Journal of Multivariate Analysis **65**, 228–244 (1998)
Article No. MV981743

Testing for Spherical Symmetry of a Multivariate Distribution

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Received December 20, 1996; revised August 20, 1997

We consider a test for spherical symmetry of a distribution in $\mathbb{R}^d$ with an unknown center. It is a multivariate version of the tests suggested by Schuster and Barker and by Arcones and Giné. The test statistic is based on the multivariate

metric alternative as well as for a sequence of local alternatives converging to a spherically symmetric distribution. We also study numerically the performance of the test for moderate sample sizes and justify a symmetrized version of bootstrap approximation of the distribution of test statistics. © 1998 Academic Press

AMS 1991 subject classification: primary 62E20, 62H15; secondary 60F05.

Key words and phrases: empirical processes, measure of asymmetry of probability distribution, spherical symmetry, symmetrized bootstrap, VC-subgraph classes.

1. INTRODUCTION

Testing a probability distribution for symmetry is a well-known problem in the univariate case (see, e.g., Shorack and Wellner, 1986; Csörgö and Heathcote, 1987; Schuster and Barker, 1987; Arcones and Giné, 1991), but it is much less known in the multivariate case, where there are many different kinds of symmetry, and only few tests have been suggested. For instance, Beran (1979) considered a test for ellipsoidal symmetry based on orthogonal projection estimators of multivariate density. Romano (1989) studied a rather general family of bootstrap tests, including the one for spherical symmetry of a multivariate distribution with a known center. Baringhaus (1991) suggested ω^2-type statistics for testing spherical symmetry, consistent against all alternatives (the center of symmetry was also supposed to be known). Heathcote, Rachev, and Cheng (1995) studied the so called "diagonal" symmetry. They suggested bootstrap test based on the empirical characteristic function. We refer also to the papers of Kariya and Eaton (1977), Blough (1989), Ghosh and Ruymgaart (1992).

0047-259X/98 $25.00

The goal of this paper is to suggest a new test for spherical symmetry of a multivariate distribution, based on the extension of the distribution and quantile functions to the multivariate case, recently introduced by Koltchinskii and Dudley (see Koltchinskii, 1994a, b) and Chaudhuri (1996). This test is much closer than previously suggested ones to the Kolmogorov–Smirnov type tests. In the context of symmetry testing, the univariate analogues can be found in Schuster and Barker (1987) and Arcones and Giné (1991). The test statistic is based on the sup-norm of certain empirical process, and main results are based on limit theorems for such processes and their bootstrapped versions. Considering other functionals of the process (instead of the sup-norm), one can get other tests for spherical symmetry (e.g., of ω^2-type). The center of symmetry is not supposed to be known, and to estimate it we use Haldane's spatial median. The test statistic is invariant with respect to the groups of all translations and orthogonal transformations of the space $\mathbb{R}^d$. We prove consistency of the test against any spherically asymmetric alternative, study the approximation of the distribution of the test statistics using a symmetrized bootstrap, and investigate numerically the performance of the test for moderate sample sizes. In what follows the problem is described in some detail.

Let P be a probability measure defined on the Borel σ-algebra $\mathscr{B}(\mathbb{R}^d)$ of the d-dimensional space $\mathbb{R}^d$. Given a vector $\theta \in \mathbb{R}^d$, denote by P_θ the θ-shift of P: $P_\theta(A) = P(A - \theta)$, $A \in \mathscr{B}(\mathbb{R}^d)$. P is called *spherically symmetric* about 0 iff, for all orthogonal transformations O of $\mathbb{R}^d$, $P \circ O^{-1} = P$. If there exists a vector $\theta \in \mathbb{R}^d$ such that $P_{-\theta}$ is spherically symmetric about 0, then P is said to be spherically symmetric about θ. The vector θ is then the center of spherical symmetry. Denote $\mathscr{SYM}(\mathbb{R}^d)$ the set of all spherically symmetric Borel probability measures on $\mathbb{R}^d$. Given a sample $(X_1, ..., X_n)$ of independent random vectors with common distribution P, our goal is to test the hypothesis H_0: $P \in \mathscr{SYM}(\mathbb{R}^d)$ against the alternative H_a: $P \notin \mathscr{SYM}(\mathbb{R}^d)$.

Preliminaries

Denote $x^{(j)}$ the jth component of $x = (x^{(1)}, ..., x^{(d)}) \in \mathbb{R}^d$; for an $\mathbb{R}^d$-valued function F, $F^{(j)}$ denotes its jth coordinate. Let $\langle \cdot, \cdot \rangle$ be the canonical inner product in $\mathbb{R}^d$ and $|\cdot| := \langle \cdot, \cdot \rangle^{1/2}$. $\|\cdot\|$ (often with indices) will stand for the norms of functions and operators. Also, $B_d := \{s: |s| < 1\}$, $S^{d-1} := \{s: |s| = 1\}$, and the uniform probability distribution on S^{d-1} will be denoted by m. Given $u, v \in \mathbb{R}^d$, $u \otimes v$ denotes the linear transformation $x \mapsto \langle v, x \rangle u$ from $\mathbb{R}^d$ into $\mathbb{R}^d$. For a differentiable function f on an open subset $U \subset \mathbb{R}^d$, $\nabla f = f'$ will denote the gradient (derivative) of f. If F is a differentiable function from U into $\mathbb{R}^d$, F' denotes its derivative. Note that, for $s \in U$, $F'(s)$ is a linear transformation of $\mathbb{R}^d$, so, F' is an operator-(matrix-)valued function on U. If A is an invertible linear transformation of $\mathbb{R}^d$, $\mathrm{inv}(A)$ denotes its inverse. Given an operator-valued function Ψ, defined on a subset of $\mathbb{R}^d$, $\mathrm{inv}(\Psi)$ is the function $s \mapsto \mathrm{inv}(\Psi(s))$ (assuming, of course, invertibility). Given a function f from a subset of $\mathbb{R}^d$ into $\mathbb{R}^d$, and an operator-valued function Ψ, defined on the domain of f, the "product" Ψf means the function $s \to \Psi(s)\, f(s)$.

The main probability space, on which the observations $X_1, X_2, \ldots$, are defined, will be denoted by $(\Omega, \Sigma, \mathrm{Pr})$. $\mathbb{E}$ is the expectation with respect to Pr. We also use the outer probability Pr* and the outer expectation $\mathbb{E}^*$. The symbol $\xrightarrow{d}$ is used to denote weak convergence of probability distribution (in $\mathbb{R}^1$ or $\mathbb{R}^d$); $\xrightarrow{P}$ denotes convergence in probability. The symbols o_P and O_P are used in a standard way.

Given a set S, denote $\ell^\infty(S)$ the space of all uniformly bounded $\mathbb{R}^d$-valued functions on S with the sup-norm $\|Y\|_S := \sup_{s \in S} |Y(s)|$, $Y \in \ell^\infty(S)$. A sequence of stochastic processes $\xi_n\colon S \mapsto \mathbb{R}^d$ is said to converge weakly in $\ell^\infty(S)$ iff there exists a Radon probability measure γ on $\ell^\infty(S)$ such that, for all bounded and $\|\cdot\|_S$-continuous functionals $\Phi\colon \ell^\infty(S) \mapsto \mathbb{R}^1$, we have $\mathbb{E}^*\Phi(\xi_n) \to \int_{\ell^\infty(S)} \Phi\, d\gamma$ as $n \to \infty$. In what follows S will be a metric space and γ the distribution of an a.e. bounded and uniformly continuous stochastic process $\xi\colon S \mapsto \mathbb{R}^d$. We use the sign $\xrightarrow{w}$ for such a convergence: $\xi_n \xrightarrow{w} \xi$ as $n \to \infty$.

Let $BL_1(\ell^\infty(S))$ be the set of all functionals $\Phi\colon \ell^\infty(S) \mapsto \mathbb{R}^1$ such that for all $Y \in \ell^\infty(S)$ $|\Phi(Y)| \leqslant 1$ and for all $Y_1, Y_2 \in \ell^\infty(S)$ $|\Phi(Y_1) - \Phi(Y_2)| \leqslant \|Y_1 - Y_2\|_S$. Given two random functions $\zeta_1, \zeta_2\colon \Omega \times S \mapsto \mathbb{R}^1$, defined the following distance:

$$d_{\mathrm{Pr}}(\zeta_1, \zeta_2) := \sup_{\Phi \in BL_1(\ell^\infty(S))} |\mathbb{E}^*\Phi(\zeta_1) - \mathbb{E}^*\Phi(\zeta_2)|.$$

It can be shown that $\xi_n \xrightarrow{w} \xi$ as $n \to \infty$ iff $d_{\mathrm{Pr}}(\xi_n; \xi) \to 0$ as $n \to \infty$ (ξ is an a.e. bounded and uniformly continuous stochastic process).

Test Statistics

In this paper we use an integral transform $P \mapsto F_P$ of the measure P, defined by (here and in what follows we assume that $0/0 = 0$)

$$F_P(s) := \int_{\mathbb{R}^d} \frac{s - x}{|s - x|}\, P(dx), \qquad s \in \mathbb{R}^d, \tag{1.1}$$

to construct a test for spherical symmetry of P. Such a transform was introduced by Koltchinskii and Dudley (see Koltchinskii, 1994a, b) and Chaudhuri (1996) in connection with their extension of quantiles to the multivariate case (the so called spatial or geometric quantiles). It was shown by these authors that, for a measure P which is not concentrated in a straight line, F_P is a one-to-one map from $\mathbb{R}^d$ into the open unit ball B_d, and, moreover, if P is nonatomic, $F_p(\mathbb{R}^d) = B_d$. F_P possesses many properties of the one-dimensional distribution function, including the fact that $F_P = F_Q$ implies $P = Q$, and its inverse F_P^{-1} is in many respects similar to the quantile function. Note, that in the one-dimensional case, F_P and F_P^{-1} are simple transformations of the distribution and quantile functions, respectively. It was also shown that $F_P^{-1}(0)$ coincides with the well-known Haldane's L_1-or spatial median of the distribution P, defined as a minimal point of the functional $f_P(s) := \int_{\mathbb{R}^d} (|s-x| - |x|)\, P(dx)$, $s \in \mathbb{R}^d$. In the case of a spherically symmetric P, the median $F_P^{-1}(0)$ is exactly the center of symmetry.

The idea of our approach is to use the following functional

$$\gamma(P) := \sup_{s \in \mathbb{R}^d} \left| F_P(s + F_P^{-1}(0)) - \psi_P(|s|) \frac{s}{|s|} \right| \tag{1.2}$$

with the function

$$\begin{aligned} \psi_P(\lambda) &:= \int_{S^{d-1}} \langle F_P(\lambda v + F_P^{-1}(0)), v \rangle\, m(dv) \\ &= \int_{\mathbb{R}^d} \int_{S^{d-1}} \frac{\langle \lambda v - x, v \rangle}{|\lambda v - x|}\, m(dv)\, \Pi(dx) \end{aligned} \tag{1.3}$$

as *a measure of spherical asymmetry* of P. Here $\Pi := P_{-\theta}$ where $\theta := F_P^{-1}(0)$. The next proposition gives the properties of the functional γ, which justify the definition. The proof easily follows from the main properties of F_P (see Koltchinskii, 1997).

1.1. PROPOSITION. *$\gamma(P)$ is invariant with respect to all shifts and all orthogonal transformation of* $\mathbb{R}^d$, i.e., *for all $\theta \in \mathbb{R}^d$ and all orthogonal transformations O of $\mathbb{R}^d$, $\gamma(P_\theta) = \gamma(P)$ and $\gamma(P \circ O^{-1}) = \gamma(P)$. $\gamma(P) = 0$ iff $P \in \mathcal{SYM}(\mathbb{R}^d)$, otherwise $\gamma(P) > 0$.*

Thus, it makes sense to base a test for spherical symmetry on the distribution of statistic $\gamma(P_n)$, which is the measure of spherical asymmetry of the empirical distribution P_n, based on a sample $(X_1, ..., X_n)$. For an absolutely continuous $P \in \mathcal{SYM}(\mathbb{R}^d)$ with a uniformly bounded density, we prove (in Section 2) the convergence in distribution of the sequence of statistics $T_n := n^{1/2} \gamma(P_n)$, investigate the limit behavior of the test statistics T_n under a sequence of asymmetric alternatives, converging to a spherically symmetric distribution, and evaluate the asymptotic power of the test against these alternatives.

A Symmetrized Bootstrap

Since the limit distribution of the test statistics depends on the unknown distribution P, we use bootstrap to estimate the critical values of our test. A version of bootstrap we suggest can be described as follows (see also Romano, 1989). Any spherically symmetric distribution P is completely characterized by its center θ and the distribution π of the radial component $|X-\theta|$, where X is a vector with distribution P. Let us call (θ, π) the parameters of P. Given a sample $(X_1, ..., X_n)$ from P, denote π_n the empirical distribution of the sample $(|X_1 - F_{P_n}^{-1}(0)|, ..., |X_n - F_{P_n}^{-1}(0)|)$. A spherically symmetric distribution with parameters $(F_{P_n}^{-1}(0), \pi_n)$ will be denoted P_n^s. Define on a probability space $(\hat{\Omega}, \hat{\Sigma}, \widehat{\Pr})$ a sample $(\hat{X}_1, ..., \hat{X}_n)$ of independent random vectors with common distribution P_n^s. It is worth noting that such a sample can be constructed as follows. Let $(\hat{R}_1, ..., \hat{R}_n)$ be a sample of independent random variables with common distribution π_n (it is produced by the regular resampling with replacement from the sample $(|X_1 - F_{P_n}^{-1}(0)|, ..., |X_n - F_{P_n}^{-1}(0)|)$, like in the usual bootstrap). Then take a sample $(U_1, ..., U_n)$ of independent random points with uniform distribution on the unit sphere S^{d-1}, which is also independent of $(X_1, ..., X_n)$ and $(\hat{R}_1, ..., \hat{R}_n)$. Set $\hat{X}_j := F_{P_n}^{-1}(0) + \hat{R}_j U_j$, $j = 1, ..., n$. Denote $\hat{P}_n$ the empirical distribution of $(\hat{X}_1, ..., \hat{X}_n)$. We show (Section 2) that, for any P with a uniformly bounded density, the sequence of bootstrapped statistics $\hat{T}_n := n^{1/2}\gamma(\hat{P}_n)$ converges in distribution to a limit random variable. Moreover, if $P \in \mathcal{SYM}(\mathbb{R}^d)$, then the limit is the same as for the sequence T_n. This allow us to justify the bootstrap version of the test.

2. ASYMPTOTICS OF THE TEST STATISTICS

In this section, we study the asymptotic behavior of the sequence of $\mathbb{R}^d$-valued stochastic processes

$$\delta_n(s) := n^{1/2} \left(F_{P_n}(s + F_{P_n}^{-1}(0)) - \psi_{P_n}(|s|) \frac{s}{|s|} \right), \qquad s \in \mathbb{R}^d.$$

Recall that the statistic T_n, suggested in Section 1 to test the hypothesis $H_0 \colon P \in \mathcal{SYM}(\mathbb{R}^d)$, is exactly the sup-norm of δ_n. In what follows we suppose that P has a uniformly bounded density in $\mathbb{R}^d$ with $d \geqslant 2$, which ensures that F_P is continuously differentiable in $\mathbb{R}^d$ with a uniformly bounded and uniformly continuous derivative F'_P. Moreover, $F'_P(s)$ is positively definite for all $s \in \mathbb{R}^d$ and F_P is a diffeomorphism of $\mathbb{R}^d$ and B_d. Denote

$$\delta_P(s) := \xi_P(s + F_P^{-1}(0)) - \int_{S^{d-1}} \langle \xi_P(|s|\, v + F_P^{-1}(0)), v \rangle\, m(dv)\, \frac{s}{|s|}$$
$$- F'_P(s + F_P^{-1}(0))\, \mathrm{inv}(F'_P(F_P^{-1}(0)))\, \xi_P(F_P^{-1}(0)),$$

where ξ_P is an $\mathbb{R}^d$-valued Gaussian process on $\mathbb{R}^d$ with zero mean and the covariance

$$\mathbb{E}\xi_P(s_1) \otimes \xi_P(s_2)$$
$$:= \int_{\mathbb{R}^d} \frac{s_1 - x}{|s_1 - x|} \otimes \frac{s_2 - x}{|s_2 - x|}\, P(dx) - F_P(s_1) \otimes F_P(s_2), \qquad s_1, s_2 \in \mathbb{R}^d.$$

2.1. Theorem. *If $P \in \mathcal{SYM}(\mathbb{R}^d)$, then $\delta_n \xrightarrow{w} \delta_P$ as $n \to \infty$. In particular, for the sequence of statistics $T_n := n^{1/2}\gamma(P_n) = \|\delta_n\|_{\mathbb{R}^d}$ we have*

$$\lim_{n \to \infty} \Pr\{T_n \leqslant t\} = \Pr\{\|\delta_P\|_{\mathbb{R}^d} \leqslant t\}, \qquad t \geqslant 0$$

On the other hand, if $P \notin \mathcal{SYM}(\mathbb{R}^d)$, then, for all $t \geqslant 0$, $\lim_{n \to \infty} \Pr\{T_n \leqslant t\} = 0$.

Recall that H_0 denotes the hypothesis $P \in \mathcal{SYM}(\mathbb{R}^d)$ and H_a denotes the alternative $P \notin \mathcal{SYM}(\mathbb{R}^d)$.

2.2. Corollary. *Given $\alpha \in (0, 1)$, let $t_\alpha := \inf\{t\colon \Pr\{\|\delta_P\|_{\mathbb{R}^d} \geqslant t\} \leqslant \alpha\}$. Then*

$$\Pr\{T_n \geqslant t_\alpha \mid H_0\} \to \alpha \qquad \text{and} \qquad \Pr\{T_n \geqslant t_\alpha \mid H_\alpha\} \to 1 \qquad \text{as} \quad n \to \infty.$$

Let now $P^{(n)}$ be a sequence of probability measures on $\mathbb{R}^d$ such that $P^{(n)} \xrightarrow{d} P$, where $P \in \mathcal{SYM}(\mathbb{R}^d)$. Let $(X_1, ..., X_n) = (X_{1,n}, ..., X_{n,n})$ be an i.i.d. sample from $P^{(n)}$, and denote by P_n the empirical measure based on this sample. Given a function $\Delta\colon \mathbb{R}^d \mapsto \mathbb{R}^d$, denote

$$\Delta_P(x) := \Delta(s + F_P^{-1}(0)) - \int_{S^{d-1}} \langle \Delta(|s|\, v + F_P^{-1}(0)), v \rangle\, m(dv)\, \frac{s}{|s|}$$
$$- F'_P(s + F_P^{-1}(0))\, \mathrm{inv}(F'_P(F_P^{-1}(0)))\, \Delta(F_P^{-1}(0)).$$

2.3. Theorem. *If*

$$\|F_{P^{(n)}} - F_P\|_{\mathbb{R}^d} = o(n^{-1/2}), \tag{2.1}$$

then

$$\delta_n \xrightarrow{w} \delta_P \qquad as \quad n \to \infty. \tag{2.2}$$

Suppose that there exists a sequence a_n of non-negative real numbers with $a_n \to \infty$, $a_n = O(n^{1/2})$ as $n \to \infty$ and a bounded uniformly continuous function $\Delta: \mathbb{R}^d \mapsto \mathbb{R}^d$ such that

$$a_n(F_{P^{(n)}} - F_P) \to \Delta \qquad as \quad n \to \infty \quad in\ \ell^\infty(\mathbb{R}^d). \tag{2.3}$$

If $a_n = n^{1/2}$, then

$$\delta_n \xrightarrow{w} \delta_P + \Delta_P \qquad as \quad n \to \infty. \tag{2.4}$$

If $a_n = o(n^{1/2})$, $n \to \infty$, then

$$\left\| \frac{a_n}{n^{1/2}} \delta_n - \Delta_P \right\|_{\mathbb{R}^d} \xrightarrow{p} 0 \qquad as \quad n \to \infty. \tag{2.5}$$

In particular, if $\Delta_P \not\equiv 0$, then for all $t \geqslant 0$, $\lim_{n\to\infty} \Pr\{\|\delta_n\|_{\mathbb{R}^d} \leqslant t\} = 0$.

Now we consider the problem of testing of the hypothesis H_0 (the unknown distribution of the sample $(X_1, ..., X_n)$ is spherically symmetric) against the sequence of alternatives $H_a^{(n)}$ (the unknown distribution is $P^{(n)}$).

2.4. Corollary. *Let $\alpha \in (0, 1)$. Then, under condition (2.1), $\Pr\{T_n \geqslant t_\alpha \mid H_a^{(n)}\} \to \alpha$ as $n \to \infty$. Under condition (2.3) with $a_n = n^{1/2}$ $\Pr\{T_n \geqslant t_\alpha \mid H_a^{(n)}\} \to \Pr\{\|\delta_P + \Delta_P\|_{\mathbb{R}^d} \geqslant t_\alpha\}$ as $n \to \infty$. Finally, under conditions (2.3) with $a_n = o(n^{(1/2})$ and $\Delta_P \not\equiv 0$, $\Pr\{T_n \geqslant t_\alpha \mid H_a^{(n)}\} \to 1$ as $n \to \infty$.*

To justify the symmetrized version of bootstrap, we study the asymptotic behavior of the bootstrap version of the process δ_n:

$$\hat{\delta}_n(s) := n^{1/2}\left(F_{\hat{P}_n}(s + F_{\hat{P}_n}^{-1}(0)) - \psi_{\hat{P}_n}(|s|)\frac{s}{|s|}\right), \qquad s \in \mathbb{R}^d.$$

The bootstrap version of T_n is $\hat{T}_n := \|\hat{\delta}_n\|_{\mathbb{R}^d}$. In what follows, we denote $\hat{\xi}_P$ a version of the process ξ_P, defined on the probability space $(\hat{\Omega}, \hat{\Sigma}, \widehat{\Pr})$; $\hat{\delta}_P$ will denote the corresponding version of the process δ_P. Let P^s denote *the spherical symmetrization* of P (more precisely, given a vector X with distribution P and a vector U with uniform distribution on the sphere S^{d-1}, independent of X, P^s is the distribution of $|X - F_P^{-1}(0)|\, U$).

2.5. THEOREM. *For all P with a uniformly bounded density in $\mathbb{R}^d$ ($d \geqslant 2$), $d_{\widehat{\mathrm{Pr}}}(\hat{\delta}_n, \hat{\delta}_{P^s}) \to 0$ as $n \to \infty$ in* Pr. *In particular, if $P \in \mathscr{SYM}(\mathbb{R}^d)$ (so that $P^s = P$), then $d_{\widehat{\mathrm{Pr}}}(\hat{\delta}_n, \hat{\delta}_P) \to 0$ as $n \to \infty$ in* Pr.

2.6. COROLLARY. *Given $\alpha \in (0, 1)$, let $t_{n,\alpha} := \inf\{t : \widehat{\Pr}\{\hat{T}_n \geqslant t\} \leqslant \alpha\}$. Then $\Pr\{T_n \geqslant t_{n,\alpha} \mid H_0\} \to \alpha$ and $\Pr\{T_n \geqslant t_{n,\alpha} \mid H_a\} \to 1$ as $n \to \infty$. Moreover, $t_{n,\alpha} \xrightarrow{P} t_\alpha$ as $n \to \infty$.*

3. SIMULATIONS AND NUMERICAL RESULTS

Next we study the performance of our test numerically for finite samples. We simulated i.i.d. samples from the following distributions in $\mathbb{R}^2$: $H_0^{(1)}$ = the standard normal distribution; $H_0^{(2)}$ = the uniform distribution in the unit ball B_2; $H_0^{(3)}$ = the uniform distribution on the unit circle S^1; $H_a^{(1)}$ = the distribution of the random vector with two independent exponential components with parameters $\lambda_1 = 1$ and $\lambda_2 = 2$, respectively; $H_a^{(2)}$ = the distribution of the random vector with two independent components, exponential with parameter $\lambda = 1$ and standard normal; $H_a^{(3)}$ = the mixture (with parameter 1/2) of two normal distributions in $\mathbb{R}^2$ with unit covariances and with means $(0, 0)$ and $(3, 0)$; $H_a^{(4)}$ = the uniform distribution in equilateral triangle with center at the point $(0, 0)$. Note that the first three distributions in this list belong to $\mathscr{SYM}(\mathbb{R}^2)$; the rest of the distributions are not *spherically* symmetric, although the last three ones have some symmetry.

In order to compute the empirical spatial median $F_{P_n}^{-1}(0)$ (the center of symmetry), we used a version of algorithms described in Gower (1974) and Bedall and Zimmermann (1979). The computation of the test statistics

$$T_n := n^{1/2}\gamma(P_n) := n^{1/2} \sup_{s \in \mathbb{R}^d} \left| F_{P_n}(s + F_{P_n}^{-1}(0)) - \psi_{P_n}(|s|) \frac{s}{|s|} \right| \tag{3.1}$$

is based on the stochastic approximation of the supremum in (3.1) by the maximum over a large sample of random points (like, e.g., in Beran and Millar, 1986). More precisely, T_n was approximated by a statistic

$$T_n(\mathscr{S}_{n,N}) := n^{1/2} \max_{s \in \mathscr{S}_{n,N}} \left| F_{P_n}(s + F_{P_n}^{-1}(0)) - \psi_{P_n}(|s|) \frac{s}{|s|} \right|, \tag{3.2}$$

where $\mathscr{S}_{n,N} := \{X_1, ..., X_n\} \cup \{Y_1, ..., Y_N\}$, with $Y_1, ..., Y_N$ being i.i.d. standard normal vectors. It can be shown that $T_n(\mathscr{S}_{n,N}) \to T_n$ as $N \to \infty$

a.e. The function $\psi_{P_n}(\lambda)$, involved in (4.1), can be represented as $\psi_{P_n}(\lambda) := n^{-1}\sum_1^n \varphi(\lambda; X_j - F_{P_n}^{-1}(0))$, where $\varphi(\lambda; x) := \int_{S^{d-1}} \langle \lambda v - x, v\rangle / |\lambda v - x|\, m(dv)$. The calculation of the function φ in the case $d=2$ can be reduced to a simple numerical integration. In general case, one can use instead Monte Carlo approximation.

We compare Monte Carlo simulation and bootstrap approximations of the distribution of the test statistic T_n in the case of hypothesis $H_0^{(1)}$, $H_0^{(2)}$, and $H_0^{(3)}$. For each of these three distributions, we simulated 1000 samples of size $n=200$. We calculated the values of the test statistics T_n for each of the samples. Then, we simulated one more sample of size $n=200$ from each of the three distributions, produced each time 1000 bootstrap samples and calculated the values of the test statistic. In Table I, we compare the critical values of the test statistic T_n, obtained (a) by Monte Carlo simulation; and (b) by bootstrap. We calculated these values for significance levels $\alpha=0.1$, $\alpha=0.05$, $\alpha=0.01$. Table II gives Monte Carlo evaluation of the power of the bootstrap test for symmetry for each of the alternatives $H_a^{(1)}$, $H_a^{(2)}$, $H_a^{(3)}$, $H_a^{(4)}$, for different sample sizes, and for different significance levels.

TABLE I
Critical Values of T_n

Distribution	Significance Level	Method	Sample Size 20	50	100	200
$H_0^{(1)}$	10%	B	1.16007	0.97156	0.95362	1.00776
		E	1.02702	1.01250	1.00776	0.97347
	5%	B	1.27394	1.03104	1.02888	1.08656
		E	1.12278	1.12390	1.08656	1.04135
	1%	B	1.54635	1.17104	1.19971	1.24562
		E	1.42764	1.31297	1.24652	1.17135
$H_0^{(2)}$	10%	B	1.58605	1.79056	2.45002	2.32483
		E	2.01084	2.35625	2.59439	2.80528
	5%	B	1.76256	2.02131	2.71941	2.61772
		E	2.14215	2.56910	2.83551	3.07841
	1%	B	2.01766	2.39789	3.20315	3.35989
		E	2.36844	2.93596	3.29352	3.62053
$H_0^{(3)}$	10%	B	1.58710	1.20667	1.14530	1.09400
		E	1.19430	1.12518	1.11741	1.10724
	5%	B	1.78296	1.36261	1.25964	1.17673
		E	1.33537	1.23256	1.22629	1.19583
	1%	B	2.02032	1.66681	1.49717	1.33904
		E	1.70394	1.45004	1.47125	1.41920

Note. (B) Bootstrap approximation; (E) Monte Carlo approximation.

TABLE II

The Power of the Bootstrap Test (Monte Carlo Evaluation)

Distribution	Sample size	Significance level 10%	5%	1%
$H_a^{(1)}$	100	23%	16%	4%
	200	94%	86%	55%
$H_a^{(2)}$	100	92%	90%	52%
	200	100%	92%	63%
$H_a^{(3)}$	100	14%	11%	2%
	200	92%	83%	44%
$H_a^{(4)}$	100	47%	21%	7%
	200	81%	57%	19%

Note. The number of samples is 100.

Based on these (and some other) experiments, we conclude that the symmetrized bootstrap approximates the distribution of the test statistic T_n reasonably well for moderate sample sizes. As to the power of the bootstrap test, it is rather low for the sample sizes $\leqslant 100$ (against all asymmetric alternatives we considered), but it becomes significantly higher for the sample sizes $\geqslant 200$. We have done similar experiments for two other tests for spherical symmetry, suggested in the literature (one of the tests of Baringhaus, 1991, and a version of Romano, 1989, bootstrap test). The performance of these tests for moderate sample sizes was rather close to ours.

4. PROOFS OF THE MAIN RESULTS

We assume that the reader is familiar with basic notions of empirical processes (VC-subgraph classes, uniformly Donsker classes, etc.; see, e.g., van der Vaart and Wellner, 1996).

4.1. PROPOSITION. *For any sequence $P^{(n)}$, such that $P^{(n)} \xrightarrow{d} P$ as $n \to \infty$, $n^{1/2}(F_{P_n} - F_{P^{(n)}}) \xrightarrow{q} \xi_P$ as $n \to \infty$.*

The proof easily follows from the fact that the class of functions $\mathscr{F}_1 := \{f^{(j)}(s;\cdot) := (s - \cdot)^{(j)}\,|s - \cdot|^{-1} : s \in \mathbb{R}^d,\ \ 1 \leqslant j \leqslant d\}$ is VC-subgraph and, hence, uniformly Donsker, and from general results on uniformity in weak convergence of measures, see Billingsley and Topsøe (1967). We need also the following proposition, which can be deduced, e.g., from the results of Koltchinskii (1994a, b; 1997).

4.2. Proposition. *If* $\|F_{P^{(n)}} - F_P\|_{\mathbb{R}^d} \to 0$, *then* $F_{P_n}^{-1}(0) \xrightarrow{p} F_P^{-1}(0)$. *Moreover, suppose that, for some sequence* a_n *of non-negative real numbers with* $a_n \to \infty$, $\|F_{P^{(n)}} - F_P\|_{\mathbb{R}^d} = O(a_n^{-1})$ *as* $n \to \infty$. *If* $a_n = o(n^{1/2})$, *then* $|F_{P_n}^{-1}(0) - F_P^{-1}(0)| = O_P(a_n^{-1})$ *and*

$$|F_{P_n}^{-1}(0) - F_P^{-1}(0) + \mathrm{inv}(F'_P(F_P^{-1}(0)))\, F_{P_n}(F_P^{-1}(0))| = o_P(a_n^{-1})$$

If $n^{1/2} = O(a_n)$, *then* $|F_{P_n}^{-1}(0) - F_P^{-1}(0)| = O_P(n^{-1/2})$ *and*

$$|F_{P_n}^{-1}(0) - F_P^{-1}(0) + \mathrm{inv}(F'_P(F_P^{-1}(0)))\, F_{P_n}(F_P^{-1}(0))| = o_P(n^{-1/2}).$$

Let S be the operator from the space of all uniformly bounded Borel functions g: $\mathbb{R}^d \mapsto \mathbb{R}^d$ into itself, defined by $Sg(s) := \int_{S^{d-1}} \langle g(|s|\, v), v\rangle\, m(dv)\, s\, |s|^{-1}$, $s \in \mathbb{R}^d$. Let $Tg := g - Sg$. Operator S is linear and bounded (moreover, it is a contraction): $\|Sg\|_{\mathbb{R}^d} \leqslant \|g\|_{\mathbb{R}^d}$. Clearly, T is also bounded. It is easily seen that if g is spherically symmetric about 0 (in the sense that it admits the representation $g(s) = \psi(|s|)\, s/|s|$), then $Sg \equiv g$ and $Tg \equiv 0$. If such a g is differentiable with uniformly bounded and uniformly continuous derivative, then we have $T(g'(s)\, x) = g'(s)\, x$ for all $s \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$. Given $\theta \in \mathbb{R}^d$, denote τ_θ the θ-shift operator in the space of all functions from $\mathbb{R}^d$ into $\mathbb{R}^d$: $\tau_\theta g(s) := g(s + \theta)$, $s \in \mathbb{R}^d$, g: $\mathbb{R}^d \mapsto \mathbb{R}^d$. For a differentiable g with uniformly continuous in $\mathbb{R}^d$ derivative g' and for any vector $s \in \mathbb{R}^d$, the function $\theta \mapsto T\tau_\theta g(s)$ is differentiable with respect to θ with the derivative such that

$$(\nabla_\theta T\tau_\theta g(\cdot))\, x = T\tau_\theta(g'(\cdot)\, x), \qquad x \in \mathbb{R}^d. \tag{4.1}$$

Moreover, the Taylor expansion of the first order holds for this function uniformly in $s \in \mathbb{R}^d$. In terms of operators τ_θ and T, the measure γ of spherical asymmetry can be written as $\gamma(P) = \|T\tau_{\theta_0} F_P\|_{\mathbb{R}^d}$ with $\theta_0 = F_P^{-1}(0)$. The process δ_n can be rewritten as $\delta_n = n^{1/2} T\tau_{\theta_n} F_{P_n}$ with $\theta_n := F_{P_n}^{-1}(0)$.

Proof of Theorems 2.1 *and* 2.3. We start with the setting of Theorem 2.3. Denote $\xi_n := n^{1/2}(F_{P_n} - F_{P^{(n)}})$. Under this notation, the following representation holds:

$$\begin{aligned}\delta_n &= n^{1/2} T\tau_{\theta_n} F_{P_n}\\ &= T\tau_{\theta_n}\xi_n + n^{1/2}(T\tau_{\theta_n} F_{P^{(n)}} - T\tau_{\theta_n} F_P)\\ &\quad + n^{1/2}(T\tau_{\theta_n} F_P - T\tau_{\theta_0} F_P) + n^{1/2} T\tau_{\theta_0} F_P.\end{aligned} \tag{4.2}$$

By Proposition 4.1, $\xi_n \xrightarrow{x} \xi_P$ as $n \to \infty$, so, the sequence ξ_n is asymptotically equicontinuous. By Proposition 4.2, we have $\theta_n = F_{P_n}^{-1}(0) \xrightarrow{p} F_P^{-1}(0) = \theta_0$ as $n \to \infty$. Thus, by asymptotic equicontinuity of ξ_n and the boundness of the operator T, we get

$$T\tau_{\theta_n}\xi_n = T\tau_{\theta_0}\xi_n + o_P(1) \qquad \text{in } \ell^\infty(\mathbb{R}^d). \tag{4.3}$$

Using differentiability of the map $\theta \mapsto T\tau_{\theta_0} F_P(\cdot)$ and the formula (4.1), we obtain

$$\begin{aligned} &a_n(T\tau_{\theta_n}F_P - T\tau_{\theta_0}F_P)(\cdot) \\ &\quad = T\tau_{\theta_0}F'_P(\cdot)\, a_n(\theta_n - \theta_0) + o(a_n(\theta_n - \theta_0)) \qquad \text{in } \ell^\infty(\mathbb{R}^d) \end{aligned} \tag{4.4}$$

(with $a_n \equiv n^{1/2}$ in the case of the condition (2.1)).

Suppose the condition (2.1) holds. Then, by Proposition 4.2, $\theta_n - \theta_0 = -\mathrm{inv}(F'_P(\theta_0))\, F_{P_n}(\theta_0) + o_P(n^{-1/2})$. Therefore

$$\begin{aligned} &n^{1/2}(T\tau_{\theta_n}F_P - T\tau_{\theta_0}F_P) \\ &\quad = -T\tau_{\theta_0}F'_P(\cdot)\,\mathrm{inv}(F'_P(\theta_0))[\xi_n(\theta_0) + n^{1/2}(F_{P^{(n)}} - F_P)(\theta_0)] + o_P(1), \end{aligned} \tag{4.5}$$

which implies (still under (2.1))

$$n^{1/2}(T\tau_{\theta_n}F_P - T\tau_{\theta_0}F_P) = -T\tau_{\theta_0}F'_P(\cdot)\,\mathrm{inv}(F'_P(\theta_0))\,\xi_n(\theta_0) + o_P(1). \tag{4.6}$$

We also have in this case $\|n^{1/2}(\tau_{\theta_n}F_{P^{(n)}} - \tau_{\theta_n}F_p)\|_{\mathbb{R}^d} = o_P(1)$, which, by the boundness of the operator T, implies $\|n^{1/2}(T\tau_{\theta_n}F_{P^{(n)}} - T\tau_{\theta_n}F_p)\|_{\mathbb{R}^d} = o_P(1)$. Now, relationships (4.2), (4.3), and (4.6) yield

$$\begin{aligned} \delta_n - n^{1/2}T\tau_{\theta_0}F_P = T\tau_{\theta_0}\xi_n - T\tau_{\theta_0}F'_P(\cdot)\,\mathrm{inv}(F'_P(\theta_0))\,\xi_n(\theta_0) + o_P(1) \\ \text{in } \ell^\infty(\mathbb{R}^d). \end{aligned} \tag{4.7}$$

If $P \in \mathcal{SYM}(\mathbb{R}^d)$, we have $T\tau_{\theta_0}F_P(\cdot) \equiv 0$ and $T\tau_{\theta_0}F'_P(\cdot)\,x \equiv \tau_{\theta_0}F'_P(\cdot)\,x$, $x \in \mathbb{R}^d$. Thus, (4.7) implies (2.2)

If the condition (2.3) holds, the Proposition 4.2 implies $\theta_n - \theta_0 = -\mathrm{inv}(F'_P(\theta_0))\, F_{P_n}(\theta_0) + o_P(a_n^{-1})$. In the case $a_n = n^{1/2}$, (4.5) yields

$$\begin{aligned} &n^{1/2}(T\tau_{\theta_n}F_P - T\tau_{\theta_0}F_P) \\ &\quad = -T\tau_{\theta_0}F'_P(\cdot)\,\mathrm{inv}(F'_P(\theta_0))[\xi_n(\theta_0) + \Delta(\theta_0)] + o_P(1). \end{aligned} \tag{4.8}$$

If (2.3) holds with $a_n = o(n^{1/2})$, we get, quite similarly,

$$a_n(T\tau_{\theta_n}F_P - T\tau_{\theta_0}F_P) = -T\tau_{\theta_0}F'_P(\cdot)\,\mathrm{inv}(F'_P(\theta_0))\,\Delta(\theta_0) + o_P(1). \tag{4.9}$$

Under the condition (2.3), we also have $\|a_n(\tau_{\theta_n} F_{P^{(n)}} - \tau_{\theta_n} F_P) - \tau_{\theta_n} \Delta\|_{\mathbb{R}^d} \xrightarrow{p} 0$. Since Δ is uniformly continuous, $\theta_n \xrightarrow{p} \theta_0$ and the operator T is bounded, we get

$$\|a_n(T\tau_{\theta_n} F_{P^{(n)}} - T\tau_{\theta_n} F_P) - T\tau_{\theta_0} \Delta\|_{\mathbb{R}^d} \xrightarrow{p} 0. \tag{4.10}$$

If $a_n = n^{1/2}$, we have (due to (4.2), (4.3), (4.8), and (4.10))

$$\begin{aligned}\delta_n - n^{1/2} T\tau_{\theta_0} F_P = T\tau_{\theta_0}\xi_n + T\theta_0 \Delta - T\tau_{\theta_0} F'_P(\cdot)\,\mathrm{inv}(F'_P(\theta_0))& \\ \times [\xi_n(\theta_0) + \Delta(\theta_0)] + o_P(1) \qquad &\text{in } \ell^\infty(\mathbb{R}^d),\end{aligned}$$

which, under the assumption $P \in \mathcal{SYM}(\mathbb{R}^d)$, implies (2.4)

Finally, under the condition (2.3) with $a_n = o(n^{1/2})$, we have (due to (4.2), (4.3), (4.9), and (4.10))

$$\begin{aligned}&\frac{a_n}{n^{1/2}}\,\delta_n - a_n T\tau_{\theta_0} F_P \\ &\qquad = T\tau_{\theta_0}\Delta - T\tau_{\theta_0} F'_P(\cdot)\,\mathrm{inv}(F'_P(\theta_0))\,\Delta(\theta_0) + o_P(1) \qquad \text{in } \ell^\infty(\mathbb{R}^d),\end{aligned}$$

which implies (2.5) (for $P \in \mathcal{SYM}(\mathbb{R}^d)$). This completes the proof of Theorem 2.3.

The first statement of Theorem 2.1 follows if $P^{(n)} \equiv P$. By the well-known Cirel'son's Theorem, (see Cirel'son, 1975) the distribution of the random variable $\|\delta_P\|_{\mathbb{R}^d}$ is absolutely continuous with a strictly positive density (on the interior of its support). Thus, any t in the interior of the support is a continuity point of this distribution and we have $\Pr\{T_n \leqslant t\} \to \Pr\{\|\delta_P\|_{\mathbb{R}^d} \leqslant t\}$ as $n \to \infty$. If P is not symmetric about θ_0, then $\|T\tau_{\theta_0} F_P\|_{\mathbb{R}^d} > 0$, and $n^{1/2}\,\|T\tau_{\theta_0} F_P\|_{\mathbb{R}^d} \to +\infty$. Since the sequence $\|\delta_n - n^{1/2} T\tau_{\theta_0} F_P\|_{\mathbb{R}^d}$ is stochastically bounded, we get $\Pr\{T_n \leqslant t\} \to 0$. ▮

To prove Theorem 2.6, we need a few more facts and some new notations. Denote

$$G(s;\theta;x;v) := \frac{s - |x-\theta|\,v}{|s - |x-\theta|\,v|}, \qquad s, \theta, x \in \mathbb{R}^d; \qquad v \in S^{d-1};$$

$$\mathcal{G}_1 := \{G^{(j)}(s;\theta;\cdot;v) \colon s \in \mathbb{R}^d, \theta \in \mathbb{R}^d, v \in S^{d-1}, 1 \leqslant j \leqslant d\},$$

$$\begin{aligned}\mathcal{G}_2 := \{&G^{(j)}(s_1;\theta_1;\cdot;v_1)\,G^{(k)}(s_2;\theta_2;\cdot;v_2);\, s_1, s_2, \theta_1, \theta_2 \in \mathbb{R}^d, \\ &v_1, v_2 \in S^{d-1}, 1 \leqslant j, k \leqslant d\}.\end{aligned}$$

We skip the proofs of the next two propositions (the first one can be proved similarly to the fact that $\mathcal{F}_1$ is a VC-subgraph class (see, e.g., Koltchinskii, 1997).

4.3. PROPOSITION. *The classes of functions* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are both* VC-*subgraph.*

4.4. PROPOSITION. *If* P *has a uniformly bounded density, then*

(i) *the map* $\mathbb{R}^d \times [-r, r]^d \times S^{d-1} \ni (s; \theta; v) \mapsto G^{(j)}(s; \theta; \cdot; v) \in L_2(P)$ *is uniformly continuous for all* $j = 1, ..., d$ *and for all* $r > 0$.

(ii) *the map* $\mathbb{R}^d \ni \theta \mapsto \rho(s, \theta) := \int_{\mathbb{R}^d} \int_{S^{d-1}} G(s; \theta; x; v)\, m(dv)\, P(dx) \in \mathbb{R}^d$ *is differentiable at the point* $\theta = \theta_0$ *for all* $s \in \mathbb{R}^d$, *and, moreover, the derivative is uniformly bounded in* $s \in \mathbb{R}^d$ *and the Taylor expansion of the first order holds uniformly in* $s \in \mathbb{R}^d$.

We denote $\Gamma(s)$ the derivative of the last map at $\theta = \theta_0$. Note that, for all $s \in \mathbb{R}^d$,

$$F_{P_n^s}(s) = \int_{\mathbb{R}^d} \frac{s-x}{|s-x|}\, P_n^s(dx) = \int_{S^{d-1}} \int_{\mathbb{R}^d} G(s; \theta_n; x; v)\, P_n(dx)\, m(dv) \tag{4.11}$$

and

$$F_{P^s}(s) = \int_{\mathbb{R}^d} \frac{s-x}{|s-x|}\, P^s(dx) = \int_{S^{d-1}} \int_{\mathbb{R}^d} G(s; \theta_0; x; v)\, P(dx)\, m(dv). \tag{4.12}$$

4.5. LEMMA. *For a probability distribution* P *with a uniformly bounded density,* $n^{1/2}(F_{P_n^s} - F_{P^s})(\cdot) \xrightarrow{w} \Lambda(\cdot\,; \theta_0) - \Gamma(\cdot)\, \mathrm{inv}(F'_P(\theta_0))\, \xi_P(\theta_0)$, *where*

$$\Lambda(s; \theta) := \int_{S^{d-1}} \zeta(s; \theta, v)\, m(dv), \qquad \zeta(s; \theta; v) := \int_{\mathbb{R}^d} G(s; \theta; x; v)\, W_P^o(dx).$$

Proof. Denote

$$\zeta_n(s; \theta; v) := n^{1/2} \int_{\mathbb{R}^d} G(s; \theta; x; v)(P_n - P)(dx),$$

$$\Lambda_n(s; \theta) := \int_{S^{d-1}} \zeta_n(s, \theta, v)\, m(dv).$$

Under these notations, it follows from (4.11) and (4.12), that

$$n^{1/2}(F_{P_n^s} - F_{P^s})(x) = \Lambda_n(s, \theta_0) + [\Lambda_n(s, \theta_n) - \Lambda_n(s, \theta_0)] \\ + n^{1/2}[\rho(s, \theta_n) - \rho(s, \theta_0)]. \tag{4.13}$$

Proposition 4.4(i) and the fact that the class $\mathcal{G}_1$ is universally Donsker yield that $\zeta_n \xrightarrow{w} \zeta$ as $n\to\infty$, which implies $\Lambda_n \xrightarrow{w} \Lambda$ as $n\to\infty$. Since $n^{1/2}(\theta_n-\theta_0) = -\mathrm{inv}(F'_P(\theta_0))\,\xi_n(\theta_0)+o_P(1)$, we get, using Proposition 4.4(ii),

$$n^{1/2}[\rho(s,\theta_n)-\rho(s,\theta_0)] = -\Gamma(s)\,\mathrm{inv}(F'_P(\theta_n))\,\xi_n(\theta_0)+o_P(1) \qquad \text{in } \ell^\infty(\mathbb{R}^d). \tag{4.14}$$

Now, the representation (4.13), the asymptotic equicontinuity of the sequence Λ_n, and the relationship (4.14) imply the result. ▮

Using representations similar to (4.11) and (4.12) and the fact that $\mathcal{G}_1$ and $\mathcal{G}_2$ are uniformly bounded VC-subgraph, and, hence, universally Donsker, it is easy to get

$$\int_{\mathbb{R}^d} \frac{s^{(j)}-x^{(j)}}{|s-x|}\,P_n^s(dx) \xrightarrow{p} \int_{\mathbb{R}^d} \frac{s^{(j)}-x^{(j)}}{|s-x|}\,P^s(dx),$$

and

$$\int_{\mathbb{R}^d} \frac{s_1^{(j)}-x^{(j)}}{|s-x|}\,\frac{s_2^{(k)}-x^{(k)}}{|s-x|}\,P_n^s(dx) \xrightarrow{p} \int_{\mathbb{R}^d} \frac{s_1^{(j)}-x^{(j)}}{|s-x|}\,\frac{s_2^{(k)}-x^{(k)}}{|s-x|}\,P^s(dx)$$

uniformly in s, s_1, s_2 as $n\to\infty$. Since $\mathcal{G}_1$ is uniformly Donsker (as any uniformly bounded VC-subgraph class), the last relationships lead to the following.

4.6. Proposition. *For a probability distribution P with uniformly bounded density,*

$$d_{\widehat{\Pr}}(n^{1/2}(F_{\hat P_n}-F_{P_n^s});\,\hat\xi_{P^s})\to 0 \qquad \textit{as} \quad n\to\infty \textit{ in } \Pr.$$

In what follows the symbol $\xrightarrow{p}$ means convergence in $\Pr\times\widehat{\Pr}$ (it applies also to the notations o_P and O_P). Finally, we need the following statement.

4.7. Proposition. *If P has a uniformly bounded density, then*

$$F_{\hat P_n}^{-1}(0)-F_{P_n^s}^{-1}(0)+\mathrm{inv}(F'_{P^s}(\theta_0))(F_{\hat P_n}(\theta_0)-F_{P_n^s}(\theta_0))$$
$$= o_P(n^{-1/2}) \qquad \textit{as} \quad n\to\infty.$$

The proof follows from Lemma 4.5, Proposition 4.6, the differentiability properties of functional inverse (see, e.g., Koltchinskii, 1994a, b; 1995), and the fact that, for a P with a uniformly bounded density, F_{P^s} is differentiable in $\mathbb{R}^d$ with uniformly bounded and uniformly continuous derivative F'_{P^s}.

Proof of Theorem 2.5. We follow the proof of Theorems 2.1 and 2.3. Denote $\hat{\theta}_n := F_{\hat{P}_n}^{-1}(0)$, $\hat{\xi}_n := n^{1/2}(F_{\hat{P}_n} - F_{P^s_n})$. The following representation is the bootstrap version of (4.2):

$$\hat{\delta}_n = T\tau_{\hat{\theta}_n}\hat{\xi}_n + (n^{1/2}(T\tau_{\hat{\theta}_n} F_{P^s_n} - T\tau_{\hat{\theta}_n} F_{P^s}) - n^{1/2}(T\tau_{\theta_n} F_{P^s_n} - T\tau_{\theta_n} F_{P^s}))$$
$$+ n^{1/2}(T\tau_{\hat{\theta}_n} F_{P^s} - T\tau_{\theta_n} F_{P^s}). \tag{4.15}$$

The asymptotic equicontinuity of $\hat{\xi}_n$ (which follows from Proposition 4.6), the fact that $\hat{\theta}_n \xrightarrow{p} \theta_0$ (see Proposition 4.7), and the boundness of the operator T imply

$$T\tau_{\hat{\theta}_n}\hat{\xi} = T\tau_{\theta_0}\hat{\xi}_n + o_P(1) \qquad \text{in } \ell^\infty(\mathbb{R}^d). \tag{4.16}$$

Since F_{P^s} is differentiable in $\mathbb{R}^d$ with a uniformly bounded and uniformly continuous derivative, formula (4.1) implies that uniformly in $s \in \mathbb{R}^d$

$$n^{1/2}(T\tau_{\hat{\theta}_n} F_{P^s} - T\tau_{\theta_n} F_{P^s})(s)$$
$$= T\tau_{\theta_0} F'_{P^s}(s)\, n^{1/2}(\hat{\theta}_n - \theta_n) + o(n^{1/2}(\hat{\theta}_n - \theta_n)). \tag{4.17}$$

It follows from Proposition 4.7 and the spherical symmetry of P^s that

$$n^{1/2}(T\tau_{\hat{\theta}_n} F_{P^s} - T\tau_{\theta_n} F_{P^s})(s)$$
$$= -\tau_{\theta_0} F'_{P^s}(s)\, \mathrm{inv}(F'_{P^s}(\theta_0))\, n^{1/2}(F_{\hat{P}_n}(\theta_0) - F_{P^s_n}(\theta_0)) + o_P(1)$$
$$\text{in } \ell^\infty(\mathbb{R}^d). \tag{4.18}$$

By Lemma 4.5, the sequence $n^{1/2}(F_{P^s_n} - F_{P^s})$ is asymptotically equicontinuous, which implies (since the operator T is bounded and, by Proposition 4.7, $\hat{\theta}_n - \theta_n = o_P(1)$)

$$(n^{1/2}(T\tau_{\hat{\theta}_n} F_{P^s_n} - T\tau_{\hat{\theta}_n} F_{P^s}) - n^{1/2}(T\tau_{\theta_n} F_{P^s_n} - T\tau_{\theta_n} F_{P^s}))$$
$$= o_P(1) \qquad \text{in } \ell^\infty(\mathbb{R}^d). \tag{4.19}$$

It follows from (4.15), (4.16), (4.18), and (4.19) that

$$\hat{\delta}_n(\cdot) = T\tau_{\theta_0}\hat{\xi}_n(\cdot) - \tau_{\theta_0} F'_{P^s}(\cdot)\, \mathrm{inv}(F'_{P^s}(\theta_0))\, \hat{\xi}_n(\theta_0) + o_P(1) \qquad \text{in } \ell^\infty(\mathbb{R}^d),$$

which, in view of Proposition 4.6 and the boundness of T, implies the statements of the theorem. ∎

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