposition 5.1. *Let R be a finite Lie conformal superalgebra. Then any non-central T-invariant ideal J of $\mathcal{A}(R)$ contains a non-zero regular ideal.*

Proposition 5.2. *Let R_1 and R_2 be two finite Lie conformal superalgebras, which are free as $\mathbb{C}[\partial]$-modules. Let $\varphi : \mathbb{C}T_1 \ltimes \mathcal{A}(R_1) \to \mathbb{C}T_2 \ltimes \mathcal{A}(R_2)$ be a homomorphism of the corresponding extended annihilation algebras such that $\varphi(T_1) = T_2$ and $\varphi(\mathcal{A}(R_1)) = \mathcal{A}(R_2)$. Then there exists a unique Lie conformal superalgebra homomorphism $\tilde{\varphi} : R_1 \to R_2$ that induces φ, i.e. $\varphi(a_i) = (\tilde{\varphi}(a))_i$ for all $a \in R$, $i \in \mathbb{Z}_+$.*

Lemma 5.3. *Let R be a finite simple Lie conformal superalgebra. Then $\mathcal{A}(R)$ is isomorphic* (*as a topological Lie superalgebra*) *to an irreducible central extension of the Lie superalgebra $\mathbb{C}[[t_1, \ldots, t_r]] \widehat{\otimes}\, \mathfrak{s}$, where $r = 0$ or 1 and $\mathfrak{s}$ is a simple linearly compact Lie superalgebra.*

Proof. Recall that $\mathcal{A}(R)$ is a linearly compact Lie superalgebra and that it is a closed ideal of codimension 1 in the extended annihilation algebra $\mathcal{A}(R)^e = \mathbb{C}T \ltimes \mathcal{A}(R)$. By Proposition 5.1, due to the simplicity of R, $\mathcal{A}(R)$ contains no non-central T-invariant ideals different from $\mathcal{A}(R)$. Let Z be the center of $\mathcal{A}(R)$. Since the derived algebra of $\mathcal{A}(R)$ is a non-central T-invariant ideal of $\mathcal{A}(R)$ (otherwise R would be nilpotent), we conclude that $\mathcal{A}(R) = [\mathcal{A}(R), \mathcal{A}(R)]$ and therefore

$$0 \to Z \to \mathcal{A}(R) \to \mathcal{A}(R)/Z \to 0$$

is an irreducible central extension. Also, the center of $\mathcal{A}(R)/Z$ is zero, since otherwise its pre-image in $\mathcal{A}(R)$ would be a proper non-central T-invariant nilpotent ideal.

It follows that $\mathcal{A}(R)/Z$ contains no non-trivial T-invariant ideals and therefore $\mathcal{A}(R)/Z$ is a minimal ideal in $\mathcal{A}(R)^e/Z$. Thus we may apply Corollary 2.8 to the Cartan–Guillemin theorem to obtain the isomorphism of linearly compact Lie superalgebras $\mathcal{A}(R)/Z \simeq (\mathbb{C}[[t_1, \ldots, t_r]] \otimes \wedge(m)) \widehat{\otimes}\, \mathfrak{s}$, where $r, m \in \mathbb{Z}_+$ and $\mathfrak{s}$ is a simple linearly compact Lie superalgebra.

Next, we show that $m = 0$. Let $\wedge_1(m)$ be the ideal of $\wedge(m)$ generated by $\xi_1, \ldots, \xi_m$. We will show that $I := (\mathbb{C}[[t_1, \ldots, t_r]] \otimes \wedge_1(m)) \widehat{\otimes}\, \mathfrak{s}$ is a T-invariant ideal of $\mathcal{A}(R)/Z$, which, of course, will imply that $m = 0$. Indeed, due to Proposition 2.12, any continuous derivation T of $(\mathbb{C}[[t_1, \dots, t_r]] \otimes \wedge(m)) \widehat{\otimes} \mathfrak{s}$ has the form

$$T = D_0 \otimes 1 + \sum_i f_i \otimes D_i, \tag{5.1}$$

where D_0 (respectively D_i) is a continuous derivation of $\mathbb{C}[[t_1, \dots, t_r]] \otimes \wedge(m)$ (respectively $\mathfrak{s}$) and $f_i \in \mathbb{C}[[t_1, \dots, t_r]] \otimes \wedge(m)$. Since T is even, D_0 is an even derivation, hence T maps the ideal I into itself.

It remains to show that $r \leqslant 1$. If D_0 leaves the ideal $(t_1, \dots, t_r)$ of $\mathbb{C}[[t_1, \dots, t_r]]$ invariant, then $(t_1, \dots, t_r) \widehat{\otimes} \mathfrak{s}$ is a non-trivial T-invariant ideal of $\mathcal{A}(R)/Z$, which is impossible. Therefore there exists a continuous automorphism of $\mathrm{Der}(\mathbb{C}[[t_1, \dots, t_r]])$ which transforms D_0 to $\partial/\partial t_1$ (see, e.g., Proposition 2.13). But in this case

$$T = \frac{\partial}{\partial t_1} \otimes 1 + \sum_i f_i \otimes D_i,$$

hence the ideal $(t_2, \dots, t_r) \widehat{\otimes} \mathfrak{s}$ of $\mathcal{A}(R)/Z$ is T-invariant. This implies that $r \leqslant 1$. □

Corollary 5.4. *Let R be a finite simple Lie conformal superalgebra and let L be an even element such that $L_{(0)}a = \partial a$ for any $a \in R$. Then $\mathcal{A}(R)$ is the universal central extension of its quotient by the center. In particular, this holds if R is one of the Lie conformal superalgebras W_N $(N \geqslant 0)$, $S_{N,a}$ $(N \geqslant 2,\ a \in \mathbb{C})$, $\widetilde{S}_N$ $(N \geqslant 2$, N even$)$, K_N $(N \geqslant 0,\ N \neq 4)$, K_4' or CK_6.*

Proof. As we have just remarked, $\mathcal{A}(R) = [\mathcal{A}(R), \mathcal{A}(R)]$, and by Lemma 4.9, $H^2(\mathcal{A}(R)) = 0$. □

By taking the completion of (3.6)–(3.9) we obtain:

$$\mathcal{A}(R) = \overline{\mathcal{L}}_0 \supseteq \overline{\mathcal{L}}_1 \supseteq \overline{\mathcal{L}}_2 \supseteq \cdots, \tag{5.2}$$

$$[\overline{\mathcal{L}}_i, \overline{\mathcal{L}}_j] \subset \overline{\mathcal{L}}_{i+j-d} \quad \text{for some } d \in \mathbb{Z}_+, \tag{5.3}$$

$$\dim \overline{\mathcal{L}}_i/\overline{\mathcal{L}}_{i+1} \leqslant \operatorname{rank} R, \tag{5.4}$$

(recall that R is a free $\mathbb{C}[\partial]$-module), and

$$\left[T, \overline{\mathcal{L}}_i\right] = \overline{\mathcal{L}}_{i-1} \quad (i \geqslant 0). \tag{5.5}$$

Proposition 5.5. *The annihilation algebra $\mathcal{A}(R)$ of a finite simple Lie conformal superalgebra R is isomorphic as a topological Lie superalgebra to one of the following linearly compact Lie superalgebras*:

(1) $\overline{W}(1,N)$, $N \geqslant 0$; $\overline{S}(1,N)'$, $N \geqslant 2$; $\overline{K}(1,N)$, $N \geqslant 0$, $N \neq 4$; $C\overline{K}(1,4)$; $\overline{E}(1,6)$;
(2) $\mathfrak{s}[\![t]\!]$, *where* $\mathfrak{s}$ *is a simple finite-dimensional Lie superalgebra.*

Proof. By Lemma 5.3, $\mathcal{A}(R)$ is an irreducible central extension of the Lie superalgebra $\mathbb{C}[\![t_1,\ldots,t_r]\!]\widehat{\otimes}\,\mathfrak{s}$, where $r=0$ or 1 and $\mathfrak{s}$ is a simple linearly compact Lie superalgebra. It follows from (5.2) and (5.4) that $\ell_j := \overline{\mathcal{L}}_{j+d}$ $(j \in -d+\mathbb{Z}_+)$ is an algebra filtration of $\mathcal{A}(R)$ such that

$$\dim \ell_j/\ell_{j+1} \leqslant \operatorname{rank} R, \quad j \in \mathbb{Z}_+, \tag{5.6}$$

hence the growth of this filtration is at most 1. This filtration induces an algebra filtration on $1\otimes\mathfrak{s}$, whose growth is therefore at most 1. We conclude (see Theorem 2.10) that $\mathrm{gw}(\mathfrak{s}) \leqslant 1$ and therefore, by Theorem 2.11, either $\dim\mathfrak{s} < \infty$ or $\mathfrak{s}$ is one of the Lie superalgebras $\overline{W}(1,N)$ $(N \geqslant 0)$, $\overline{S}(1,N)'$ $(N \geqslant 2)$, $\overline{K}(1,N)$ $(N \geqslant 0)$, $\overline{E}(1,6)$.

If $r=0$, then $\dim\mathfrak{s}=\infty$, since $\dim\mathcal{A}(R)=\infty$. In this case, due to Corollary 5.4, $\mathcal{A}(R)$ is the universal central extension of $\mathfrak{s}$. Consequently, by the results of Section 3.2, $\mathcal{A}(R)$ is one of the Lie superalgebras $\overline{W}(1,N)$, $\overline{S}(1,N)'$, $\overline{K}(1,N)$, $C\overline{K}(1,4)$, $\overline{E}(1,6)$.

If $r=1$ and $\dim\mathfrak{s}<\infty$, then, by Lemma 5.3, $\mathcal{A}(R)$ is an irreducible central extension of $\mathfrak{s}[\![t]\!]$. This gives us an embedding $\mathrm{Der}(\mathcal{A}(R)) \subset \mathrm{Der}(\mathfrak{s}[\![t]\!])$. By Proposition 4.6, the universal central extension of $\mathfrak{s}[\![t]\!]$ is $\hat{\mathfrak{s}}[\![t]\!]$, where $\hat{\mathfrak{s}}$ is the universal central extension of $\mathfrak{s}$. Hence we have a surjective homomorphism $\varphi_1 : \hat{\mathfrak{s}}[\![t]\!] \to \mathcal{A}(R)$ such that $\ker\varphi_1$ is a central ideal. Also, since any derivation of a Lie superalgebra lifts uniquely to the universal central extension, we obtain an embedding (see Proposition 2.12)

$$\mathrm{Der}\big(\mathcal{A}(R)\big) \subset \mathrm{Der}\big(\hat{\mathfrak{s}}[\![t]\!]\big) = \mathrm{Der}\big(\mathbb{C}[\![t]\!]\big)\otimes 1 + \mathbb{C}[\![t]\!]\otimes\mathrm{Der}(\hat{\mathfrak{s}}).$$

Thus, the derivation T of $\mathcal{A}(R)$ induces a derivation of $\hat{\mathfrak{s}}[\![t]\!]$, which we denote by $\widehat{T}$. We have:

$$\widehat{T} = P(t)\frac{\partial}{\partial t} + T_1, \quad \text{where } P(t)\in\mathbb{C}[\![t]\!],\ T_1 \in \mathbb{C}[\![t]\!]\otimes\mathrm{Der}(\hat{\mathfrak{s}}).$$

Define a filtration on $L = \mathbb{C}\widehat{T} \ltimes \hat{\mathfrak{s}}[\![t]\!]$ by letting

$$\deg t = -\deg\frac{\partial}{\partial t} = 1, \qquad \deg\hat{\mathfrak{s}} = 0: \qquad L \supset L_0 \supset L_1 \supset \cdots.$$

Then $\widehat{T}(L_0) \not\subset L_0$, otherwise $\widehat{T}$, being surjective on $\mathcal{A}(R)$, is surjective on L_0, hence on $L_0/L_1 \supset \hat{\mathfrak{s}}$, which is impossible by Proposition 2.14 since $\hat{\mathfrak{s}}$ is not solvable. Hence we may assume that $\widehat{T} = \partial/\partial t + T_0$, where $T_0 \in L_0$. Applying Proposition 2.13 to L, $V=0$, $D=\partial/\partial t$, and $g_0 = T_0$, we may find a continuous automorphism ψ of $\hat{\mathfrak{s}}[\![t]\!]$ that transforms $\widehat{T}$ to $\partial/\partial t$. But

$$\mathbb{C}\frac{\partial}{\partial t} \ltimes \hat{\mathfrak{s}}[\![t]\!]$$

is the extended annihilation algebra of the Lie conformal superalgebra $\mathrm{Cur}\,\hat{\mathfrak{s}}$ and the homomorphism $\varphi_1 \circ \psi$ extends to a surjective homomorphism of extended annihilation algebras

$$\varphi : \frac{\partial}{\partial t} \ltimes \hat{\mathfrak{s}}[\![t]\!] \to T \ltimes \mathcal{A}(R)$$

satisfying the conditions of Proposition 5.2. Hence φ is induced by a surjective homomorphism $\tilde{\varphi} : \mathrm{Cur}\,\hat{\mathfrak{s}} \to R$. But R is simple, hence $\tilde{\varphi}$ induces an isomorphism $\mathrm{Cur}\,\mathfrak{s} \to R$.

It remains to consider the case $\mathcal{A}(R)/Z \simeq \mathbb{C}[\![t]\!] \mathbin{\widehat{\otimes}} \mathfrak{s}$, where $\dim \mathfrak{s} = \infty$ and Z is a central ideal. Recall that we may assume, from the proof of Lemma 5.3 that

$$T = \frac{\partial}{\partial t} \otimes 1 + \sum_i f_i \otimes D_i, \quad D_i \in \mathrm{Der}(\mathfrak{s}),\ f_i \in \mathbb{C}[\![t]\!]. \tag{5.7}$$

Note that $L_i = \overline{\mathcal{L}}_{(i+1)d}$ $(i \geqslant 0)$ is an algebra filtration of the extended annihilation algebra $\mathcal{A}(R)^e$ $(= L_{-1})$. Denote again by $\{L_i\}_{i \geqslant -1}$ the induced filtration on $\mathcal{A}(R)^e/Z \simeq \mathbb{C}T \ltimes (\mathbb{C}[\![t]\!] \mathbin{\widehat{\otimes}} \mathfrak{s})$. Let $1 \otimes \mathfrak{s}_0 = L_0 \cap (1 \otimes \mathfrak{s})$ and consider the canonical filtration of $\mathfrak{s}$ associated with the subalgebra $\mathfrak{s}_0$:

$$\mathfrak{s} = \mathfrak{s}_{-1} \supset \mathfrak{s}_0 \supset \mathfrak{s}_1 \supset \cdots.$$

Consider the following filtration of $\mathbb{C}T \ltimes (\mathbb{C}[\![t]\!] \mathbin{\widehat{\otimes}} \mathfrak{s}) =: \widetilde{L}_{-1}$:

$$\widetilde{L}_m = \sum_{\substack{i \geqslant 0,\ j \geqslant -1 \\ i+j=m}} t^i \mathbb{C}[\![t]\!] \mathbin{\widehat{\otimes}} \mathfrak{s}_j \quad (m \geqslant 0).$$

Since $\{\mathfrak{s}_j\}_{j \geqslant -1}$ is the canonical filtration of $\mathfrak{s}$ associated to $\mathfrak{s}_0$ and T has the form (5.7), it is easy to see that $\{\widetilde{L}_m\}_{m \geqslant -1}$ is the canonical filtration of $\mathcal{A}(R)^e/Z$ associated to $\widetilde{L}_0$. By Chevalley's principle, $\widetilde{L}_0 \supset L_N$ for some $N > 0$, and since $\{\widetilde{L}_m\}$ is a canonical filtration, we conclude that $\widetilde{L}_j \supset L_{N+j}$ for all $j \geqslant 0$. It follows that

$$\mathrm{gw}\big(\{\widetilde{L}_j\}\big) \leqslant \mathrm{gw}\big(\{L_j\}\big) \leqslant 1$$

(the last inequality follows from (5.4)). But $\mathrm{gw}(\{\widetilde{L}_j\}) = \mathrm{gw}(\{t^j \mathbb{C}[\![t]\!]\}) + \mathrm{gw}(\mathfrak{s}) = 2$. Thus, the remaining case is impossible. □

5.2. Derivations of the annihilation algebra

Proposition 5.6. *Let L be a linearly compact Lie superalgebra, and let $\mathfrak{s}$ be a reductive finite-dimensional Lie subalgebra of L such that $L = \prod_i V_i$, where the V_i's are finite-dimensional irreducible $\mathfrak{s}$-modules. Then there exists a closed $\mathfrak{s}$-submodule V of* $\mathrm{Der}(L)$, *complementary to the space of inner derivations* $\mathrm{ad}(L)$; *one has $[v, \mathrm{ad}(a)] = 0$ for all $v \in V$ and $a \in \mathfrak{s}$.*

Proof. We have: $\mathrm{Der}(L) \subset \mathrm{Hom}(\bigoplus_i V_i, \prod_j V_j) = \prod_{i,j} \mathrm{Hom}(V_i, V_j)$, hence $\mathrm{Der}(L)$ is an $\mathfrak{s}$-invariant closed (hence linearly compact) subspace of a direct product of finite-dimensional irreducible $\mathfrak{s}$-modules. The subspace $\mathrm{ad}(L)$ of $\mathrm{Der}(L)$ is $\mathfrak{s}$-invariant and closed too (by Proposition 2.1(1)). Hence there exists a closed $\mathfrak{s}$-invariant complementary subspace V. But $[\mathrm{ad}(\mathfrak{s}), D] \subset \mathrm{ad}(L)$ for any $D \in \mathrm{Der}(L)$, hence $[\mathrm{ad}(\mathfrak{s}), D] = 0$ if $D \in V$. □

We will be working with the following *standard* $\mathbb{Z}$-gradation of the Lie superalgebras that occur in Proposition 5.5(1):

$$\overline{W}(1,N) = \prod_{j \geqslant -1} W(1,N)_j,$$
$$\overline{S}(1,N)' = \prod_{j \geqslant -1} S(1,N)'_j, \qquad \overline{S}(1,2)' = \prod_{j \geqslant -2} S(1,2)^0_j,$$
$$\overline{K}(1,N) = \prod_{j \geqslant -2} K(1,N)_j, \qquad C\overline{K}(1,4) = \prod_{j \geqslant -2} CK(1,4)_j,$$
$$\overline{E}(1,6) = \prod_{j \geqslant -2} E(1,6)_j; \tag{5.8}$$

here the gradations of depth 1 (respectively 2) are defined by letting

$$\deg x = -\deg \partial_0 = 1 \quad \text{(respectively 2)}, \qquad \deg \xi_i = -\deg \partial_i = 1,$$

and in the $C\overline{K}(1,4)$ case we let $\deg(\text{center}) = 0$.

Lemma 5.7 [K5]. *All continuous derivations of* $\overline{W}(1,N)$, $\overline{K}(1,N)$, $C\overline{K}(1,4)$, *and* $\overline{E}(1,6)$ *are inner.*

Proof. Let $\mathfrak{s}$ be the even part of $W(1,N)_0$, $K(1,N)_0$, $E(1,6)_0$. We have $\mathfrak{s} = \mathbb{C} \oplus \mathfrak{gl}_N$, $\mathfrak{cso}_N$, $\mathfrak{cso}_6$, respectively, and the representation of $\mathfrak{s}$ on $W(1,N)_{-1}$, $K(1,N)_{-1}$, $E(1,6)_{-1}$ is the direct sum of the standard $\mathfrak{gl}_N$- and $\mathfrak{gl}_1$-modules, the standard $\mathfrak{cso}_N$- and $\mathfrak{cso}_6$-module, respectively. By Proposition 5.6, in all cases, $\mathrm{Der}(L) = \mathrm{ad}(L) \oplus V$ as $\mathfrak{s}$-module, and any $D \in V$ is an $\mathfrak{s}$-module homomorphism.

We remark that $x\partial_0 + \sum_i \xi_i \partial_i \in \mathfrak{s}$ for $\overline{W}(1,N)$ and $2x\partial_0 + \sum_i \xi_i \partial_i \in \mathfrak{s}$ for $\overline{K}(1,N)$, $C\overline{K}(1,4)$, and $\overline{E}(1,6)$. It follows that any $D \in V$ preserves the standard gradation of L.

By Schur Lemma, $D = \mathrm{diag}(\lambda, \mu, \ldots, \mu)$ on $W(1,N)_{-1}$. It follows that the grading preserving derivation $D' = D - \mathrm{ad}(\lambda x \partial_0 + \mu \sum_i \xi_i \partial_i)$ is zero on $W(1,N)_{-1}$ and it is an $\mathfrak{s}$-module homomorphism.

Let $y \in W(1,N)_k$ and $g_{-1} \in W(1,N)_{-1}$. By induction on $k \geqslant -1$, we have $0 = D'([y, g_{-1}]) = [D'y, g_{-1}]$. Hence by transitivity we conclude that $D'y = 0$, i.e. $D' = 0$ on $W(1,N)_k$. Consequently, $D' = 0$ on $\overline{W}(1,N)$ and $D = \mathrm{ad}(\lambda x \partial_0 +$

$\mu\sum_i \xi_i\partial_i)$. On the other hand, $D \in V$, hence $\lambda = \mu = 0$. Therefore $D = 0$ and every derivation of $\overline{W}(1,N)$ is inner.

A similar method can be used in the remaining cases. Notice that we may exclude the case $\overline{K}(1,2)$, which is isomorphic to $\overline{W}(1,1)$, so that $K(1,N)_{-1}$ (respectively $E(1,6)_{-1}$) is irreducible as a $\mathfrak{so}_N$-module (respectively $\mathfrak{so}_6$-module), hence any derivation $D \in V$ acts on this subspace as a scalar matrix. Now we proceed as above using also that in both cases the -2nd component is the bracket of the -1st component with itself. The result for $C\overline{K}(1,4)$ easily follows once one has established it for $\overline{K}(1,4)$. □

In the next two lemmas we will use the following outer derivations of $\overline{S}(1,N)'$ $(\subset \overline{W}(1,N))$: $E = \mathrm{ad}(\xi_1\dots\xi_N\partial_0)$, $H = \mathrm{ad}(\sum_{i=1}^N \xi_i\partial_i)$.

Lemma 5.8 [K5]. $\mathrm{Der}(\overline{S}(1,N)') = \mathrm{ad}(\overline{S}(1,N)') \oplus \mathbb{C}E \oplus \mathbb{C}H$ *for* $N > 2$.

Proof. The Lie algebra $\mathfrak{gl}_N$ is the even part of $S(1,N)'_0$. By Lemma 5.6, we have $\mathrm{Der}(\overline{S}(1,N)') = \mathrm{ad}(\overline{S}(1,N)') \oplus V$ as $\mathfrak{gl}_N$-modules and any $D \in V$ is a $\mathfrak{gl}_N$-module homomorphism. Let us denote by $(R(\pi), m)$ the irreducible $\mathfrak{sl}_N$-module $R(\pi)$ whose eigenvalue with respect to the operator $N\,\mathrm{ad}(x\partial_0) + H$ is m. The irreducible $\mathfrak{gl}_N$-modules which appear more than once in the standard gradation (5.8) are: $(R(\pi_{N-1}), -1)$, which occurs in $S(1,N)'_{-1}$, and $S(1,N)'_{N-2}$ and $(R(\pi_{N-1}), N-1)$ which occurs in $S(1,N)'_0$ and $S(1,N)'_{N-1}$.

Let $v_1 = \partial_N$ (respectively $v_2 = \xi_1\dots\xi_{N-1}\partial_0$) be the highest weight vector of the module $(R(\pi_{N-1}), -1)$ in $S(1,N)'_{-1}$ (respectively in $S(1,N)'_{N-2}$). Then $[\xi_1\dots\xi_N\partial_0, \partial_N] = \xi_1\dots\xi_{N-1}\partial_0$. Now, D is a $\mathfrak{gl}_N$-module homomorphism, hence $D(v_1) = \alpha v_1 + \beta v_2$ and $D - \beta E$ maps $S(1,N)'_{-1}$ into itself. Also, $S(1,N)'_{-1}$ is sum of two non-isomorphic, irreducible $\mathfrak{sl}_N$-modules, so by Schur Lemma we have that $D - \beta E = \mathrm{diag}(\lambda, \mu, \dots, \mu)$. Consequently, $D' = D - \beta E - \mathrm{ad}(\lambda x\partial_0 + \mu\sum_i \xi_i\partial_i)$ acts as 0 on $S(1,N)'_{-1}$. Suppose $y \in S(1,N)'_k$ and $g_{-1} \in S(1,N)'_{-1}$. By induction on $k \geqslant 1$, we have $0 = D'([y, g_{-1}]) = [D'y, g_{-1}]$. Note that the homogeneous components of $D'y$ have degree greater or equal than 0. By transitivity, we conclude that $D'y = 0$, hence $D' = 0$ on $\overline{S}(1,N)'$, and therefore D can be expressed as a linear combination of E, $\mathrm{ad}(x\partial_0) + H$ and some inner derivation. Since $\mathrm{ad}(Nx\partial_0) + H$ is an inner derivation, the lemma is proved. □

Lemma 5.9 [K5]. $\mathrm{Der}(\overline{S}(1,2)') = \mathrm{ad}(\overline{S}(1,2)') \oplus \mathfrak{sl}_2$, *where the standard basis of* $\mathfrak{sl}_2$ *consists of* E, H *as above and* F, *defined as follows*:

$$F\big(P(x)\xi_2\partial_0 - \partial_0 P(x)\xi_1\xi_2\partial_1\big) = -P(x)\partial_1,$$

$$F\big(P(x)\xi_1\partial_0 + \partial_0 P(x)\xi_1\xi_2\partial_2\big) = P(x)\partial_2,$$

$$F\big(P(x)S(2)\big) = 0, \quad \textit{and} \quad F\big(P(x)\partial_0 - 1/2\partial_0 P(x)(\xi_1\partial_1 + \xi_2\partial_2)\big) = 0.$$

Proof. With respect to the action of H, $\bar{S}(1,2)'$ decomposes into eigenspaces relative to the eigenvalues $\{-1,0,1\}$. The even part of $\bar{S}(1,2)'$ is contained in the zero eigenspace. Also, E (respectively F) transforms the -1 (respectively $+1$) eigenspace into the $+1$ (respectively -1) eigenspace and kills the other eigenspaces. We will use the standard depth 2 gradation of $\bar{S}(1,2)'$, see (5.8). The even part of $\bar{S}(1,2)^0_0$ is $\mathfrak{gl}_2$. Lemma 5.6 implies that $\mathrm{Der}(\bar{S}(1,2)') = \mathrm{ad}(\bar{S}(1,2)') \oplus V$ as $\mathfrak{gl}_2$-modules and any $D \in V$ is a $\mathfrak{gl}_2$-module homomorphism. The modules occurring more than once are $(R(\pi_1),-1)$ (in $S(1,2)^0_{-1}$ and $S(1,2)^0_0$) and $(R(\pi_1),1)$ (in $S(1,2)^0_0$ and $S(1,2)^0_1$).

Let $v_1 = \partial_2$ (respectively $v_2 = \xi_1\partial_0$) be the highest weight vector of $(R(\pi_1),-1)$ in $S(1,2)^0_{-1}$ (respectively $S(1,2)^0_0$). Then $E(v_1) = v_2$. The derivation D is a $\mathfrak{gl}_2$-module homomorphism, so $D(v_1) = \alpha v_1 + \beta v_2$, and $D - \beta E$ maps $S(1,2)^0_{-1}$ into itself. By Schur Lemma, $D - \beta E = \mathrm{diag}(\lambda,\mu,\mu)$. It follows that the derivation $D' = D - \beta E - \mathrm{ad}(\lambda x\partial_0 + \mu\sum_i \xi_i\partial_i)$ acts as 0 on $S(1,2)^0_{-1}$.

Let $y \in S(1,2)^0_0$. We have:

$$S(1,2)^0_0 = \big(R(\pi_1),-1\big) \oplus \big(\big(R(2\pi_1),0\big) \oplus \big(R(0),0\big)\big) \oplus \big(R(\pi_1),1\big),$$

so that $y = y_{-1} + y_0 + y_1$. For any $g_{-1} \in S(1,2)^0_{-1}$, $0 = D'([y,g_{-1}]) = [D'y_1, g_{-1}]$, hence $[D'y, g_{-1}] + [D'y_0, g_{-1}] + [D'y_1, g_{-1}] = 0$. Now, $(R(2\pi_1),0)$ and $(R(0),0)$ occur only in $S(1,2)^0_0$, hence $D'y_0 \in S(1,2)^0_0$. The module $(R(\pi_1),1)$ occurs in $S(1,2)^0_0$ and $S(1,2)^0_1$, so $D'y_1 \in S(1,2)^0_0 \oplus S(1,2)^0_1$. Also, $D'y_{-1} \in S(1,2)^0_{-1} \oplus S(1,2)^0_0$. We may assume that $y_{-1} = \xi_1\partial_0$. Then $D'y_{-1} = \alpha\partial_2 + \beta\xi_1\partial_0$. F kills $((R(2\pi_1),0) \oplus (R(0),0)) \oplus (R(\pi_1),1)$ and $F(\xi_1\partial_0) = \partial_2$. It follows that $(D' - \alpha F)(\xi_1\partial_0) = \beta\xi_1\partial_0 \in S(1,2)^0_0$. Also, $D'' = D' - \alpha F$ is still identically zero on $S(1,2)^0_{-1}$. Hence we have $0 = D''([y,g_{-1}]) = [D''y, g_{-1}]$, and $D''y$ has homogeneous components of degree greater or equal to zero. Transitivity implies that $D'' = 0$ on $S(1,2)^0_0$. Induction on $k \geqslant 1$ shows that $D'' = 0$ on $S(1,2)^0_k$. Therefore, D can be written as a linear combination of E, H, F and some inner derivation. □

5.3. *The classification theorem*

Lemma 5.10. *Let $L = \prod_{j \geqslant -d} \mathfrak{g}_j$ be one of the linearly compact Lie superalgebras that occur in Proposition* 5.5(1) *with the standard gradation* (5.8) *and let $L_0 = \prod_{j \geqslant 0} \mathfrak{g}_j$. Let T be an even surjective derivation of L. Then $T(L_0) \not\subset L_0$.*

Proof. In the contrary case, we also have $T(L_1) \subset L_1$ and therefore T induces a surjective derivation of $\mathfrak{g}_0 \simeq L_0/L_1$. But for $\overline{W}(1,N)$, $\bar{S}(1,N)'$, $\overline{K}(1,N)$, and $\overline{E}(1,6)$ we have $\mathfrak{g}_0 \simeq \mathfrak{gl}(1,N)$, $sl(1,N)$, $\mathfrak{cso}_N$, $\mathfrak{cso}_6$, respectively. Hence by Proposition 2.14 we reach a contradiction, unless $L = \overline{W}(1,0)$, $\overline{W}(1,1) \simeq \overline{K}(1,2)$, or $\overline{K}(1,1)$. The second case is also excluded since $\mathfrak{gl}(1,1)$ has only inner derivations (this is immediate by Proposition 5.6) and the third case is reduced to the first one since the even part of $\overline{K}(1,1)$ is $\overline{W}(1,0)$. But if $T=\sum_{j\geqslant 0} a_j x^j \partial_0 \in \overline{W}(1,0)$, it is easy to see that one of the elements ∂_0 or $x\partial_0$ does not lie in the image of $\mathrm{ad}(T)$. □

Theorem 5.11. *Let R be a finite simple Lie conformal superalgebra. The following list gives all the linearly compact Lie superalgebras that can occur as extended annihilation algebras $\mathcal{A}(R)^e$:*

(1) $\mathbb{C}\,\mathrm{ad}(\partial_0) \ltimes \overline{W}(1,N),\ N\geqslant 0$;
(2) $(\mathbb{C}\,\mathrm{ad}(\partial_0 - a\sum_{i=1}^N \xi_i\partial_i)) \ltimes \overline{S}(1,N)'$, $N\geqslant 2$;
(3) $(\mathbb{C}\,\mathrm{ad}(\partial_0 - \xi_1\ldots\xi_N\partial_0) \ltimes \overline{S}(1,N)'$, *$N$ even*, $N\geqslant 2$;
(4) $\mathbb{C}\,\mathrm{ad}(\partial_0) \ltimes \overline{K}(1,N),\ N\geqslant 0,\ N\neq 4$;
(5) $\mathbb{C}\,\mathrm{ad}(\partial_0) \ltimes C\overline{K}(1,4)$;
(6) $\mathbb{C}\,\mathrm{ad}(\partial_0) \ltimes \overline{E}(1,6)$;
(7) $\mathbb{C}\frac{\partial}{\partial t} \ltimes \mathfrak{s}[\![t]\!]$, *where $\mathfrak{s}$ is a finite-dimensional simple Lie superalgebra.*

Proof. Recall that $\mathcal{A}(R)^e = \mathbb{C}T \ltimes L$, where $L=\mathcal{A}(R)$ is one of the linearly compact Lie superalgebras listed in Proposition 5.5 and T is an even surjective derivation of L.

If L is one of the Lie superalgebras listed in Proposition 5.5(1), consider the filtration $\{L_i\}$ of L corresponding to the standard gradation $\prod_{j\geqslant -d}\mathfrak{g}_j$ of L (cf. (5.8)). One checks directly that in all cases one has:

$$[\partial_0,\mathfrak{g}_j]=\mathfrak{g}_{j-d}\quad \text{for all } j\geqslant d. \tag{5.9}$$

Furthermore, due to Lemmas 5.7–5.9 and 5.10 we have

$$T = c\,\mathrm{ad}(\partial_0) + v + \mathrm{ad}(g_0), \tag{5.10}$$

where $c\in\mathbb{C}$ is non-zero, $g_0\in L_0$, and $v\in V$, where V is one of the following subspaces of $\mathrm{Der}(L)$:

$$V=0 \quad \text{if } L=\overline{W}(1,N),\ \overline{K}(1,N),\ C\overline{K}(1,4),\ \text{or } \overline{E}(1,6);$$
$$V=\mathbb{C}H \quad \text{if } L=\overline{S}(1,N)',\ N \text{ odd} \quad \text{(cf. Lemma 5.8)};$$
$$V=\mathbb{C}E\oplus\mathbb{C}H \quad \text{if } L=\overline{S}(1,N)',\ N \text{ even},\ N>2 \quad \text{(cf. Lemma 5.8)};$$
$$V=\mathbb{C}E\oplus\mathbb{C}H\oplus\mathbb{C}F \quad \text{if } L=\overline{S}(1,2)' \quad \text{(cf. Lemma 5.9)}.$$

We may apply now Proposition 2.13 to $D=c\partial_0$ (cf. (5.10)) and the above V since (2.4) holds due to (5.9) and (2.5) also obviously holds. Hence by an inner automorphism of L we can bring T to the form:

$$T=c\,\mathrm{ad}(\partial_0)+v,\quad \text{where } c\in\mathbb{C}\backslash\{0\},\ v\in V.$$

By rescaling we can make $c=1$ and, using an inner automorphism of the Lie algebra V, T can be brought further, in all $\overline{S}(1,N)'$ cases, to the form $\mathrm{ad}(\partial_0)-E$ (if N is even) or $\mathrm{ad}(\partial_0)-aH$, $a\in\mathbb{C}$.

The case of $L = \mathfrak{s}[\![t]\!]$ has been treated in a similar fashion in the proof of Proposition 5.5. □

Theorem 5.12. *Any finite simple Lie conformal superalgebra is isomorphic to one of the Lie conformal superalgebras of the following list*:

(1) W_N, $N \geqslant 0$;
(2) $S_{N,a}$, $N \geqslant 2$, $a \in \mathbb{C}$;
(3) $\widetilde{S}_N$, *N even*, $N \geqslant 2$;
(4) K_N, $N \geqslant 0$, $N \neq 4$;
(5) K_4';
(6) CK_6;
(7) $\mathrm{Cur}\,\mathfrak{s}$, *where* $\mathfrak{s}$ *is a simple finite-dimensional Lie superalgebra.*

Proof. It follows from Theorem 5.11 and Proposition 5.2. □

A formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$ is called *simple* if it contains no non-trivial regular ideals; it is called *finite* if the $\mathbb{C}[\partial_z]$-module $\overline{\mathcal{F}}$ is finitely generated. Two formal distribution Lie superalgebras $(\mathfrak{g}, \mathcal{F})$ and $(\mathfrak{g}_1, \mathcal{F}_1)$ are called *isomorphic* if there exists an isomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}_1$ such that $\varphi(\overline{\mathcal{F}}) = \overline{\mathcal{F}_1}$.

The correspondence between Lie conformal superalgebras and formal distribution Lie superalgebras implies the following corollary of Theorem 5.12.

Corollary 5.13 [K7]. *A complete list of finite simple formal distribution Lie superalgebras consists of quotients of loop algebras* $(\mathfrak{s}[t, t^{-1}]/(P), \mathcal{F}_{\mathfrak{s}})$, *where* P *is a non-invertible polynomial of* $\mathbb{C}[t, t^{-1}]$ *and* $\mathfrak{s}$ *is a simple finite-dimensional Lie superalgebra, and the following examples*:

$$\big(W(\!|1, N|\!), \mathcal{F}_W\big) \quad (N \geqslant 0),$$
$$\big(S(\!|1, N, a|\!)', \mathcal{F}_{S,a}\big) \quad (N \geqslant 2,\ a \in \mathbb{C}),$$
$$\big(\widetilde{S}(\!|1, N|\!), \mathcal{F}_{\widetilde{S}}\big) \quad (N \geqslant 2,\ N \text{ even}),$$
$$\big(K(\!|1, N|\!), \mathcal{F}_K\big) \quad (N \geqslant 0,\ N \neq 4),$$
$$\big(CK(\!|1, 4|\!)', \mathcal{F}_K\big), \qquad \big(K(\!|1, 4|\!)', \mathcal{F}_{K'}\big), \qquad \big(CK(\!|1, 6|\!), \mathcal{F}_{CK}\big).$$

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