Local Fusions in Block Source Algebras

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In [3, Sect. 3], we extend the concept of “source” to any interior G-algebra $A$ (i.e., to any $O$-algebra $A$ endowed with a group homomorphism $\varphi: G \to A^*$) such that $1_A$ is primitive in $A^*$, a source algebra of $A$ being the interior $P$-algebra $B = iA_i$, where $P$ is a defect group of $A$ in Green’s sense [2]. $i$ is a primitive idempotent of $A^*$ such that $i \notin A_Q^*$ for any proper subgroup $Q$ of $P$, and the group homomorphism $P \to B^*$ maps $u \in P$ on $\varphi(u) i$.

Obviously the most interesting new case where this generalization applies occurs when $A$ is a block algebra of $G$ (i.e., when $A = OGe$ where $e$ is a primitive idempotent of $ZOG$ and $\varphi: G \to A^*$ maps $x \in G$ on $xe$). In this case, the source algebra $B$ is likely to contains all the “local” information about the block; for instance, by Corollary 3.5 in [3], the module categories of $A$ and $B$ are equivalent through restriction, and it is easy to see that vertices and sources of indecomposable $A$-modules can be computed from the corresponding $B$-modules; similarly, it is not difficult to show from Corollary 4.4 and Theorem 1.2 in [3] that the matrix of generalized decomposition numbers can be computed from $B$.

Our main result here (Theorem 3.1) implies the analogous statement concerning the local category of the block where objects are the local pointed groups on $A$ (see Sect. 1 or Definition 1.1 in [3]) and morphisms are the $G$-exomorphisms (see Definition 2.1), namely the equivalence class of the local category can be computed from $B$. In order to prove that, we introduce the so called local fusion category of any interior $G$-algebra (see Definition 2.15) and then we show that

(i) local fusion categories of $A$ and $B$ are equivalent (see Corollary 2.17),

(ii) local and local fusion categories of $A$ coincide (see Corollary 3.6).

A last comment. The local category probably supplies a better indication on the “difficulty” of the block than the structure of defect groups, and
therefore should be a better "invariant" to classify blocks. At least this turns out to be true in the easiest case: if the local category is equivalent to the corresponding one for a defect group, the block can be studied from all points of view [1, 4], regardless of the structure of its defect groups.

1. Notations and Preliminary Results

Throughout the paper \( p \) is a prime number and \( O \) a complete discrete valuation ring with residual field of characteristic \( p \). All the \( O \)-algebras we consider here are associative with unit element and \( O \)-free of finite rank as \( O \)-modules. For any \( O \)-algebra \( A \) we denote by \( A^* \) the group of invertible elements of \( A \), by \( \text{Aut}(A) \) the group of automorphisms of \( A \) and by \( J(A) \) the Jacobson radical of \( A \).

Let \( G \) be a finite group. Following Green [2], a \( G \)-algebra \( A \) is an \( O \)-algebra endowed with a group homomorphism \( \varphi : G \to \text{Aut}(A) \); write \( a^x \) instead of \( \varphi(x)^{-1}a \) for any \( x \in G \) and \( a \in A \). If \( H \) is a subgroup of \( G \), \( A^H \) is the subalgebra of \( H \)-fixed elements of \( A \) and, for any subgroup \( K \) of \( H \), we denote by \( \text{Tr}^H_K : A^K \to A^H \) the relative trace mapping \( a \in A^K \) on \( \sum_x a^x \) where \( x \) runs over a right transversal to \( K \) in \( H \), and by \( A^H \) its image.

But in most of the \( G \)-algebras \( A \) which we have to deal with, the action of \( G \) on \( A \) comes by conjugation from an explicit group homomorphism from \( G \) to \( A^* \); in such a case, we lose information by considering \( A \) just as a \( G \)-algebra; so, it is worth to consider the more precise (although less general) concept of interior \( G \)-algebra. Following [3, Definition 3.11], an interior \( G \)-algebra \( A \) is an \( O \)-algebra endowed with a group homomorphism \( \varphi : G \to A^* \); as above, simplify notation writing \( x \cdot a \) and \( a \cdot x \) instead of \( \varphi(x)a \) and \( a\varphi(x) \) for any \( x \in G \) and \( a \in A \), and set \( a^x = x^{-1} \cdot a \cdot x \).

Note that an interior \( G \)-algebra has an \( (OG, OG) \)-bimodule structure such that \( (a \cdot x) a' = a(x \cdot a') \) for any \( x \in G \) and \( a, a' \in A \); conversely,

1.1. An \( O \)-algebra \( A \) endowed with an \( (OG, OG) \)-bimodule structure such that \( (a \cdot x) a' = a(x \cdot a') \) for any \( x \in G \) and \( a, a' \in A \) has an interior \( G \)-algebra structure mapping \( x \in G \) on \( x \cdot 1 \).

Indeed, if \( a = 1 = a' \) we get \( 1 \cdot x = x \cdot 1 \) for any \( x \in G \) and therefore, for any \( y \in G \) we have

\[
(x \cdot 1)(y \cdot 1) = (x \cdot 1) \cdot y = x \cdot (1 \cdot y) = x \cdot (y \cdot 1) = xy \cdot 1.
\]

If \( \psi : H \to G \) is a group homomorphism, denote by \( \text{Res}_\psi(A) \) (cf. [3, Definition 3.1]) the interior \( H \)-algebra defined by the group homomorphism \( \varphi \cdot \psi : H \to A^* \); when \( H \) is a subgroup of \( G \) and \( \psi \) the inclusion map, set \( \text{Res}_\psi(A) = \text{Res}_H^G(A) \).
We will point out two special features of interior $G$-algebras in comparison with $G$-algebras. The first one regards induction: if $H$ is a subgroup of $G$ and $B$ an interior $H$-algebra, the induced algebra $\text{Ind}_H^G(B)$ (cf. [3, Definition 3.3]) is the interior $G$-algebra formed by the $(OG, OG)$-bimodule $OG \otimes_{OH} B \otimes_{OH} OG$ endowed with the distributive product defined by the formula
\[
(x \otimes b \otimes y)(x' \otimes b' \otimes y') = \begin{cases} 
0 & \text{if } yx' \notin H \\
x \otimes b, yx', b' \otimes y' & \text{if } yx' \in H,
\end{cases}
\]
where $x, y, x', y' \in G$ and $b, b' \in B$; this product is clearly associative with unit element $\sum_x x \otimes 1_B \otimes x^{-1}$ where $x$ runs over a left transversal to $H$ in $G$; moreover, for any $z \in G$ we have
\[
(x \otimes b \otimes yz)(x' \otimes b' \otimes y') = (x \otimes b \otimes y)(zx' \otimes b' \otimes y')
\]
and we apply 1.1. We do not know an analogous construction for $G$-algebras.

The second one regards conjugate homomorphisms. Let $A$ and $A'$ be interior $G$-algebras; a homomorphism $f$ from $A$ to $A'$ (cf. [3, Definition 3.1]) is both an $(OG, OG)$-bimodule and an $O$-algebra homomorphism possibly non unitary. But in noncommutative algebra it is handy to consider homomorphisms up to inner automorphisms: an exomorphism (or exterior homomorphism; cf. [3, Definition 3.1]) $\tilde{f}$ from $A$ to $A'$ is the set of homomorphisms obtained by composing $f$ with all the inner automorphisms of $A$ and $A'$; actually, to obtain $\tilde{f}$ it suffices to compose $f$ with all the inner automorphisms of $A'$ and therefore

1.2. Exomorphisms of interior $G$-algebras can be composed.

Denote by $\tilde{\text{Hom}}(A, A')$ the set of exomorphisms from $A$ to $A'$. If $\psi : H \to G$ is a group homomorphism, denote by $\text{Res}_\psi(\tilde{f})$ the exomorphism from $\text{Res}_\psi(A)$ to $\text{Res}_\psi(A')$ containing $\tilde{f}$; when $H$ is a subgroup of $G$ and $\psi$ the inclusion map, set $\text{Res}_\psi(\tilde{f}) = \text{Res}_\psi(f)$; then

1.3. If $\tilde{f}, \tilde{g} \in \tilde{\text{Hom}}(A, A')$, $\text{Res}_\psi(\tilde{f}) = \text{Res}_\psi(\tilde{g})$ is equivalent to $\tilde{f} = \tilde{g}$.
(cf. [3, Lemma 3.7]). We do not know an analogous result for $G$-algebras.

We will consider also exomorphisms between groups: similarly, the exomorphism $\tilde{\psi}$ defined by a group homomorphism $\psi : H \to G$ is the set of homomorphisms from $H$ to $G$ obtained by composing $\psi$ with all the inner automorphisms of $H$ and $G$; as above, it suffices to compose $\tilde{\psi}$ with the inner automorphisms of $G$ and therefore
1.4. Exomorphisms of groups can be composed.

Denote by $\text{Hom}(H, G)$ the set of exomorphisms from $H$ to $G$ and set $\tilde{\text{Aut}}(G) = \text{Aut}(G) / \text{Int}(G)$.

To extend to interior $G$-algebras the concept of direct summand in module theory, consider the following definition: an embedding $\mathcal{f}$ from $A$ to $A'$ is an exomorphism of interior $G$-algebras such that

$$\text{Ker}(f) = \{0\} \quad \text{and} \quad \text{Im}(f) = f(1_A) A' f(1_A).$$

For instance, if $H$ is a subgroup of $G$ and $B$ an interior $H$-algebra, the canonical embedding $\mathcal{f}_{\mathcal{f}}(B) : B \to \text{Res}_{H}^{G} \text{Ind}_{H}^{G}(B)$ is the exomorphism defined by the homomorphism mapping $b \in B$ on $1 \otimes b \otimes 1$. We will employ often the following easy fact about embeddings: let $\mathcal{f} : A \to A'$ and $\mathcal{g} : A' \to A''$ be exomorphisms of interior $G$-algebras and set $\tilde{h} = \mathcal{g} \circ \mathcal{f}$; then

1.5. If $\mathcal{g}$ is an embedding, $\mathcal{f}$ is uniquely determined by $\tilde{h}$ and is an embedding if and only if $\tilde{h}$ is an embedding.

Let $A$ be an interior $G$-algebra. Following [3, Definition 1.1], a pointed group $H_{\beta}$ on $A$ is a pair formed by a subgroup $H$ of $G$ and an $(A^n)^*$-conjugate class $\beta$ of primitive idempotents of $A^n$; we say that $\beta$ is a point of $H$ on $A$. If $H = \langle x \rangle$, the pair $x_{\beta}$ is a pointed element on $A$.

In module theory (i.e., if $A = \text{End}_{O}(M)$ where $M$ is an $OG$-module) $\beta$ corresponds to an isomorphism class of indecomposable $OH$-direct summands. Similarly, in our general situation, an embedded algebra $(B, \tilde{g})$ of $H_\beta$ is a pair formed by an interior $H$-algebra $B$ and an embedding $\tilde{g} : B \to \text{Res}_{H}^{G}(A)$ such that $g(1_B) \in \beta$, and we claim that

1.6. There exists an embedded algebra of $H_{\beta}$ and, if $(B, \tilde{g})$ and $(B', \tilde{g}')$ are embedded algebras of $H_{\beta}$, there is an unique exoisomorphism $\tilde{h} : B' \cong B$ such that $\tilde{g} \circ \tilde{h} = \tilde{g}'$.

Indeed, to show the existence it suffices to consider the $O$-algebra $B = i_A i$, where $i \in \beta$, endowed with the group homomorphism mapping $x \in H$ on $x \cdot i$, and the embedding $\tilde{g} : B \to \text{Res}_{H}^{G}(A)$ induced by the inclusion map; moreover, if $(B', \tilde{g}')$ is an embedded algebra of $H_{\beta}$ it is clear that $g'$ induces an isomorphism $B' \cong i' A i'$ where $i' = g'(1_B)$; then there is $\alpha \in (A^n)^*$ such that $(i')^\alpha = i$ and the isomorphism $h : B' \to B$ mapping $b' \in B'$ on $g'(b')^\alpha$ fulfills $\tilde{g} \circ \tilde{h} = \tilde{g}'$; finally, the uniqueness of $\tilde{h}$ follows from 1.4. Denote by $(A_{\beta}, \mathcal{f}_{\beta})$ an embedded algebra of $H_{\beta}$ (as Paul Fong pointed out, the definition of $(A_{\beta}, \mathcal{f}_{\beta})$ in [3] was unclear).

Let $\tilde{\mathcal{f}} : A \to A'$ be an embedding of interior $G$-algebras; it is not difficult to see that
1.7. For any pointed group $H_\beta$ on $A$, $f(\beta)$ is contained in a unique point $\beta'$ of $H$ on $A'$ and $\beta = f^{-1}(\beta')$.

Moreover, $(A_\beta, f^\#(A_\beta))$ is clearly an embedded algebra of $H_\beta$. Often we will denote $\beta$ and $\beta'$ with the same letter; for instance, $\beta$ could be the name of the unique point of $H$ on $A_\beta$.

Let $H_\beta$ and $K_\gamma$ pointed groups on $A$; write $K_\gamma \subset H_\beta$ and say that $K_\gamma$ is contained in $H_\beta$ (cf. [3, Definition 1.11]) if $K \subset H$ and for any $i \in \beta$ there is $j \in \gamma$ such that $ij = ji$. If $K = \langle x \rangle$ write $x_\gamma \in H_\beta$ whenever $K_\gamma \subset H_\beta$. Clearly, the relation $\subset$ between pointed groups is a transitive one and is compatible with embeddings (cf. 1.7). It is not difficult to restate this relation in terms of embedded algebras:

1.8. If $K \subset H$, we have $K_\gamma \subset H_\beta$ if and only if there is an exomorphism

$$\overline{f}_\gamma: A_\gamma \to \text{Res}_\lambda^H(A_\lambda)$$

such that $\overline{f}_\gamma = \text{Res}_\lambda^H(f_\beta) \circ \overline{f}_\gamma$.

Moreover, in this case, it follows from 1.5 above that

1.9. The exomorphism $\overline{f}_\gamma^\#$ is an embedding uniquely determined.

In particular, $K_\gamma$ is still a pointed group on $A_\beta$ (cf. 1.7) and $(A_\gamma, \overline{f}_\gamma^\#)$ an embedded algebra of this pointed group.

A pointed group $P_\gamma$ on $A$ is local (or $\gamma$ is a local point of $P$ on $A$)(cf. [3, Definition 1.11]) if $\gamma \not\subset A_\gamma^0$ for any proper subgroup $Q$ of $P$. Clearly, localness is compatible with embeddings (cf. 1.7). Let $H_\beta$ be a pointed group on $A$; a defect pointed group $P_\gamma$ of $H_\beta$ is a local pointed group on $A$ which is maximal fulfilling $P_\gamma \subset H_\beta$; by Theorem 1.2(iii) in [3].

1.10. The group $H$ acts transitively on the set of defect pointed groups of $H_\beta$.

In particular, identifying any local pointed group on $A_\gamma$ with the corresponding local pointed group on $A_\beta$ through $\overline{f}_\gamma^\#$ (cf. 1.7), it follows from 1.10 that

1.11. Local pointed groups on $A_\beta$ are the $H$-conjugate of local pointed groups on $A_\gamma$.

Similarly, if $H$ is a subgroup of $G$ and $B$ an interior $H$-algebra, identifying any pointed group on $B$ with the corresponding pointed group on $\text{Ind}_H^G(B)$ through $\overline{d}_H^G(B)$ (cf. 1.7), we get

1.12. Local pointed groups on $\text{Ind}_H^G(B)$ are the $G$-conjugate of local pointed groups on $B$. 
If \( 1_\beta \) is primitive in \( B^H \), this statement follows from Proposition 3.9 in [3], but it is easy to see that a similar argument holds without any hypothesis.

2. **G-EXOMORPHISMS AND A-FUSIONS**

Let \( G \) be a finite group and \( A \) an interior \( G \)-algebra. It is clear that \( G \) acts by conjugation on the set of pointed groups on \( A \); to formalize this action, we consider the following definition.

**Definition 2.1.** Let \( H_\beta \) and \( K_\gamma \) be pointed groups on \( A \). A \( G \)-exomorphism \( \tilde{\phi} \) from \( K_\gamma \) to \( H_\beta \) is a group exomorphism \( \phi: K \to H \) such that there is \( x \in G \) fulfilling \( (K_\gamma)^x \subset H_\beta \) and \( \phi(y) = y^x \) for any \( y \in K \). Denote by \( E_G(K_\gamma, H_\beta) \) the set of \( G \)-exomorphisms from \( K_\gamma \) to \( H_\beta \) and set \( E_G(H_\beta) = E_G(H_\beta, H_\beta) \). Note that \( \bigcup_{\gamma \in \mathfrak{A}(A)} E_G(K_\gamma, H_\beta) = \bigcup_{\beta \in \mathfrak{A}(A)} E_G(K_\gamma, H_\beta) \) since both members coincide with the set \( E_G(K,H) \) of all the group exomorphisms \( \phi: K \to H \) such that there is \( x \in G \) fulfilling \( \phi(y) = y^x \) for any \( y \in K \).

Evidently this definition does not depend on the choice of \( \phi \) in \( \tilde{\phi} \). It is also clear that

2.2. **Composition of G-exomorphisms are G-exomorphisms.**

2.3. If \( \phi \) is onto then \( \tilde{\phi}^{-1} \in E_G(H_\beta, K_\gamma) \).

In particular, \( E_G(H_\beta) \) is a subgroup of \( \tilde{\text{Aut}}(H) \) and the action of \( N_G(H_\beta) \) on \( H \) induces an isomorphism \( N_G(H_\beta)H/C_G(H) \cong E_G(H_\beta) \).

We are interested on the possibility of recognizing \( E_G(K_\gamma, H_\beta) \) as a subset of \( \tilde{\text{Hom}}(K,H) \) without the presence of \( G \); so, we should exhibit some property of \( \tilde{\phi} \in E_G(K_\gamma, H_\beta) \) independent of \( G \); with such a purpose we observe that

2.4. *Any \( x \in G \) fulfilling \( (K_\gamma)^x \subset H_\beta \) and \( \phi(y) = y^x \) for any \( y \in K \) induces an exomorphism \( \tilde{f}_x : A_\gamma \to \text{Res}_\phi(A_\beta) \) of interior \( K \)-algebras such that*

\[
\text{Res}_{\tilde{f}_x}(\tilde{f}_x) = \text{Res}_{\tilde{f}}(\tilde{f}_{\beta}) \circ \text{Res}_{\tilde{f}}(\tilde{f}_x).
\]

Indeed, if \( i \in \beta \) there is \( j \in \gamma \) such that \( ij^x = j^x = j^x i \) and therefore we have \( (jA_\beta)^x \subset iA_\beta \) but we may assume that \( A_\beta = iA_\beta \) and \( A_\gamma = jA_\beta \), and that \( \tilde{f}_x \) and \( \tilde{f}_x \) are induced by the inclusions \( iA_\beta \subset A \) and \( jA_\beta \subset A \); in this case it suffices to consider \( f_x : jA_\beta = iA_\beta \) mapping \( a \in jA_\beta \) on \( a^x \).

This statement suggests the following definition.

**Definition 2.5.** Let \( H_\beta \) and \( K_\gamma \) be pointed groups on \( A \). An \( A \)-fusion \( \tilde{\phi} \)
from $K$, to $H$ is a group exomorphism $\phi : K \to H$ such that $\phi$ is into and there is an exomorphism $f_\phi : A_\gamma \to \text{Res}_\phi (A_\beta)$ of interior $K$-algebras fulfilling
\[
\text{Res}_\phi (f_\phi) = \text{Res}_\phi (f_\phi) \circ \text{Res}_\phi (f_\phi).
\]
Denote by $F_A(K_\gamma, H_\beta)$ the set of $A$-fusions from $K_\gamma$ to $H_\beta$ and set $F_A(H_\beta) = F_A(H_\beta, H_\beta)$. Note that if $\beta' \in \mathcal{P}_A(H)$ and $\text{id}_{H} \in F_A(H_\beta, H_\beta)$ then $H_\beta \subset H_\beta$ (cf. 1.3 and 1.8) and so $\beta = \beta'$.

Again, this definition does not depend on the choice of $\phi$ in $\tilde{\phi}$; indeed, if $\phi' \in \mathcal{P}_A(H)$ there is $x \in H$ such that $\phi'(y) = \phi(y)^x$ for any $y \in K$ and it suffices to consider the homomorphism $f_\phi : A_\gamma \to \text{Res}_\phi (A_\beta)$ mapping $a \in A_\gamma$ on $f_\phi (a)^x \in A_\beta$. It is also clear that the definition above does not depend on the choice of the embedded algebras of $H_\beta$ and $K_\gamma$.

Moreover, the analogous of statements 2.2 and 2.3 are still true.

2.6. **Compositions of $A$-fusions are $A$-fusions.**

Indeed, if $L_\alpha$ is a pointed group on $A$ and $\tilde{\psi} \in F_A(L_\alpha, K_\gamma)$, there is an exomorphism $f_\psi : A_\alpha \to \text{Res}_\psi (A_\gamma)$ of interior $L$-algebras such that $\text{Res}_\psi (f_\psi) = \text{Res}_\psi (f_\psi) \circ \text{Res}_\psi (f_\psi)$ and therefore, considering the exomorphism of interior $L$-algebras $\text{Res}_\psi (f_\psi) : A_\alpha \to \text{Res}_\psi (A_\beta)$ we have $\text{Res}_\psi (f_\psi) = (\text{Res}_\psi (f_\psi) \circ \text{Res}_\psi (f_\psi)) \circ \text{Res}_\psi (f_\psi) = \text{Res}_\psi (f_\psi) \circ \text{Res}_\psi (f_\psi) \circ f_\psi$; hence $\tilde{\psi} \in F_A(K_\gamma, H_\beta)$.

2.7. **If $\phi$ is onto then $\tilde{\phi}^{-1} \in F_A(H_\beta, K_\gamma).**

Indeed, first of all notice that, by 1.3 and 1.5 above,

2.8. **The exomorphism $f_\phi$ is an embedding uniquely determined.**

(Hence, if $K$ maps into $A_\gamma$, the existence of $f_\phi$ forces $\phi$ to be into.) Now, $\phi(K) = H$ implies $f_\phi (A_\gamma) \subset A_\beta$ and therefore $f_\phi$ is an unitary embedding, so that $f_\phi$ is an isomorphism; then, statement 2.7 follows from the existence of the exomorphism $\text{Res}_{\phi^{-1}} (f_\phi^{-1}) : A_\beta \to \text{Res}_{\phi^{-1}} (A_\gamma)$. In particular, $F_A(H_\beta)$ is a subgroup of $\text{Aut}(H)$.

On the other hand, the following trivial fact shows the independence of $A$-fusions with regard to the whole group $G$,

2.9. **If $G'$ is a subgroup of $G$ containing $H$ and $K$, and $A' = \text{Res}_{G'}^G (A)$, we have $F_A(K_\gamma, H_\beta) = F_A(K_\gamma, H_\beta)$.**

However, statement 2.4 above shows that $E_G(K_\gamma, H_\beta) \subset F_A(K_\gamma, H_\beta)$ and precisely

2.10. **We have $E_G(K_\gamma, H_\beta) \subset F_A(K_\gamma, H_\beta) \cap E_G(K, H)$.**

Indeed, by 2.11 we may assume that $H \cong K$, and since $E_G(K, H) =
$\bigcup_{\gamma \in \mathcal{S}(K)} E_\phi(K_{\gamma}, H_{\beta})$ (cf. 2.1), it suffices to prove that if $\phi \in F_A(K_{\gamma}, H_{\beta}) \cap E_\phi(K_{\gamma}, H_{\beta})$ then $\gamma' = \gamma$; but by 2.3 we have $\phi^{-1} \in E_\phi(H_{\beta}, K_{\gamma'} \subset F_A(H_{\beta}, K_{\gamma'})$ and by 2.6 $\exists K \in F_A(K_{\gamma}, K_{\gamma'})$, so that $\gamma = \gamma'$ (cf. 2.5).

In particular, if $K_{\gamma} \subset H_{\beta}$ the exomorphism defined by the inclusion $K \subset H$ is an $A$-fusion from $K_{\gamma}$ to $H_{\beta}$ (this follows also from 1.8). Conversely,

2.11. Any $A$-fusion decomposes in an $A$-fusion which is onto and the $A$-fusion defined by an inclusion.

Indeed, if $\phi \in F_A(K_{\gamma}, H_{\beta})$ set $L = \varphi(K)$ and denote by $\psi : K \to L$ the isomorphism defined by $\varphi$; as far as $f_{\phi}$ is an embedding, the image by $\text{Res}_\phi(f_{\phi}) \circ f_{\phi}$ of the unit element of $A_i$ is contained in a point $\delta$ of $L$ on $A$ and it is easy to see that $L_{\delta} \subset H_{\beta}$ and $f_{\phi} = \text{Res}_\phi(f_{\phi}) \circ f_{\phi}$ where $f_{\phi}$ is an exomorphism from $A_{\gamma}$ to $\text{Res}_\phi(A_{\delta})$; then we have

$$\text{Res}_\phi^k(f_{\phi}) = \text{Res}_\phi^h(f_{\phi}) \circ \text{Res}_\phi^t(f_{\phi}) \circ \text{Res}_\phi^\delta(f_{\phi}) = \text{Res}_\phi^t(f_{\phi}) \circ \text{Res}_\phi^\delta(f_{\phi})$$

and therefore $\psi \in F_A(K_{\gamma}, L_{\delta})$.

But the $A$-fusions which are onto are easily related with $A^*$-conjugation.

**Proposition 2.12.** Let $H_{\beta}$ and $K_{\gamma}$ be pointed groups on $A$ such that $H \cong K$; choose $i \in \beta$ and $j \in \gamma$. A group exoisomorphism $\phi : K \to H$ is an $A$-fusion from $K_{\gamma}$ to $H_{\beta}$ if and only if there is $a \in A^*$ such that

$$(v \cdot j)^a = \varphi(v) \cdot i \quad \text{for any} \quad v \in K.$$

In particular, if $H$ and $K$ map into $A^*_\beta$ and $A^*_\gamma$, any $a \in A^*$ such that $(K \cdot j)^a = H \cdot i$ induces an $A$-fusion from $K_{\gamma}$ to $H_{\beta}$.

**Proof.** We may assume that $A_{\beta} = iA_i$ and $A_{\gamma} = jA_j$, and that $f_{\beta}$ and $f_{\gamma}$ come from the inclusions $iA_i \subset A$ and $jA_j \subset A$. Assume that $\phi \in F_A(K_{\gamma}, H_{\beta})$ and let $f_{\phi} : A_{\gamma} \to \text{Res}_\phi(A_{\beta})$ be the embedding of interior $K$-algebras fulfilling $\text{Res}_\phi^c(f_{\phi}) = \text{Res}_\phi^t(f_{\phi}) \circ \text{Res}_\phi^\delta(f_{\phi})$; by our choice of embedded algebras, this equality means that there is $a \in A^*$ such that $f_{\phi}(b) = b^a$ for any $b \in jA_j$ and, as far as $f_{\phi}(j) \in (iA_i)^{\mu}$, we have $f_{\phi}(j) = i$; by consequent, for any $y \in K$, we get $(y \cdot j)^a = f_{\phi}(y \cdot j) = \varphi(y) \cdot i$.

Conversely, if there is $a \in A^*$ such that $(v \cdot j)^a = \varphi(v) \cdot i$ for any $v \in K$, we have $(jA_j)^a = iA_i$ and the $O$-algebra isomorphism $f_{\phi} : jA_j \to iA_i$ mapping $b \in jA_j$ on $b^a$ is an isomorphism of interior $K$-algebras from $A_{\gamma}$ onto $\text{Res}_\phi(A_{\beta})$ fulfilling $\text{Res}_\phi^c(f_{\phi}) = \text{Res}_\phi^t(f_{\phi}) \circ \text{Res}_\phi^\delta(f_{\phi})$; hence $\phi \in F_A(K_{\gamma}, H_{\beta})$.

**Corollary 2.13.** Let $H_{\beta}$ be a pointed group on $A$ and denote by $N_A(H)$ the set of $a \in A^*_\beta$ such that $H \cdot a = a \cdot H$. The action of $N_A(H)$ on the image of $H$ in $A^*_\beta$ induces a group homomorphism from $F_A(H_{\beta})$ to
$N_{A^*_\beta}(H)/(H \cdot (A^*_\beta)^*)$. This homomorphism is an isomorphism when $H$ maps into $A^*_\beta$.

**Proof.** Choose $i \in \beta$ and assume that $A^*_\beta = iA^*_i$ and $\overline{f}_\beta$ comes from $iA^*_i \subset A^*_i$; by Proposition 2.12, for any $\phi \in F_A(H)$ there is $a \in A^*$ such that $(x \cdot i)^\phi = \phi(x) \cdot i$ for any $x \in H$; hence, $ai$ belongs to $N_{A^*_\beta}(H)$ and it is clear that its image in $N_{A^*_\beta}(H)/(H \cdot (A^*_\beta)^*)$ depends only on $\phi$. Conversely, if $a \in N_{A^*_\beta}(H)$ we have $a + (1_A - i) \in A^*$ and $(H \cdot i)^{a + (1_A - i)} = H \cdot i$; hence, if $H$ maps into $A^*_\beta$, it follows again from proposition 2.12 that $a$ induces an $A$-fusion from $H, \beta$ to $H, \beta$.

Next proposition shows that $G$-exomorphisms and $A$-fusions are both compatible with embeddings (cf. 1.7).

**Proposition 2.14.** Let $\gamma : A \to A'$ be an embedding of interior $G$-algebras, $H, \gamma$ and $K, \gamma$ pointed groups on $A$, and denote respectively by $b'$ and $\gamma'$ the points of $H$ and $K$ on $A'$ such that $\gamma(b) \subset b'$ and $\gamma(b') \subset \gamma'$. Then,

$$E_{A}(K, \gamma, H, \gamma) = E_{A'}(K', \gamma', H, \gamma') \quad \text{and} \quad F_{A}(K, \gamma, H, \gamma) = F_{A'}(K', \gamma', H, \gamma').$$

**Remark.** There is still no confusion if we denote $\beta$ and $\beta'$ by the same letter.

**Proof.** The first equality follows from the fact that, for any $x \in G$, we have $\gamma(x') \subset (\gamma')^x$ and therefore the inclusions $(K, \gamma)^x \subset H, \gamma$ and $(K, \gamma')^x \subset H, \gamma'$ are equivalent. To prove the second equality, we may assume that $A'_{\gamma'} = A_{\beta}$ and $A'_{\gamma} = A_{\gamma}$, and that $\overline{f}_{\gamma} = \text{Res}^{\gamma}_{\gamma}(\overline{f}) \circ \overline{f}_{\gamma}$ and $\overline{f}_{\gamma'} = \text{Res}^{\gamma}_{\gamma}(\overline{f}) \circ \overline{f}_{\gamma'}$; in particular we have

$$\text{Res}^{\gamma}_{\gamma}(\overline{f}) = \text{Res}^{\gamma}_{\gamma}(\overline{f}) \circ \text{Res}^{\gamma}_{\gamma}(\overline{f}) \quad \text{and} \quad \text{Res}^{\gamma}_{\gamma}(\overline{f}) = \text{Res}^{\gamma}_{\gamma}(\overline{f}) \circ \text{Res}^{\gamma}_{\gamma}(\overline{f}).$$

By consequence, if $\phi \in \text{Hom}(K, H)$ and $\overline{f}_{\phi} : A, \gamma \to \text{Res}_{\phi}(A, \beta)$ is an exomorphism of interior $K$-algebras, statement 1.5 above implies that the equalities

$$\text{Res}^{\gamma}_{\gamma}(\overline{f}) = \text{Res}^{\gamma}_{\gamma}(\overline{f}) \circ \text{Res}^{\gamma}_{\gamma}(\overline{f}) \quad \text{and} \quad \text{Res}^{\gamma}_{\gamma}(\overline{f}) = \text{Res}^{\gamma}_{\gamma}(\overline{f}) \circ \text{Res}^{\gamma}_{\gamma}(\overline{f})$$

are equivalent.

The two corollaries below are easier to formulate with the following definition.

**Definition 2.15.** The local fusion category of $A$ is the category where the objects are the local pointed groups on $A$ and the morphisms are the $A$-fusions.

**Corollary 2.16.** Let $H$ be a subgroup of $G$ and $B$ an interior $H$-algebra. The canonical embedding $\overline{f}_{\gamma}^H(B) : B \to \text{Res}^{\gamma}_{\gamma}(\text{Ind}^H_{\gamma}(B))$ induces an equivalence between the local fusion categories of $B$ and $\text{Ind}^H_{\gamma}(B)$. 
Proof: By Proposition 2.14, $\tilde{\alpha}_G^\mu(B)$ induces a faithful full functor from the local fusion category of $B$ into the local one of $\text{Ind}_G^\mu(B)$; hence, by 1.12 above, this functor is an equivalence of categories.

Corollary 2.17. Let $x$ be a point of $G$ on $A$ and $P_x$ a defect pointed group of $G_x$. The embedding $\tilde{f}_x^\mu: A_x \rightarrow \text{Res}_G^\mu(A_x)$ induces an equivalence between the local fusion categories of $A_x$ and $A_x$.

Proof: As above, by Proposition 2.14, $\tilde{f}_x^\mu$ induces a faithful full functor from the local fusion category of $A_x$ into the local one of $A_x$; hence, by 1.11, this functor is an equivalence of categories.

If $M$ is an $OG$-module, $O$-free of finite rank, $\text{End}_O(M)$ has an evident structure of interior $G$-algebra and the $\text{End}_O(M)$-fusions have an easy translation in module terms.

Proposition 2.18. Assume that $A = \text{End}_O(M)$ where $M$ is an $OG$-module $O$-free of finite rank. Let $H_\beta$ and $K_\gamma$ be pointed groups on $A$, choose $i \in \beta$ and $j \in \gamma$, and consider $i \cdot M$ and $j \cdot M$ as $OH$-module and $OK$-module, respectively. A group homomorphism $\phi: K \rightarrow H$ is an $A$-fusion from $K_\gamma$ to $H_\beta$ if and only if $\phi$ is into and $j \cdot M$ is a direct summand of $\text{Res}_\phi(i \cdot M)$.

Proof: Clearly we may assume that $A_\beta = \text{End}_O(i \cdot M)$ and $A_\gamma = \text{End}_O(j \cdot M)$; hence, $j \cdot M$ is a direct summand of $\text{Res}_\phi(j \cdot M)$ if and only if there is an embedding $\tilde{f}_\phi: A_\gamma \rightarrow \text{Res}_\phi(A_\beta)$ of interior $K$-algebras; but, in this case, $\text{Res}_\phi^K(\tilde{f}_\gamma)$ and $\text{Res}_\phi^H(\tilde{f}_\mu) \circ \text{Res}_\phi^K(\tilde{f}_\phi)$ are embeddings from $\text{Res}_\phi^K(A_\gamma)$ into $\text{Res}_\phi^H(A_\beta)$ which are both full matrix algebras over $O$; then, by Proposition 2.3(ii) in [3], these embeddings are equal.

3. The Group Algebra Case

Let $G$ be a finite group and set $A = OG$ endowed with the obvious structure of interior $G$-algebra. The $A$-fusions from local pointed groups are quite easy to describe.

Theorem 3.1. Let $H_\beta$ and $Q_\delta$ pointed groups on $A$ and assume that $Q_\delta$ is local. Then

$$E_G(Q_\delta, H_\beta) = F_A(Q_\delta, H_\beta).$$

To prove the theorem, we need some preliminary results. First,

3.2. The kernel $K$ of the structural group homomorphism from $H$ to $A_\beta^\times$ is a $p'$-group.

Indeed, it is clear that $A$ becomes a projective $OH$-module by left multiplication; but we may assume that $A_\beta = iAi$ where $i \in \beta$; hence $A_\beta$ becomes
also a projective \( OH \)-module by left multiplication and the restriction to a Sylow \( p \)-subgroup \( R \) of \( K \) should be both projective and trivial, which forces \( R = 1 \).

In particular, if \( P \) is a pointed \( p \)-group on \( A \), \( P \) maps into \( A^*_\gamma \). Actually, in this case we have a stronger statement, namely

3.3. \textit{There is an} \( O \)-basis \( B \) of \( A^*_\gamma \) \textit{such that} \( P \cdot B \cdot P = B \) \textit{and} \( |P \cdot b| = |P| = |b \cdot P| \) \textit{for any} \( b \in B \).

Indeed, it is clear that \( A \) becomes a permutation \( O(P \times P) \)-module by left and right multiplication; hence, as above, \( A^*_\gamma \) becomes also a permutation \( O(P \times P) \)-module by left and right multiplication; that is, there exists an \( O \)-basis \( B \) of \( A^*_\gamma \) such that \( P \cdot B \cdot P = B \). Moreover, as above, \( A^*_\gamma \) is a projective \( OP \)-module by either left or right multiplication and therefore, for any \( b \in B \), \( |P \cdot b| = |P| = |b \cdot P| \).

But when \( \gamma \) is local, we need a slight more precise result.

3.4. \textit{If} \( P \) \textit{is a local pointed group on} \( A \), \textit{there is an} \( O \)-basis \( B \) of \( A^*_\gamma \) \textit{containing the unit element and fulfilling} \( P \cdot B \cdot P = B \) \textit{and} \( |P \cdot b| = |P| = |b \cdot P| \) \textit{for any} \( b \in B \).

Indeed, choose \( B \) as above and set \( 1 = \sum_{b \in B} \lambda_b \cdot b \) where \( \lambda_b \in \mathbb{C} \); as far as \( \gamma \) is local, \( 1 \notin (A^*_\gamma)^o \) for any proper subgroup \( Q \) of \( P \); on the other hand, \( A^*_\gamma / J(A^*_\gamma) \) is a (possibly noncommutative) field; by consequent, there is \( b \in B \cap A^*_\gamma \) such that \( \lambda_b \cdot b \notin J(A^*_\gamma) \); in particular, we have \( b \in (A^*_\gamma)^* \) and \( Bb^{-1} \) is still an \( O \)-basis of \( A^*_\gamma \) fulfilling the above conditions.

\textit{Proof of Theorem 3.1.} By 2.10, it suffices to prove that any \( \phi \in F_A(Q_0, H_0) \) belongs to \( E_{\phi,Q,H} \) and by 2.11, we may assume that \( \phi \) is onto. Choose \( i \in \beta \) and \( j \in \delta \); then, by Proposition 2.12, there is \( a \in A^* \) such that \( (y)_a = \phi(y) \cdot i \) for any \( y \in Q \). In particular, we have \( jAi = j\alpha j a \) and \( yja = ja\phi(y) \) for any \( y \in Q \); as far as we may assume that \( A^*_\gamma = j\alpha j \), it follows now from 3.4 that there is an \( O \)-basis \( B \) of \( j\alpha j A \) such that \( Q \cdot B \cdot H = B, j\alpha j A \in B, Q\alpha j = j\alpha j A, \) and \( |Qj\alpha j| = |Q| \).

Consider the structure of \( O(Q \times H) \)-module on \( A \) defined by \( (y, z) \cdot b = yb^{-1} \) for any \( b \in A \) and \( (y, z) \in Q \times H \). On one hand, setting \( Q_\gamma = Q \cap H^{-1} \) and denoting by \( \sigma_\gamma : Q_\gamma \to Q \times H \) the group homomorphism mapping \( y \in Q_\gamma \) on \( (y, y^{-1}) \) for any \( y \in G \), it is clear that

\[ A \cong \bigoplus_\lambda \text{Ind}_{\sigma_\gamma(Q_\gamma)}^Q(H)(O), \]

where \( x \) runs over a set of representatives for the double cosets of \( G \) with respect to \( Q \) and \( H \), and \( O \) denotes the trivial module; moreover, any direct summand above is an indecomposable \( O(Q \times H) \)-module.
On the other hand, it is also clear that \( jAi \) is a direct summand of \( A \) as \( O(Q \times H) \)-modules, and the existence of the \( O \)-basis \( B \) above shows that \( \sum_{y \in Q} O \cdot yja \) is a direct summand of \( jAi \) as \( O(Q \times H) \)-modules, isomorphic to \( \text{Ind}_{Q \times H}^{Q \times H}(O) \) where \( \sigma_{\phi} : Q \to Q \times H \) is the homomorphism of groups mapping \( y \in Q \) to \( (y, \phi(y)) \) and \( O \) is the trivial module.

By consequent, there is \( x \in G \) such that \( \text{Ind}_{Q \times H}^{Q \times H}(O) \cong \text{Ind}_{Q \times H}^{Q \times H}(O) \), or equivalently \( Q' = Q \) and the groups \( \sigma_x(Q) \) and \( \sigma_{\phi}(Q) \) are conjugate in \( Q \times H \); hence \( Q' = H \) and, up to a suitable choice of \( x \), we may assume that \( \sigma_x(Q) = \sigma_{\phi}(Q) \), or equivalently \( \phi(y) = y^x \) for any \( y \in Q \), so that \( \tilde{\phi} \) belongs to \( E_G(Q, H) \).

**Corollary 3.5.** Let \( Q_\delta \) be a local pointed group on \( A \). The respective actions of \( N_G(Q_\delta) \) and \( N_{A_\mu}(Q) \) on \( Q \) and the image of \( Q \) in \( A_\mu^* \) induce a group isomorphism \( N_G(Q_\delta)/C_G(Q) \cong N_{A_\mu}(Q)/(A_\mu^*)^* \).

**Proof.** By Theorem 3.1, \( E_G(Q_\delta) = F_A(Q_\delta) \); but the action of \( N_G(Q_\delta) \) on \( Q \) induces \( N_G(Q_\delta)/Q \). \( C_G(Q) \cong E_A(Q_\delta) \) and by 3.2 and Corollary 2.13, \( Q \) maps into \( A_\mu^* \) and the action of \( N_{A_\mu}(Q) \) on this image induces an isomorphism \( N_{A_\mu}(Q)/(Q \cdot (A_\mu^*)^*) \cong F_A(Q_\delta) \).

**Corollary 3.6.** Let \( H_\beta \) be a pointed group on \( A \). If \( Q_\delta \) and \( R_\epsilon \) are local pointed groups on \( A_\mu \) then,

\[ E_G(R_\epsilon, Q_\delta) = F_{A_\mu}(R_\epsilon, Q_\delta). \]

In particular, choosing \( i \in \delta \) and \( j \in \epsilon \), \( Q_\delta \) and \( R_\epsilon \) are \( G \)-conjugate on \( A \) if and only if \( Q \cdot i \) and \( R \cdot j \) are \( A_\mu^* \)-conjugate; similarly, if \( Q = \langle u \rangle \) and \( R = \langle v \rangle \), \( u_\delta \) and \( v_\epsilon \) are \( G \)-conjugate on \( A \) if and only if \( u \cdot i \) and \( v \cdot j \) are \( A_\mu^* \)-conjugate.

**Proof.** The equality above follows from Theorem 3.1, statement 2.9 and Proposition 2.14. The last statements follow from this equality, Proposition 2.12 and statement 3.1.

**References**