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The Distribution of Estimates of Parameters of Multidimensional Stationary AR Processes

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Abstract—It is shown that the suitably normalized maximum likelihood estimates of the parameters of periodical type of some multidimensional stationary AR processes have exactly normal distribution. This provides a generalization of the well-known behaviour of the estimate of period of a complex 1-dimensional AR process (see [1–6]).

Keywords—Multidimensional stationary autoregressive processes, Radon-Nikodym derivative, Itô's formula.

1. INTRODUCTION

Consider the complex-valued stationary autoregressive process $\xi(t) = \xi_1(t) + i\xi_2(t)$, $t \geq 0$, given by the stochastic differential equation

$$d\xi(t) = -\gamma\xi(t)\,dt + dw(t),$$

where $w(t) = w_1(t) + iw_2(t)$, $t \geq 0$, is a standard complex Wiener process (i.e., $w_1(t)$ and $w_2(t)$ are independent standard real-valued Wiener processes) and $\gamma = \lambda - i\omega$ with $\lambda > 0$, $\omega \in \mathbb{R}$.

Consider the statistics

$$s_\xi^2(t) = \int_0^t |\xi(u)|^2\,du, \qquad r_\xi(t) = \int_0^t |\xi(u)|^2\,d\theta(u),$$

where $\theta(t), t \geq 0$, is defined by

$$\xi(t) = |\xi(t)|e^{i\theta(t)}.$$

The process

$$r_\xi(t) = \int_0^t (\xi_1(u)\,d\xi_2(u) - \xi_2(u)\,d\xi_1(u)), \qquad t \geq 0,$$

is called Lévy's stochastic area process. (It is interesting to remark that in case $\gamma = 0$, i.e., $\xi(t) = w(t)$, $r_w(t) = \int_0^t (w_1(u)\,dw_2(u) - w_2(u)\,dw_1(u))$ the process $(w_1(t), w_2(t), r_w(t))$, $t \geq 0$, is just the standard Wiener process on the Heisenberg group, see e.g., [7,8].)

It is known that the maximum likelihood estimate of the period ω is

$$\hat{\omega}_\xi(t) = \frac{r_\xi(t)}{s_\xi^2(t)},$$

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and

$$\sqrt{s_\xi^2(t)}\,(\hat{\omega}_\xi(t)-\omega) \stackrel{\mathcal{D}}{=} \mathcal{N}(0,1) \qquad \text{for all } t>0,$$

where $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution. Surprisingly, we have exact distribution, not only an asymptotic property! This statement was first formulated in [4]. Complicated proofs can be found in [1,5,6,9]. Using Novikov's method, Arató [3] gave an elegant new proof.

The above mentioned result can be reformulated in the following way. Let us consider the 2-dimensional real-valued stationary autoregressive process $X(t)$, $t \geq 0$, given by the stochastic differential equation

$$\begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \begin{pmatrix} -\lambda & -\omega \\ \omega & -\lambda \end{pmatrix} \begin{pmatrix} X_(t)\,dt \\ X_2(t)\,dt \end{pmatrix} + \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix},$$

where $W(t)$, $t \geq 0$, is a standard 2-dimensional Wiener process, and $\lambda > 0$, $\omega \in \mathbb{R}$. Consider the statistics

$$s_X^2(t) = \int_0^t (X_1^2(u)+X_2^2(u))\,du, \qquad r_X(t) = \int_0^t (X_1(u)\,dX_2(u) - X_2(u)\,dX_1(u)).$$

Then the maximum likelihood estimate of the period ω is

$$\hat{\omega}_X(t) = \frac{r_X(t)}{s_X^2(t)},$$

and

$$\sqrt{s_X^2(t)}\,(\hat{\omega}_X(t)-\omega) \stackrel{\mathcal{D}}{=} \mathcal{N}(0,1) \qquad \text{for all } t>0.$$

The aim of the present paper is to generalize this result for some 3- and 4-dimensional stationary AR processes.

2. PRELIMINARIES

Let $X(t) = (X_1(t),\dots,X_k(t))$, $t \geq 0$, be the k-dimensional stationary (continuous) process given by the stochastic differential equation

$$dX(t) = AX(t)\,dt + dW(t),$$

where $W(t)$, $t \geq 0$, is a standard k-dimensional Wiener process, A is a $k \times k$ matrix with eigenvalues having negative real parts. It is known that $X(t)$, $t \geq 0$ is a Gaussian process with

$$\mathbb{E}X(t) = 0, \qquad \mathbb{E}X(s+t)X^*(s) = R(t) = e^{tA}R(0),$$

where $R(0) = \mathbb{E}X(s)X^*(s)$ is the unique solution of the matrix equation

$$AR(0) + R(0)A^* = -I, \tag{1}$$

where I is the unit matrix.

Let $\mathbb{P}_{t,X}$ and $\mathbb{P}_{t,W}$ be the measures generated on $((C[0,t])^k, \mathcal{B}((C[0,t])^k))$ by the processes $X(s)$, $0 \leq s \leq t$ and $W(s)$, $0 \leq s \leq t$, respectively. If $R(0)$ is nonsingular, the the measures $\mathbb{P}_{t,X}$ and $\mathbb{P}_{t,W}$ are equivalent and the Radon-Nikodym derivative has the form

$$\frac{d\mathbb{P}_{t,X}}{d\mathbb{P}_{t,W}}(X) = (2\pi)^{-k/2}\,(\operatorname{Det} R(0))^{-1/2} \exp\left\{-\frac{1}{2}\langle R^{-1}(0)X(0), X(0)\rangle + \int_0^t \langle AX(u), dX(u)\rangle \right.$$
$$\left. -\frac{1}{2}\int_0^t \langle AX(u), AX(u)\rangle\,du\right\}.$$

For an arbitrary symmetric matrix M, Itô's formula implies

$$\int_0^t \langle MX(u), dX(u)\rangle = \frac{1}{2}\left(\langle MX(t), X(t)\rangle - \langle MX(0), X(0)\rangle\right) - \frac{1}{2}t \operatorname{Tr} M, \tag{2}$$

where Tr denotes the trace. Formula (2) can be applied for evaluation of the Radon-Nikodym derivative splitting the matrix A into its symmetric and skew-symmetric parts

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A + A^*).$$

For investigation of the distribution of functionals of integral type we shall use the following statement (see e.g., [10,11]). Let $\psi : [0,T]^k \to \mathbb{R}$ be a function such that ψ, $\frac{\partial \psi}{\partial x_i}$, $\frac{\partial^2 \psi}{\partial x_i \partial x_j}$, $1 \le i,j \le k$ are bounded continuous functions. Then the function $u(t,x)$, $0 \le t \le T$, $x \in \mathbb{R}^k$ defined by

$$u(t,x) = \mathbb{E}\left(\exp\left\{\int_0^t \psi(X(s))\, ds\right\} \middle| X(0) = x\right)$$

is the solution of the Cauchy problem

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \langle AX, \nabla u\rangle + \psi(x)u, \qquad u(0,x) = 1 \text{ for } x \in \mathbb{R}^k. \tag{3}$$

For computation of expected value of a random variable the following simple formula holds. Let ξ and η be random variables with distributions $\mathbb{P}_\xi$ and $\mathbb{P}_\eta$ such that $\mathbb{P}_\xi \ll \mathbb{P}_\eta$ ($\mathbb{P}_\xi$ is absolutely continuous with respect to $\mathbb{P}_\eta$). Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded Borel-measurable function. Then

$$\mathbb{E}g(\xi) = \mathbb{E}\left(g(\eta)\frac{d\mathbb{P}_\xi}{d\mathbb{P}_\eta}(\eta)\right).$$

We shall make use of the conditional version of the above formula.

LEMMA 1. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(\mathbb{X}, \mathcal{X})$ a measurable space. Let $\xi, \eta : \Omega \to \mathbb{X}$ be random elements with distributions $\mathbb{P}_\xi$ and $\mathbb{P}_\eta$ such that $\mathbb{P}_\xi \ll \mathbb{P}_\eta$. Let $g, h : \mathbb{X} \to \mathbb{R}$ be measurable functions and suppose that g is bounded and $\mathbb{P}_{h(\eta)} \ll \mathbb{P}_{h(\xi)}$. Then*

$$\mathbb{E}\left(g(\xi) \mid h(\xi) = x\right) = \mathbb{E}\left(g(\eta)\frac{d\mathbb{P}_\xi}{d\mathbb{P}_\eta}(\eta) \middle| h(\eta) = x\right)\frac{d\mathbb{P}_{h(\eta)}}{d\mathbb{P}_{h(\xi)}}(x) \qquad \left(\mathbb{P}_{h(\xi)} \text{ a.s. }\right).$$

PROOF. It is only to show that for any Borel set $B \in \mathcal{B}(\mathbb{R})$

$$\int_B \mathbb{E}\left(g(\eta)\frac{d\mathbb{P}_\xi}{d\mathbb{P}_\eta}(\eta) \middle| h(\eta) = x\right)\frac{d\mathbb{P}_{h(\eta)}}{d\mathbb{P}_{h(\xi)}}(x)\mathbb{P}_{h(\xi)}(dx) = \int_{\{\omega : h(\xi(\omega)) \in B\}} g(\xi(\omega))\,\mathbb{P}(d\omega).$$

The left-hand side is equal to

$$\int_B \mathbb{E}\left(g(\eta)\frac{d\mathbb{P}_\xi}{d\mathbb{P}_\eta}(\eta) \middle| h(\eta) = x\right)\mathbb{P}_{h(\eta)}(dx) = \int_{\{\omega : h(\eta(\omega)) \in B\}} g(\eta(\omega))\,\frac{d\mathbb{P}_\xi}{d\mathbb{P}_\eta}(\eta(\omega))\,\mathbb{P}(d\omega)$$

$$= \mathbb{E}\left(g(\eta)\chi_{h^{-1}(B)}(\eta)\frac{d\mathbb{P}_\xi}{d\mathbb{P}_\eta}(\eta)\right) = \mathbb{E}\left(g(\xi)\chi_{h^{-1}(B)}(\xi)\right) = \int_{\{\omega : h(\xi(\omega)) \in B\}} g(\xi(\omega))\,\mathbb{P}(d\omega),$$

where $\chi_{h^{-1}(B)}$ denotes the indicator function of the set $h^{-1}(B) = \{x \in \mathbb{R} : h(x) \in B\}$. Hence the assertion.

3. A 3-DIMENSIONAL AR PROCESS

Consider the 3-dimensional stationary AR process $X(t)$, $t \geq 0$ given by

$$\begin{pmatrix} dX_1(t) \\ dX_2(t) \\ dX_3(t) \end{pmatrix} = \begin{pmatrix} -\lambda & -\omega & \omega \\ \omega & -\lambda & -\omega \\ -\omega & \omega & -\lambda \end{pmatrix} \begin{pmatrix} X_1(t)dt \\ X_2(t)dt \\ X_3(t)dt \end{pmatrix} + \begin{pmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{pmatrix},$$

where $W(t)$, $t \geq 0$ is a standard 3-dimensional Wiener process, and $\lambda > 0$, $\omega \in \mathbb{R}$. Consider the statistics

$$\begin{aligned} s_X^2(t) &= 2\int_0^t \left(X_1^2(u) + X_2^2(u) + X_3^2(u) - X_1(u)X_2(u) - X_1(u)X_3(u) - X_2(u)X_3(u)\right) du \\ &= \int_0^t \left((X_1(u) - X_2(u))^2 + (X_1(u) - X_3(u))^2 + (X_2(u) - X_3(u))^2\right) du, \end{aligned}$$

$$\begin{aligned} r_X(t) = \int_0^t \big(&X_1(u)\,dX_2(u) - X_2(u)\,dX_1(u) + X_1(u)\,dX_3(u) - X_3(u)\,dX_1(u) \\ &+ X_2(u)\,dX_3(u) - X_3(u)\,dX_2(u)\big). \end{aligned}$$

First we investigate the distribution of the statistic $s_X^2(t)$.

LEMMA 2. *The distribution of $s_X^2(t)$ does not depend on the parameter ω (it depends only on the parameter λ).*

PROOF. We present two methods for proving this important statement.

First we examine the conditional Laplace transform

$$u(t, x_1, x_2, x_3) = \mathbb{E}\left(\exp\{-\alpha s_X^2\} \,\middle|\, X_1(0) = x_1, X_2(0) = x_2, X_3(0) = x_3\right).$$

Applying (3) we obtain that $u : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ is the solution of

$$\begin{aligned} \frac{\partial u}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) &+ (-\lambda x_1 - \omega x_2 + \omega x_3)\frac{\partial u}{\partial x_1} + (\omega x_1 - \lambda x_2 - \omega x_3)\frac{\partial u}{\partial x_2} \\ &+ (-\omega x_1 + \omega x_2 - \lambda x_3)\frac{\partial u}{\partial x_3} - 2\alpha\left(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3\right)u \end{aligned}$$

with $u(0, x_1, x_2, x_3) = 1$ for $x_1, x_2, x_3 \in \mathbb{R}$. It can be shown easily that there is a function $v(t, \varrho)$, $t, \varrho \geq 0$ such that

$$u(t, x_1, x_2, x_3) = v\left(t, x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3\right)$$

and v is the solution of

$$\frac{\partial v}{\partial t} = 3\varrho\frac{\partial^2 v}{\partial \varrho^2} + 2(3 - \lambda\varrho)\frac{\partial v}{\partial \varrho} - 2\alpha\varrho v$$

with $v(0, x_1, x_2, x_3) = 1$ for $x_1, x_2, x_3 \in \mathbb{R}$. Consequently, we obtain that the conditional distribution of the statistic $s_X^2(t)$ under the initial condition

$$X_1(0) = x_1, \quad X_2(0) = x_2, \quad X_3(0) = x_3$$

does not depend on the parameter ω. Solving the equation (1) we obtain $\mathbb{E}X(0)X^*(0) = R(0) = (2\lambda)^{-1}I$, thus the distribution of the variable $X(0) = (X_1(0), X_2(0), X_3(0))$ also does not depend on ω. Hence the assertion.

The second method for proving the lemma is based on Itô's formula. Let us examine the process

$$Z_X(t) = X_1^2(t) + X_2^2(t) + X_3^2(t) - X_1(t)X_2(t) - X_1(t)X_3(t) - X_2(t)X_3(t).$$

Using

$$d(X_1^2) = [2X_1(-\lambda X_1 - \omega X_2 + \omega X_3) + 1]\, dt + 2X_1\, dW_1$$
$$d(X_1X_2) = [X_1(\omega X_1 - \lambda X_2 - \omega X_3) + X_2(-\lambda X_1 - \omega X_2 + \omega X_3)]\, dt + X_2\, dW_1 + X_1\, dW_2$$

and similar formulas for the other terms of $Z_X(t)$ we obtain

$$dZ_X = (3 - 2\lambda Z_X)\, dt + (2X_1 - X_2 - X_3)\, dW_1 + (2X_2 - X_1 - X_3)\, dW_2 + (2X_3 - X_1 - X_2)\, dW_3.$$

Consider now the process $\widetilde{W}(t)$, $t \geq 0$ defined by

$$\widetilde{W}(t) = \int_0^t \frac{(2X_1 - X_2 - X_3)\, dW_1 + (2X_2 - X_1 - X_3)\, dW_2 + (2X_3 - X_1 - X_2)\, dW_3}{\sqrt{6(X_1^2 + X_2^2 + X_3^2 - X_1X_2 - X_1X_3 - X_2X_3)}}.$$

The paths of the process $\widetilde{W}(t)$, $t \geq 0$ are continuous with probability 1, and we have for all $0 \leq s \leq t$

$$\mathbb{E}\left(\widetilde{W}(t) \mid \mathcal{F}_s^W\right) = \widetilde{W}(s)$$
$$\mathbb{E}\left(\left(\widetilde{W}(t) - \widetilde{W}(s)\right)^2 \mid \mathcal{F}_s^W\right) = t - s,$$

where $\mathcal{F}_s^W = \sigma\{W(u) : 0 \leq u \leq s\}$. Thus Lévy's theorem implies that $\widetilde{W}(t)$, $t \geq 0$ is a standard Wiener process. Consequently, the process $Z_X(t)$, $t \geq 0$ is a weak solution of the stochastic differential equation

$$dZ_X(t) = (3 - 2\lambda Z_X(t))\, dt + \sqrt{6Z_X(t)}\, d\widetilde{W}(t).$$

Using again that the covariance matrix $\mathbb{E}X(0)X^*(0) = R(0) = (2\lambda)^{-1}I$ depends only on the parameter λ, we obtain that the distribution of $Z_X(0)$ depends only on λ. Hence the distribution of the whole process $Z_X(t)$, $t \geq 0$ does not depend on the parameter ω. Obviously this implies the statement of the lemma.

It should be remarked that the Laplace transform of $s_X^2(t)$ can be explicitly computed using the general result in Koncz [12], and it serves as a third proof of Lemma 2. One could also use Lemma 1 to find this Laplace transform as it is described in [3,13].

Now we are ready to investigate the maximum likelihood estimate.

THEOREM 1. *The maximum likelihood estimate of the parameter* ω *is*

$$\hat{\omega}_X(t) = \frac{r_X(t)}{s_X^2(t)},$$

and

$$\sqrt{s_X^2(t)}\,(\hat{\omega}_X(t) - \omega) \stackrel{\mathcal{D}}{=} \mathcal{N}(0,1) \qquad \textit{for all } t > 0.$$

PROOF. Using the covariance matrix $\mathbb{E}X(0)X^*(0) = R(0) = (2\lambda)^{-1}I$ we obtain that the Radon-Nikodym derivative has the form

$$\frac{d\mathbb{P}_{t,X}}{d\mathbb{P}_{t,W}}(X) = \left(\frac{\lambda}{\pi}\right)^{3/2} \exp\left\{-\frac{1}{2}\lambda^2 \int_0^t \left(X_1^2(u) + X_2^2(u) + X_3^2(u)\right) du - \frac{1}{2}\omega^2 s_X^2(t) + \omega r_X(t) \right.$$
$$\left. - \frac{\lambda}{2}\left(X_1^2(0) + X_2^2(0) + X_3^2(0) + X_1^2(t) + X_2^2(t) + X_3^2(t)\right) + \frac{3}{2}\lambda t\right\}.$$

Consequently, the maximum likelihood estimate of the parameter ω is $\hat{\omega}_X(t) = r_X(t)/s_X^2(t)$.

Let us consider now another 3-dimensional stationary AR process $Y(t)$, $t \geq 0$ given by

$$\begin{pmatrix} dY_1(t) \\ dY_2(t) \\ dY_3(t) \end{pmatrix} = \begin{pmatrix} -\lambda & -\omega + c & \omega - c \\ \omega - c & -\lambda & -\omega + c \\ -\omega + c & \omega - c & -\lambda \end{pmatrix} \begin{pmatrix} Y_1(t)dt \\ Y_2(t)dt \\ Y_3(t)dt \end{pmatrix} + \begin{pmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{pmatrix}$$

with the same Wiener process $W(t)$, $t \geq 0$, and arbitrary $c \in \mathbb{R}$. Then the measures $\mathbb{P}_{t,X}$ and $\mathbb{P}_{t,Y}$ are equvivalent and

$$\frac{d\mathbb{P}_{t,X}}{d\mathbb{P}_{t,Y}}(X) = \exp\left\{\frac{1}{2}(c^2 - 2c\omega)s_X^2(t) + cr_X(t)\right\}.$$

By the help of Lemma 1 and Lemma 2 we obtain

$$\begin{aligned} \mathbb{E}\left(\exp\left\{-c\left(r_X(t) - \omega s_X^2(t)\right)\right\} \mid s_X^2(t) = \sigma^2\right) &= \mathbb{E}\left(\exp\left\{\frac{1}{2}c^2 s_X^2(t)\right\} \,\middle|\, s_X^2(t) = \sigma^2\right) \frac{d\mathbb{P}_{s_Y^2(t)}}{d\mathbb{P}_{s_X^2(t)}}(\sigma^2) \\ &= \exp\left\{\frac{1}{2}\sigma^2 c^2\right\}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\mathbb{E}\left(\exp\left\{-c\sqrt{s_X^2(t)}\,(\hat{\omega}_X(t) - \omega)\right\} \,\middle|\, s_X^2(t) = \sigma^2\right) \\ &= \mathbb{E}\left(\exp\left\{-\frac{c}{\sigma}\left(r_X(t) - \omega s_X^2(t)\right)\right\} \,\middle|\, s_X^2(t) = \sigma^2\right) = \exp\left\{\frac{1}{2}c^2\right\}. \end{aligned}$$

Hence we obtain

$$\mathbb{E}\exp\left\{-c\sqrt{s_X^2(t)}\,(\hat{\omega}_X(t) - \omega)\right\} = \exp\left\{\frac{1}{2}c^2\right\}$$

for all $c \in \mathbb{R}$, which proves the assertion.

4. A 4-DIMENSIONAL AR PROCESS

Consider the 4-dimensional stationary AR process $X(t)$, $t \geq 0$ given by

$$\begin{pmatrix} dX_1(t) \\ dX_2(t) \\ dX_3(t) \\ dX_4(t) \end{pmatrix} = \begin{pmatrix} -\lambda & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & -\lambda & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & -\lambda & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & -\lambda \end{pmatrix} \begin{pmatrix} X_1(t)dt \\ X_2(t)dt \\ X_3(t)dt \\ X_4(t)dt \end{pmatrix} + \begin{pmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \\ dW_4(t) \end{pmatrix}$$

where $W(t)$, $t \geq 0$ is a standard 4-dimensional Wiener process, and $\lambda > 0$, $\omega_1, \omega_2, \omega_3 \in \mathbb{R}$. Consider the statistics

$$\begin{aligned} s_X^2(t) &= \int_0^t \left(X_1^2(u) + X_2^2(u) + X_3^2(u) + X_4^2(u)\right) du \\ r_X^{(1)}(t) &= \int_0^t \left(X_1(u)\,dX_2(u) - X_2(u)\,dX_1(u) + X_4(u)\,dX_3(u) - X_3(u)\,dX_4(u)\right) \\ r_X^{(2)}(t) &= \int_0^t \left(X_1(u)\,dX_3(u) - X_3(u)\,dX_1(u) + X_2(u)\,dX_4(u) - X_4(u)\,dX_2(u)\right) \\ r_X^{(3)}(t) &= \int_0^t \left(X_1(u)\,dX_4(u) - X_4(u)\,dX_1(u) + X_3(u)\,dX_2(u) - X_2(u)\,dX_3(u)\right). \end{aligned}$$

First we investigate the distribution of the statistic $s_X^2(t)$.

LEMMA 3. *The distribution of $s_X^2(t)$ does not depend on the parameter ω (it depends only on the parameter λ).*

PROOF. The first method of the proof of Lemma 2 gives that for the conditional Laplace transform

$$u(t,x_1,x_2,x_3,x_4) = \mathbb{E}\left(\exp\left\{-\alpha s_X^2\right\} \middle| X_1(0)=x_1,\ X_2(0)=x_2,\ X_3(0)=x_3,\ X_4(0)=x_4\right),$$

we have

$$u(t,x_1,x_2,x_3,x_4) = v\left(t, x_1^2+x_2^2+x_3^2+x_4^2\right),$$

where v is the solution of

$$\frac{\partial v}{\partial t} = 2\varrho\frac{\partial^2 v}{\partial \varrho^2} + 2(2-\lambda\varrho)\frac{\partial v}{\partial \varrho} - \alpha\varrho v$$

with $v(0,x_1,x_2,x_3,x_4) = 1$ for $x_1,x_2,x_3,x_4 \in \mathbb{R}$, which proves the assertion. Of course, the second method can also be applied, which shows that even the distribution of the whole process $X_1^2(t)+X_2^2(t)+X_3^2(t)+X_4^2(t)$, $t \geq 0$ does not depend on the parameters $\omega_1,\omega_2,\omega_3$.

Now we investigate the maximum likelihood estimate.

THEOREM 2. *The maximum likelihood estimates of the parameters $\omega_1,\omega_2,\omega_3$ are*

$$\hat\omega_X^{(j)}(t) = \frac{r_X^{(j)}(t)}{s_X^2(t)}, \qquad j=1,2,3,$$

and

$$\sqrt{s_X^2(t)}\left(\hat\omega_X^{(1)}(t)-\omega_1, \hat\omega_X^{(2)}(t)-\omega_2, \hat\omega_X^{(3)}(t)-\omega_3\right) \stackrel{\mathcal{D}}{=} \mathcal{N}(0,I) \qquad \textit{for all } t>0.$$

PROOF. Using the covariance matrix $\mathbb{E}X(0)X^*(0) = R(0) = (2\lambda)^{-1}I$ we obtain that the Radon-Nikodym derivative has the form

$$\frac{d\mathbb{P}_{t,X}}{d\mathbb{P}_{t,W}}(X) = \frac{\lambda^2}{\pi^2}\exp\left\{-\frac{1}{2}(\lambda^2+\omega_1^2+\omega_2^2+\omega_3^2)s_X^2(t) + \omega_1 r_X^{(1)}(t) + \omega_2 r_X^{(2)}(t) + \omega_3 r_X^{(3)}(t)\right.$$
$$\left. - \frac{\lambda}{2}\left(X_1^2(0)+X_2^2(0)+X_3^2(0)+X_4^2(0)+X_1^2(t)+X_2^2(t)+X_3^2(t)+X_4^2(t)\right) + 2\lambda t\right\}.$$

Consequently the maximum likelihood estimates of the parameters $\omega_1,\omega_2,\omega_3$, are $\hat\omega_X^{(j)}(t) = r_X^{(j)}(t)/s_X^2(t)$, $j=1,2,3$.

Let us consider now another 4-dimensional stationary AR process $Y(t)$, $t \geq 0$ given by

$$\begin{pmatrix} dY_1(t)\\ dY_2(t)\\ dY_3(t)\\ dY_4(t)\end{pmatrix} = \begin{pmatrix} -\lambda & -\omega_1+c_1 & -\omega_2+c_2 & -\omega_3+c_3\\ \omega_1-c_1 & -\lambda & \omega_3-c_3 & -\omega_2+c_2\\ \omega_2-c_2 & -\omega_3+c_3 & -\lambda & \omega_1-c_1\\ \omega_3-c_3 & \omega_2-c_2 & -\omega_1+c_1 & -\lambda\end{pmatrix}\begin{pmatrix} Y_1(t)\,dt\\ Y_2(t)\,dt\\ Y_3(t)\,dt\\ Y_4(t)\,dt\end{pmatrix} + \begin{pmatrix} dW_1(t)\\ dW_2(t)\\ dW_3(t)\\ dW_4(t)\end{pmatrix}$$

with the same Wiener process $W(t)$, $t\geq 0$, and arbitrary $c_1,c_2,c_3 \in \mathbb{R}$. Then the measures $\mathbb{P}_{t,X}$ and $\mathbb{P}_{t,Y}$ are equvivalent and

$$\frac{d\mathbb{P}_{t,X}}{d\mathbb{P}_{t,Y}}(X) = \exp\left\{\frac{1}{2}(c_1^2+c_2^2+c_3^2-2c_1\omega_1-2c_2\omega_2-2c_3\omega_3)s_X^2(t)\right.$$
$$\left. + c_1 r_X^{(1)}(t) + c_2 r_X^{(2)}(t) + c_3 r_X^{(3)}(t)\right\}.$$

By the help of Lemma 1 and Lemma 3, we obtain

$$\mathbb{E}\left(\exp\left\{-c_1\left(r_X^{(1)}(t)-\omega_1 s_X^2(t)\right) - c_2\left(r_X^{(2)}(t)-\omega_2 s_X^2(t)\right)\right.\right.$$
$$\left.\left. -c_3\left(r_X^{(3)}(t)-\omega_3 s_X^2(t)\right)\right\} \middle| s_X^2(t) = \sigma^2\right) = \exp\left\{\frac{1}{2}\sigma^2\left(c_1^2+c_2^2+c_3^2\right)\right\}.$$

Finally,

$$\mathbb{E}\exp\left\{-\sqrt{s_X^2(t)}\left(c_1\left(\hat{\omega}_X^{(1)}(t)-\omega_1\right)+c_2\left(\hat{\omega}_X^{(2)}(t)-\omega_2\right)+c_3\left(\hat{\omega}_X^{(3)}(t)-\omega_3\right)\right)\right\}$$
$$=\exp\left\{\frac{1}{2}\left(c_1^2+c_2^2+c_3^2\right)\right\}$$

for all $c_1, c_2, c_3 \in \mathbb{R}$, which proves the assertion.

REMARK 1. The 4-dimensional real-valued stationary autoregressive process $X(t),\ t \geq 0$ can be identified with the 2-dimensional complex-valued stationary process

$$(\xi(t), \eta(t)) = (\xi_1(t) + i\xi_2(t), \eta_1(t) + i\eta_2(t)), \qquad t \geq 0,$$

given by the stochastic differential equation

$$\begin{pmatrix} d\xi(t) \\ d\eta(t) \end{pmatrix} = \begin{pmatrix} -\lambda + i\omega_1 & \omega_2 - i\omega_3 \\ \omega_2 + i\omega_3 & -\lambda - i\omega_1 \end{pmatrix} \begin{pmatrix} \xi(t)\,dt \\ \eta(t)\,dt \end{pmatrix} + \begin{pmatrix} du(t) \\ dv(t) \end{pmatrix},$$

where $u(t)$ and $v(t)$ are independent standard complex Wiener processes.

REMARK 2. A similar result holds for the 4-dimensional stationary AR process with coefficient matrix

$$A = \begin{pmatrix} -\lambda & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & -\lambda & -\omega_3 & \omega_2 \\ \omega_2 & \omega_3 & -\lambda & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & -\lambda \end{pmatrix}.$$

Other 4-dimensional stationary AR processes with one or two parameters of periodic type can be found, having structure different from the above mentioned examples, for which the suitably normalized maximum likelihood estimates of these parameters are exactly normally distributed.

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