Spanning eulerian subgraphs, the Splitting Lemma, and Petersen’s Theorem

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Abstract


In this paper we show that a bridgeless graph without 2-valent vertices has a spanning eulerian subgraph without isolated vertices. This result is obtained by applying the Splitting Lemma and Petersen’s Theorem. On the other hand, it can be viewed as a generalization of this famous theorem.

Preliminaries. All concepts not defined in this paper can be found in [1]. We point out, however, that graphs may contain loops and/or multiple edges. Also, an eulerian graph need not be connected in this context (thus an eulerian graph is what others call an even graph). For the sake of completeness we define the ‘splitting operation’ at a vertex \( v \in V(G) \) of degree \( d(v) \geq 3 \), roughly as follows. Let \( e_1, e_2, e_3 \) be three edges incident with \( v \). Insert a new vertex \( v_{1,j}, j \in \{2, 3\} \), let \( e_1 \) and \( e_j \) be incident with \( v_{1,j} \), and leave all the other incidences unchanged.

The graph thus obtained is denoted by \( G_{1,j} \); we say \( G_{1,j} \) has been obtained from \( G \) by ‘splitting away \( e_1 \) and \( e_j \)’. Conversely, one can say that \( G \) results from \( G_{1,j} \) by identifying \( v_{1,j} \) and \( v \). Note that \( v_{1,j} \) and \( v \) are adjacent in \( G_{1,j} \) if and only if \( v \) is incident in \( G \) with a loop for which one half-edge has been split away, whereas the other half-edge has retained its incidence with \( v \). We shall make use of the following two results.

Petersen’s Theorem. If \( G \) is a bridgeless cubic graph, then it can be decomposed into a 1-factor and a 2-factor.

Splitting Lemma ([1, Lemma III.26]). Let \( G \) be a connected bridgeless graph having a vertex \( v \) with \( d(v) \geq 4 \). Let \( e_1, e_2, e_3 \) be three edges incident with \( v \) and
chosen in such a way that $e_1$ and $e_3$ belong to different blocks of $G$ if $v$ is a cutvertex. Then at least one of $G_{1,2}$ and $G_{1,3}$ is connected and bridgeless; and if $v$ is a cutvertex, then $G_{1,3}$ has this property.

**Note.** In our understanding of blocks and cutvertices, a loop $vu$ is viewed as a block and $v$ as a cutvertex of $G$ unless $v$ is 2-valent. Thus in the formulation of the Splitting Lemma we permit $e_1 = e_2$ to hold provided $v$ is a cutvertex and $e_1$ is a loop.

The Splitting Lemma has proved to be a useful tool for proving many results and/or reducing unsolved problems to special classes of graphs. Here is another application of this lemma.

**Theorem.** Let $G$ be a bridgeless graph without 2-valent vertices. Then $G$ has a spanning (not necessarily connected) eulerian subgraph $G_0$ without isolated vertices.

We note in passing that $G_0$ can be chosen to be connected provided $G$ is 4-edge-connected (see [5–6] and [4, Proposition 8.(b)], results by Polesskii–Kundu and Jaeger, respectively. Jaeger used them to prove his 4-Flow Theorem [4, Proposition 10]). On the other hand, if $G$ is just a 2-edge-connected graph, then $G$ has a connected spanning subgraph which is the edge-disjoint union of an eulerian graph and a path-forest, [3, Theorem 1]. Thus the above Theorem is the best one can hope for under the given hypothesis. In fact, when applied to 3-regular (= cubic) graphs it reduces to Petersen’s Theorem as stated above. However, we shall make use of the latter theorem to prove the former. Finally, we would like to point out that Catlin has studied various classes of graphs and conditions which admit connected spanning eulerian subgraphs (see e.g., [7]).

**Proof of the Theorem.** Since the eulerian subgraph $G_0 \subset G$ need not be connected we assume w.l.o.g. that $G$ is connected (otherwise, apply the Theorem to each component of $G$). If $\Delta(G) \leq 4$, set $G_1 := G$. If $v \in V(G)$ with $d(v) = 5$ exists, then the (if necessary, repeated) application of the Splitting Lemma yields the reduction of $v$ to a 3-valent vertex (if $d(v) = 1 \mod 2$) or a 4-valent vertex (if $d(v) = 0 \mod 2$) and the creation of $\lceil (d(v) - 3)/2 \rceil$ 2-valent vertices, while the property of being connected and bridgeless remains unchanged. Doing this for all vertices whose valency exceeds 4, we arrive at the graph $G_2$ which is connected and bridgeless. Every $x \in V(G_2)$ satisfies $d(x) \in \{2, 3, 4\}$. Now denote by $G_i$ the graph homeomorphic to $G_2$ such that $G_i$ has no 2-valent vertices. Clearly, $G_i$ is connected and bridgeless.

If $\Delta(G_i) = 3$, i.e., if $G$ is 3-regular, set $G_3 = G_i$. Otherwise, choose $y \in V(G_i)$ with $d(y) = 4$ and apply the Splitting Lemma to $y$ to obtain a new connected and bridgeless graph $G'_i$. Join the two 2-valent vertices of $G'_i$ by a new edge thus
obtaining $G_r$ which has one 4-valent vertex less and two 3-valent vertices more than $G_1$. It follows that $G_r$ is also connected and bridgeless (regardless of whether the addition of the new edge creates a multiple edge, i.e., whether $y$ is incident with a loop). Repeating this operation if necessary we finally arrive at a connected and bridgeless graph which arises from $G_1$ by replacing appropriately every 4-valent vertex of $G_1$ with two 3-valent vertices. Let $G_3$ denote this new graph. Thus $G_3$ is a connected, bridgeless, cubic graph in any case.

Applying Petersen’s Theorem to $G_3$ we obtain a 2-factor $Q \subset G_3$. At every $z \in V(G_3)$, at most one of the edges of $Q$ is a new edge joining two vertices which stem from the splitting of a 4-valent vertex of $G_1$. Define $Q_1 := Q - E_0$, where $E_0$ is the set of new edges arising in the transition from $G_1$ to $G_3$. Since we view edges as being independent from vertices as such (but related to them via incidence functions), we can define $G_1^0$ to be the edge-induced subgraph of $G_1$ with edge set $E(Q_1)$. However, $G_1 = G_1/E_0$, the graph obtained by contracting the edges of $E_0$. It follows that every $u \in V(G_1)$ is incident to 2 or 4 edges of $G_1^0$. That is, $G_1^0$ is a spanning eulerian subgraph of $G_1$ without isolated vertices. By subdividing edges of $G_1^0$ if necessary we transform $G_1^0$ into an eulerian subgraph $G_2^0$ of $G_2$. $G_2^0$ may not be spanning, but $t \in V(G_2) - V(G_2^0)$ implies $d_{G_2}(t) = 2$. Consequently, if we identify the 2-valent vertices of $G_2$ with the respective 3- or 4-valent vertices of $G_2$, we not only arrive at $G$, but we also transform $G_2^0$ into a subgraph $G_0$ of $G$: $G_0$ is eulerian because $G_2^0$ is eulerian; and since $G_2^0$ misses at most 2-valent vertices of $G_2$, whereas $G$ has no 2-valent vertices, we conclude that $G_0$ contains all vertices of $G$. That is, $G_0$ is a spanning eulerian subgraph of $G$ without isolated vertices. This finishes the proof of the theorem. \[\square\]

**Corollary.** If $G$ is a bridgeless graph, then $G$ can be written in the form $G = G_e \cup G_w$, where $G_e$ is eulerian, $G_w$ is a forest, $E(G_e) \cap E(G_w) = \emptyset$, such that $x \in V(G_w) - V(G_e)$ satisfies $d_G(x) = 2$.  

Fig. 1.
We note, however, that if $G$ is connected and bridgeless, then $G_w$ cannot, in general, be modified by edge deletion so as to become a path-forest $P$ such that $G_v \cup P$ is connected (see the paragraph following the statement of the Theorem). This can be seen in the following case: let $G$ consist of seven digons $D_i = (a_i, b_i)$ and of two vertices $x, y$ such that $x$ is joined to each $a_i$ and $y$ is joined to each $b_i$, $1 \leq i \leq 7$; see Fig. 1. $G_3$ is obtained by the following sequence of splitting operations: first split away $xa_1$ and $xa_2$, then $yb_2$ and $yb_3$, then $xa_3$ and $xa_4$, and finally $yb_4$ and $yb_5$; then suppress the 2-valent vertices arising in the course of the splitting operation; see Fig. 2. Note that after each operation, the respective graph is connected and bridgeless. One conceivable 2-factor $Q$ of $G_3$ consists of $D_i, 1 \leq i \leq 5$, and the hexagon containing the vertices of $D_6, D_7$, and $x, y$ (in Fig. 2, the edges of $Q$ are drawn as thick lines). Note that $Q$ is also a 2-factor of $G$. Thus $G_v = Q$ and $G_w = \langle E(G) - E(Q) \rangle$ satisfy the corollary ($G_w$ consists of four components: two are isomorphic to $K_2$ and two are isomorphic to $K_{1,5}$). Consequently, to connect the five digons of $Q$ to the hexagon of $Q$ it takes at least three edges of $G_w$ incident to $z$, $z \in \{x, y\}$. That is, the above construction of $G_v$ via the Splitting Lemma and Petersen's Theorem cannot be used, in general, for the construction of EPS-graphs $S = E \cup P$ such that $E = G_v$. We note that EPS-graphs play an essential role in the study of hamiltonian cycles in the square of graphs (see [3]). We also observe that the proof of the theorem gives rise to a polynomial time algorithm for constructing spanning eulerian graphs in bridgeless graphs without 2-valent vertices. Note that the respective decision problem regarding connected spanning eulerian graphs is NP-complete since it includes the decision problem regarding hamiltonian cycles in 2-connected cubic graphs.

References

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