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Naturally graded quasi-filiform Lie algebras

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Abstract

We present the classification of one type of graded nilpotent Lie algebras. We start from the gradation related to the filtration which is produced in a natural way by the descending central sequence in a Lie algebra. These gradations were studied by Vergne [Bull. Soc. Math. France 98 (1970) 81–116] and her classification of the graded filiform Lie algebras is here extended to other algebras with a high nilindex. We also show how symbolic calculus can be useful in order to obtain results in a similar classification problem.

Keywords: Nilpotent Lie algebras; Gradations

1. Introduction

In the cohomological study of the variety of laws of nilpotent Lie algebras established by Vergne [4] the classification of the "naturally" graded filiform Lie algebras plays a fundamental role. The achieved classification allows an easy expression of a filiform Lie algebra, that is, of an algebra with a maximal nilindex among those in the same dimension. Vergne proves that, up to isomorphisms, there is only one graded filiform Lie algebra in odd dimensions and two of them when the dimension is even. This fact also allows other authors to deal with different aspects of the theory. For example, using the graded filiform Lie

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algebras, Goze and Khakimdjanov give in [2] the geometric description of the characteristically nilpotent filiform Lie algebras.

Thus, the knowledge of the graded algebras of a particular Lie algebra class gives a valuable information about the structure of that class. That knowledge can later facilitate the study of several problems that can appear within the whole of the class. The aim of this paper is to expand Vergne's results, obtaining the classification of graded Lie algebras for a particular type of non-filiform nilpotent Lie algebras.

This paper is structured in the following way. In Section 2 a brief description of Vergne's results is given. The basic structure of graded quasi-filiform Lie algebras is explained in Section 3, where some results about their classification have already been obtained. The final section deals with the classification of quasi-filiform Lie algebras in any dimension, whose list was introduced in Section 2. We describe also how symbolic calculus has been used, concretely by way of the software Mathematica.

2. Graded nilpotent Lie algebras

In this paper Lie algebras $\mathfrak{g}$ will be considered over the field of complex numbers $\mathbb{C}$, and with finite dimension n. Let $\mathfrak{g}$ be a complex Lie algebra. Then, $\mathfrak{g}$ is naturally filtered by the descending central sequence $C^0\mathfrak{g} = \mathfrak{g}$, $C^i\mathfrak{g} = [\mathfrak{g}, C^{i-1}\mathfrak{g}]$, $i \geqslant 1$. Indeed, we consider the filtration given by (S_i), where $S_i = \mathfrak{g}$, if $i \leqslant 1$; $S_i = C^{i-1}\mathfrak{g}$, if $2 \leqslant i \leqslant k$; and $S_i = \{0\}$, if $i > k$. Thus, with any nilpotent Lie algebra $\mathfrak{g}$ of nilindex $k = \inf\{i \in \mathbb{N}\colon\ C^i\mathfrak{g} = 0\}$, we can associate naturally a graded Lie algebra with the same nilindex, noted by $\operatorname{gr}\mathfrak{g}$ and defined by

$$\operatorname{gr}\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i, \quad \text{with } \mathfrak{g}_i = \frac{S_i}{S_{i+1}}.$$

Because of the nilpotency of the algebra, the gradation is finite, i.e.,

$$\operatorname{gr}\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$$

with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, for $i + j \leqslant k$, verifying that $\dim\mathfrak{g}_1 \geqslant 2$ and $\dim\mathfrak{g}_i \geqslant 1$, for $2 \leqslant i \leqslant k$. A Lie algebra $\mathfrak{g}$ is said to be *naturally graded* if $\operatorname{gr}\mathfrak{g}$ is isomorphic to $\mathfrak{g}$, which from now on will be noted by $\operatorname{gr}\mathfrak{g} = \mathfrak{g}$ [4]. Examples of these algebras are L_n, Q_n defined as follows. The undefined brackets, except for those expressing antisymmetry, are supposed to vanish.

Example 2.1. The algebra L_n is defined in the basis $(X_0, X_1, \ldots, X_{n-1})$ by

$$[X_0, X_i] = X_{i+1}, \quad 1 \leqslant i \leqslant n-2.$$

The algebra Q_n is defined in the basis $(X_0, X_1, \ldots, X_{n-1})$, with $n = 2q$, by

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-2, \\ [X_i, X_{n-1-i}] = (-1)^{i-1}X_{n-1}, & 1 \leqslant i \leqslant q-1. \end{cases}$$

A Lie algebra $\mathfrak{g}$ with dimension n is said to be *filiform* if its nilindex is $k = n-1$, i.e., $\dim(C^i\mathfrak{g}) = n-i-1$, for $1 \leqslant i \leqslant n-1$. The above Lie algebras L_n and Q_n are filiform. In [4] Vergne proved the following theorem.

Theorem 2.1 [4]. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ be a naturally graded filiform Lie algebra with dimension n. Then, there exists a homogeneous basis $(X_0, X_1, \ldots, X_{n-1})$ of $\mathfrak{g}$, with $X_0, X_1 \in \mathfrak{g}_1$ and $X_i \in \mathfrak{g}_i$ for $i \geqslant 2$ such that $\mathfrak{g} = L_n$, if n is odd, and $\mathfrak{g} = L_n$ or $\mathfrak{g} = Q_n$, if n is even.*

Thus, any naturally graded filiform Lie algebra with dimension n is isomorphic to L_n, if n is odd, and isomorphic to L_n or Q_n, if n is even. The aim of this paper is to determine the n-dimensional naturally graded Lie algebras with nilindex $k = n-2$. We will obtain in Theorem 4.1 a locally finite family of algebras (depending on one parameter) defined in a basis $(X_0, X_1, \ldots, X_{n-1})$ as follows.

2.1. Naturally graded quasi-filiform Lie algebras

Split: $L_{n-1} \oplus \mathbb{C} \quad (n \geqslant 4)$:

$$[X_0, X_i] = X_{i+1}, \quad 1 \leqslant i \leqslant n-3;$$

$Q_{n-1} \oplus \mathbb{C}, \quad (n \geqslant 7,\ n \text{ odd})$:

$$[X_0, X_i] = X_{i+1}, \qquad 1 \leqslant i \leqslant n-3,$$
$$[X_i, X_{n-2-i}] = (-1)^{i-1} X_{n-2}, \qquad 1 \leqslant i \leqslant \frac{n-3}{2}.$$

Principal: $\mathcal{L}_{(n,r)} \quad \left(n \geqslant 5,\ r \text{ odd},\ 3 \leqslant r \leqslant 2\left\lfloor \frac{n-1}{2} \right\rfloor - 1\right)$:

$$[X_0, X_i] = X_{i+1}, \qquad 1 \leqslant i \leqslant n-3,$$
$$[X_i, X_{r-i}] = (-1)^{i-1} X_{n-1}, \qquad 1 \leqslant i \leqslant \frac{r-1}{2};$$

$\mathcal{Q}_{(n,r)} \quad (n \geqslant 7,\ n \text{ odd},\ r \text{ odd},\ 3 \leqslant r \leqslant n-4)$:

$$[X_0, X_i] = X_{i+1}, \qquad 1 \leqslant i \leqslant n-3,$$
$$[X_i, X_{r-i}] = (-1)^{i-1} X_{n-1}, \qquad 1 \leqslant i \leqslant \frac{r-1}{2},$$
$$[X_i, X_{n-2-i}] = (-1)^{i-1} X_{n-2}, \qquad 1 \leqslant i \leqslant \frac{n-3}{2}.$$

Terminal: $\mathcal{T}_{(n,n-3)} \quad (n \text{ even}, n \geqslant 6)$:

$$[X_0, X_i] = X_{i+1}, \qquad 1 \leqslant i \leqslant n-3,$$
$$[X_{n-1}, X_1] = \frac{n-4}{2} X_{n-2},$$

$$[X_i, X_{n-3-i}] = (-1)^{i-1}(X_{n-3} + X_{n-1}), \qquad 1 \leqslant i \leqslant \frac{n-4}{2},$$
$$[X_i, X_{n-2-i}] = (-1)^{i-1}\frac{n-2-2i}{2}X_{n-2}, \qquad 1 \leqslant i \leqslant \frac{n-4}{2};$$

$\mathcal{T}_{(n,n-4)}$ (n odd, $n \geqslant 7$):

$$[X_0, X_i] = X_{i+1}, \qquad 1 \leqslant i \leqslant n-3,$$
$$[X_{n-1}, X_i] = \frac{n-5}{2}X_{n-4+i}, \qquad 1 \leqslant i \leqslant 2,$$
$$[X_i, X_{n-4-i}] = (-1)^{i-1}(X_{n-4} + X_{n-1}), \qquad 1 \leqslant i \leqslant \frac{n-5}{2},$$
$$[X_i, X_{n-3-i}] = (-1)^{i-1}\frac{n-3-2i}{2}X_{n-3}, \qquad 1 \leqslant i \leqslant \frac{n-5}{2},$$
$$[X_i, X_{n-2-i}] = (-1)^{i}(i-1)\frac{n-3-i}{2}X_{n-2}, \qquad 2 \leqslant i \leqslant \frac{n-3}{2}.$$

The algebras $\mathcal{E}_{(7,3)}$, $\mathcal{E}^1_{(9,5)}$, and $\mathcal{E}^2_{(9,5)}$ of Remark 4.1 must also be considered.

3. Graded quasi-filiform Lie algebras

A key concept in this paper will be that of quasi-filiform Lie algebra defined in the following way.

Definition 3.1. An n-dimensional nilpotent Lie algebra $\mathfrak{g}$ is said to be *quasi-filiform* if its nilindex is $k = n-2$, i.e., $C^{n-3}\mathfrak{g} \neq 0$ and $C^{n-2}\mathfrak{g} = 0$, where $C^0\mathfrak{g} = \mathfrak{g}$, $C^i\mathfrak{g} = [\mathfrak{g}, C^{i-1}\mathfrak{g}]$.

Moreover, from now on the term *graded* will be used if the Lie algebra is naturally graded. Thus, for a graded quasi-filiform the gradation will be $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_{n-2}$, where $\mathfrak{g}_i = C^{i-1}\mathfrak{g}/C^i\mathfrak{g}$.

3.1. *Graded quasi-filiform Lie algebra structure*

Let $\mathfrak{g}$ be an n-dimensional graded quasi-filiform Lie algebra. If we obtain the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_{n-2}$ with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, for $i+j \leqslant n-2$, then either $\dim \mathfrak{g}_1 = 3$ and $\dim \mathfrak{g}_i = 1$, $2 \leqslant i \leqslant n-2$, or $\dim \mathfrak{g}_1 = 2$ and there exists r, $2 \leqslant r \leqslant n-2$, such that $\dim \mathfrak{g}_r = 2$, with $\dim \mathfrak{g}_i = 1$ for $i \neq 1, r$.

In order to make the structure that the family of graded quasi-filiform Lie algebras presents easier, we must find a basis where the expression of the algebras is suitable. The following proposition gives a step forward in this way.

Proposition 3.1. *Let $\mathfrak{g}$ be a graded quasi-filiform Lie algebra with dimension n. Then, there exists a homogeneous basis $(X_0, X_1, \ldots, X_{n-1})$ of the algebra such* *that* $X_0, X_1 \in \mathfrak{g}_1$, $X_i \in \mathfrak{g}_i$, $2 \leqslant i \leqslant n-2$; *and* $X_{n-1} \in \mathfrak{g}_r$ *with* $1 \leqslant r \leqslant n-2$, *verifying*

$$[X_0, X_i] = X_{i+1}, \quad 1 \leqslant i \leqslant n-3;$$
$$[X_0, X_{n-2}] = 0, \qquad [X_0, X_{n-1}] = 0.$$

A basis $(X_0, X_1, \ldots, X_{n-1})$ *verifying the above conditions will be referred to as an adapted basis of algebra* $\mathfrak{g}$.

Proof. Suppose firstly that $\dim \mathfrak{g}_1 = 3$. Let $Y_1, \ldots, Y_{n-2}$, with $Y_i \in \mathfrak{g}_i$, where $1 \leqslant i \leqslant n-2$. If $Y \in \mathfrak{g}_1$, we can obtain

$$[Y, Y_i] = f_i(Y) Y_{i+1},$$

with $f_i(Y) \in \mathbb{C}$. Since $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$, the linear function $f_i : \mathfrak{g}_i \to \mathbb{C}$ is non-null. Therefore, there is an element $X_0 \in \mathfrak{g}_1$ for which $f_i(X_0) \neq 0$, $1 \leqslant i \leqslant n-2$. Then, one can choose vectors $X_i = \lambda_i Y_i \in \mathfrak{g}_i$, $\lambda_i \neq 0$, so that the following relations are obtained

$$[X_0, X_i] = X_{i+1}, \quad 1 \leqslant i \leqslant n-3; [X_0, X_{n-2}] = 0. \tag{1}$$

On the other hand, expanding the basis of $\mathfrak{g}_1$ with a vector Y_{n-1}, we have $[X_0, Y_{n-1}] = \alpha X_2$ and one can take $X_{n-1} = Y_{n-1} - \alpha X_1$, finally proving

$$[X_0, X_{n-1}] = 0.$$

Suppose now that $\dim \mathfrak{g}_1 = 2$ and $\dim \mathfrak{g}_r = 2$. Let $Y_1, \ldots, Y_{n-2}$ be any nonzero elements, with $Y_i \in \mathfrak{g}_i$ for $1 \leqslant i \leqslant n-2$. If $[\mathfrak{g}_1, Y_r] = 0$, we choose as a no null vector of $\mathfrak{g}_r$ an element from the supplementary subspace of $\langle Y_r \rangle$ in $\mathfrak{g}_r$. Thus, since $[\mathfrak{g}_1, \mathfrak{g}_r] = \mathfrak{g}_{r+1}$, from definition for each $Y \in \mathfrak{g}_1$ of the number $f_r(Y)$ by the equation $[Y, Y_r] = f_r(Y) Y_{r+1}$, follows that the linear form $f_r : \mathfrak{g}_1 \to \mathbb{C}$ is not null. Then, a similar argument to the previous case allows us to choose a vector $X_0 \in \mathfrak{g}_1$ and vectors $X_i \in \mathfrak{g}_i$, $1 \leqslant i \leqslant n-2$ proving the relations (1). Expanding now the basis of $\mathfrak{g}_r$ with a vector Y_{n-1}, we have $[X_0, Y_{n-1}] = \alpha X_{r+1}$ and taking $X_{n-1} = Y_{n-1} - \alpha X_r$, it is proved that $[X_0, X_{n-1}] = 0$. □

If we denote by $\langle X, Y, \ldots \rangle$ the subspace of $\mathfrak{g}$ generated by the vectors $X, Y, \ldots$, we can express the structure of the decomposition of the algebra in terms of an adapted basis.

Corollary 3.1. *If* $\mathfrak{g}$ *is a graded quasi-filiform lie algebra with dimension* n, *its decomposition in an adapted basis* $(X_0, X_1, \ldots, X_{n-1})$ *is one of the following types*:

$$\mathfrak{t}_1 = \langle X_0, X_1, X_{n-1} \rangle \oplus \langle X_2 \rangle \oplus \cdots \oplus \langle X_{n-2} \rangle,$$
$$\mathfrak{t}_2 = \langle X_0, X_1 \rangle \oplus \langle X_2, X_{n-1} \rangle \oplus \cdots \oplus \langle X_{n-2} \rangle,$$

$$\vdots$$

$$\mathfrak{t}_{n-2} = \langle X_0, X_1\rangle \oplus \langle X_2\rangle \oplus \cdots \oplus \langle X_{n-2}, X_{n-1}\rangle.$$

3.2. Split graded quasi-filiform Lie algebras

We will consider now the case in which $\mathfrak{g}$ is an algebra of type $\mathfrak{g} = \mathfrak{t}_1$. This case, in which $\dim \mathfrak{g}_1$ is maximal, is the only possible to be considered for filiform Lie algebras. The following proposition shows that the graded quasi-filiform Lie algebras of such a type are just trivial extensions of the filiform Lie algebras. Their classification is directly given as a corollary of Theorem 2.1.

Proposition 3.2. *Let $\mathfrak{g}$ be a graded quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{t}_1$ with dimension n. Then, $\mathfrak{g}$ is isomorphic to $L_{n-1} \oplus \mathbb{C}$, if n is even, and to $L_{n-1} \oplus \mathbb{C}$ or $Q_{n-1} \oplus \mathbb{C}$, if n is odd.*

Proof. Let $(X_0, X_1, \ldots, X_{n-1})$ be an adapted basis of $\mathfrak{g}$. Then, denoting by $\lfloor x\rfloor$ the floor function of x and since the algebra has type $\mathfrak{g} = \mathfrak{t}_1$, the algebra $\mathfrak{g}$ belongs to the family

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_{n-1}, X_i] = a_i X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_j] = a_{ij} X_{i+j}, & 1 \leqslant i \leqslant \left\lfloor \frac{n-3}{2} \right\rfloor,\ i < j \leqslant n-2-i, \end{cases}$$

with $\{a_i\}$, $\{a_{ij}\}$ verifying the algebraic relations obtained from the Jacobi identities

$$\big[X_i, [X_j, X_k]\big] + \big[X_j, [X_k, X_i]\big] + \big[X_k, [X_i, X_j]\big] = 0$$

which from now on will be denoted as $J(X_i, X_j, X_k) = 0$. In particular, the relations $J(X_0, X_{n-1}, X_i) = 0$, $1 \leqslant i \leqslant n-4$, prove that $a_i = a_1$ for $1 \leqslant i \leqslant n-3$. The change of basis defined by the equations $X'_{n-1} = X_{n-1} - a_1 X_0$, $X'_i = X_i$ $(i \neq n-1)$, allows to suppose $a_1 = 0$. Hence, the algebra $\mathfrak{g}$ can be expressed as

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_j] = a_{ij} X_{i+j}, & 1 \leqslant i \leqslant \left\lfloor \frac{n-3}{2} \right\rfloor,\ i < j \leqslant n-2-i; \end{cases}$$

and therefore X_{n-1} is an element of the center of the algebra $\mathcal{Z}(\mathfrak{g}) = \{X \in \mathfrak{g}: [X, \mathfrak{g}] = 0\}$. Thus, it has been proved that $\mathfrak{g} = \mathfrak{g}' \oplus \langle X_{n-1}\rangle$, where the subalgebra $\mathfrak{g}' = \bigoplus \mathfrak{g}_i / \langle X_{n-1}\rangle$ is a graded filiform Lie algebra. The only thing left is to use Theorem 2.1 in order to finish the proof. □

The previous result indicates that the only graded quasi-filiform Lie algebras of type $\mathfrak{g} = \mathfrak{t}_1$ are direct sums of graded filiform algebras and $\mathbb{C}$. They will be referred to as split from now on, and their only interest is to emphasize the naturally gradation underlying the filiform subalgebra of the extension as we can see in Section 2.

Remark 3.1. If $\mathfrak{g}$ is a graded quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{t}_2$, then $\dim \mathfrak{g}_2 = 2$. On the one hand, from the natural gradation, it follows $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$. But, on the other hand, from the Corollary 3.1 and the properties that the Proposition 3.1 determines for the adapted basis $(X_0, X_1, \dots, X_{n-1})$, it is obtained

$$\big[\langle X_0, X_1\rangle, \langle X_0, X_1\rangle\big] = \langle X_2\rangle.$$

Therefore, there cannot exist any graded quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{t}_2$.

The following proposition allows us to reduce in half the task about the classification of graded quasi-filiform Lie algebras. Indeed, we prove that there exists no algebras of type $\mathfrak{g} = \mathfrak{t}_r$ when r is even.

Proposition 3.3. *Let $\mathfrak{g}$ be a non-split graded quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{t}_r$ with dimension n. Then r is odd and $\mathfrak{g}$ belongs to the parameterized family*

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_{n-1}, X_i] = aX_{i+r}, & 1 \leqslant i \leqslant n-2-r, \\ [X_i, X_j] = a_{ij} X_{i+j}, & 1 \leqslant i \leqslant \lfloor \frac{n-3}{2} \rfloor,\ i < j \leqslant n-2-i, \\ & (j \neq r-i), \\ [X_i, X_{r-i}] = a_{i,r-i} X_r + (-1)^i X_{n-1}, & 1 \leqslant i \leqslant \lfloor \frac{r-1}{2} \rfloor \ (r \geqslant 3), \end{cases}$$

where $a \in \mathbb{C}$, if $3 \leqslant r \leqslant n-3$, and $a = 0$, if $r = n-2$.

Proof. Let $\mathfrak{g}$ be an n-dimensional graded quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{t}_r$. The observation in Remark 3.1 tells us that $r \neq 2$. Assume that $3 \leqslant r \leqslant n-2$. According to Corollary 3.1 one can obtain, if $r \neq n-2$, that

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_{n-1}, X_i] = a_i X_{i+r}, & 1 \leqslant i \leqslant n-2-r \ (r \leqslant n-3), \\ [X_i, X_j] = a_{ij} X_{i+j}, & 1 \leqslant i \leqslant \lfloor \frac{n-3}{2} \rfloor,\ i < j \leqslant n-2-i, \\ & (j \neq r-i), \\ [X_i, X_{r-i}] = a_{i,r-i} X_r + b_i X_{n-1}, & 1 \leqslant i \leqslant \lfloor \frac{r-1}{2} \rfloor. \end{cases}$$

In the extreme position $r = n-2$, it is $X_{n-1} \in \mathcal{Z}(\mathfrak{g})$ and we have

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_j] = a_{ij} X_{i+j}, & 1 \leqslant i \leqslant \lfloor \frac{n-3}{2} \rfloor,\ i < j < n-2-i, \\ [X_i, X_{n-2-i}] = a_{i,n-2-i} X_{n-2} + b_i X_{n-1}, & 1 \leqslant i \leqslant \lfloor \frac{n-3}{2} \rfloor. \end{cases}$$

Since $r \geqslant 3$, we obtain $\lfloor (r-1)/2 \rfloor \geqslant 1$. Then, condition $[\mathfrak{g}_1, \mathfrak{g}_{r-1}] = \mathfrak{g}_r$ of the gradation guarantees that $b_1 \neq 0$, because of $[X_0, X_{r-1}] = X_r$ and $\mathfrak{g}_r =$

$\langle X_r, X_{n-1}\rangle$. If we take i such that $1 \leqslant i \leqslant \lfloor (r-1)/2 \rfloor - 1$, from the Jacobi relations $J(X_0, X_i, X_{r-1-i}) = 0$, we can deduce that

$$b_i = (-1)^{i-1} b_1, \quad 1 \leqslant i \leqslant \left\lfloor \frac{r-1}{2} \right\rfloor.$$

Then, if r is even, the Jacobi identity $J(X_0, X_{r/2-1}, X_{r/2}) = 0$, implies that

$$b_1 = 0,$$

and therefore we have $X_{n-1} \notin C^1\mathfrak{g}$. Thus $\dim(C^0\mathfrak{g}/C^1\mathfrak{g}) = \dim \mathfrak{g}_1 = 3$, and then $\dim \mathfrak{g}_r = 1$ contradicting the hypothesis that $\mathfrak{g}$ is an algebra of type $\mathfrak{g} = \mathfrak{t}_r$, with $r \geqslant 3$. Thus r is odd, and when $3 \leqslant r \leqslant n-3$, the Jacobi relations $J(X_0, X_{n-1}, X_i) = 0$, with $1 \leqslant i \leqslant n-3-r$, implies

$$a_i = a_1, \quad 2 \leqslant i \leqslant n-2-r.$$

Hence, the proof is completed by writing $a = a_1$ and by choosing $b_1 X_{n-1}$ as the vector of $\mathfrak{g}_r$ in the adapted basis. □

The following section deals with the classification of non-split graded quasi-filiform Lie algebras. By the results obtained in Section 3 we have to consider just the gradation of type $\mathfrak{g} = \mathfrak{t}_r$, with r odd.

4. Classification of graded quasi-filiform Lie algebras

So far we have not obtained graded quasi-filiform Lie algebras, apart from the split ones $L_{n-1} \oplus \mathbb{C}$ and $Q_{n-1} \oplus \mathbb{C}$. One can check easily that the algebras $\mathcal{L}_{(n,r)}$ and $\mathcal{Q}_{(n,r)}$, introduced as *principal* algebras in Section 2, are naturally graded quasi-filiform Lie algebras of type $\mathfrak{g} = \mathfrak{t}_r$.

We will later see, in Section 4.2, how $\mathcal{L}_{(n,r)}$ and $\mathcal{Q}_{(n,r)}$ will play the role that algebras L_n and Q_n had in the filiform case. That is, in the sense of that an n-dimensional graded quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{t}_r$ can only be isomorphic to one of the principal algebras (having similar condition to the filiform case depending on the parity of n), except for certain extreme values of r in comparison with n. Those cases in which r is close to the dimension of the algebra are now studied.

4.1. *Algebras of type* $\mathfrak{g} = \mathfrak{t}_r$*, with* $r \geqslant n-4$

When the dimension of the algebra is odd, the fact that the element X_{n-1} of an adapted basis is central makes this case a little bit special.

Proposition 4.1. *If* $\mathfrak{g}$ *is an* n*-dimensional graded quasi-filiform Lie algebra of type* $\mathfrak{g} = \mathfrak{t}_{n-2}$*,* n *odd, then* $\mathfrak{g} = \mathcal{L}_{(n,n-2)}$.

Proof. If $(X_0, X_1, \dots, X_{n-1})$ is an adapted basis of algebra $\mathfrak{g}$, we can deduce from Proposition 3.3 that $r = n-2$ is odd and $\mathfrak{g}$ belongs to the family

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_j] = a_{ij} X_{i+j}, & 1 \leqslant i \leqslant \frac{n-3}{2},\ i < j < n-2-i; \\ [X_i, X_{n-2-i}] = a_{i,n-2-i} X_{n-2} + (-1)^{i-1} X_{n-1}, & 1 \leqslant i \leqslant \frac{n-3}{2}. \end{cases}$$

We have

$$\mathfrak{g} = \langle X_0, X_1 \rangle \oplus \langle X_2 \rangle \oplus \cdots \oplus \langle X_{n-3} \rangle \oplus \langle X_{n-2}, X_{n-1} \rangle$$

with $\dim(C^1\mathfrak{g}) = n-2$ and $C^{n-3}\mathfrak{g} = \langle X_{n-2}, X_{n-1} \rangle$. Then, denoting each class by the corresponding element, $(X_0, \dots, X_{n-3})$ is an adapted basis of the quotient

$$\frac{\mathfrak{g}}{C^{n-3}\mathfrak{g}} = \langle X_0, X_1 \rangle \oplus \langle X_2 \rangle \oplus \cdots \oplus \langle X_{n-3} \rangle.$$

Since $\mathfrak{g}/\langle X_{n-2}, X_{n-1} \rangle$ is a graded filiform algebra with dimension $n-2$, odd, the Theorem 2.1 stated that

$$\frac{\mathfrak{g}}{C^{n-3}\mathfrak{g}} = L_{n-2}.$$

Thus, we have

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_{n-2-i}] = a_{i,n-2-i} X_{n-2} + (-1)^{i-1} X_{n-1}, & 1 \leqslant i \leqslant \frac{n-3}{2}. \end{cases}$$

The Jacobi relations $J(X_0, X_i, X_{n-3-i}) = 0$, when $1 \leqslant i \leqslant (n-5)/2$, implies $a_{i+1,n-2-(i+1)} = -a_{i,n-2-i}$ and denoting $a_{1,n-3} = \alpha$, and varying i from 1 to $(n-5)/2$, we have

$$a_{i,n-2-i} = (-1)^{i-1}\alpha, \quad 1 \leqslant i \leqslant \frac{n-3}{2}.$$

These equations prove that

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_{n-2-i}] = (-1)^{i-1}(\alpha X_{n-2} + X_{n-1}), & 1 \leqslant i \leqslant \frac{n-3}{2}. \end{cases}$$

Thus, an adequate change of bases provides that $\mathfrak{g} = \mathcal{L}_{(n,n-2)}$. □

Depending on the dimension n of the algebra $\mathfrak{g}$, for each acceptable type of gradation $\mathfrak{g} = \mathfrak{t}_r$, we will obtain that the only possible graded quasi-filiform Lie algebras are $\mathcal{L}_{(n,r)}$ when n is even, and $\mathcal{L}_{(n,r)}$ or $\mathcal{Q}_{(n,r)}$ when n is odd. However, when $r = n-3$ (n even) or when $r = n-4$ (n odd), we have also to consider the algebras $\mathcal{T}_{(n,n-3)}$ and $\mathcal{T}_{(n,n-4)}$, defined in Section 2. Those naturally graded quasi-filiform Lie algebras have types $\mathfrak{g} = \mathfrak{t}_{n-3}$ and $\mathfrak{g} = \mathfrak{t}_{n-4}$, respectively. In each dimension n, the appropriate $\mathcal{T}_{(n,r)}$ algebra will be called the *terminal* algebra.

We can now prove that the graded quasi-filiform Lie algebras so far considered are really different.

Proposition 4.2. *The n-dimensional graded quasi-filiform Lie algebras $L_{n-1} \oplus \mathbb{C}$, $Q_{n-1} \oplus \mathbb{C}$, $\mathcal{L}_{(n,r)}$, $\mathcal{Q}_{(n,r)}$, $\mathcal{T}_{(n,n-4)}$, $\mathcal{T}_{(n,n-3)}$ are pairwise non-isomorphic.*

Proof. The Split Lie algebras $L_{n-1} \oplus \mathbb{C}$, $Q_{n-1} \oplus \mathbb{C}$ are obviously graded quasi-filiform algebras of type $\mathfrak{g} = \mathfrak{t}_1$. For any other algebra it is deduced from its definition that $(X_0, X_1, \ldots, X_{n-1})$ is a homogeneous adapted basis of a quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{t}_r$. It is obvious that the dimensions of the ideals of the descending central sequence of algebras from different types are different and, therefore, they are non-isomorphic. Among those which belong to the same type, when n is odd, $\mathcal{L}_{(n,r)}$ and $\mathcal{Q}_{(n,r)}$ are not isomorphic since $\dim(D^2\mathfrak{g})$ is different for each one of them, being $D^2(\mathfrak{g}) = [C^1\mathfrak{g}, C^1\mathfrak{g}]$ the second characteristic ideal of the derived sequence of the algebra. Finally, in the terminal algebras $\mathcal{T}_{(n,r)}$, the dimension of the center $\mathcal{Z}(\mathfrak{g})$ is different from the corresponding $\mathcal{L}_{(n,r)}$ and $\mathcal{Q}_{(n,r)}$, which should be considered depending on the parity of the dimension n of the algebra. □

In dimensions $n = 7$ and $n = 9$ can be proved that, besides algebras of type $\mathfrak{g} = \mathfrak{t}_{n-4}$ in the previous Proposition 4.2, we have to consider the following algebras.

Remark 4.1. The algebra with dimension 7

$$\mathcal{E}_{(7,3)} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant 4, \\ [X_6, X_i] = X_{3+i}, & 1 \leqslant i \leqslant 2, \\ [X_1, X_2] = X_3 + X_6, & \\ [X_1, X_i] = X_{i+1}, & 3 \leqslant i \leqslant 4, \end{cases}$$

and the following algebras with dimension 9 denoted as $\mathcal{E}^i_{(9,5)}$ $(i = 1, 2)$ and defined by

$$\mathcal{E}^1_{(9,5)} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant 6, \\ [X_8, X_i] = 2X_{5+i}, & 1 \leqslant i \leqslant 2, \\ [X_1, X_4] = X_5 + X_8, & \\ [X_1, X_5] = 2X_6, & \\ [X_1, X_6] = 3X_7, & \\ [X_2, X_3] = -X_5 - X_8, & \\ [X_2, X_4] = -X_6, & \\ [X_2, X_5] = -X_7, & \end{cases}$$

$$\mathcal{E}^2_{(9,5)} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant 6, \\ [X_8, X_i] = 2X_{5+i}, & 1 \leqslant i \leqslant 2, \\ [X_1, X_4] = X_5 + X_8, \\ [X_1, X_5] = 2X_6, \\ [X_1, X_6] = X_7, \\ [X_2, X_3] = -X_5 - X_8, \\ [X_2, X_4] = -X_6, \\ [X_2, X_5] = X_7, \\ [X_3, X_4] = -2X_7 \end{cases}$$

are graded quasi-filiform Lie algebras.

We are now in a position to obtain the classification of graded quasi-filiform Lie algebras of type $\mathfrak{g} = \mathfrak{t}_r$ in the cases $r = n - 3$ and $r = n - 4$, which will be the only ones in which terminal algebras can appear.

Proposition 4.3. *If $\mathfrak{g}$ is a graded quasi-filiform Lie algebra with dimension n, then*:

(1) *If the algebra is of type $\mathfrak{g} = \mathfrak{t}_{n-3}$ (n even) and $n \geqslant 6$, then $\mathfrak{g} = \mathcal{L}_{(n,n-3)}$ or $\mathfrak{g} = \mathcal{T}_{(n,n-3)}$.*
(2) *If the algebra is of type $\mathfrak{g} = \mathfrak{t}_{n-4}$ (n odd) and $n \geqslant 7$, then $\mathfrak{g} = \mathcal{L}_{(n,n-4)}$, $\mathfrak{g} = \mathcal{Q}_{(n,n-4)}$, or $\mathfrak{g} = \mathcal{T}_{(n,n-4)}$. In those cases in which the dimension is $n = 7$ or $n = 9$, those algebras in Remark* 4.1 *must be added.*

Proof. According to Proposition 3.3, if the algebra is of type $\mathfrak{g} = \mathfrak{t}_{n-3}$, then n is even and if it is of type $\mathfrak{g} = \mathfrak{t}_{n-4}$, n is odd. Thus, two cases in the proof will be distinguished.

First case: n even. If the dimension of the algebra is 6, and $\mathfrak{g} = \mathfrak{t}_3$ is naturally graded, then it can be check that $\mathfrak{g} = \mathcal{L}_{(6,3)}$ or $\mathfrak{g} = \mathcal{T}_{(6,3)}$. Suppose that the dimension of the algebra is $n \geqslant 8$. If $(X_0, X_1, \ldots, X_{n-2}, X_{n-1})$ is an adapted basis of the algebra $\mathfrak{g}$, the algebra of type $\mathfrak{g} = \mathfrak{t}_{n-3}$ presents the following decomposition:

$$\mathfrak{g} = \langle X_0, X_1\rangle \oplus \langle X_2\rangle \oplus \cdots \oplus \langle X_{n-3}, X_{n-1}\rangle \oplus \langle X_{n-2}\rangle.$$

Starting from the family according to Proposition 3.3, and working as in Proposition 4.1, we come to the conclusion that algebra $\mathfrak{g}$ belongs to the family parameterized by

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_{n-1}, X_1] = aX_{n-2}, \\ [X_i, X_{n-3-i}] = (-1)^{i-1}(\alpha X_{n-3} + X_{n-1}), & 1 \leqslant i \leqslant \frac{n-4}{2}, \\ [X_i, X_{n-2-i}] = a_{i,n-2-i} X_{n-2}, & 1 \leqslant i \leqslant \frac{n-4}{2}. \end{cases}$$

From the Jacobi relations $J(X_0, X_i, X_{n-3-i}) = 0$, when $1 \leqslant i \leqslant (n-4)/2$, we obtain

$$a_{i,n-2-i} = (-1)^{i-1}\frac{n-2-2i}{2}\alpha, \quad 1 \leqslant i \leqslant \frac{n-4}{2}.$$

Since the dimension of $\mathfrak{g}$ is greater than or equal to 8 we have that at least one of the Jacobi relations $J(X_1, X_i, X_{n-3-i}) = 0$ with $2 \leqslant i \leqslant (n-4)/2$ is non-trivial. From such non-trivial Jacobi relations the following equations are obtained:

$$(-1)^{i-1}(\alpha a_{1,n-3} - a) = 0,$$

and substituting the value of $a_{1,n-3}$, previously found, we can express that

$$a = \frac{n-4}{2}\alpha^2.$$

The remaining Jacobi identities are trivially verified. Thus, if $\alpha = 0$, we have $\mathfrak{g} = \mathcal{L}_{(n,n-3)}$. If $\alpha \neq 0$, the change of bases determined by the equations $X'_0 = X_0$, $X'_{n-1} = \alpha^{-2}X_{n-1}$, $X'_i = \alpha^{-1}X_i$ $(i \neq 0, n-1)$, proves that $\mathfrak{g} = \mathcal{T}_{(n,n-3)}$. With this, the proof of case n even is completed.

Second case: n odd. In the dimensions $n = 7$ and $n = 9$, it can be directly checked that the results stated in item (2) are true. Therefore, suppose that the dimension of the algebra is $n \geqslant 9$ and let $(X_0, X_1, \ldots, X_{n-2}, X_{n-1})$ be an adapted basis of the algebra of type $\mathfrak{g} = \mathfrak{t}_{n-4}$. Proceeding as in the previous case, we arrive now to the fact that if $\mathfrak{g}/\langle X_{n-2}\rangle = \mathcal{L}_{(n-1,n-4)}$, then we have $\mathfrak{g} = \mathcal{L}_{(n,n-4)}$ or $\mathfrak{g} = \mathcal{Q}_{(n,n-4)}$. If the algebra $\mathfrak{g}/\langle X_{n-2}\rangle = \mathcal{T}_{(n-1,n-4)}$, then the algebra $\mathfrak{g}$ belongs to the family

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_{n-1}, X_i] = \frac{n-5}{2}X_{n-4+i}, & 1 \leqslant i \leqslant 2, \\ [X_i, X_{n-4-i}] = (-1)^{i-1}(X_{n-4} + X_{n-1}), & 1 \leqslant i \leqslant \frac{n-5}{2}, \\ [X_i, X_{n-3-i}] = (-1)^{i-1}\frac{n-3-2i}{2}X_{n-3}, & 1 \leqslant i \leqslant \frac{n-5}{2}, \\ [X_i, X_{n-2-i}] = a_{i,n-2-i}X_{n-2}, & 1 \leqslant i \leqslant \frac{n-3}{2}. \end{cases}$$

Now, since $\dim \mathfrak{g} \geqslant 11$, we have that at least one of the Jacobi relations $J(X_2, X_i, X_{n-4-i}) = 0$, with $3 \leqslant i \leqslant (n-5)/2$, is non-trivial. Then, from such non-trivial Jacobi relations, the following equations are obtained:

$$(-1)^{i-1}\left(a_{2,n-4} - \frac{n-5}{2}\right) = 0;$$

so $a_{2,n-4} = (n-5)/2$. The Jacobi identity $J(X_0, X_1, X_{n-4}) = 0$ provides the value $a_{1,n-3} = 0$, and from $J(X_0, X_i, X_{n-3-i}) = 0$, $2 \leqslant i \leqslant (n-5)/2$, we have $a_{i+1,n-2-(i+1)} = (-1)^{i-1}(n-3-2i)/2 - a_{i,n-2-i}$. Finally, by considering i from 2 to $(n-5)/2$ we obtain

$$a_{i,n-2-i} = (-1)^i(i-1)\frac{n-3-i}{2}, \quad 2 \leqslant i \leqslant \frac{n-3}{2}.$$

The remaining Jacobi relations are trivially verified, so we have $\mathfrak{g} = \mathcal{T}_{(n,n-4)}$. With this, the proof of case n odd is completed, and we conclude the proof of the proposition. □

4.2. Algebras of type $\mathfrak{g} = \mathfrak{t}_r$ with $3 \leqslant r \leqslant n-5$

Next, we will show that if a graded quasi-filiform Lie algebra is of type $\mathfrak{g} = \mathfrak{t}_r$, with $3 \leqslant r \leqslant n-5$ (that is, in most of the cases if the dimension considered is high), then $\mathfrak{g}$ can only be one of the principal algebras. The proof lies in the parameterization that can be obtained for the family of algebras of type $\mathfrak{g} = \mathfrak{t}_r$, and that is formulated in the following lemma.

Lemma 4.1. *Let $\mathfrak{g}$ be an n-dimensional graded quasi-filiform Lie algebra of type $\mathfrak{g} = \mathfrak{t}_r$, with r odd and $3 \leqslant r \leqslant n-5$. Then, if $(X_0, \ldots, X_{n-1})$ is an adapted basis of $\mathfrak{g}$, there exists $\alpha \in \mathbb{C}$ such that*

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_i, X_{r-i}] = (-1)^{i-1} X_{n-1}, & 1 \leqslant i \leqslant \frac{r-1}{2}, \\ [X_i, X_{n-2-i}] = (-1)^{i-1} \alpha X_{n-2}, & 1 \leqslant i \leqslant \frac{n-3}{2}, \end{cases}$$

where $\alpha = 0$ if n is even.

Proof. The proof will be done by induction in the dimension n of algebra $\mathfrak{g}$, distinguishing if n is even or odd. For the first meaningful cases $n = 8$ and $n = 9$, it can be directly checked that the lemma is verified.

Admitting the induction hypothesis, we will prove that if Lie algebra $\mathfrak{g}$ with dimension $n+1$, is a graded quasi-filiform of type $\mathfrak{g} = \mathfrak{t}_r$, with r odd and $3 \leqslant r \leqslant n-4$, then in the adapted basis $(X_0, X_1, \ldots, X_{n-1}, X_n)$ for some $\alpha \in \mathbb{C}$, we have

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-2, \\ [X_i, X_{r-i}] = (-1)^{i-1} X_n, & 1 \leqslant i \leqslant \frac{r-1}{2}, \\ [X_i, X_{n-1-i}] = (-1)^{i-1} \alpha X_{n-1}, & 1 \leqslant i \leqslant \frac{n-2}{2}, \end{cases}$$

with $\alpha = 0$ if $n+1$ is odd.

We have to consider those two cases that can appear, depending on the parity of the dimension of the algebra $\mathfrak{g}$. When $n+1$ is odd, the induction hypothesis could be directly applied to the algebra obtained from the quotient over the last subspace in the gradation. When $n+1$ is even, we have also to distinguish the particular case $r = n-4$, in which the obtained quotient can, at first, be a terminal algebra.

Case $n+1$ odd. Let $\mathfrak{g}$ be the algebra with dimension $n+1$ and of type $\mathfrak{g} = \mathfrak{t}_r$, with $3 \leqslant r \leqslant n-4$. We have the decomposition

$$\begin{aligned} &\langle X_0, X_1 \rangle \oplus \langle X_2 \rangle \oplus \cdots \oplus \langle X_r, X_n \rangle \oplus \cdots \\ &\quad \oplus \langle X_{n-4} \rangle \oplus \langle X_{n-3} \rangle \oplus \langle X_{n-2} \rangle \oplus \langle X_{n-1} \rangle, \end{aligned}$$

where $r \neq n-4$, since $n-4$ is even. Therefore, since $C^{n-2}\mathfrak{g} = \langle X_{n-1}\rangle$, denoting each class by the corresponding element, $(X_0, \dots, X_{n-2}, X_n)$ is an adapted basis of the quotient

$$\frac{\mathfrak{g}}{C^{n-2}\mathfrak{g}} = \langle X_0, X_1\rangle \oplus \langle X_2\rangle \oplus \cdots \oplus \langle X_r, X_n\rangle \oplus \cdots \oplus \langle X_{n-4}\rangle \oplus \langle X_{n-3}\rangle \oplus \langle X_{n-2}\rangle.$$

Thus, the quotient $\mathfrak{g}/\langle X_{n-1}\rangle$ is a graded quasi-filiform Lie algebra, with dimension n even, to which the induction hypothesis can be applied. Then,

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-2, \\ [X_i, X_{r-i}] = (-1)^{i-1} X_n, & 1 \leqslant i \leqslant \frac{r-1}{2}, \\ [X_i, X_{n-1-i}] = a_{i,n-1-i} X_{n-1}, & 1 \leqslant i \leqslant \frac{n-2}{2}. \end{cases}$$

Next, let $a_{1,n-2} = \alpha$. From the Jacobi relations $J(X_0, X_i, X_{n-2-i}) = 0$, when $1 \leqslant i \leqslant (n-4)/2$, we have that

$$a_{i,n-1-i} = (-1)^{i-1}\alpha, \quad 1 \leqslant i \leqslant \frac{n-2}{2},$$

and the lemma is proved for $3 \leqslant r \leqslant n-4$, when $n+1$ is odd.

Case $n+1$ even. We will deal with the case of $r = n-4$ aside, since on computing the quotient $\mathfrak{g}/\langle X_{n-1}\rangle$, we have a graded algebra with dimension n to which the induction hypothesis cannot be directly applied.

(1) Let $\mathfrak{g}$ be a graded quasi-filiform Lie algebra with dimension $n+1$ of type $\mathfrak{g} = \mathfrak{t}_r$, with $3 \leqslant r \leqslant n-6$. We have the following decomposition for algebra $\mathfrak{g}$

$$\langle X_0, X_1\rangle \oplus \langle X_2\rangle \oplus \cdots \oplus \langle X_r, X_n\rangle \oplus \cdots \oplus \langle X_{n-3}\rangle \oplus \langle X_{n-2}\rangle \oplus \langle X_{n-1}\rangle.$$

Proceeding as above, we have now that $\mathfrak{g}/\langle X_{n-1}\rangle$ is an algebra, with dimension n odd, to which the induction hypothesis can be applied. Then, from the Jacobi relation $J(X_0, X_i, X_{n-2-i}) = 0$, $1 \leqslant i \leqslant (n-3)/2$, follows

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-2, \\ [X_i, X_{r-i}] = (-1)^{i-1} X_n, & 1 \leqslant i \leqslant \frac{r-1}{2}, \\ [X_i, X_{n-2-i}] = (-1)^{i-1}\alpha X_{n-2}, & 1 \leqslant i \leqslant \frac{n-3}{2}, \\ [X_i, X_{n-1-i}] = (-1)^{i-1}\frac{n-1-2i}{2}\alpha X_{n-1}, & 1 \leqslant i \leqslant \frac{n-3}{2}. \end{cases}$$

Hence, the Jacobi identity $J(X_1, X_{(n-3)/2}, X_{(n-1)/2}) = 0$ implies $\alpha = 0$, so the lemma is proved for $3 \leqslant r \leqslant n-6$, when $n+1$ is even.

(2) Let now $\mathfrak{g}$ be an algebra of type $\mathfrak{g} = \mathfrak{t}_{n-4}$, with the decomposition

$$\langle X_0, X_1\rangle \oplus \cdots \oplus \langle X_{n-4}, X_n\rangle \oplus \langle X_{n-3}\rangle \oplus \langle X_{n-2}\rangle \oplus \langle X_{n-1}\rangle.$$

Then, $\mathfrak{g}/\langle X_{n-1}\rangle$ is a graded quasi-filiform Lie algebra, with dimension n odd, to which Proposition 4.3 states that it is $\mathcal{L}_{(n,n-4)}$, $\mathcal{Q}_{(n,n-4)}$ or $\mathcal{T}_{(n,n-4)}$ (in the particular case of $\dim(\mathfrak{g}/\langle X_{n-1}\rangle) = 9$ it can be directly checked that $\mathfrak{g}$ of type $\mathfrak{g} = \mathfrak{t}_6$, with dimension 10, verifies the induction hypothesis). But, algebra $\mathfrak{g}/\langle X_{n-1}\rangle$, when $n \geqslant 11$, cannot be $\mathcal{T}_{(n,n-4)}$, since getting the corresponding expression for $\mathfrak{g}$, the Jacobi relations $J(X_0, X_i, X_{n-2-i}) = 0$, $1 \leqslant i \leqslant (n-5)/2$, $J(X_0, X_{(n-3)/2}, X_{(n-1)/2}) = 0$ and $J(X_1, X_{(n-3)/2}, X_{(n-1)/2}) = 0$ prove that $\mathfrak{g}$ is not a Lie algebra. Therefore, algebra $\mathfrak{g}/\langle X_{n-1}\rangle$ must be either $\mathcal{L}_{(n,n-4)}$ or $\mathcal{Q}_{(n,n-4)}$ and then it is obtained that

$$\mathfrak{g} = \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-2, \\ [X_i, X_{n-4-i}] = (-1)^{i-1} X_n, & 1 \leqslant i \leqslant \frac{n-5}{2}, \\ [X_i, X_{n-2-i}] = (-1)^{i-1} \alpha X_{n-2}, & 1 \leqslant i \leqslant \frac{n-3}{2}, \\ [X_i, X_{n-1-i}] = a_{i,n-1-i} X_{n-1}, & 1 \leqslant i \leqslant \left\lfloor \frac{n-2}{2} \right\rfloor = \frac{n-3}{2}. \end{cases}$$

Now, the same Jacobi relations considered above implies that $\alpha = 0$ and $a_{i,n-1-i} = 0$ for all $1 \leqslant i \leqslant (n-3)/2$, so the lemma is proved when $\mathfrak{g}$ is an algebra of type $\mathfrak{g} = \mathfrak{t}_{n-4}$ and the proof for the dimension $n+1$ even is completed. □

The immediate consequence from Lemma 4.1 is the classification of the considered algebras. When n is odd, we only need to take αX_{n-1} in the adapted basis.

Proposition 4.4. *If $\mathfrak{g} = \mathfrak{t}_r$, in the conditions of Lemma* 4.1, *it is verified that $\mathfrak{g} = \mathcal{L}_{(n,r)}$, if n is even, and $\mathfrak{g} = \mathcal{L}_{(n,r)}$ or $\mathfrak{g} = \mathcal{Q}_{(n,r)}$ if n is odd.*

The following theorem summarizes the results obtained in this section and provides the classification of naturally graded quasi-filiform Lie algebras.

Theorem 4.1. *Every naturally graded quasi-filiform Lie algebra with dimension n is isomorphic to one of the following algebras*:

- *If n is even to $L_{n-1} \oplus \mathbb{C}$, $\mathcal{T}_{(n,n-3)}$ or $\mathcal{L}_{(n,r)}$, with r odd and $3 \leqslant r \leqslant n-3$.*
- *If n is odd to $L_{n-1} \oplus \mathbb{C}$, $Q_{n-1} \oplus \mathbb{C}$, $\mathcal{L}_{(n,n-2)}$, $\mathcal{T}_{(n,n-4)}$, $\mathcal{L}_{(n,r)}$, or $\mathcal{Q}_{(n,r)}$, with r odd and $3 \leqslant r \leqslant n-4$. In the cases of $n = 7$ and $n = 9$, we add those algebras in Remark* 4.1.

4.3. Symbolic calculus in Lie algebras

The use of a formal language that allows programming and symbolic computing as the software Mathematica [5] was very useful in order to obtain the classification of graded quasi-filiform Lie algebras. Mathematica was used as an assistant when getting examples (in a reliable way) of specific graded algebras in dimensions high enough. The observation of these examples turned out to be extremely valuable for the understanding of the structure, letting us prove or dismiss the new conjectures derived from the new results. In order to obtain graded algebras in a specific dimension, the process to carry out can be summarized in the following steps:

(1) To generate the set of algebras laws, defined by the structure constants in a basis that is supposed to be adapted.
(2) To determine the polynomial equations between structure constants obtained from the Jacobi identities.
(3) To reduce the equations of item 2.

With the reduction obtained in item (3), we proceed until obtaining the classification. In the three items above we use *Mathematica*. We use its programming language face to automatize items (1) and (2), and, then, we switch to the symbolic calculator for simplifying the unhandy output obtained in the previous step. The *Mathematica* notebook interface takes advantage of the system's graphical interfaces by manipulating input and output at interactive documents friendly.

The key rule-function code in the program developed for solving the tasks described by items (1) and (2) above is now considered. A general bilinear alternate law μ can be stated by the rules

```
mu[0, x_]:= 0;
mu[x_, 0]:= 0;
mu[x_, x_]:= 0;
mu[x_, y_]:= Simplify[-mu[y, x]]/; OrderedQ[{y,x}];
mu[x_+y_, z_]:= Simplify[mu[x, z]+mu[y, z]];
mu[z_, x_+y_]:= Simplify[mu[z, x]+mu[z, y]];
mu[a_ x_, y_]:= a mu[x,y];
mu[x_, a_ y_]:= a mu[x,y];
```

If $\mathfrak{g}_r = \langle X_r, X_{n-1}\rangle$, bracket products $[X_i, X_j]$ of a graded quasi-filiform Lie algebra in an adapted basis can be obtained (`grad`$= r$, `dim`$= n$) by

```
mu[x[0], x[dim-2]] = 0;
mu[x[0], x[dim-1]] = 0;
For[i=1, i <= dim-3, i++, mu[x[0], x[i]] = x[i+1]];
For[i=1, i <= dim-2, i++,
  If[i <= dim-grad-2,
    mu[x[dim-1],x[i]] = a[i] x[i+grad],
    mu[x[dim-1],x[i]] = 0
  ]
];
```

```
For[i=1, i <= dim-3, i++,
  For[j=i, j <= dim-2, j++,
    If[i+j <= dim-2 && i != j ,
      If[i+j != grad,
        mu[x[i],x[j]] = a[i,j] x[i+j],
        mu[x[i],x[j]] = a[i,j] x[i+j] + b[i] x[dim-1]
      ],
      mu[x[i],x[j]]=0
    ]
  ]
];
```

Also it is needed to determine the polynomial relations between the structure constants (automatized in the program) given by Jacobi's identity which are obtained from "select" nonzero coefficients in expressions

```
mu[mu[x[i],x[j]],x[k]] + mu[mu[x[j],x[k]],x[i]]
 +mu[mu[x[k],x[i]],x[j]].
```

As results that allowed the simplification of the considered family were obtained, the algorithms were modified in order to get examples in greater dimensions in those cases in which relevant information could be applied. For example, when Proposition 3.3 has been proved (at first, it was only a conjecture) the code can be updated, replacing `a[i]` by `a`, and `b[i]` by `(-1)^i`.

This approach can be used for those analogous problems of classification because it allows a significant progress. Moreover, the symbolic formulation of a problem and its treatment can point out the interest for studying certain "similar" objects to those that are being studied and that become apparent during computations. Thus, for instance, quasi-filiform Lie algebras with dimension n, $\mathcal{A}_{(n,r)}$ and $\mathcal{B}_{(n,r)}$ defined as

$\mathcal{A}_{(n,r)}$ $\quad (n \geqslant 5,\ 2 \leqslant r \leqslant n-3)$:

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_{n-1}, X_i] = X_{i+r}, & 1 \leqslant i \leqslant n-2-r; \end{cases}$$

$\mathcal{B}_{(n,r)}$ $\quad (n \geqslant 7,\ n$ odd; $3 \leqslant r \leqslant n-4$ (r odd), and $r = n-3)$:

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-3, \\ [X_{n-1}, X_i] = X_{i+r}, & 1 \leqslant i \leqslant n-2-r, \\ [X_i, X_{n-2-i}] = (-1)^i X_{n-2}, & 1 \leqslant i \leqslant \frac{n-3}{2}, \end{cases}$$

are the only non-split algebras admitting the decomposition

$$\mathfrak{g} = \langle X_0, X_1 \rangle \oplus \langle X_2 \rangle \oplus \cdots \oplus \langle X_r, X_{n-1} \rangle \oplus \cdots \oplus \langle X_{n-2} \rangle$$

when $(X_0, \ldots, X_{n-1})$ is a homogeneous adapted basis of $\mathfrak{g}$ and strict relation $[\mathfrak{g}_1, \mathfrak{g}_{r-1}] \subset \mathfrak{g}_r$ is considered.

In [3], the algebras $\mathcal{A}_{(n,r)}$ and $\mathcal{B}_{(n,r)}$, defined above, are obtained from a "quasi-natural" filtration (F_i), in terms of the descending central sequence of $\mathfrak{g}$, of which the natural filtration constitutes a particular case. In fact, the associated graded Lie algebras $\operatorname{gr} \mathfrak{g} = \bigoplus \mathfrak{g}_i$, with $\mathfrak{g}_i = F_i / F_{i+1}$, are the algebras $\mathcal{A}_{(n,r)}$, $\mathcal{B}_{(n,r)}$, and the algebras stated in Theorem 4.1. In dimension 8, we can find those algebras in [1], where they separate the irreducible components on the variety. So, it could be an interesting problem to study other non-natural graded Lie algebras in a similar way.

One can wonder then about gradations of that type, considering the problem from the point of view of obtaining the decomposition of an algebra $\mathfrak{g}$ into as many subspaces $\mathfrak{g}_i$ as possible with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, without using the natural filtration. In the study of those gradations, homogeneous adapted bases could be found, and new graded Lie algebras obtained could provide us relevant information for the study of nilpotent Lie algebras.

References

[1] J.M. Ancochea, J.R. Gómez, M. Goze, G. Valeiras, Sur les composantes irreductibles de la variété des lois nilpotentes, J. Pure Appl. Algebra 106 (1996) 11–22.

[2] M. Goze, Y. Hakimjanov, Sur les algèbres de Lie nilpotentes admettant un tore de dérivations, Manuscripta Math. 84 (1994) 115–224.

[3] A. Jiménez-Merchán, Familias de leyes de álgebras de Lie nilpotentes, PhD thesis, Universidad de Sevilla, 1995.

[4] M. Vergne, Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes, Bull. Soc. Math. France 98 (1970) 81–116.

[5] S. Wolfram, Mathematica: A System for Doing Mathematics by Computer, Addison–Wesley, 1991.