Measures of Non-compactness of Operators in Banach Lattices

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Let $E$ and $F$ be complex Banach lattices. Then a measure of non-compactness $\rho(T)$ is introduced for an order bounded operator $T$ from $E$ into $F$. If $E^*$ and $F$ have order continuous norm then for every $AM$-compact operator $\rho(T) = \beta(T)$, where $\beta(T)$ denotes the ball measure of non-compactness of $T$. From this result monotonicity properties of $\beta(T)$ and the essential spectral radius $r_{\text{ess}}(T)$ are derived for $AM$-compact operators. Also shown is that $r_{\text{ess}}(T) \leq \sigma_{\text{ess}}(T)$ for positive $AM$-compact operators. In addition properties of the essential spectrum of norm bounded disjointness preserving operators are proved.

INTRODUCTION

In the present paper we study measures of non-compactness of operators in Banach lattices and their applications to the essential spectrum of such operators. There are many quantities related to bounded linear operators in Banach spaces which have been considered in the literature (see, e.g., [14, 22, 24, 27, 31]), but in this paper we will be concerned mainly with the ball measure of non-compactness of an operator $T$, which will be denoted by $\beta(T)$ (see Section 2 for the definition). We recall that for a

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bounded operator $T$ in a Banach space the relation of $\beta$ to the essential spectral radius $r_{\text{ess}}(T)$ is given by the formula (Nussbaum) $r_{\text{ess}}(T) = \lim_{n \to \infty} \|\beta(T^n)\|^{1/n}$. There are different notions of essential spectrum of an operator, but all of them have the same radius (see, e.g., [9, Sect. 53; 5, Sect. 9.8]). In this paper we define $\sigma_{\text{ess}}(T)$ to be the complement of the Fredholm domain of $T$, i.e., the spectrum of the canonical image of $T$ in the Calkin-algebra (this is sometimes called the Wolf essential spectrum).

There are three questions which play a central role: For which operators and for which Banach lattices $E$ is it true that (a) $\beta$ is monotone, i.e., $0 \leq S \leq T$ in $\mathcal{L}(E)$ implies $\beta(S) \leq \beta(T)$; (b) $r_{\text{ess}}$ is monotone, i.e., $0 \leq S \leq T$ in $\mathcal{L}(E)$ implies $r_{\text{ess}}(S) \leq r_{\text{ess}}(T)$; (c) $r_{\text{ess}}(T)$ belongs to $\sigma_{\text{ess}}(T)$? As is well known, if $T$ is a positive operator in a Banach lattice, then $r(T) \in \sigma(T)$, and the monotonicity of the spectral radius is evident. Furthermore, in connection with (a) we mention that, by a theorem of P. G. Dodds and D. H. Fremlin [6] (see also [33, Theorem 124.10]), if $E$ and $E^*$ have order continuous norms, then $0 \leq S \leq T$ in $\mathcal{L}(E)$ with $T$ compact implies that $S$ is compact; i.e., in this situation $\beta(T) = 0$ implies that $\beta(S) = 0$. As we will see, the measure of non-compactness is monotone for certain classes of operators, e.g., integral operators in Banach function spaces, but in general it is not monotone.

In Section 2 we introduce another measure of non-compactness, the so-called measure of non-semicompactness $\rho(T)$, for order bounded operators between Banach lattices. It turns out that $\rho(T)$ is a useful quantity in the study of $\beta(T)$ and $r_{\text{ess}}(T)$ for such operators. In fact, for $AM$-compact operators (in particular, absolute integral operators) we show that $\beta(T) = \rho(T)$. Furthermore, we derive formulas for $\rho(T)$, which can be used for the actual computation of the measure of non-compactness of certain integral operators.

In Section 3 we will use $\rho(T)$ for the investigation of $r_{\text{ess}}(T)$ and $\sigma_{\text{ess}}(T)$. In particular, we show that questions (b) and (c) have affirmative answers for $AM$-compact operators. We end the paper with some results on disjointness preserving operators (e.g., weighted composition operators). Among other things, we will show that if $T$ is a norm bounded disjointness preserving operator in a Banach lattice with non-atomic dual space, then $\beta(T) \geq 1/2 \|T\|$, $r_{\text{ess}}(T) = r(T)$, and $r(T) \in \sigma_{\text{ess}}(T)$.

Our results extend results of L. Weis, who studied measures of noncompactness and related quantities of operators on $L_p$-spaces in [31].

1. Preliminaries

We assume that the reader has some familiarity with the terminology and theory of Banach lattices, as can be found in the books [19, 26, 33].
All Banach lattices $E$ considered are assumed to be complex, i.e., $E = \text{Re } E \oplus i \text{Re } E$, the complexification of a real Banach lattice $\text{Re } E$. The absolute value in $\text{Re } E$ is extended to $E$ by means of the formula $|z| = \sup\{(\cos \theta) x + (\sin \theta) y: 0 \leq \theta \leq 2\pi\}$, where $z = x + iy$ with $x, y \in \text{Re } E$ (see, e.g., [33, Sect. 91]), and the norm in $E$ satisfies $||z|| = |||z||$ for all $z \in E$. For any $0 \leq u \in E$ we will denote $[-u, u]_C = \{z \in E: |z| \leq u\}$.

Given $f \in E$, the principal ideal $E_f = \{z \in E: |x| \leq n |f|\}$ for some $n \in \mathbb{N}$ is, by the Yosida representation theorem ([19, Theorem 45.4]), isomorphic to a space $C(K)$ of all complex valued continuous functions on a compact Hausdorff space $K$, such that the element $|f|$ corresponds to the function $1$. The multiplication by $f$ in $C(K)$ now induces a corresponding operator in $E_f$, which will be denoted by $\sigma_f$. Then $\sigma_f(|f|) = f$ and $|\sigma_f(g)| = |g|$ for all $g \in E_f$. Note that $\sigma_f \in Z(E_f)$, the center of $E_f$, and $|\sigma_f| = I$ (for the definition of the center $Z(E_f)$ see [33, Chap. 20] or [17]). In general, $\sigma_f$ cannot be extended to the whole space $E$. If $E$ is Dedekind complete, however, then $\sigma_f$ can be extended to an element of the center $Z(E)$ of $E$ (with absolute value equal to $I$).

2. THE MEASURE OF NON-SEMICOMPACTNESS

In this section we study a quantity for order bounded operators on Banach lattices, which turns out to be useful in the study of measures of non-compactness of positive operators. As before, let $E = \text{Re } E \oplus i \text{Re } E$ be a complex Banach lattice.

Recall that the subset $D$ of $E$ is called almost order bounded if for every $\varepsilon > 0$ there exists $0 \leq u \in E$ such that $\|(f - u)^+\| \leq \varepsilon$ for all $f \in D$ (see [33, Sect. 122]). It should be observed that $\|(f - u)^+\| \leq \varepsilon$ for all $f \in D$ if and only if $D \subseteq [-u, u]_C + \varepsilon B_E$. Indeed, if $D \subseteq [-u, u]_C + \varepsilon B_E$ and $f \in D$, then $f = f_1 + f_2$ with $|f_1| \leq u$ and $\|f_2\| \leq \varepsilon$, so $\|(f - u)^+\| \leq \|(f_1 + |f_2| - u)^+\| \leq \|(f_1 - u)^+\| + \|f_2\| \leq \varepsilon$. Conversely, assume that $\|(f - u)^+\| \leq \varepsilon$ for all $f \in D$, and take $f \in D$. Let $\sigma_f \in Z(E_f)$ be such that $|\sigma_f| = I$ and $\sigma_f(|f|) = f$. Then $f = \sigma_f(|f|) = \sigma_f(|f| \wedge u) + \sigma_f(|f - u|^+)$, with $|\sigma_f(|f| \wedge u)| \leq |f \wedge u| \leq u$, so $\sigma_f(|f| \wedge u) \in [-u, u]_C$, and $\sigma_f(|f| - u)^+ \| \leq \|(|f| - u)^+\| \leq \varepsilon$, so $\sigma_f(|f| - u)^+ \in \varepsilon B_E$. For a norm bounded set $D \subseteq E$ we define

$$\rho(D) = \inf\{\delta > 0: \exists 0 \leq u \in E \text{ such that } \|(f - u)^+\| \leq \delta \text{ for all } f \in D\}.$$ 

From the above observations it is clear that

$$\rho(D) = \inf\{\delta > 0: \exists 0 \leq u \in E \text{ such that } D \subseteq [-u, u]_C + \delta B_E\}.$$
We list some simple properties of $\rho$. If $D, D_1, D_2$ are norm bounded subsets of $E$, then

(i) $\rho(D) = 0$ if and only if $D$ is almost order bounded;
(ii) $\rho(D_1 + D_2) \leq \rho(D_1) + \rho(D_2)$ and $\rho(\lambda D) = |\lambda| \rho(D)$ for all $\lambda \in \mathbb{C}$;
(iii) if $D_1 \subseteq D_2$, then $\rho(D_1) \leq \rho(D_2)$;
(iv) $\rho(\overline{D}) = \rho(D)$, where $\overline{D}$ denotes the norm closure of $D$.

Remark. Let $E$ be a Banach lattice and, as before, $B_E = \{ f \in E: \| f \| \leq 1 \}$. We assert that $\rho(B_E) < 1$ if and only if there exists an order unit $0 \leq u \in E$ and the norm in $E$ is equivalent to the order unit norm $\| \cdot \|_u$. Indeed, suppose $\rho(B_E) < \delta < 1$, then there exists $0 \leq u \in E$ such that $B_E \subseteq [-u, u]_C + \delta B_E$. Now it follows that $B_E \subseteq [-u, u]_C + \delta[-u, u]_C + \ldots + \delta^n[-u, u]_C + \delta^{n+1}B_E$, and hence

$$B_E \subseteq (1 - \delta)^{-1}[-u, u]_C + \delta^{n+1}B_E.$$ 

Letting $n \to \infty$ we get $B_E \subseteq (1 - \delta)^{-1}[-u, u]_C$, and the result follows. We note that thus $\rho(B_E) = 0$ or $1$.

Recall that for a norm bounded subset $D$ of $E$ the ball measure of non-compactness is defined by

$$\beta(D) = \inf \left\{ \delta > 0: \exists f_1, \ldots, f_n \in E \text{ such that } D \subseteq \bigcup_{j=1}^{n} B(f_j, \delta) \right\}$$

(see, e.g., [5, Sect. 7.3]). Since $D \subseteq \bigcup_{j=1}^{n} B(f_j, \delta)$ implies that $\|(f - u)^+\| \leq \delta$ for all $f \in D$, with $u = \sup(|f_1|, \ldots, |f_n|)$, it is clear that $\rho(D) \leq \beta(D)$. Next we will show that for a certain class of subsets $D$ of a Banach lattice $E$ the equality $\rho(D) = \beta(D)$ holds. For any $0 \leq \phi \in E^*$ we define the Riesz seminorm $p_\phi$ on $E$ by $p_\phi(f) = \langle |f|, \phi \rangle$. Furthermore, for $f \in E$ and $\varepsilon > 0$ we denote $B_\phi(f, \varepsilon) = \{ g \in E: p_\phi(f - g) \leq \varepsilon \}$. The set $D \subseteq E$ is called PL-compact if for every $0 \leq \phi \in E^*$ and every $\varepsilon > 0$ there exist $f_1, \ldots, f_n \in E$ such that

$$D \subseteq \bigcup_{j=1}^{n} B_\phi(f_j, \varepsilon)$$

(see [6, Definition 4.1]). Observe that $D$ is PL-compact if and only if $D$ is $p_\phi$-precompact for every $0 \leq \phi \in E^*$ ([33, Definition 124.7]).

**Proposition 2.1.** Let $E$ be a complex Banach lattice with order continuous norm and let $D$ be a PL-compact subset of $E$. Then $\beta(D) = \rho(D)$.

**Proof.** Since $E$ is Dedekind complete, there exists for every $f \in E$ a
unique \( \sigma_f \in Z(E) \) such that \( \sigma_f(\{|f|\}) = f \) and \( |\sigma_f| = I \). First observe that, for \( f, g \in E \) and \( 0 \leq u \leq E \) we have

\[
|\sigma_f(\{|f| \wedge u\}) - \sigma_g(\{|g| \wedge u\}| \leq |f - g|.
\]

To prove this inequality, represent the principal ideal in \( E \) generated by \( \{f\} + \{|g|\} \) by way of the Yosida representation theorem, as the space of complex continuous functions on a compact Hausdorff space. Now the above inequality is an immediate consequence of the inequality

\[
|\text{sgn } z| \min(|z|, a) - |\text{sgn } w| \min(|w|, a)| \leq |z - w|
\]

for \( z, w \in C \) and \( 0 \leq a \leq R \), where \( \text{sgn } z = z |z|^{-1} \) if \( z \neq 0 \) and \( \text{sgn}(0) = 1 \).

For the proof of the proposition, take \( \delta > \rho(D) \). Then there exists \( 0 \leq u \in E \) such that \( \|\{f| - u\}^+\| \leq \delta \) for all \( f \in D \). Take \( \varepsilon > 0 \). Since \( E \) has order continuous norm, there exists \( 0 \leq \phi \in E^* \) such that 

\[
\langle u, (|\psi| - \phi)^+ \rangle \leq \varepsilon \text{ for all } \psi \in B_{E^*} \text{ (see [33, Theorem 152.2]).}
\]

Since \( D \) is PL-compact, there exist \( f_1, ..., f_n \in E \) such that \( D \subseteq \bigcup_{j=1}^n B_{E^*}(f_j, \varepsilon) \). Let \( g_j = \sigma_f(\{|f| \wedge u\}) \). Take \( f \in D \) and \( f_j \) such that \( \langle |f - f_j|, \phi \rangle \leq \varepsilon \). For \( 0 \leq \psi \in B_{E^*} \) we then have

\[
\langle |f - g_j|, \psi \rangle \leq \langle |\sigma_f(\{|f| \wedge u\}) - g_j, \psi \rangle + \langle |\sigma_f(\{|f| \wedge u\}) - g_j, \psi \rangle
\]

\[
\leq \|\{f| - u\}^+\| + \langle |\sigma_f(\{|f| \wedge u\}) - g_j, \psi \rangle
\]

\[
\leq \delta + \langle |\sigma_f(\{|f| \wedge u\}) - g_j, \psi \rangle.
\]

Furthermore, using the above inequality, we find

\[
\langle |\sigma_f(\{|f| \wedge u\}) - g_j, \psi \rangle
\]

\[
\leq \langle |\sigma_f(\{|f| \wedge u\}) - g_j, \psi \wedge \phi \rangle + \langle |\sigma_f(\{|f| \wedge u\}) - g_j, (|\psi| - \phi)^+ \rangle
\]

\[
\leq \langle |f - f_j|, \phi \rangle + 2 \langle u, (|\psi| - \phi)^+ \rangle \leq 3\varepsilon.
\]

Hence \( \langle |f - g_j|, \psi \rangle \leq \delta + 3\varepsilon \) for all \( 0 \leq \psi \in B_{E^*} \), so \( \|f - g_j\| \leq \delta + 3\varepsilon \), which shows that

\[
D \subseteq \bigcup_{j=1}^n B(g_j, \delta + 3\varepsilon).
\]

Since this holds for all \( \varepsilon > 0 \) and all \( \delta > \rho(D) \), we conclude that \( \beta(D) \leq \rho(D) \). As observed before, \( \rho(D) \leq \beta(D) \) always holds, and so the proposition is completely proved.

Now suppose that \( E \) and \( F \) are complex Banach lattices. The space of order bounded linear operators from \( E \) into \( F \) will be denoted by \( \mathcal{L}_b(E, F) \), which is a linear subspace of the space \( \mathcal{L}(E, F) \) of all norm bounded linear operators.
operators. Recall that the operator $T \in \mathcal{L}_b(E, F)$ is called semi-compact if $T$ maps norm bounded sets onto almost order bounded sets (see [33, Sect. 1251]).

**Definition 2.2.** For $T \in \mathcal{L}_b(E, F)$ the measure of non-semicompactness is defined by

$$\rho(T) = \inf\{k \geq 0: \rho(TD) \leq k\rho(D) \text{ for all norm bounded } D \subseteq E\}.$$ 

We list some simple properties of $\rho$. If $T \in \mathcal{L}_b(E, F)$, then

1. $\rho(TD) \leq \rho(T) \rho(D)$ for all norm bounded $D \subseteq E$;
2. $\rho(T) = 0$ if and only if $T$ is semi-compact;
3. $\rho(T) \leq \|T\|$;
4. $\rho$ is a semi-norm on $\mathcal{L}_b(E, F)$;
5. $\rho(TB_E) = \rho(T)$;
6. if $0 < s < \rho(T)$, then $\rho(T) < \rho(TB_E)$.

We indicate the proof of (v) and (vi). Since $\rho(TB_E) \leq \rho(T) \rho(B_E)$, it is clear that $\rho(TB_E) \leq \rho(T)$. Now take a norm bounded set $D \subseteq E$ and $\delta > \rho(D)$. Then there exists $0 \leq y \in E$ such that $D \subseteq [u, u]_c + \delta B_E$, so that $TD \subseteq T[-u, u]_c + \delta TB_E$. Since $T$ is order bounded, $\rho(T[-u, u]_c) = 0$, hence $\rho(TD) \leq \delta \rho(TB_E)$. This holds for all $\delta > \rho(D)$, so $\rho(TD) \leq \rho(TB_E) \rho(D)$, which shows that $\rho(T) \leq \rho(TB_E)$.

For the proof of (vi), suppose that $0 \leq S \leq T: E \to F$ and take $\delta > \rho(T)$. Then there exists $0 \leq w \in F$ such that $\|(T|f| - w)^+\| \leq \delta$ for all $f \in B_E$. Now it follows from $\|(S|f| - w)^+\| \leq \|(T|f| - w)^+\|$ for all $f \in B_E$, that $\rho(S) = \rho(SB_E) \leq \delta$, and hence $\rho(S) \leq \rho(T)$.

For any $T \in \mathcal{L}(E, F)$ we denote by $\beta(T)$ the ball measure of noncompactness of $T$ (see, e.g., [5, Sect. 9.7]), i.e.,

$$\beta(T) = \inf\{k \geq 0: \beta(TD) \leq k\beta(D) \text{ for all norm bounded } D \subseteq E\}.$$ 

Since $\beta(T) = \beta(TB_E)$, it is evident that $\rho(T) \leq \beta(T)$ for all $T \in \mathcal{L}_b(E, F)$. We will show that $\rho(T) = \beta(T)$ holds for some important classes of operators. First, however, we will derive some formulas for $\rho(T)$, which are useful for the computation of $\rho(T)$ (cf. Example 2.8).

**Theorem 2.3.** Let $E$ and $F$ be complex Banach lattices such that $E^*$ and $F$ have order continuous norms. For any $T \in \mathcal{L}_b(E, F)$ we have
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\[ \rho(T) = \sup \left\{ \lim_{n \to \infty} \| T_{f_n} \| : \{ f_n \}_{n=1}^{\infty} \text{ disjoint sequence in } B_E \right\} \]

\[ = \sup \left\{ \lim_{n \to \infty} \| P_n T \| : \{ P_n \}_{n=1}^{\infty} \text{ band projections in } F, P_n \downarrow 0 \right\} \]

\[ = \sup \left\{ \lim_{n \to \infty} \langle T_{f_n}, \phi_n \rangle : \{ f_n \}_{n=1}^{\infty} \text{ and } \{ \phi_n \}_{n=1}^{\infty} \text{ disjoint sequences in } B_E \text{ and } B_{F^*}, \text{ respectively} \right\}. \]

Moreover, if \( E \) has in addition the principal projection property, then

\[ \rho(T) = \sup \left\{ \lim_{n \to \infty} \| P_n T Q_n \| : \{ P_n \}_{n=1}^{\infty} \text{ and } \{ Q_n \}_{n=1}^{\infty} \text{ band projections in } F \text{ and } E, \text{ respectively, } P_n \downarrow 0 \text{ and } Q_n \downarrow 0 \right\}. \]

The proof of the above theorem is patterned on the proofs of Theorems 127.4 and 128.3 in [33], but there are many differences. Before proving the above result we recall the following facts which will be used. If \( E \) is a (complex) Banach lattice, then

(i) \( E \) has order continuous norm if and only if for every \( 0 \leq u \in E \) and for every \( \varepsilon > 0 \) there exists \( 0 \leq \phi \in E^* \) such that \( \langle u, (|\psi| - \phi)^+ \rangle \leq \varepsilon \) for all \( \psi \in B_{E^*} \) [33, Theorem 125.2];

(ii) \( E^* \) has order continuous norm if and only if for every \( 0 \leq \phi \in E^* \) for every \( \varepsilon > 0 \) there exists \( 0 \leq u \in E \) such that \( \langle (f - u)^+, \phi \rangle \leq \varepsilon \) for all \( f \in B_E \) [33, Theorem 125.1];

(iii) \( E^* \) has order continuous norm if and only if every norm bounded disjoint sequence in \( E \) converges weakly to zero [33, Theorem 116.1].

Furthermore, we will use the following result, which was inspired by Lemma 4.4 of [21].

**Lemma 2.4.** Let \( E \) be a Banach lattice and suppose \( 0 \leq u_n \in E(n = 1, 2, \ldots) \) such that \( u_n \to 0 \) weakly. Then there exists a subsequence \( \{ u_{n_k} \}_{k=1}^{\infty} \) in \( B_E^+ \) such that \( \lim_{k \to \infty} \langle u_{n_k}, \phi_k \rangle = \lim_{n \to \infty} \| u_n \| \).

**Proof.** For \( n = 1, 2, \ldots \) there exists \( 0 \leq \psi_n \in B_{E^*} \) such that \( \langle u_n, \psi_n \rangle \geq (1 - 2^{-n}) \| u_n \| \). We define the sequence \( \{ n_k \}_{k=1}^{\infty} \) inductively as follows. Take \( n_1 = 1 \). Now assume that \( n_1 < n_2 < \cdots < n_{k-1} \) have been defined. Since \( u_n \to 0 \) weakly, there exists \( n_k > n_{k-1} \) such that \( \langle u_{n_k}, 2^k (\psi_{n_1} + \cdots + \psi_{n_{k-1}}) \rangle < 2^{-k} \) and \( \| u_{n_k} \| \geq (1 - 2^{-k}) \lim_{n \to \infty} \| u_n \| \).

Now define

\[ \phi_k = \left( \psi_{n_k} - 2^k \sum_{j=1}^{k-1} \psi_{n_j} - \sum_{j=k+1}^{\infty} 2^{-j} \psi_{n_j} \right)^+. \]
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Since \( 0 \leq \phi_k \leq \psi_{n_k} \) and \( \psi_{n_k} \in B_{F^*} \), it is clear that \( \phi_k \in B_{F^*} \). Furthermore, if \( k < m \), then \( 0 \leq \phi_k \leq (\psi_{n_k} - 2^{-m} \psi_{n_m})^+ \) and \( 0 \leq \phi_m \leq (\psi_{n_m} - 2^{-m} \psi_{n_m})^+ = 2^{-m}(\psi_{n_k} - 2^{-m} \psi_{n_m})^+ \), so \( \{ \phi_k \}_{k=1}^{n} \) is a disjoint sequence. Moreover,

\[
\langle u_{n_k}, \phi_k \rangle \geq \langle u_{n_k}, \psi_{n_k} \rangle - \left( u_{n_k}, 2^k \sum_{j=1}^{k-1} \psi_{n_j} \right) - \left( u_{n_k}, \sum_{j=k+1}^{\infty} 2^{-j} \psi_{n_j} \right) \\
\geq (1 - 2^{-n_k}) \| u_{n_k} \| - 2^{-k+1} \geq (1 - 2^{-k})^2 \lim_{n \to \infty} \| u_{n_k} \| - 2^{-k+1},
\]

which shows that \( \lim_{k \to \infty} \langle u_{n_k}, \phi_k \rangle \geq \lim_{n \to \infty} \| u_{n_k} \| \). The converse inequality being trivial, the lemma is proved.

**Proof of Theorem 2.3.** For the sake of convenience, we call the suprema on the right hand side \( A, B, C, \) and \( D \), respectively. We show first that \( \rho(T) = A = B = C \). It is clear that \( C \leq A \).

**Claim.** \( C \leq B \). Indeed, suppose that \( \{ f_n \}_{n=1}^{\infty}, \{ \phi_n \}_{n=1}^{\infty} \subseteq B_{E^*} \) are disjoint sequences. Let \( N(\phi_n) = \{ g \in F : \langle |g|, \phi_n \rangle = 0 \} \), the null ideal of \( \phi_n \), and \( C(\phi_n) = N(\phi_n)^\perp \), the carrier of \( \phi_n \) (see [33, Sect. 90]). Then \( F = N(\phi_n) \oplus C(\phi_n) \) and \( C(\phi_n) \perp C(\phi_m) \) for \( n \neq m \) (note that \( F^* = F_{E^*} \), as \( F \) has order continuous norm). Let \( R_n \) be the band projection in \( F \) onto \( C(\phi_n) \), and let \( P_n = \bigvee_{k=m}^n R_k \) \((n = 1, 2, \ldots) \). Now it is easy to see that \( P_n \downarrow 0 \), and since \( |\langle T_n f_n, \phi_n \rangle| = |\langle P_n T f_n, \phi_n \rangle| \leq \| P_n T \| \), we may conclude that \( C \leq B \).

**Claim.** \( A \leq \rho(T) \). Let \( \{ f_n \}_{n=1}^{\infty} \) be a disjoint sequence in \( B_E \) and take \( \delta > \rho(T) \). Then there exists \( 0 \leq w \in F \) such that \( \| (|T f_n| - w)^+ \| \leq \delta \) for all \( f \in B_E \). Take \( \epsilon > 0 \). Since \( F \) has order continuous norm, there exists \( 0 \leq \phi \in F^* \) such that \( \langle w, (|\psi| - \phi)^+ \rangle \leq \epsilon \) for all \( \psi \in B_{E^*} \). Now, if \( \psi \in B_{E^*} \), then

\[
\langle T f_n, \psi \rangle \leq \langle |T f_n|, \psi \rangle = \langle (|T f_n| - w)^+, \psi \rangle + \langle |T f_n| \wedge w, (|\psi| - \phi)^+ \rangle + \langle |T f_n| \wedge w, \psi \wedge \phi \rangle \\
\leq \delta + \langle w, (|\psi| - \phi)^+ \rangle + \langle |T| (|f_n|), \phi \rangle \\
\leq \delta + \epsilon + \langle |T| (|f_n|), \phi \rangle,
\]

and therefore \( \| T f_n \| \leq \delta + \epsilon + \langle |T| (|f_n|), \phi \rangle \) for all \( n = 1, 2, \ldots \). It follows from the order continuity of the norm in \( E^* \), that \( |f_n| \to 0 \) (weakly), and hence \( |T| (|f_n|) \to 0 \) (weakly) in \( F \), so in particular \( \lim_{n \to \infty} |\langle T| (|f_n|), \phi \rangle| = 0 \) as \( n \to \infty \), and therefore \( \lim_{n \to \infty} \| T f_n \| \leq \delta + \epsilon \). This holds for all \( \epsilon > 0 \) and all \( \delta > \rho(T) \), so \( \lim_{n \to \infty} \| T f_n \| \leq \rho(T) \), which shows that \( A \leq \rho(T) \).

**Claim.** \( B \leq \rho(T) \). Let \( \{ P_n \}_{n=1}^{\infty} \) be a sequence of band projections in \( F \) such that \( P_n \downarrow 0 \). Take \( \delta > \rho(T) \) and \( \epsilon > 0 \). Then there exists \( 0 \leq w \in F \) such
that $\|(|Tf| - w)^+\| \leq \delta$ for all $f \in B_E$. Since $F$ has order continuous norm, there exists $0 \leq \phi \in F^*$ such that $\langle w, (|\psi| - \phi)^+ \rangle \leq \varepsilon$ for all $\psi \in B_{F^*}$. For $f \in B_E$ and $\psi \in B_{F^*}$ we have

$$\langle P_n T f, \psi \rangle \leq \langle P_n |Tf|, \psi \rangle = \langle P_n (|Tf| - w)^+, \psi \rangle + \langle P_n (|Tf| \wedge w), (|\psi| - \phi)^+ \rangle + \langle P_n (|Tf| \wedge w), \psi \wedge \phi \rangle \leq \delta + \varepsilon + \langle P_n w, \phi \rangle,$$

and hence $\|P_n T\| \leq \delta + \varepsilon + \langle P_n w, \phi \rangle$ for all $n = 1, 2, \ldots$. Since $F$ has order continuous norm, $P_n \downarrow 0$ implies that $\langle P_n w, \phi \rangle \downarrow 0$ as $n \to \infty$, and so $\lim_{n \to \infty} \|P_n T\| \leq \delta + \varepsilon$. This holds for all $\varepsilon > 0$ and all $\delta > \rho(T)$, so $\lim_{n \to \infty} \|P_n T\| \leq \rho(T)$, and hence $B \not\subset \rho(T)$.

Claim. $\rho(T) \leq C$. We may assume that $\rho(T) > 0$. As before, for $f \in E$ we denote by $\sigma_f$ the unique element in $Z(E_f)$ with $\sigma_f(|f|) = f$ and $|\sigma_f| = I$. If $f \in E$ and $0 \leq u \in E$, then

$$Tf = T[\sigma_f(|f| \wedge u)] + T[\sigma_f(|f| - u)^+]$$

and so

$$\langle |Tf| - |T| u \rangle^+ \leq \langle |Tf| - |T[\sigma_f(|f| \wedge u)]| \rangle^+ \leq |T[|f - \sigma_f(|f| \wedge u)|] = |T[|\sigma_f(|f| - u)^+]|.$$

Therefore, if there exists $0 \leq u \in E$ such that $\|T[\sigma_f(|f| - u)^+]\| \leq \delta$ for all $f \in B_E$, then $\rho(T) \leq \delta$.

Now take $0 < \delta < \rho(T)$. By the above observations, for any $0 \leq u \in E$ there exists $f \in B_E$ such that $\|T[\sigma_f(|f| - u)^+]\| > \delta$. Define the sequence $\{f_n\}_{n=1}^\infty$ in $B_E$ inductively as follows. Take $f_1 \in B_E$ such that $\|Tf_1\| > \delta$. Now assume that $f_1, \ldots, f_{n-1}$ ($n \geq 2$) have been defined. Take $f_n \in B_E$ such that

$$\|T[\sigma_{f_n} \left( |f_n| - 2^n \sum_{k=1}^{n-1} |f_k| \right)^+]\| > \delta.$$

Define

$$g_n = \sigma_{f_n} \left( |f_n| - 2^n \sum_{k=1}^{n-1} |f_k| - \sum_{k=n+1}^\infty 2^{-k} |f_k| \right)^+$$

for $n = 1, 2, \ldots$. Since $|g_n| \leq |f_n|$, it is clear that $g_n \in B_E$, and it follows from $|g_n| \leq (|f_n| - 2^n \sum_{k=1}^{n-1} |f_k| - \sum_{k=n+1}^\infty 2^{-k} |f_k|)^+$ that $\{g_n\}_{n=1}^\infty$ is a disjoint sequence (cf. the proof of Lemma 2.3). Now
\[ \|Tg_n\| \geq \|T\left[\sigma_{f_n}\left(|f_n| - 2^n \sum_{k=1}^{n-1} |f_k|\right)^+\right] - \|T\left[\sigma_{f_n}\left(|f_n| - 2^n \sum_{k=1}^{n-1} |f_k|\right)^+ - g_n\right]\| \]
\[ \geq \delta - \|T\left[\sigma_{f_n}\left(|f_n| - 2^n \sum_{k=1}^{n-1} |f_k|\right)^+ - g_n\right]\|. \]

Furthermore
\[ |T\left[\sigma_{f_n}\left(|f_n| - 2^n \sum_{k=1}^{n-1} |f_k|\right)^+ - g_n\right]| \leq |T|\left[\sigma_{f_n}\left(|f_n| - 2^n \sum_{k=1}^{n-1} |f_k|\right)^+ - g_n\right] \leq |T|\left(\sum_{k=n+1}^{\infty} 2^{-k} |f_k|\right). \]

and so
\[ \|T\left[\sigma_{f_n}\left(|f_n| - 2^n \sum_{k=1}^{n-1} |f_k|\right)^+ - g_n\right]\| \leq \|T\|\left(\sum_{k=n+1}^{\infty} 2^{-k} \|f_k\|\right) \leq 2^{-n} \|T\|_. \]

Therefore \(\|Tg_n\| \geq \delta - 2^{-n} \|T\|_r, (n = 1, 2, \ldots)\), so \(\lim_{n \to \infty} \|Tg_n\| \geq \delta\). Since \(\{g_n\}_{n=1}^{\infty}\) is a disjoint sequence in \(B_E\), and since \(E^*\) has order continuous norm, we have \(g_n \to 0\) (weakly) and hence \(|Tg_n| \to 0\) (weakly) in \(F\). Now it follows from Lemma 2.3 that there exists a subsequence \(\{g_{n_k}\}_{k=1}^{\infty}\) and a disjoint sequence \(\{\phi_k\}_{k=1}^{\infty}\) in \(B_{F^*}\) such that
\[ \lim_{k \to \infty} \langle |Tg_{n_k}|, \phi_k \rangle = \lim_{n \to \infty} \|g_n\| \geq \delta. \]

Now let \(\pi_k \in Z(F)\) be such that \(\pi_k(|Tg_{n_k}|) = Tg_{n_k}\) and \(\pi_k = I\), and define \(\psi_k = \pi_k^* \phi_k\) \((k = 1, 2, \ldots)\). Then \(\{\psi_k\}_{k=1}^{\infty}\) is a disjoint sequence in \(B_{F^*}\) and \(\langle Tg_{n_k}, \psi_k \rangle = \langle |Tg_{n_k}|, \phi_k \rangle\), so \(\lim_{k \to \infty} \|\langle Tg_{n_k}, \psi_k \rangle\| \geq \delta\). This shows that \(C \geq \delta\) for all \(0 < \delta < \rho(T)\), hence \(C \geq \rho(T)\).

We thus have shown that \(\rho(T) = A = B = C\). Now assume in addition that \(E\) has the principal projection property. It is clear that \(D \leq B\). Using an argument similar to the first part of the present proof, it is not difficult to show that \(C \leq D\), and we conclude that \(D = \rho(T)\). \(\Box\)

We end this section by showing that there is an interesting class of operators \(T\) for which \(\rho(T) = \beta(T)\) holds (other types of operators for which such an equality holds will be discussed in the next section). Recall that, if \(E\) and \(F\) are Banach lattices, then the operator \(T\) from \(E\) into \(F\) is called \(AM\)-compact if \(T\) maps order bounded sets onto relatively compact sets (equivalently, \(T\) maps almost order bounded sets onto relatively compact sets; see [33, Sect. 123]). We note already that if \(E\) and \(F\) are Banach
function spaces \((F\) with order continuous norm\), then any absolute integral operator from \(E\) into is \(AM\)-compact.

**THEOREM 2.5.** Let \(E\) and \(F\) be complex Banach lattices such that \(E^*\) and \(F\) have order continuous norms. If \(T \in \mathcal{L}_b(E, F)\) is \(AM\)-compact, then \(\beta(T) = \rho(T)\).

**Proof.** Take any \(0 < \phi \in F^*\) and let \(\psi = |T|^* \phi\). Take \(\varepsilon > 0\). Since \(E^*\) has order continuous norm, there exists \(0 < u \in E\) such that \(\langle\langle |f| - u \rangle, \psi \rangle \leq \varepsilon\) for all \(f \in B_E\), or, equivalently, \(B_E \subseteq [-u, u]_C + \varepsilon B_\psi\), where \(B_\psi = \{f \in E : \langle\langle |f|, \psi \rangle \leq \varepsilon\}\) (see the beginning of this section). Now it follows from \(TB_\psi \subseteq B_\phi\) that \(TB_E \subseteq T([-u, u]_C) + \varepsilon B_\phi\). Since \(T\) is \(AM\)-compact, \(T([-u, u]_C)\) is relatively compact, and hence \(TB_E\) can be covered with finitely many \(\phi\)-balls of radius \(2\varepsilon\). This shows that \(TB_E\) is \(PL\)-compact, and so Proposition 2.1 implies that \(\beta(TB_E) = \rho(TB_E)\). Therefore \(\beta(T) = \beta(TB_E) = \rho(TB_E) = \rho(T)\).

Combined with the properties of \(\rho\), the above theorem immediately yields the following monotonicity result for \(\beta\).

**COROLLARY 2.6.** Let \(E\) and \(F\) be complex Banach lattices such that \(E^*\) and \(F\) have order continuous norms. If \(S, T \in \mathcal{L}_b(E, F)\) are such that \(0 < S \leq T\), and if \(S\) is \(AM\)-compact, then \(\beta(S) < \beta(T)\).

**Proof.** Since \(S\) is \(AM\)-compact, it follows from the above theorem that \(\beta(S) = \rho(S)\). Furthermore, as observed earlier, \(0 < S \leq T\) implies that \(\rho(S) \leq \rho(T)\), and the inequality \(\rho(T) \leq \beta(T)\) always holds. A combination of these inequalities shows that \(\beta(S) \leq \beta(T)\).

It should be observed that if we replace in the above corollary the assumption that \(S\) is \(AM\)-compact by the \(AM\)-compactness of \(T\), then \(\beta(S) \leq \beta(T)\) holds as well. Indeed, in that case the \(AM\)-compactness of \(T\) implies that \(S\) is likewise \(AM\)-compact, as \(F\) has order continuous norm (see, e.g., [33, Theorem 123.4]). In the next section we will show, by way of an example, that the \(AM\)-compactness of \(S\) in the above corollary cannot be omitted.

A combination of Theorems 2.5 and 2.3 yields formulas for \(\beta(T)\) of \(AM\)-compact operators. Next we will single out this result for integral operators between Banach function spaces. We recall some of the relevant notions. Let \((Y, A, \nu)\) be a \(\sigma\)-finite measure space and denote by \(L_0(Y, \nu)\) the space of all complex valued \(\nu\)-measurable functions on \(Y\) (with the usual identification mod \(\nu\)). A Banach lattice \(E\) is called a Banach function space on \((Y, A, \nu)\) if \(E\) is an ideal of measurable functions in \(L_0(Y, \nu)\), which is a Banach space with respect to a function norm (see [33, Sect. 112]). Now assume that \(E\) and \(F\) are Banach function spaces on
(Y, A, v) and (X, Σ, μ), respectively. The operator \( T: E \to F \) is called an integral operator if there exists a \( μ \times ν \)-measurable function \( T(x, y) \) on \( X \times Y \) such that

\[
(Tf)(x) = \int_Y T(x, y)f(y) \, dv(y) \quad \mu\text{-a.e.}
\]

for all \( f \in E \). Furthermore, \( T \) is called an absolute integral operator if \( |T(x, y)| \) defines an integral operator from \( E \) into \( F \) (see [33, Sect. 93]). If \( F \) has order continuous norm, then any absolute integral operator from \( E \) into \( F \) is AM-compact [33, Theorem 123.9]. Furthermore, if the norm in \( F \) is order continuous, then the Banach dual \( F^* \) can be identified with the associate space \( F' \) of \( F \); i.e., for every \( \psi \in F^* \) there exists a (unique) \( g \in L_0(X, μ) \) such that \( \langle f, \psi \rangle = \int fg \, dμ \) for all \( f \in F \). For a measurable set \( A \), we denote by \( χ_A \) the operator defined by multiplication with the characteristic function of \( A \). From Theorems 2.3 and 2.5 we obtain the following result.

**Corollary 2.7.** Let \( E \) and \( F \) be Banach function spaces on (Y, A, v) and (X, Σ, μ), respectively, such that \( E^* \) and \( F \) have order continuous norms. If \( T \) is an absolute integral operator from \( E \) into \( F \), then

\[
\beta(T) = \sup \left\{ \lim_{n \to \infty} \| T f_n \| : \{ f_n \}_{n=1}^\infty \text{ disjoint sequence in } E \right\}
\]

\[
= \sup \left\{ \lim_{n \to \infty} \left\| \int_X (T f_n) g_n \, dμ \right\| : \{ f_n \}_{n=1}^\infty \text{ and } \{ g_n \}_{n=1}^\infty \text{ disjoint sequences in } E \text{ and } F' \right\}
\]

\[
= \sup \left\{ \lim_{n \to \infty} \| χ_{A_n} T \| : A_n \in Σ, A_n \downarrow \emptyset \right\}
\]

\[
= \sup \left\{ \lim_{n \to \infty} \| χ_{A_n} T_{|B_n} \| : A_n \in Σ, A_n \downarrow \emptyset \text{ and } B_n \in A, B_n \downarrow \emptyset \right\}.
\]

For the case that \( E = L_p(Y, v) \) and \( F = L_p(X, μ) \) (\( 1 < p < \infty \)) the above formulas were obtained by L. Weis [31, Theorem 4.3]. In the next example we will illustrate how these formulas can be used to compute the measure of non-compactness of a certain type of integral operator.

**Example 2.8.** Let \( 1 \leq p \leq q \leq \infty \), and \( -\infty \leq a < b \leq \infty \). Suppose that \( u \) and \( v \) are complex valued Lebesgue measurable functions on \((a, b)\), and define for \( f \in L_p(a, b) \)

\[
(Tf)(x) - u(x) \int_a^x v(y) f(y) \, dy \quad \text{a.e. on } (a, b);
\]
i.e., $T$ is a weighted Volterra operator. We assume that $u \in L_q(c, b)$ and $v \in L_p(a, c)$ for all $a < c < b(1/p + 1/p' = 1)$. For $a < c < d < b$ define

$$K(c, d) = \sup_{c < x < d} \left( \int_c^x |v(y)|^{p'} dy \right)^{1/p} \left( \int_c^d |u(y)|^q dy \right)^{1/q}.$$  

Then $T$ is a bounded operator from $L_p(a, b)$ into $L_q(a, b)$ if and only if $K(a, b) < \infty$, and then $K(a, b) \leq ||T|| \leq A(p, q) K(a, b)$, where $A(p, q) = \min(p^{1/p}(p')^{1/p'}, (q')^{1/q'} q', 1)$ (see [3, 30]). In particular, if $u \in L_q(a, b)$ and $v \in L_p(a, b)$, then $T$ is bounded with $||T|| \leq A(p, q) ||v||_{p'} ||u||_q$, and hence, if $1 < p \leq q < \infty$ then $||x_A T|| \to 0$ as $A_n \downarrow \emptyset$, so that $T$ is compact in this case (see, e.g., [33, Sect. 128]; this follows, of course, also from the above corollary). From now on we assume that $1 < p \leq q < \infty$. Now for any $a < c < d < b$ we have

$$T = \chi_{[c,d]} T \chi_{[c,d]} + \chi_{[d,b]} T \chi_{[c,d]} + \chi_{[c,d]} T \chi_{[a,d]}.$$  

The last two operators in this expression are now weighted Volterra operators with weights in $L_q(a, b)$ and $L_p(a, b)$, respectively, so that, by the above observation, these operators are compact. Hence

$$\beta(T) = \beta(\chi_{[c,d]} T \chi_{[c,d]}) \leq \|\chi_{[c,d]} T \chi_{[c,d]}\| \leq A(p, q) \max\{K(a, c), K(d, b)\}.$$  

Therefore, with $K = \max\{\lim_{\epsilon \downarrow 0} K(a, c), \lim_{\epsilon \uparrow b} K(d, b)\}$, we get $\beta(T) \leq A(p, q) K$. On the other hand, by the above corollary

$$\beta(T) = \sup\{\lim_{n \to \infty} ||x_{A_n} T||; A_n \downarrow \emptyset \text{ in } (a, b)\}.$$  

Considering the sets $A_n = (a, c_n) \cup (d_n, b)$, with $c_n \downarrow a$ and $d_n \uparrow b$, we obtain $K \leq \beta(T) \leq A(p, q) K$.

This reproves, essentially the main result of R. K. Juberg [11]. Our approach is similar to Weis' approach [31], who dealt with the case $p = q$, reproving an earlier result of Juberg [10].

3. The Essential Spectrum

In this section we will show that the essential spectrum of $AM$-compact operators and disjointness preserving operators has special properties. The main tools we use are the measures of non-compactness and of non-compactness. First recall some of the relevant facts. If $E$ is a complex Banach space, and $T \in \mathcal{L}(E)$, then we denote, as usual, the spectrum by
σ(T) and the spectral radius by \( r(T) \). The Fredholm domain of \( T \in \mathcal{L}(E) \) is defined by

\[
\Phi_T = \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is a Fredholm operator} \}
\]

(see, e.g., [9, 29]). The set \( C \setminus \Phi_T \) will be called the (Wolf) essential spectrum of \( T \), and is denoted by \( \sigma_{\text{ess}}(T) \). The essential spectrum of \( T \) is equal to the spectrum of the canonical image of \( T \) in the Calkin algebra \( \mathcal{C}(E) = \mathcal{L}(E)/\mathcal{K}(E) \), where \( \mathcal{K}(E) \) denotes the ideal of compact operators in \( E \). In particular, \( \sigma_{\text{ess}}(T) \neq \emptyset \) when \( \dim E = \infty \). Throughout this section we will assume that \( E \) is infinite dimensional. The essential spectral radius of \( T \) is defined by \( r_{\text{ess}}(T) = \sup \{ |\lambda| : \lambda \in \sigma_{\text{ess}}(T) \} \). Clearly, \( \sigma_{\text{ess}}(T) \subseteq \sigma(T) \) and \( r_{\text{ess}}(T) \leq r(T) \). The relation between \( r_{\text{ess}}(T) \) and the measures of noncompactness was noticed by R. D. Nussbaum [22], who proved that

\[
r_{\text{ess}}(T) = \lim_{n \to \infty} \left[ \beta(T^n) \right]^{1/n}
\]

(see also [5, Sect. 9.8]).

It is now an immediate consequence of Corollary 2.6 that, if \( E \) is a Banach lattice with order continuous norms in \( E \) and \( E^* \), and if \( 0 < S \leq T \) in \( \mathcal{L}(E) \) with \( S \) AM-compact, then \( r_{\text{ess}}(S) \leq r_{\text{ess}}(T) \). We will show next that this result holds without any conditions on the norms in \( E \) and \( E^* \).

**Lemma 3.1.** Let \( E, F, \) and \( G \) be complex Banach lattices and suppose that \( S_1 \in \mathcal{L}_b(E, F) \), \( S_2 \in \mathcal{L}_b(F, G) \) and that \( S_2 \) is AM-compact. Then \( \beta(S_2 S_1) \leq \rho(S_1) \beta(S_2) \).

**Proof:** For \( \delta > \rho(S_1) \) there exists \( 0 < u \in F \) such that \( S_1 B_E \subseteq [-u, u]_c + \delta B_F \), and so \( S_2 S_1 B_E \subseteq S_2([-u, u]_c) + \delta S_2 B_F \). By the AM-compactness of \( S_2 \) we have \( \beta(S_2 [-u, u]_c) = 0 \), hence \( \beta(S_2 S_1) = \beta(S_2 S_1 B_E) \leq \delta \beta(S_2 B_F) = \delta \beta(S_2) \). This holds for all \( \delta > \rho(S_1) \), so \( \beta(S_2 S_1) \leq \rho(S_1) \beta(S_2) \).

**Theorem 3.2.** Let \( E \) be a complex Banach lattice and suppose that \( S, T \in \mathcal{L}(E) \) such that \( 0 \leq S \leq T \) and \( S \) is AM-compact. Then \( r_{\text{ess}}(S) \leq r_{\text{ess}}(T) \).

**Proof:** It follows from the above lemma that \( \beta(S^2) \leq \rho(S) \beta(S) \), and, as observed in the previous section, \( 0 \leq S \leq T \) implies that \( \rho(S) \leq \rho(T) \). Hence

\[
\beta(S^2) \leq \rho(S) \beta(S) \leq \rho(T) \beta(S) \leq \beta(T) \beta(S).
\]

Applying this to \( 0 \leq S^n \leq T^n \) we find that

\[
\beta(S^{2n})^{1/n} \leq \beta(T^n)^{1/n} \beta(S^n)^{1/n} \quad (n = 1, 2, \ldots),
\]
and so \( r_{\text{ess}}(S^2) \leq r_{\text{ess}}(T) r_{\text{ess}}(S) \). Furthermore, \( r_{\text{ess}}(S^2) = r_{\text{ess}}(S)^2 \) (e.g., by the spectral mapping theorem in the Calkin algebra), hence

\[
r_{\text{ess}}(S) \leq r_{\text{ess}}(T).
\]

In Example 3.7 we will show that the \( AM \)-compactness of \( S \) in the above theorem cannot be omitted. In connection with the above result we note that if \( E \) has order continuous norm and if \( 0 \leq S \leq T \) in \( \mathcal{L}(E) \) with \( T \) \( AM \)-compact, then \( S \) is likewise \( AM \)-compact (see [33, Theorem 123.4]).

As is well known, if \( T \) is a positive operator on a Banach lattice, then \( r(T) \in \sigma(T) \) (see, e.g., [26, V.4.1] or [33, Lemma 135.1]). It is natural question to ask whether an analogous result is valid for the essential spectrum of positive operators. As we will see in Example 3.7, in general \( r_{\text{ess}}(T) \notin \sigma_{\text{ess}}(T) \). For certain classes of positive operators, however, the question has an affirmative answer. The next proposition shows that there is a connection with the monotonicity of the essential spectral radius.

**Proposition 3.3.** Let \( E \) be a complex Banach lattice and assume that \( 0 \leq S \in \mathcal{L}(E) \) is such that \( 0 \leq S^n \leq T \) in \( \mathcal{L}(E) \) implies that \( r_{\text{ess}}(S^n) \leq r_{\text{ess}}(T) \) for all \( n = 1, 2, \ldots \). Then \( r_{\text{ess}}(S) \in \sigma_{\text{ess}}(S) \).

**Proof.** Without loss of generality we assume that \( r_{\text{ess}}(S) = 1 \). Suppose that \( \sigma_{\text{ess}}(S) \subseteq \{ z \in \mathbb{C} : \text{Re } z \leq \alpha \} \) for some \( \alpha \in \mathbb{R} \). Now consider the operators \( e^{itS} \), \( t \geq 0 \). By the spectral mapping theorem (in the Calkin algebra)

\[
\sigma_{\text{ess}}(e^{itS}) = e^{it\sigma_{\text{ess}}(S)} \subseteq \{ z \in \mathbb{C} : |z| \leq e^{it\alpha} \}
\]

and hence \( r_{\text{ess}}(e^{itS}) \leq e^{it\alpha} \) for all \( t \geq 0 \). On the other hand

\[
0 \leq \frac{t^n}{n!} S^n \leq e^{itS}
\]

for all \( n = 1, 2, \ldots \) and all \( t \geq 0 \), so by hypothesis

\[
r_{\text{ess}}\left( \frac{t^n}{n!} S^n \right) \leq r_{\text{ess}}(e^{itS}).
\]

Since \( r_{\text{ess}}((t^n/n!) S^n) = (t^n/n!) r_{\text{ess}}(S^n) = t^n/n! \), we get \( 0 \leq (t^n/n!) \leq e^{nt} \) for all \( n = 1, 2, \ldots \) and all \( t \geq 0 \). Substituting \( t = n/\alpha \), it follows from Stirling's formula that \( \alpha \geq 1 \), and we may conclude that \( r_{\text{ess}}(S) \in \sigma_{\text{ess}}(S) \).

Combining the above proposition with Theorem 3.2, we obtain the following result.

**Theorem 3.4.** If \( E \) is a complex Banach lattice, then \( r_{\text{ess}}(T) \in \sigma_{\text{ess}}(T) \) for any positive \( AM \)-compact operator \( T \) in \( E \).
Remark 3.5. In connection with the above result we note that the essential spectrum of $AM$-compact operators has another special property. If $E$ is a complex Banach lattice without atoms, then $0 \in \sigma_{\text{ess}}(T)$ for any $AM$-compact operator $T$ in $E$. Indeed, if $0 \notin \sigma_{\text{ess}}(T)$, then there exist $S \in \mathcal{L}(E)$ and a compact $K \in \mathcal{L}(E)$ such that $I = ST + K$, and hence $I$ maps order bounded sets onto relatively compact sets. This implies that order intervals in $E$ are compact, which is impossible since $E$ is non-atomic (see, e.g., [2, Theorem 21.12]).

In the next corollary we specialize the above theorem to integral operators on Banach function spaces.

Corollary 3.6. Let $E$ be a Banach function space on $(X, \Sigma, \mu)$ with order continuous norm.

(i) If $0 \leq S \leq T$ in $\mathcal{L}(E)$, and if $S$ is an integral operator, then $r_{\text{ess}}(S) \leq r_{\text{ess}}(T)$.

(ii) If $T$ is a positive integral operator in $E$, then $r_{\text{ess}}(T) \in \sigma_{\text{ess}}(T)$.

Independently, and by different methods, part (ii) of the above corollary in the case $E = L_p(1 \leq p < \infty)$ was obtained by L. Weis (oral communication), and part (i) was proved by V. Caselles [4, Theorem 4.2] in the special case that $E$ is a reflexive rearrangement invariant Banach function space. Furthermore we mention that the essential spectrum of operators in $L_1$-spaces has been studied by L. Weis and M. Wolff in [32], proving, among other results, that $r_{\text{ess}}(T)$ belongs to $\sigma_{\text{ess}}(T)$ for any positive operator in $L_1$.

Example 3.7. In this example we will show that the $AM$-compactness in the above results cannot be dropped. Let $\mathcal{A} = \{-1, 1\}^N$ be the Cantor group, with Haar measure $\lambda$. For $n = 1, 2, \ldots$ let the probability measures $\mu_n$ on $\{-1, 1\}$ be defined by $\mu_n(\{-1\}) = \alpha_n, \mu_n(\{1\}) = 1 - \alpha_n$ with $0 < \alpha_n < 1$, and let $\mu = \otimes_{n=1}^{\infty} \mu_n$ be the corresponding product measure on $\mathcal{A}$ (note that for the choice $\alpha_n = 1/2$ for all $n$ we have $\mu = \lambda$). By a theorem of S. Kakutani [12], the measure $\mu$ is either absolutely continuous or singular with respect to $\lambda$, according as the series $\sum_{n=1}^{\infty} (2\alpha_n - 1)^2$ is convergent or divergent. As usual, the Rademacher functions $\{r_n\}_{n=0}^{\infty}$ on $\mathcal{A}$ are defined by $r_0 = 1$ and $r_n(t_1, t_2, \ldots) = t_n$ $(n \geq 1)$, and for each finite subset $F = \{0, 1, 2, \ldots\}$ the corresponding Walsh function on $\mathcal{A}$ is defined by $w_F = \prod_{n \in F} r_n$ (see, e.g., [7, Sect. 14.1]). The Walsh functions constitute an orthonormal basis in $E = L_2(\mathcal{A}, \lambda)$. For any product measure $\mu = \otimes_{n=1}^{\infty} \mu_n$ the convolution operator $T_\mu$ in $E$ is defined by $T_\mu f = \mu \ast f$. Then $T_\mu w_F = \prod_{n \in F} (2\alpha_n - 1) \cdot w_F$ for any Walsh function $w_F$. Furthermore we note that $T_\mu$ is an integral operator if and only if $\mu \ll \lambda$ (see [25]). Using
that \( T_\mu \) is a diagonal operator with respect to the Walsh functions, it is not
difficult to show that
\[
\beta(T_\mu) = \lim_{F} \prod_{n \in F} |2x_n - 1| = \lim_{n \to \infty} |2x_n - 1|.
\]

Now let \( \mu \) be the product measure on \( \Lambda \) corresponding to the particular
choice \( x_\mu = 1/3 \) for all \( n \). It is easy to verify that \( \sigma_{ess}(T_\mu) = \{-1/3, 1/9, -1/27, \ldots\} \cup \{0\} \), so \( r_{ess}(T_\mu) = 1/3 \) and \( r_{ess}(T_\nu) \not\in \sigma_{ess}(T_\mu) \). Let \( \nu \) be the product measure on \( \Lambda \) corresponding to the choice \( x_\nu = 2/3 \) for all \( n \),
and consider \( T_\mu + T_\nu \), which is diagonal as well. It is readily seen that
\( \sigma_{ess}(T_\mu + T_\nu) = \{2/9, 2/81, \ldots\} \cup \{0\} \), so \( r_{ess}(T_\mu + T_\nu) = 2/9 \). We thus have
\( 0 \leq T_\mu \leq T_\mu + T_\nu \) and \( r_{ess}(T_\mu + T_\nu) < r_{ess}(T_\mu) \). Moreover, \( \beta(T_\mu) = 1/3 \) and
\( \beta(T_\mu + T_\nu) = 2/9 \).

Next we will discuss another class of operators in Banach lattices for
which the measures of non-compactness and the essential spectrum have
special properties. We start with a lemma, in which we compute the ball
measure of non-compactness of order intervals in certain Banach lattices.

**Lemma 3.8.** Let \( E \) be a Dedekind complete non-atomic complex Banach
lattice such that \( \|f\| = \sup \{ \langle f, \phi \rangle : 0 \leq \phi \in E^*_\mu, \|\phi\| = 1 \} \) for all \( f \in E \). Then
\( \beta([-u,u]_C) = \|u\| \) for all \( 0 \leq u \in E \).

**Proof.** Without loss of generality, we may restrict ourselves to the case
that \( E \) is real. Take \( \delta > \beta([-u,u]) \) and \( \delta > \delta' > \beta([-u,u]) \). Given \( \varepsilon > 0 \)
there exists \( 0 \leq \phi \in E^*_\mu, \|\phi\| = 1 \) such that \( \langle u, \phi \rangle \geq \|u\| - \varepsilon \). By the choice
of \( \delta' \), there exist \( f'_1, \ldots, f'_n \in E \) such that \( [-u,u] \subseteq \bigcup_{j=1}^n B(f'_j, \delta') \), and we
may assume that \( f'_1, \ldots, f'_n \subseteq [-u,u] \). Now it follows from the Freudenthal
spectral theorem [19, Theorem 40.2] that there exist disjoint elements
\( u_1, \ldots, u_m \in E \) such that \( \sum_{j=1}^m u_j = u \), and real numbers \( \{x_{ij} : 1 \leq i \leq n, 1 \leq j \leq m \} \in [-1,1] \) such that the elements
\[
f_i = \sum_{j=1}^m x_{ij} u_j \quad (i = 1, \ldots, n)
\]
satisfy \( [-u,u] \subseteq \bigcup_{j=1}^n B(f_i, \delta) \). Since \( E \) is non-atomic and \( 0 \leq \phi \in E^*_\mu \),
there exist for each \( j = 1, \ldots, m \) disjoint elements \( u_{j1}, u_{j2} \in E \) such that
\( u_j = u_{j1} + u_{j2} \) and \( \langle u_{j1}, \phi \rangle = \langle u_{j2}, \phi \rangle = 1/2 \langle u, \phi \rangle \). Now define
\[
g = \sum_{j=1}^m (u_{j1} - u_{j2}).
\]
Since \( g \in [-u, u] \), there exists an \( f_j \) such that \( \|g - f_j\| \leq \delta \). Using that the elements \( \{u_{jk}\} \) are mutually disjoint we get

\[
\langle |g - f_i|, \phi \rangle = \sum_{j=1}^{m} \langle (1 - \alpha_{ij})u_{j1} + (1 + \alpha_{ij})u_{j2}, \phi \rangle = \sum_{j=1}^{m} \langle u_j, \phi \rangle = \langle u, \phi \rangle \geq \|u\| - \varepsilon,
\]

and hence \( \delta \geq \|u\| - \varepsilon \). This holds for all \( \delta > \beta([-u, u]) \) and all \( \varepsilon > 0 \), so \( \beta([-u, u]) \geq \|u\| \). The converse inequality being obvious, the lemma is proved.

Recall that the positive linear operator \( T \) from Banach lattice \( E \) into Banach lattice \( F \) is called a Maharam operator (or interval preserving) if \( T[0, u] = [0, Tu] \) for all \( 0 \leq u \in E \) \([18, 17]\). The next proposition follows immediately from the above lemma.

**Proposition 3.9.** Let \( E \) and \( F \) be complex Banach lattices, with \( F \) Dedekind complete and non-atomic, and such that \( \|f\| = \sup\{ \langle |f|, \phi \rangle : 0 \leq \phi \in E_n, \|\phi\| \leq 1 \} \) for all \( f \in F \). If \( 0 \leq T : E \to F \) is a Maharam operator, then \( \beta(T) = \|T\| \).

**Proof.** We may restrict ourselves to real spaces. Given \( \varepsilon > 0 \), there exists \( 0 < M \) such that \( \|u\| = 1 \) and \( \|Tu\| \geq \|T\| - \varepsilon \). Since \( [-Tu, Tu] = T[-u, u] \subseteq TB_E \), we have \( \beta([-Tu, Tu]) \leq \beta(TB_E) = \beta(T) \). By the above lemma, \( \beta([-Tu, Tu]) = \|Tu\| \), so \( \beta(T) \geq \|T\| - \varepsilon \). This shows that \( \beta(T) \geq \|T\| \), and hence \( \beta(T) = \|T\| \).

By means of duality, we will apply the above proposition to disjointness preserving operators. We recall the following facts. Suppose that \( E \) and \( F \) are complex Banach lattices.

1. The linear operator \( T \) is called disjointness preserving if \( f \perp g \) in \( E \) implies that \( Tf \perp Tg \) in \( F \). If \( T \) is disjointness preserving and norm bounded, then \( T \) is order bounded, the absolute value \( |T| \) exists, satisfying \( |Tf| = \|T\| |f| \) for all \( f \in E \), and \( |T| \) is a Riesz homomorphism (see \([1]\) and also \([23]\); use \([20]\) for adaptation to the complex case).

2. If \( T : E \to F \) is a Riesz homomorphism, then the adjoint \( T^* : F^* \to E^* \) is a Maharam operator (see, e.g., \([16]\)).

3. Now assume in addition that \( E \) and \( F \) are Dedekind complete, and suppose that \( T : E \to F \) is an order bounded linear operator such that \( |T| \) is an order continuous Maharam operator. Then there exists \( \pi \in Z(E) \) such that \( T = |T| \circ \pi \) and \( |\pi| = I \). Indeed, writing \( T = T_1 + iT_2 \), with \( T_1 \) and \( T_2 \) real operators (see \([33, \text{Sect.} 92]\)) we have \( |T_1|, |T_2| \leq |T| \), and so it
follows from [18, Theorem 3.1] that there exist \( \pi_1, \pi_2 \in \mathcal{Z}(E) \) such that \( T_j = |T| \circ \pi_j \) and \( |\pi_j| \leq I \) (\( j = 1, 2 \)). Hence \( T = |T| \circ \pi \) with \( \pi = \pi_1 + i\pi_2 \). Now it is not difficult to see that \( |\pi| = I \) on the carrier \( C_{|T|} \) of \( |T| \). Setting \( \pi = I \) on the null ideal \( N_{|T|} \), we obtain a desired \( \pi \in \mathcal{Z}(E) \).

**Theorem 3.10.** Let \( E \) and \( F \) be complex Banach lattices such that \( E^* \) is non-atomic. If \( T: E \to F \) is a norm bounded disjointness preserving operator, then \( \beta(T) \geq 1/2 \|T\| \).

**Proof.** As noted above, \( |T| \) exists and is a Riesz homomorphism. Hence, the adjoint \( |T|^* \) is an order continuous Maharam operator. Since \( |T^*| \leq |T| \), \( |T^*| \) is an order continuous Maharam operator as well, and hence there exists \( \pi \in \mathcal{Z}(F^*) \) such that \( T^* = |T^*| \circ \pi \) and \( |\pi| = I \). Furthermore, \( E^* \) satisfies the conditions of Corollary 3.9, so \( \beta(|T^*|) = \|T^*\| \).

Since \( \pi \) is an isometry in \( F^* \), we have \( \beta(T^*) = \beta(|T^*|) = \|T^*\| \). Moreover, \( \beta(T) \geq 1/2\beta(T^*) \) (see [22]), hence \( \beta(T) \geq 1/2 \|T\| \).

Clearly, the condition that \( E^* \) is non-atomic in the above theorem cannot be omitted. Indeed, the atoms in \( E^* \) are precisely the real valued Riesz homomorphisms on \( E \) (see, e.g., [26, II.4.4]), so any atom in \( E^* \) gives rise to a rank one Riesz homomorphism from \( E \) into \( F \). At present, however, we do not know an example of a Riesz homomorphism \( T: E \to F \), with \( E^* \) non-atomic, for which \( \beta(T) < \|T\| \). In this connection we mention the following result, the proof of which goes along the same lines as the proof of Lemma 3.8. Let \( E \) and \( F \) be Banach lattices, such that \( E \) is non-atomic and has order continuous norm, and suppose that \( T \) is a Riesz homomorphism such that \( \overline{T(E)} \) is the range of a contractive projection, then \( \beta(T) = \|T\| \). In particular, in \( L_p(1 \leq p < \infty) \) or \( (c_0) \), every closed sublattice is the range of a positive contractive projection (see, e.g., [15, Theorem 1.6.8]), so that we can apply the above mentioned result with \( F = L_p(1 \leq p < \infty) \) or \( (c_0) \).

The above theorem combined with Nussbaum's formula for the essential spectral radius yields the following.

**Corollary 3.11.** Let \( E \) be a complex Banach lattice with non-atomic dual space. If \( T \) is a norm bounded disjointness preserving operator in \( E \), then \( r_{ess}(T) = r(T) \).

**Proof.** If \( T \) is disjointness preserving, then \( T^n \) is likewise disjointness preserving for all \( n = 1, 2, ... \), and so, by the above theorem \( 1/2 \|T^n\| \leq \beta(T^n) \leq \|T^n\| \). Using the formulas for the spectral radii we get \( r(T) = r_{ess}(T) \).

Next we will discuss the relation between \( \|T\|, \beta(T), \) and \( \rho(T) \) for disjointness preserving operators. First we recall some terminology. Let \( E \) be a
Banach lattice and suppose \( 1 \leq p < \infty \). As usual, we say that \( E \) satisfies a lower \( p \)-estimate if there exists a constant \( C_1 > 0 \) such that 
\[
(\sum_{j=1}^n \|u_j\|^p)^{1/p} \leq C_1 \|\sum_{j=1}^n u_j\| \quad \text{for all disjoint } 0 \leq u_1, \ldots, u_n \in E
\] (see, e.g., [15, Definition 1.f.4]). Note that \( E \) satisfies a lower \( p \)-estimate (with constant \( C_1 \)) if and only if there exists an equivalent lattice norm \( \|\cdot\|_1 \) in \( E \), which satisfies a lower \( p \)-estimate with constant 1, and \( \|f\|_1 \leq \|f\|_1 \leq C_1 \|f\| \) for all \( f \in E \). If \( E \) satisfies a lower \( p \)-estimate, then \( E \) has order continuous norm. If there exists a constant \( C_2 > 0 \) such that \( \|\sum_{j=1}^n u_j\| \leq C_2(\sum_{j=1}^n \|u_j\|^p)^{1/p} \) for all disjoint \( 0 \leq u_1, \ldots, u_n \in E \) then \( E \) is said to satisfy an upper \( p \)-estimate, and a similar renorming statement holds. If \( E \) satisfies an upper \( p \)-estimate for some \( 1 < p < \infty \), then \( E \) and \( E^* \) have order continuous norm.

The following two lemmas will be used in the proof of the next theorem.

**Lemma 3.12.** Suppose that \( E \) is a non-atomic Banach lattice with order continuous norm. For any \( 0 \leq u \in E \) there exist \( 0 < u_1, u_2 \in E \) such that 
\[
u_1 \wedge u_2 = 0, \quad u_1 + u_2 = u, \quad \text{and } \|u_1\| = \|u_2\|.
\]

**Proof.** First observe that if \( 0 \leq w \in E \) and \( 0 < \alpha \leq \|w\| \), then there exists a component \( z \neq 0 \) of \( w \) such that \( \|z\| \leq \alpha \). Indeed, since \( E \) is non-atomic, there exists a disjoint sequence \( \{w_n\}_{n=1}^\infty \) of components of \( w \) such that \( w_n \neq 0 \) for all \( n \). Now the order continuity of the norm implies that 
\[
\|w_n\| \to 0 \quad (n \to \infty)
\] (see, e.g., [33, Theorem 104.2]), and so there exists \( w_n \) with \( \|w_n\| \leq \alpha \).

Now take \( 0 < u \in E \) and let \( V = \{v \in [0, u]: v \text{ component of } u \text{ and } \|v\| \leq \|u - v\|\} \). By the order continuity of the norm, any chain in \( V \) has a supremum, hence \( V \) has a maximal element \( v_m \). We assert that 
\[
\|v_m\| = \|u - v_m\|.
\]

Indeed, suppose on the contrary that \( \|v_m\| < \|u - v_m\| \) and let \( \alpha = 1/2(\|u - v_m\| - \|v_m\|) \). By the above, there exists a component \( v_0 \neq 0 \) of \( u - v_m \) such that \( \|v_0\| \leq \alpha \). Then \( v_m + v_0 \) is a component of \( u \), and it is easy to see that \( v_m + v_0 \in V \), contradicting the maximality of \( v_m \).

**Lemma 3.13.** Let \( E \) and \( F \) be Banach lattices with order continuous norms. Suppose that \( T \) is a Riesz homomorphism from \( E \) into \( F \). If \( 0 \leq u \in E \) and \( \{v_n\}_{n=1}^\infty \) is a sequence of components of \( u \) such that \( v_n \uparrow \), then 
\[
\|T(\lambda_n v_n) \wedge w\| \to 0 \quad \text{as } n \to \infty \quad \text{for any } 0 \leq w \in F \text{ and any sequence } \{\lambda_n\}_{n=1}^\infty \text{ in } R^+.
\]

**Proof.** Let \( P_n \) be the band projection in \( F \) onto the band \( \{Tv_n\}_{n=1}^\infty \). Since \( P_n w = \sup_k (kTv_n) \wedge w \), it is clear that \( (\lambda_n Tv_n) \wedge w \leq P_n w \) for all \( n = 1, 2, \ldots \). Since \( F \) has order continuous norm, it is sufficient to show that \( P_n w \downarrow 0 \). To this end, suppose that \( 0 \leq x \leq P_n w \) for all \( n \). Then \( 0 \leq x \wedge (Tu) \leq (P_n w) \wedge (Tu) \leq (P_n w) \wedge (Tv_n) + (P_n w) \wedge T(u - v_n) \). Now \( v_n \wedge (u - v_n) = 0 \) implies that \( (Tv_n) \wedge T(u - v_n) = 0 \), and so \( (P_n w) \wedge T(u - v_n) = 0 \). Therefore,
Let $E$ and $F$ be complex Banach lattices, $E$ non-atomic and $F$ with order continuous norm. Suppose that there exists $1 \leq p < \infty$ such that $E$ satisfies a lower $p$-estimate (with constant $C_1$) and $F$ satisfies an upper $p$-estimate (with constant $C_2$). If $T$ is a norm bounded disjointness preserving operator from $E$ into $F$, then $\|T\| \leq C_1 C_2 \rho(T)$.

Proof. It follows from $|Tf| = |T|(|f|)$ for all $f \in E$ that $\|T\| = \|T\|$ and $\rho(T) = \rho(|T|)$, and therefore we may restrict ourselves to real Banach lattices and assume that $T$ is a Riesz homomorphism. First renorm $E$ and $F$ such that the lower and upper $p$-estimate constants are both 1. Given $\varepsilon > 0$ there exists $0 \leq u \in B_E$ such that $\|Tu\| \geq \|T\| - \varepsilon$. By Lemma 3.12 there exist $u_1, u_2 \in [0, u]$ such that $u = u_1 + u_2$, $u_1 \wedge u_2 = 0$ and $\|u_1\| = \|u_2\|$. Then $\|u\| \leq (\|u_1\|^p + \|u_2\|^p)^{1/p}$, so $\|u_1\| = \|u_2\| \leq 2^{-1/p} \|u\|$. Since $T$ is a Riesz homomorphism we have $(Tu_1) \wedge (Tu_2) = 0$, so $(\|Tu_1\|^p + \|Tu_2\|^p)^{1/p} \geq \|Tu\|$ and hence there exists $u_i (i = 1 \text{ or } 2)$ such that $\|Tu_i\| \geq 2^{-1/p} \|Tu\|$. We thus have that there exists a component $u_1$ of $u$ such that $\|v_i\| \leq 2^{-1/p} \|u\|$ and $\|Tv_i\| \geq 2^{-1/p} \|Tu\|$. Repeating the argument we obtain a sequence $u \uparrow v_n \downarrow 0$ of components of $u$ such that $\|v_n\| \leq 2^{-n/p} \|u\|$ and $\|Tv_n\| \geq 2^{-n/p} \|Tu\|$. Note that $\|v_n\| \downarrow 0$, so $v_n \downarrow 0$ in $E$. Now take $\delta > \rho(T)$, then there exists $0 \leq w \in F$ such that $\|(Tf) - w\| \leq \delta$ for all $f \in B_E$. Then

$$\|T\| - \varepsilon \leq \|T(2^{n/p}v_n)\| \leq \|(2^{n/p}v_n) \wedge w\| + \|(2^{n/p}v_n) - w\|.$$

Now $2^{n/p}v_n \leq \|u\| \leq 1$, so $\|(2^{n/p}v_n) - w\|^+ \leq \delta$ for all $n$, and it follows from Lemma 3.13 that $\|T(2^{n/p}v_n) \wedge w\| \to 0$ as $n \to \infty$. This shows that $\|T\| - \varepsilon \leq \delta$. This holds for all $\delta > \rho(T)$ and all $\varepsilon > 0$, so $\|T\| \leq \rho(T)$ and hence $\rho(T) = \|T\|$. Taking the renorming of $E$ and $F$ into account we get $\|T\| \leq C_1 C_2 \rho(T)$.

The following corollary was obtained by a different method by L. Weis [13, Theorem 4.3 and Example 4.4].

Corollary 3.15. Suppose that $E = L_p$ on some non-atomic measure space and $1 \leq p < \infty$. If $T$ is a norm bounded disjointness preserving operator in $E$, then $\|T\| - \rho(T) - \beta(T)$.\]

Note that the above results, combined with Theorem 2.3, yield formulas for the norm of disjointness preserving operators. In the next example we show that the conditions on the norms in Theorem 3.14 cannot be omitted.
Example 3.16. Let $1 \leq p < q < \infty$ and define the operator $T$ from $L_p[0, 1] \oplus L_q[0, 1]$ into itself by $T(f \oplus g) = g \oplus 0$. Clearly, $T$ is a Riesz homomorphism and $\|T\| = 1$. Since the unit ball in $L_q[0, 1]$ is almost order bounded in $L_p[0, 1]$, we have $\rho(T) = 0$ (this follows also from the formulas in Theorem 2.3).

It follows immediately from Corollary 3.15 that if $E = L_p$ ($1 \leq p < \infty$) on a non-atomic measure space, and if $0 \leq S \leq T : E \to E$, with $S$ a Riesz homomorphism, then $\beta(S) \leq \beta(T)$. In view of Proposition 3.3, this implies that $r_{\text{ess}}(T)$ belongs to $\sigma_{\text{ess}}(T)$ for any Riesz homomorphism $T$ on such a space $E$. We will show next that the latter result holds in fact for a much larger class of Banach lattices. First the following observation.

Proposition 3.17. If $E$ is a complex Banach lattice and $0 \leq T \in \mathcal{L}(E)$ is such that $r_{\text{ess}}(T) = r(T)$, then $r(T) \in \sigma_{\text{ess}}(T)$.

Proof. Suppose that $r = r(T) \notin \sigma_{\text{ess}}(T)$, then $r$ is a Fredholm point of $T$. Since $r \in \partial \sigma(T)$ this implies that $r$ is a pole of the resolvent $R(\lambda, T)$ with finite rank residue (see, e.g., [9, Sects. 50 and 51]). Now it follows from a result in [26, V.5.5] that all points in $\sigma(T) \cap \{\lambda : |\lambda| = r\}$ are poles of $R(\lambda, T)$. Moreover, a close inspection of the proof of this theorem shows that the residues in these points have finite rank, and hence all points of $\sigma(T) \cap \{\lambda : |\lambda| = r\}$ are Fredholm points of $T$. This implies that $r_{\text{ess}}(T) < r(T)$, which is a contradiction.

Combining the above proposition with Corollary 3.11 we obtain the following result.

Corollary 3.18. If $E$ is a complex Banach lattice with non-atomic dual space, then $r(T) \in \sigma_{\text{ess}}(T)$ for any Riesz homomorphism in $E$.

We end the paper by mentioning another property of the (essential) spectrum of disjointness preserving operators in Banach lattices with non-atomic dual space, which can be derived from the following lemma.

Lemma 3.19. Let $E$ be a non-atomic complex Banach lattice and let $0 \leq T \in \mathcal{L}(E)$ be a Maharam operator. Then $\dim N(T) = 0$ or $\infty$.

Proof. We may assume that $E$ is real. Suppose that $N(T) \neq \{0\}$, then $Tf = 0$ for some $f \neq 0$, so $Tf^+ = Tf^-$. First assume that $Tf^+ = Tf^- = 0$, then $T|f| = 0$, and hence the ideal generated by $f$ is contained in $N(T)$. Since $E$ does not contain atoms, this implies that $\dim N(T) = \infty$. Now assume that $Tf^+ = Tf^- > 0$. Since $E$ is non-atomic, there exist non-zero disjoint elements $g_n \in [0, Tf^+] = [0, Tf^-]$ ($n = 1, 2, \ldots$). Since $T$ is a Maharam operator, there exist $0 \leq u_n \leq f^+$ and $0 \leq v_n \leq f^-$ such that $Tu_n = Tv_n = g_n$ ($n = 1, 2, \ldots$). Now define $f_n = u_n - v_n$, then $Tf_n = 0$ for all $n$. 


Since $g_n \perp g_m (n \neq m)$, we have $u_n \perp u_m$ and $v_n \perp v_m (n \neq m)$, and furthermore $u_n \perp v_m$ for all $n, m$. Hence $f_n \perp f_m$ for all $n \neq m$, which implies that \( \{f_n\}_{n=1}^{\infty} \) is a linearly independent system in $N(T)$, so $\dim N(T) = \infty$.

**Proposition 3.20.** Let $E$ be a complex Banach lattice with non-atomic dual space, and let $T$ be a norm bounded disjointness preserving operator in $E$. Then $0 \in \sigma(T)$ if and only if $0 \in \sigma_{\text{ess}}(T)$.

**Proof.** Since $T$ is norm bounded and disjointness preserving, $|T|$ exists and is a Riesz homomorphism. Then $|T|^*$ is an order continuous Maharam operator, and since $|T|^* \leq |T|^*$, it follows that $|T|^*$ is order continuous and Maharam. Hence, there exists $\pi \in Z(E^*)$ such that $T^* = |T|^* \circ \pi$, $|\pi| = I$. Since $\pi$ is an isometry, this implies that $\dim N(T^*) = \dim N(|T|^*)$, and so by the above lemma, $\dim N(T^*) = 0$ or $\infty$. Furthermore, $|T| = |T|(|f|)$ for all $f \in E$, and hence $N(T) = N(|T|)$, is an ideal in $E$. Evidently, $E$ is non-atomic, so $\dim N(T) = 0$ or $\infty$. Now the statement of the proposition is clear.

**Remarks.** Let $E = L_j^p (1 \leq p \leq \infty)$ on some non-atomic measure space $(X, \Sigma, \mu)$ and let $T$ be a weighted composition operator in $E$, i.e., $Tf(x) = a(x)f(\sigma(x)) \mu$-a.e., where $a$ is a $\mu$-measurable function and $\sigma$ is a null preserving transformation in $X$. It follows immediately from Lemma 3.19 that the closure of the range of $T$ has zero or infinite codimension. In case $p = 2$ and $a(x) \equiv 1$, this was shown by D. J. Harrington [8]. For $p = 2$ and $X = [0, 1]$ with Lebesgue measure, the result of Proposition 3.20 was proved by A. Kumar in [13].

Furthermore we note that if $T$ is the left shift in $l_2$, then $0 \in \sigma(T)$ whereas $0 \notin \sigma_{\text{ess}}(T)$, which shows that the conclusion of Proposition 3.20 is false in the presence of atoms. Finally, if $E$ is a complex Banach lattice and $\pi \in Z(E)$, then $\lambda T - \pi$ is disjointness preserving for all $\lambda \in C$, and therefore, if $E^*$ is non-atomic, then Proposition 3.20 shows that $\sigma(\pi) = \sigma_{\text{ess}}(\pi)$. This result was proved by the second author in [28, Theorem 1.11].

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*Note added in proof.* The second author has solved the problem raised in the remarks following Theorem 3.10, by showing that $\beta(T) = \|T\|$ under the hypotheses of Theorem 3.10. The details will appear in “The measure of non-compactness of a disjointness preserving operator.”
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