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International Journal of Solids and Structures 43 (2006) 6855–6868

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

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Numerical form-finding of tensegrity structures

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Received 26 August 2005; received in revised form 16 February 2006
Available online 6 March 2006

Communicated by Stelios Kyriakides

Abstract

A novel and versatile numerical form-finding procedure that requires only a minimal knowledge of the structure is presented. The procedure only needs the type of each member, i.e. either compression or tension, and the connectivity of the nodes to be known. Both equilibrium geometry and force densities are iteratively calculated. A condition of a maximal rank of the force density matrix and minimal member length, were included in the form-finding procedure to guide the search of a state of self-stress with minimal elastic potential energy. It is indeed able to calculate novel configurations, with no assumptions on cable lengths or cable-to-strut ratios. Moreover, the proposed approach compares favourably with all the leading techniques in the field. This is clearly exemplified through a series of examples.

Keywords: Tensegrity; Form-finding; Expanded octahedron; Truncated tetrahedron; Truncated icosahedron; *n*-Plex

1. Tensegrity structures

1.1. Introduction

Tensegrity structures are usually defined as a discontinuous set of compressional members inside a continuous network of tensile members (see e.g. Motro, 2003 for a review of the field). They are pin-jointed structures which are mechanically stabilized by the action of pre-stress and are self-equilibrated without the application of an external force. The tensegrities are commonly modelled with frictionless joints, and the self-weight of cables (tension) and struts (compression) is neglected.

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0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2006.02.012

Tensegrities are of interest in structural design studies because of their aesthetic value and light weight. The last property favours their application as space and deployable structures. There is also an interesting link between tensegrities and rigidity. For instance, a generalization of the physical model of joints (e.g. atoms) and members (e.g. bonds) is exploited in the computation of rigidity of molecular structures (Whiteley, 1999) or molecular conformations (Havel, 1998). Furthermore, there is an ongoing debate as to whether the tensegrity principle (self-stress, tension and compression) is used as a general design principle in biological materials (e.g. Zanotti and Guerra, 2003) and to what extent and how this principle can be used in the construction of smart materials (e.g. Luo and Bewley, 2005; Skelton and Sultan, 1997).

1.2. Form-finding

The analysis of tensegrities, which belong to the class of statically indeterminate structures, requires an initial form finding procedure (e.g. Motro, 2003; Tibert and Pellegrino, 2003). This consists in computing detailed information needed to create a state of self-stress, and usually set in self-equilibrium. A form-finding procedure could typically compute a critical parameter such as (i) a twisting angle, (ii) a cable-to-strut ratio or (iii) a force-to-length ratio, which is also known as the tension coefficient or the force density coefficient.

In most existing form-finding procedures, however, assumptions on either the tension coefficients, the element lengths or the symmetry of the whole structure must be imposed a priori (see Motro, 2003 or Tibert and Pellegrino, 2003 for a survey). For instance, (i) tension coefficients are imposed in a symbolic analysis, (ii) element lengths have to be predefined in a dynamic relaxation procedure and non-linear programming, and (iii) a global symmetry is assumed in a group-theory based form-finding procedure. But this information may not always be available or easy to estimate beforehand. The evaluation of complex tensegrity structures with these procedures therefore remains difficult.

In this work, we present a numerical form-finding procedure for statically indeterminate structures in self-equilibrium. The procedure only requires the type of each member (i.e. either compression or tension) and the connectivity between the nodes to be known. With this knowledge we approximate the self-equilibrium geometry and the member forces. We iteratively adjust them until a state of self-stress is found.

The paper is organized as follows: the relevant definitions and rank conditions for tensegrity structures are reviewed in Sections 2 and 3, respectively. The overall form-finding approach is outlined in Section 4. The core of our form-finding procedure is presented in Sections 5 and 6. The stability of tensegrity structures is briefly discussed in Section 7. Examples of two-dimensional (2D) and three-dimensional (3D) structures are presented in Sections 8 and 9. Finally, conclusions drawn from this work appear in Section 10.

2. Geometry and tension coefficients

Here and in the following sections we shall use the same standard terminology as found in Motro (2003) or Tibert and Pellegrino (2003). The equation of static equilibrium of an unconstrained reference node i connected to nodes j and k are given by

$$\begin{aligned}(x_i - x_j)f_{i,j}/l_{i,j} + (x_i - x_k)f_{i,k}/l_{i,k} &= f^{\text{ext}}_{i,x},\\ (y_i - y_j)f_{i,j}/l_{i,j} + (y_i - y_k)f_{i,k}/l_{i,k} &= f^{\text{ext}}_{i,y},\\ (z_i - z_j)f_{i,j}/l_{i,j} + (z_i - z_k)f_{i,k}/l_{i,k} &= f^{\text{ext}}_{i,z},\end{aligned} \tag{1}$$

where any member (A, B), that connects nodes A and B, has an internal force $f_{A,B}$ and a length $l_{A,B}$; and f^{ext} is the external force. A simplified linearised notation $q_{A,B} = f_{A,B}/l_{A,B}$ known as *tension coefficient* or *force density* is often used. Eq. (1) can thus either be written as

$$\begin{aligned}(x_i - x_j)q_{i,j} + (x_i - x_k)q_{i,k} &= f^{\text{ext}}_{i,x},\\ (y_i - y_j)q_{i,j} + (y_i - y_k)q_{i,k} &= f^{\text{ext}}_{i,y},\\ (z_i - z_j)q_{i,j} + (z_i - z_k)q_{i,k} &= f^{\text{ext}}_{i,z}\end{aligned} \tag{2}$$

or as

$$\begin{aligned}(q_{i,j}+q_{i,k})x_i - q_{i,j}x_j - q_{i,k}x_k &= f^{\text{ext}}_{i,x},\\ (q_{i,j}+q_{i,k})y_i - q_{i,j}y_j - q_{i,k}y_k &= f^{\text{ext}}_{i,y},\\ (q_{i,j}+q_{i,k})z_i - q_{i,j}z_j - q_{i,k}z_k &= f^{\text{ext}}_{i,z}.\end{aligned} \tag{3}$$

We study structures in a state of self-stress with unconstrained nodes and zero external load in d dimensions. Let $\mathbf{x}=[x_1\ \cdots\ x_{\mathsf{n}}]^{\mathrm{T}}$, $\mathbf{y}=[y_1\ \cdots\ y_{\mathsf{n}}]^{\mathrm{T}}$, and $\mathbf{z}=[z_1\ \cdots\ z_{\mathsf{n}}]^{\mathrm{T}}$ be the vectors of coordinates for n nodes along the x, y, and z directions, respectively. Here $(\bigstar)^{\mathrm{T}}$ represents a transpose operation. We use an incidence matrix $\mathscr{C}\in\Re^{\mathsf{b}\times\mathsf{n}}$, which has one row per member (A,B), and its columns A and B have entries $+1$ and -1, respectively. Therefore, the projected lengths, e.g. (x_A-x_B), along the x, y, z-directions are written in vector form as $\mathscr{C}\mathbf{x}$, $\mathscr{C}\mathbf{y}$, and $\mathscr{C}\mathbf{z}$, respectively.

Let $\mathbf{q}\in\Re^{\mathsf{b}}$ be a vector of tension coefficients, with one entry for each of the b members. We can write the matrix form of Eq. (2) by factorizing the projected lengths in the equilibrium matrix $\mathscr{A}\in\Re^{\mathsf{dn}\times\mathsf{b}}$ and a vector $\mathbf{q}$ of tension coefficients:

$$\mathscr{A}\mathbf{q}=\underbrace{\begin{pmatrix}\mathscr{C}^{\mathrm{T}}\mathrm{diag}(\mathscr{C}\mathbf{x})\\ \mathscr{C}^{\mathrm{T}}\mathrm{diag}(\mathscr{C}\mathbf{y})\\ \mathscr{C}^{\mathrm{T}}\mathrm{diag}(\mathscr{C}\mathbf{z})\end{pmatrix}}_{\text{Equilibrium matrix}}\mathbf{q}=\mathbf{0}, \tag{4}$$

where $\mathrm{diag}(\bigstar)$ is a square matrix with the vector $(\bigstar)$ along its diagonal.

Similarly, if we factorize the tension coefficients of Eq. (3), its matrix representation relates a symmetric matrix $\mathscr{D}\in\Re^{\mathsf{n}\times\mathsf{n}}$, known as the force density matrix (FDM), or stress matrix in mathematics (Connelly, 1982) and the nodal coordinates:

$$\mathscr{D}[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]=\underbrace{\left(\mathscr{C}^{\mathrm{T}}\mathrm{diag}(\mathbf{q})\mathscr{C}\right)}_{\text{FDM}}[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]=[\mathbf{0}\ \mathbf{0}\ \mathbf{0}]. \tag{5}$$

Eq. (4) relates the projected lengths to tension coefficients, whereas Eq. (5) relates tension coefficients to nodal coordinates. In the following sections, we shall assume however that neither element lengths nor tension coefficients are known a priori.

3. Rank conditions

Two necessary but not sufficient rank conditions have to be satisfied in a d-dimensional structure that is in a state of self-stress (e.g. Motro, 2003; Graver et al., 1993; Connelly, 1982). The first one ensures the existence of at least one state of self-stress, if

$$\mathsf{r}=\mathrm{rank}(\mathscr{A})<\mathsf{b}, \tag{6}$$

which is necessary for a non trivial solution of Eq. (4). This rank deficiency provides the number of independent states of self-stress $\mathsf{s}=\mathsf{b}-\mathsf{r}\geqslant 1$ and the number of infinitesimal mechanisms $\mathsf{m}=\mathsf{dn}-\mathsf{r}$, as explained in Calladine (1978).

The second rank condition is related to the semi-definite matrix $\mathscr{D}$ of Eq. (5) as follows:

$$\mathrm{rank}(\mathscr{D})<\mathsf{n}-\mathsf{d} \tag{7}$$

for a geometric embedding into $\Re^{\mathsf{d}}$. For an embedding of maximal affine space (Graver et al., 1993; Connelly, 1982; Connelly, 1993) the largest possible rank of $\mathscr{D}$ is $(\mathsf{n}-\mathsf{d}-1)$. In other words, the nullity is $(\mathsf{d}+1)$. For instance, the affine space spanned by a line has a dimension two, for a plane it is three and for a geometric solid it is four. Our proposed form-finding procedure uses this maximal rank condition in order to find a tensegrity structure.

4. Form-finding with prototypes

Contrary to most existing form-finding procedures (e.g. Motro, 2003; Tibert and Pellegrino, 2003), we propose one that does not require any initial assumption about the lengths of the elements, geometry or the symmetry of the structure. Instead our procedure only requires to know the dimensions (d) of the structure, the incidence matrix $\mathscr{C}$, and initial information about the type of each member. This information is contained in a vector $\mathbf{q}^0 \in \Re^b$ that is seen as a "prototype" of the tension coefficient for each member. We assign the tension coefficient $+1$ or -1 to members that are chosen to be in tension or in compression, respectively:

$$\mathbf{q}^0 = [\underbrace{+1 \quad +1 \quad +1 \quad \ldots}_{\text{tension}} \quad \underbrace{-1 \quad -1 \quad -1 \quad \ldots}_{\text{compression}}]^{\mathrm{T}}.$$

Subsequently, the matrix $\mathscr{D}$ is calculated from $\mathbf{q}^0$, i.e. $\mathscr{D} = \mathscr{C}^{\mathrm{T}}\mathrm{diag}(\mathbf{q}^0)\mathscr{C}$, and iteratively modified as to satisfy the rank conditions equation (7). In the course of these iterations, we try to deviate as little as possible from the signs of the initial prototypes in $\mathbf{q}^0$.

An outline of the form-finding process is shown in Fig. 1. In the initialization, shown in Fig. 1a, we define the incidence matrix ($\mathscr{C}$), the number of dimensions (d) and a prototype of tension coefficients ($\mathbf{q}^0$). At this stage, the vector of tension coefficients $\mathbf{q}^0$ does not produce a state of self-stress ($s = 0$) in the structure.

Subsequently, we approximate the nodal coordinates as a function of the tension coefficients (Fig. 1b); from these coordinates we approximate a new vector of tension coefficients (Fig. 1c). These two steps represent the

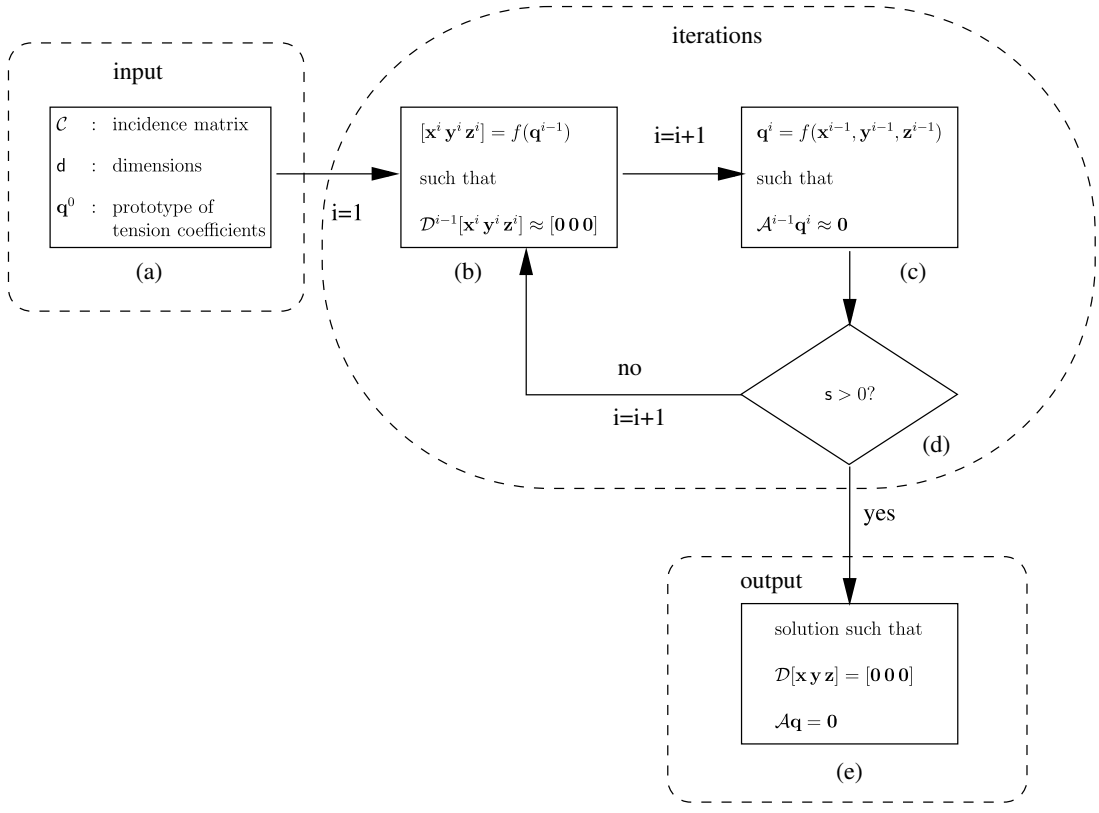

Fig. 1. Outline of the proposed numerical form-finding procedure for tensegrity structures: (a) initialization, no state of self-stress. Only a vector of prototypes of tension coefficients, the number of dimensions and the incidence matrix are needed; (b) Approximated coordinates as a function of tension coefficients. Coordinates are calculated from the decomposition of a force density matrix with the current tension coefficients. (c) Approximated tension coefficients as a function of coordinates. Tension coefficients are calculated from the decomposition of an equilibrium matrix with the current coordinates. (d) The previous decomposition is used to test whether or not the structure has a state of self-stress; (e) Final coordinates and tension coefficients that create at least one state of self-stress.

core of the proposed form-finding procedure. They are repeated until a solution is found (Fig. 1d). At this point we have at least one state of self-stress, $\mathsf{s} \geqslant 1$, and it is the end of our numerical procedure (Fig. 1e). The procedure shown in Figs. 1b and c will be described in detail in Sections 5.2 and 6.2, respectively.

Finally, for a structure with $\mathsf{s} \geqslant 1$ we can verify that the tension coefficients satisfy $\mathscr{A}\mathbf{q} = \mathbf{0}$ and the coordinates satisfy $\mathscr{D}[\mathbf{x}\ \mathbf{y}\ \mathbf{z}] = [\mathbf{0}\ \mathbf{0}\ \mathbf{0}]$, as well as that both rank conditions given in Eqs. (6) and (7) are satisfied. If the calculated structure is in a stable equilibrium, see Section 7, we may say that a signed vector of tension coefficients $\mathbf{q}^0$ provides *enough* information to compute a stable self-equilibrium with $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{q}$.

5. From tension coefficients to coordinates

5.1. Equilibrium

Let $\mathbf{q}$ be the vector of tension coefficients and $\mathscr{C}$ be the incidence matrix of a $\mathsf{d} = 3$ dimensional tensegrity structure in a state of self-stress. The FDM is then given by $\mathscr{D} = \mathscr{C}^{\mathrm{T}}\mathrm{diag}(\mathbf{q})\mathscr{C}$. If the matrix $\mathscr{D}$ is positive semi-definite of maximal rank (Connelly, 1999), i.e. satisfies the rank condition equation (7), we can perform a Schur decomposition:

$$\mathscr{D} = \mathscr{C}^{\mathrm{T}}\mathrm{diag}(\mathbf{q})\mathscr{C} = UVU^{\mathrm{T}},$$

where the first $(\mathsf{d}+1)$ columns of the matrix $U = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_{\mathsf{n}}]$, contain the basis of the nodal coordinates; and the diagonal matrix V has $\mathsf{d}+1 = 4$ zero eigenvalues. In this context, the matrix U has the basis for the nodal coordinates as columns of the nullspace each of which solves the homogeneous Eq. (5). It follows from the definition of $\mathscr{D}$ that the 1-vector, $[1\ 1\ 1\ \cdots]$, solves the homogeneous Eq. (5). However, this vector provides little information for the nodal coordinates and it should be avoided in the subsequent calculations. See Meyer (2000) for more information on the null spaces and decompositions. The general solution,

$$[\mathbf{x}\ \mathbf{y}\ \mathbf{z}] = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_{\mathsf{d}+1}]T$$

requires a transformation matrix $T \in \Re^{(\mathsf{d}+1)\times\mathsf{d}}$. We can use T to specify the final shape of the structure. An infinite number of geometrically different self-stressed configurations can thus be found for a single vector $\mathbf{q}$. Since the tension coefficients do not change under affine transformations (Tarnai, 1989), we directly use the eigenvectors of the null space. Thus, we set for simplicity and without loss of generality

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

5.2. Non-equilibrium

A tensegrity structure generated by a vector of tension coefficients $\mathbf{q}^0$ is unlikely to satisfy Eq. (7) and therefore is not in self-equilibrium. In this case it is necessary to circumvent the lack of a useful nullspace in $\mathscr{D}$, from which the basis of the nodal coordinates can be computed. Here, we show how an approximation of the nodal coordinates can be computed if the rank condition Eq. (7) is not satisfied. We present a matrix operation, $\Re^{\mathsf{b}} \to \Re^{\mathsf{n}\times\mathsf{d}}$, that uses $\mathbf{q}$ to compute an approximation of $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]$.

Let $\mathbf{q}$ be a vector of tension coefficients such that the FDM is factorized as

$$\mathscr{D} = \mathscr{C}^{\mathrm{T}}\mathrm{diag}(\mathbf{q})\mathscr{C} = UVU^{\mathrm{T}}, \tag{8}$$

does not satisfy Eq. (7) and may not be positive semi-definite. In general, a FDM with prototypes of tension coefficients $\mathbf{q}^0$ falls within this definition. The nullspace is therefore useless and other techniques need to be applied to identify a triplet, $\{\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k\} \in U$, of column vectors that satisfies the equation of static equilibrium,

$$\mathscr{D}[\mathbf{u}_i\ \mathbf{u}_j\ \mathbf{u}_k] \approx [\mathbf{0}\ \mathbf{0}\ \mathbf{0}].$$

This is an approximated solution of static equilibrium in which the column vectors $\{\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k\}$ do *not* correspond to the zero eigenvalues of Eq. (8).

One possibility we could use to select these columns is to favour the triplet that produces a dominating structure (e.g. Connelly, 1993; Connelly and Terrell, 1995), in which struts are of non-zero length but as short as possible.

This minimal length condition has some advantages, for instance with respect to Euler buckling and globally rigid tensegrities (Connelly and Terrell, 1995; Connelly, 1999). It is easy to choose the triplet $\{\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k\}$ if we consider the problem in terms of lengths, in which we can minimize the total squared length of the entire structure,

$$\sum_{p=1}^{\mathsf{b}} l_p^2 = \|\mathscr{C}\mathbf{u}_i + \mathscr{C}\mathbf{u}_j + \mathscr{C}\mathbf{u}_k\|^2. \tag{9}$$

To this end, the matrix $U = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\ \cdots\ \mathbf{u}_{\mathsf{n}}]$ computed in Eq. (8), is used to calculate all the projected lengths,

$$L = \mathscr{C}U = [(\mathbf{u}_1^A - \mathbf{u}_1^B)\ (\mathbf{u}_2^A - \mathbf{u}_2^B)\ \cdots\ (\mathbf{u}_{\mathsf{n}}^A - \mathbf{u}_{\mathsf{n}}^B)],$$

along all n directions for each pair (A, B) of connected nodes. We can remove the column vectors $\mathbf{u}_i$ for which $\mathscr{C}\mathbf{u}_i = \mathbf{0}$. In order to find the $\{\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k\}$, Eq. (9), we identify the columns of L according to their 2-norm,

$$[c_1\ c_2\ \cdots] = \arg\min_i ||L_i||. \tag{10}$$

where the elements of the sorted list $[c_1\ c_2\ \cdots]$ returned by the minimal argument function, *argmin*(★), correspond to the indices (i) in which the vector norm $\|L_i\|$ is minimized. Notice that there may be multiple indices that correspond to the minimal 2-norm. Therefore, we take the minimal indices as well as their multiplicity to have a pool of at least d candidate columns. Eq. (10) helps to narrow the search space to only a few columns. The reduced matrix $\tilde{U} = [\mathbf{u}_{c_1}\ \mathbf{u}_{c_2}\ \cdots]$ is then used to factorize the projected lengths by means of a QR (or LU) decomposition,

$$\mathscr{C}\tilde{U} = QR, \tag{11}$$

to get access to the upper triangular matrix R. In such a case, the projections along each direction are handled columnwise, i.e. $\mathscr{C}\mathbf{u}_i = QR_i$, and linearly dependent projected lengths have their pivot set to zero. Finally, these pivots help to choose the d linearly independent columns as nodal coordinates $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]$.

This selection procedure, Eqs. (10) and (11), helps to find configurations with minimal but non-zero projections, Eq. (9), with a maximal rank condition of $\mathscr{D}$, Eq. (7), and which have linearly independent projections $[\mathscr{C}\mathbf{x}\ \mathscr{C}\mathbf{y}\ \mathscr{C}\mathbf{z}]$. In other words, we pick a configuration that dominates all other equivalent ones (see Connelly, 1993, 1999; Connelly and Terrell, 1995). At this point of our form-finding procedure, we have an approximated equilibrium configuration, such that $\mathscr{D}[\mathbf{x}\ \mathbf{y}\ \mathbf{z}] \approx [\mathbf{0}\ \mathbf{0}\ \mathbf{0}]$.

6. From coordinates to tension coefficients

6.1. Equilibrium

Let $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]$ be the nodal coordinates of a structure in a state of self-stress, $\mathsf{s} \geqslant 1$. It is known (Pellegrino, 1993) that the basis of vector spaces of tension coefficients and mechanisms of a self-stressed structure are calculated from the null spaces of the equilibrium matrix. We can decompose the equilibrium matrix:

$$\mathscr{A} = \begin{pmatrix} \mathscr{C}^{\mathrm{T}}\mathrm{diag}(\mathscr{C}\mathbf{x}) \\ \mathscr{C}^{\mathrm{T}}\mathrm{diag}(\mathscr{C}\mathbf{y}) \\ \mathscr{C}^{\mathrm{T}}\mathrm{diag}(\mathscr{C}\mathbf{z}) \end{pmatrix} = UVW^{\mathrm{T}}. \tag{12}$$

In this context, the matrices U and W have the following null spaces,

$$U = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots | \mathbf{m}_1\ \cdots\ \mathbf{m}_{\mathsf{dn}-\mathsf{r}}],$$

and

$$W = [\mathbf{w}_1\ \mathbf{w}_2\ \cdots | \mathbf{q}_1\ \cdots\ \mathbf{q}_{\mathsf{b}-\mathsf{r}}],$$

where r is the rank of the diagonal matrix V; the vectors $\mathbf{m} \in \mathfrak{R}^{\mathsf{dn}}$ denote the $\mathsf{m} = \mathsf{dn} - \mathsf{r}$ infinitesimal mechanisms; and the vectors $\mathbf{q} \in \mathfrak{R}^{\mathsf{b}}$ the states of self-stress, each of which solves the homogeneous Eq. (4). We define $M \in \mathfrak{R}^{\mathsf{dn}\times\mathsf{dn}-\mathsf{r}}$ as the matrix of mechanisms, $M = [\mathbf{m}_1 \cdots \mathbf{m}_{\mathsf{dn}-\mathsf{r}}]$.

6.2. Non-equilibrium

However, if the structure is not in a state of self-stress, we do not have access to the null spaces of $\mathscr{A}$, as defined in Eq. (12). That is the case when the matrix $\mathscr{A}$ is calculated with an approximation of the nodal coordinates (see Section 5.2). We can, however, modify $\mathscr{A}$ to be rank deficient, and apply a matrix operation, $\mathfrak{R}^{\mathsf{n}\times\mathsf{d}} \to \mathfrak{R}^{\mathsf{b}}$, that uses $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]$ to compute an approximation of $\mathbf{q}$.

Let $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]$ be a matrix of estimated nodal coordinates and $\mathsf{d} = 3$. Using, for instance, the singular value decomposition of $\mathscr{A}$, we have $\mathscr{A} = UVW^{\mathrm{T}}$, where the matrix $W = [\mathbf{w}_1\ \mathbf{w}_2 \cdots \mathbf{w}_{\mathsf{b}}]$. In general, if we assume $\mathsf{s} = 1$, the signs of the last column of W and the prototypes in $\mathbf{q}^0$ may not match, i.e. $\mathrm{sgn}(\mathbf{w}_{\mathsf{b}}) \neq \mathrm{sgn}(\mathbf{q}^0)$. So we may need to use more than one column of W to compute a vector $\mathbf{q}$ that best matches $\mathbf{q}^0$. We can, for instance, use a least square fit to calculate the vector of coefficients $\tilde{\mathbf{q}}$ that minimises the quantity

$$\|[\mathbf{w}_{\mathsf{j}}\ \cdots\ \mathbf{w}_{\mathsf{b}}]\tilde{\mathbf{q}} - \mathbf{q}^0\|^2 \tag{13}$$

for a block of column vectors $[\mathbf{w}_{\mathsf{j}} \cdots \mathbf{w}_{\mathsf{b}}]$, such that $\mathbf{q} = [\mathbf{w}_{\mathsf{j}}\ \cdots\ \mathbf{w}_{\mathsf{b}}]\tilde{\mathbf{q}}$. We start by setting $j = \mathsf{b} - 1$ and verify if the signs match, $\mathrm{sgn}(\mathbf{q}) = \mathrm{sgn}(\mathbf{q}^0)$. If so, the procedure stops, otherwise we decrease j by one, and repeat it. This procedure imposes the existence of at least one state of self-stress that matches in signs with $\mathbf{q}^0$. At the end, the vector $\tilde{\mathbf{q}}$ is the unique least square solution for the system $[\mathbf{w}_{\mathsf{j}}\ \cdots\ \mathbf{w}_{\mathsf{b}}]\tilde{\mathbf{q}} = \mathbf{q}^0$, such that the product of the equilibrium matrix and the vector of approximated tension coefficients $\mathbf{q} = [\mathbf{w}_{\mathsf{j}}\ \cdots\ \mathbf{w}_{\mathsf{b}}]\tilde{\mathbf{q}}$ gives

$$\mathscr{A}\mathbf{q} = \begin{pmatrix} \mathscr{C}^{\mathrm{T}}\mathrm{diag}(\mathscr{C}\mathbf{x}) \\ \mathscr{C}^{\mathrm{T}}\mathrm{diag}(\mathscr{C}\mathbf{y}) \\ \mathscr{C}^{\mathrm{T}}\mathrm{diag}(\mathscr{C}\mathbf{z}) \end{pmatrix} \mathbf{q} \approx \mathbf{0}.$$

We therefore approximate a state of equilibrium and preserve $\mathrm{sgn}(\mathbf{q}^0)$. This closes the loop in the form-finding procedure because with the vector $\mathbf{q}$ we calculate $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]$, as described in Section 5.2.

In summary, the form-finding procedure consists in iterating equations (10), (11) and (13) until the rank condition Eq. (6) is satisfied. The potential null spaces in $\mathscr{D}$ and $\mathscr{A}$ are exploited as to impose the existence of at least one solution. The number of geometrically different configurations and tension coefficients is infinite, but the structural parameter s does not change under affine or projective transformations (Crapo and Whiteley, 1982; Tarnai, 1989). That is why we are focusing on a certain number of eigenvectors in $\mathscr{D}$ and $\mathscr{A}$, and using the bases of vector spaces for both coordinates and tension coefficients. In fact, it is Eq. (13) that forces the current equilibrium matrix to have at least one state of self-stress. The auxiliar equations Eqs. (10) and (11) update the coordinates of the next equilibrium matrix, and so on until $\mathsf{s} > 0$. At the end, we have a vector $\mathbf{q}$ such that $\mathscr{A}\mathbf{q} = \mathbf{0}$, and this single vector $\mathbf{q}$ is our solution. The rest of the states of self-stress, if $\mathsf{s} > 1$, can be calculated from the right null space of $\mathscr{A}$. The procedure guides both $\mathscr{D}$ and $\mathscr{A}$ to be rank deficient, with the proper rank, by selecting the appropriate eigenvector(s) in each decomposition and imposing the existence of at least one solution. We should remark, however, that the final vector of tension coefficients $\mathbf{q}$ is not unique for the given structure. Other vectors $\mathbf{q}$, equally valid, may exist for the same $\mathscr{C}$ and the same signs of $\mathbf{q}^0$. For instance, in Section 9.2 we present an example in which our calculated tension coefficients are different from previously published research. To calculate the examples presented in Sections 8 and 9, we implemented the form-finding procedure in Matlab version 7.

7. Initial stiffness

The form-finding procedure stops when a state of self-stress is found. It is convenient to verify that [**x y z**] and **q** create a *stable* self-equilibrated structure in a kinematically indeterminate assembly. We can check if the state of self-stress stiffens all the infinitesimal mechanisms.

It is known that a vector of tension coefficients contributes to the tangent stiffness matrix,

$$K_{\mathrm{T}} = K_{\mathrm{E}} + K_{\mathrm{G}},$$
$$= \mathscr{A}\,\mathrm{diag}\left(\frac{e_i a_i}{l_i^0} - \mathbf{q}\right)\mathscr{A}^{\mathrm{T}} + \begin{pmatrix} \mathscr{D} & & \\ & \mathscr{D} & \\ & & \mathscr{D} \end{pmatrix},$$

of pre-stressed, kinematically indeterminate structures (see, e.g. Guest, 2006; Murakami, 2001). Here $\mathbf{e} \in \mathfrak{R}^{\mathsf{b}}$ is the vector of Young's moduli, $\mathbf{a} \in \mathfrak{R}^{\mathsf{b}}$ the vector of cross-sectional areas and $\mathbf{l^0} \in \mathfrak{R}^{\mathsf{b}}$ the vector of initial lengths of each of the b members of the structure; and $\mathscr{D} = \mathscr{C}^{\mathrm{T}}\mathrm{diag}(\mathbf{q})\mathscr{C}$ for a single state of self-stress.

If an infinitesimal and inextensional mechanism $\mathbf{m} \in M$, $M = [\mathbf{m}_1 \cdots \mathbf{m}_{\mathsf{dn}-\mathsf{r}}]$ as described in Eq. (12), is applied to K_{T}, the elastic part K_{E} vanishes, i.e. $\mathscr{A}^{\mathrm{T}}\mathbf{m} = \mathbf{0}$. The stability of this initial state, or initial stiffness, of a structure therefore only involves the geometric stiffness K_{G} (e.g. Murakami, 2001; Guest, 2006; Volokh and Vilnay, 2000).

We can thus assess the stability of a pin-jointed structure in $\mathsf{d} = 3$ dimensions, subject to infinitesimal and inextensional mechanisms M, with the quadratic form of K_{G},

$$\Lambda = M^{\mathrm{T}}\begin{pmatrix} \mathscr{D} & & \\ & \mathscr{D} & \\ & & \mathscr{D} \end{pmatrix}M,$$

which is positive semi-definite if and only if $\mathscr{D}$ is positive semi-definite, for all $\mathbf{m} \in \mathfrak{R}^{\mathsf{dn}}$. For a positive $\mathscr{D}$ we verify whether K_{G} imparts positive stiffness to all directions within the inextensional displacements except for the six rigid body motions,

$$\mathrm{eig}(\Lambda) = [\underbrace{\lambda_1 = \lambda_2 = \cdots = \lambda_6 = 0}_{\text{rigid body motions}} \quad \underbrace{\lambda_7 > 0 \ \cdots \ \lambda_{\mathsf{dn}-\mathsf{r}} > 0}_{\text{positive stiffness}}], \tag{14}$$

that appear in the analysis of free standing structures in $\mathsf{d} = 3$ dimensions, for eigenvalues in increasing order and repeated according to their multiplicity. Here, it is assumed that there is a single state of self-stress, which is the typical example discussed in the related literature. The general case, $\mathsf{s} \geqslant 2$, however, is more problematic to verify whether the assembly possesses stiffness in all considered states of self-stress (e.g. Calladine and Pellegrino, 1991, 1992).

8. Examples in 2D

We demonstrate the usefulness of our procedure by presenting a few examples of 2D tensegrity units. The procedure works remarkably well for super stable planar tensegrities (Connelly, 1982, 1999), in which the matrix $\mathscr{D}$ is (i) positive semi-definite of (ii) maximal rank $\mathsf{n} - \mathsf{d} - 1$. All examples presented in this paper fulfill these two conditions. Notice that no struts intersect in any tensegrity structure.

Fig. 2a depicts a tensegrity hexagon. We can analyse this small tensegrity in exact symbolic form (e.g. as in Tibert and Pellegrino, 2003). The solution is a force density of $q = 1$ and $q = -0.5$ in cables and struts, respectively. Here the struts, which are the elements in compression, are depicted by thick grey lines. One of the earliest examples of 2D tensegrity structures is shown in Fig. 2b. It was studied by Mohr (1885). An example found in Graver et al. (1993) is shown in Fig. 2c. Each one of these three tensegrities (Figs. 2a–c) has six nodes and nine members, and each tensegrity structure has $\mathsf{s} = 1$ and $\mathsf{m} = 1$.

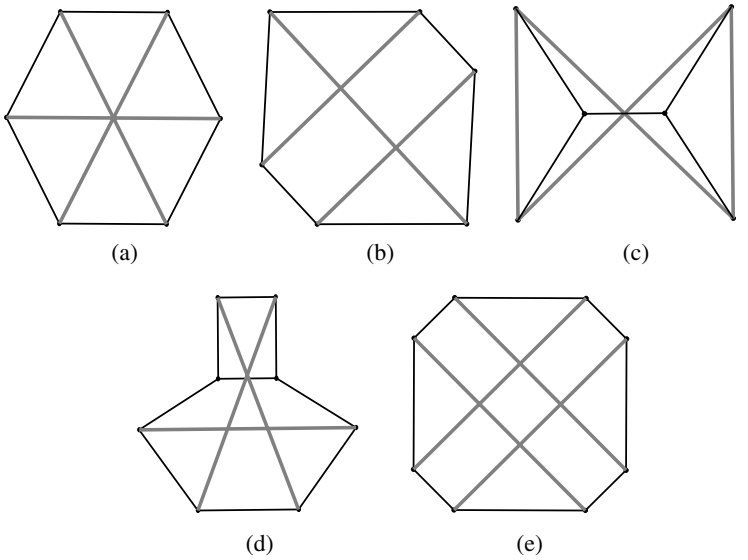

Fig. 2. Examples of 2D tensegrities calculated with the new form-finding procedure. Thick gray lines represent struts.

Fig. 2d is a compound of a square and a hexagon, in which the struts are placed such that there is no nodal symmetry, and two nodes lack incident struts. Finally, Fig. 2e shows an example found in Connelly and Back (1998). Both tensegrities (Figs. 2d–e) have $\mathsf{s} = 1$ and $\mathsf{m} = 2$, and they have 8 nodes and 12 members.

9. Examples in 3D

9.1. Expanded octahedron

Using a symbolic analysis, Tibert and Pellegrino (2003) calculated two solutions for the expanded octahedron, of which only one produces a positive semi-definite matrix $\mathscr{D}$. Applying our form-finding procedure, we find that the expanded octahedron has $\mathsf{s} = \mathsf{m} = 1$. The stable structure is shown in Fig. 3a. The 2-norm of $\mathscr{A}\mathbf{q}$ as a function of the number of iterations is plotted in a logarithmic scale in Fig. 3b. It can be seen that the iterative procedure converges in 14 iterations to one state of self-stress.

The ratio between strut and cable lengths is $l_s/l_c \approx 1.63299$, which is in agreement with other studies (Motro, 2003). By normalising $\mathbf{q}$ with respect to the tension coefficient of the cables, we compute ($q_c = 1$) and ($q_s = -3/2$) for 24 cables and 6 struts, respectively. These values correspond to the ones found by Tibert and Pellegrino (2003) using a symbolic analysis.

Moreover, Eq. (14) shows that the state of self-stress imparts a positive stiffness to all but six directions. This example demonstrates that the proposed form-finding procedure is able to efficiently calculate the solution that produces a positive semi-definite matrix $\mathscr{D}$.

9.2. Truncated tetrahedron

The truncated tetrahedron analysed by several authors (Tibert and Pellegrino, 2003; Pellegrino, 1986; Motro, 2003) has one cable length as well as a single length for all struts. This structure is calculated by imposing these conditions a priori, e.g. in a dynamic relaxation (Motro, 1990) or in non-linear programming (Pellegrino, 1986).

However, there is no such constraint in our proposed form-finding procedure. We have found a solution with *two* different cable lengths and all struts being of equal length (Fig. 3c). We compared both solutions by looking at their elastic potential energies. Global equilibrium is attained in the state of least energy. Therefore configurations with less potential energy are preferred to other local minima with higher energy.

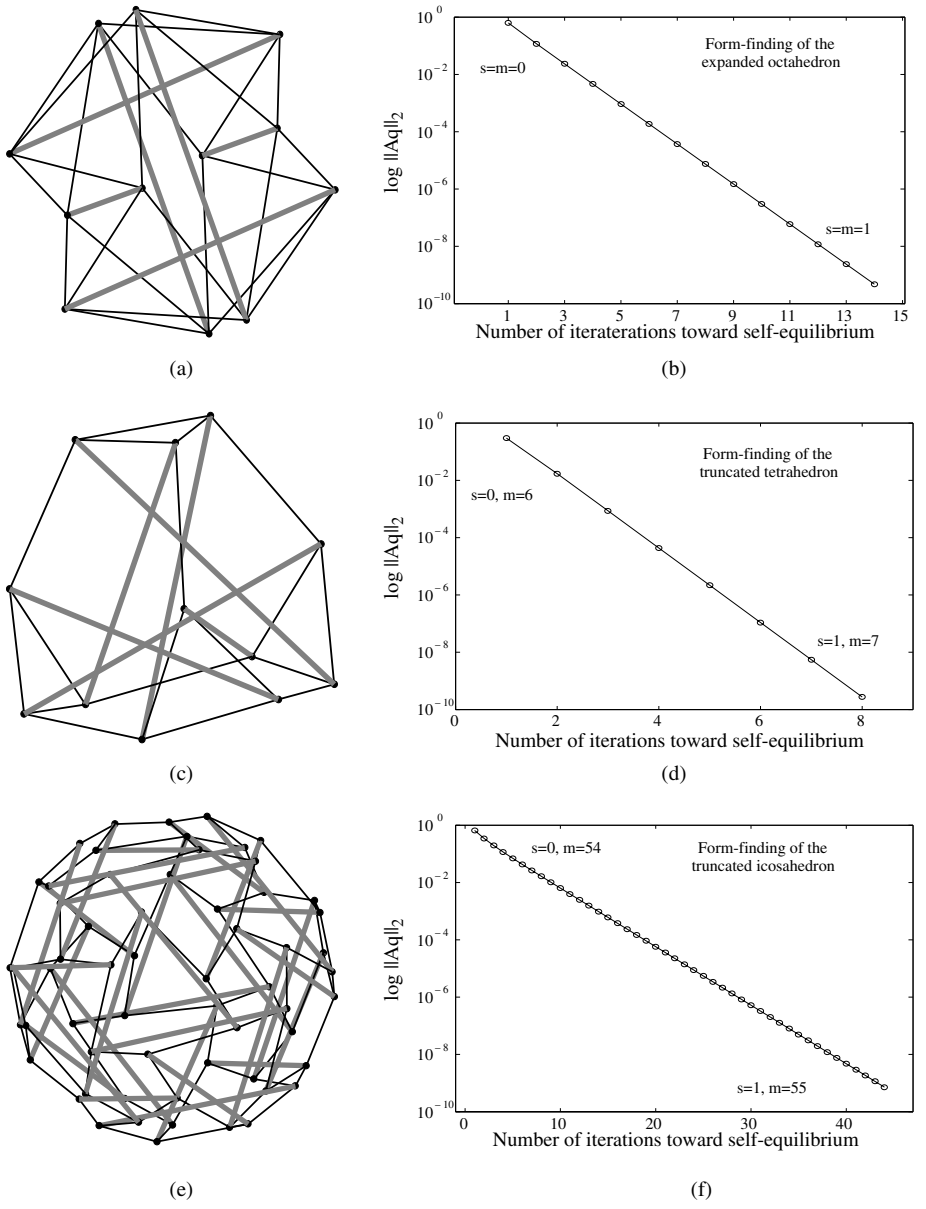

Fig. 3. Examples of tensegrities calculated with the proposed form-finding procedure: (a) expanded octahedron and (b) convergence; (c) truncated tetrahedron and (d) convergence; (e) truncated icosahedron and (f) convergence. Thick gray lines represent elements in compression.

We can calculate the elastic potential energy of a tensegrity structure by describing its members as Hookean springs that store energy upon experiencing a strain $\boldsymbol{\varepsilon}$. Let $\mathbf{e} \in \mathfrak{R}^{\mathrm{b}}$, $\boldsymbol{\varepsilon} \in \mathfrak{R}^{\mathrm{b}}$ and $\mathbf{a} \in \mathfrak{R}^{\mathrm{b}}$ be the vectors containing Young's modulus, the strain and the cross-sectional area per member, respectively. Then, if we define the stress and length per member i as $\sigma_i = e_i \varepsilon_i$ and l_i, respectively, the elastic energy per unit volume of the structure is given by

$$U = \frac{1}{2}\sum_{i=1}^{\mathrm{b}} \sigma_i \varepsilon_i = \frac{1}{2}\sum_{i=1}^{\mathrm{b}} f_i^2/e_i a_i^2 = \frac{1}{2}\sum_{i=1}^{\mathrm{b}} l_i^2 q_i^2/e_i a_i^2.$$

If we compare two or more structures with the same value of **ea**, the difference in the elastic potential energy density is $U \propto \sum_{i=1}^{b} l_i^2 q_i^2$. We compared the structure calculated by Tibert and Pellegrino (2003), Pellegrino (1986), Motro (2003) with the one found by us. The solution with a single cable length was calculated with the non-linear programming approach (Pellegrino, 1986). For the comparison, we assumed that both structures have the same value for **ea**, rescaled both structures to fit inside a box of unit lengths, and normalised their tension coefficients. We found

- 12 cables with $q = 1$, 6 cables with $q = 1.3794$ and 6 struts with $q = -0.66714$ in the structure calculated by Tibert and Pellegrino (2003).
- 12 cables with $q = 1$, 6 cables with $q = 1.1546$ and 6 struts with $q = -0.6170$ in our structure.

The calculation of the elastic energy density shows that the truncated tetrahedron calculated with a single cable length has an energy 20% *higher* than the solution with two cable lengths. In terms of equilibrium and stability the configuration with two cable lengths is therefore advantageous. The structure with only one cable length possessing longer struts and larger tension coefficients is suppressed by our form-finding procedure in favour of the low-energy structure with two different cable lengths.

The form-finding procedure takes eight iterations to start from a configuration with $\mathsf{s} = 0$ and $\mathsf{m} = 6$ to converge towards the equilibrium configuration with $\mathsf{s} = 1$ and $\mathsf{m} = 7$ (Fig. 3c). Fig. 3d shows the convergence in a logarithmic scale.

Moreover, the state of self-stress imparts positive stiffness to every infinitesimal mechanism of this structure, except for the six rigid-body displacements. This demonstrates that the proposed form-finding procedure is capable of selecting a configuration with minimal elastic potential energy.

9.3. Truncated icosahedron

Murakami and Nishimura (2001) have analysed the truncated icosahedron with a rather complicated form-finding procedure based on group theory. Their form-finding takes advantage of the symmetry and is developed for the truncated versions of the five Platonic solids.

After 45 iterations with our form-finding procedure, we calculate $\mathsf{s} = 1$ and $\mathsf{m} = 55$, which is in agreement with their assumptions (Murakami and Nishimura, 2001). Figs. 3e and f show a truncated icosahedron and its convergence to $\mathsf{s} = 1$. By normalising the tension coefficients with respect to a reference cable, we have 60 cables with $q = 1$, 30 cables with $q = 0.6775$ and 30 struts with $q = -0.3285$. This solution provides positive stiffness to all mechanisms, see Eq. (14), except for the six rigid-body displacements.

9.4. Cylindrical tensegrities

Finally, the ability of the proposed procedure to find structures that correspond to global energy minima is further exemplified with cylindrical tensegrities. They are described by a set of two parallel planes, "top" and "bottom" (see Gomez Estrada et al., 2005). Here, each plane consists of n nodal points, and both planes are connected by n brace cables and n struts. Two well known examples are the triplex and the quadruplex for which $n = 3$ and $n = 4$, respectively. The form-finding of cylindrical tensegrities has been studied using several approaches, for instance, the dynamic relaxation procedure by Motro (2003) and non-linear programming by Pellegrino (1986). But these approaches impose the condition of a unique cable length.

We calculated cylindrical tensegrities with the proposed form-finding procedure. For instance, the triplex and the quadruplex are shown in Figs. 4a and b, respectively. The pentaplex, or 5-plex for short, is shown in Fig. 4c. Finally, we show three examples (Figs. 4d–f) of tensegrity cylinders with a high degree of elements, such as the 20-plex, 50-plex, and 100-plex.

An analysis of their stability shows that the structures possess positive stiffness with the given state of self-stress. Moreover, the solutions have *two* cable lengths and one strut length, which is in stark contrast to the above-mentioned studies. The form-finding procedure is able to selectively calculate the stable configuration among several sub-optimal configurations (Gomez Estrada et al., 2005).

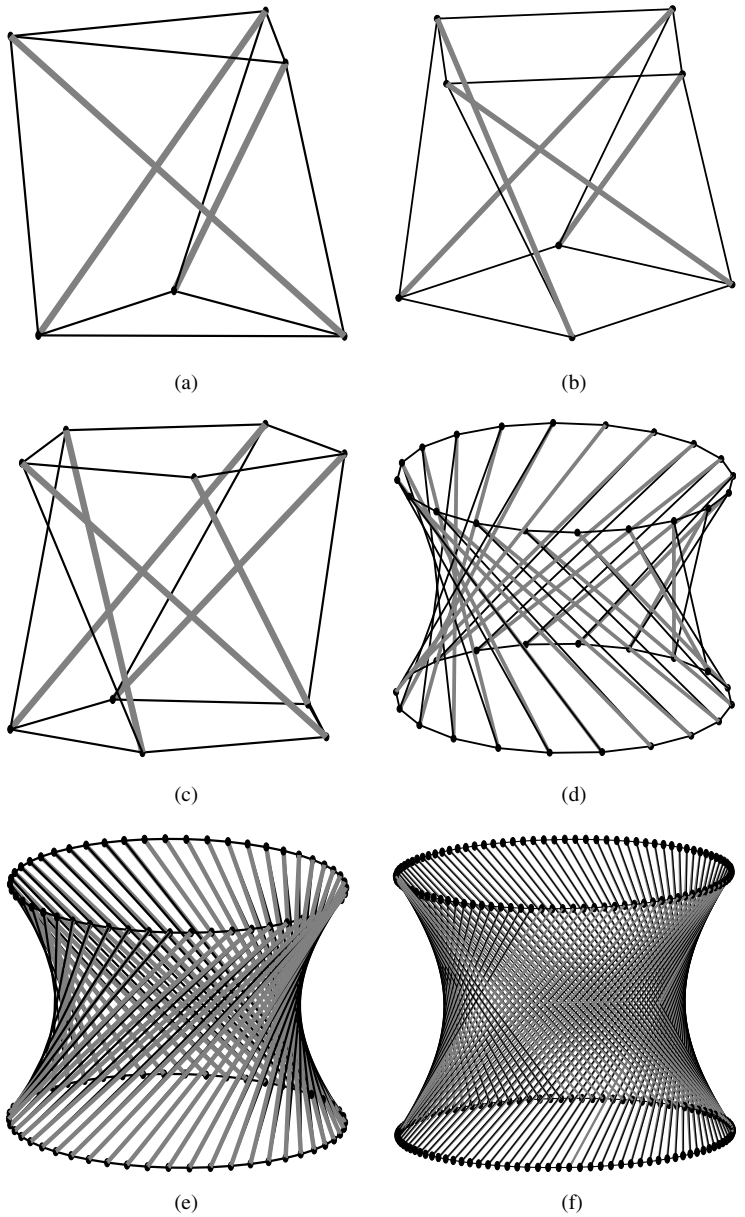

Fig. 4. Examples of cylindrical tensegrities: (a) $n = 3$, the triplex; (b) $n = 4$, the quadruplex; (c) $n = 5$; (d) $n = 20$; (e) $n = 50$; (f) $n = 100$. Thick gray lines represent elements in compression.

10. Conclusions

We have presented a multi-parameter form-finding procedure for tensegrity structures. Geometry and tension coefficients are iteratively computed from an incidence matrix and a vector of prototypes of member forces. The elements of this vector consist of unitary entries $+1$ and -1 for members in tension and compression, respectively. The conditions of a maximal rank of the force density matrix and minimal member lengths, were included in the form-finding procedure to guide the search of a state of self-stress.

As opposed to most existing form-finding procedures, our procedure requires neither the nodal coordinates, the symmetry, the element lengths, nor the tension coefficients to be imposed a priori. This lack of assumptions leads to a versatile form-finding procedure that can calculate, for instance, (i) the expanded octahedron with-

out assuming any tension coefficients, (ii) the truncated tetrahedron without assuming any cable lengths, (iii) the truncated icosahedron without assuming any global symmetry, and (iv) the cylindrical tensegrities without assuming any cable lengths. We are indeed able to calculate novel configurations.

We showed that our numerical form-finding, (i) reproduces solutions obtained with techniques based on symbolic calculations, non-linear programming, dynamic relaxation and group theory, (ii) selects configurations of minimal elastic potential energy, as well as (iii) reduces the knowledge needed in the design process to $+1/-1$ depending on whether the member is in tension or compression. Previous and new structures are therefore easily handled with our proposed numerical procedure.

Moreover, the proposed form-finding procedure does not require tuning variables nor thresholds. However, it remains to study in greater detail structures with multiple state of self-stress. Finally, the application of our form-finding procedure to topology optimization problems is regarded as an interesting research direction.

Acknowledgements

The first author wishes to thank Simon Guest, Thomas Huckle and Gunnar Tibert who offered helpful comments on previous versions of this work.

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