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Minimal Covers of S_n by Abelian Subgroups and Maximal Subsets of Pairwise Noncommuting Elements, II

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Let β_n denote the minimum possible cardinality of a cover of the symmetric group S_n by abelian subgroups and let α_n denote the maximum possible cardinality of a set of pairwise noncommuting elements of S_n. Then $\alpha_n \neq \beta_n$ for all $n \geqslant 15$. This implies that the sequence $(\beta_n/\alpha_n := 1, 2, ...)$ takes on infinitely many distinct values and does not converge. © 1991 Academic Press, Inc.

The above results were not entirely expected since the sequence $(\beta_n/\alpha_n)_{n>0}$ was known to take all its values fairly close to 1 (between 1 and 1.22) and to have all terms equal to 1 if it converged, and since all terms of the sequence whose values are known (to me) are equal to 1 [1, Theorem 7, Corollary 9, 5(A)]. The main result of this article is as follows.

THEOREM. *$\alpha_n \neq \beta_n$ for $n = 8$, $n = 13$ and all $n \geqslant 15$.*

COROLLARY. *The sequence $(\beta_n/\alpha_n)_{n>0}$ has no limit and takes on infinitely many distinct values arbitrarily close to* 1.

The corollary follows from the above theorem and [1, Theorem 8].

The theorem corrects an erroneous assertion in [1, Sect. 5] that $\alpha_8 = \beta_8$. The proof of the theorem leads directly to a computation of β_8 and a similar argument yields an easy computation of β_{10}. We will briefly sketch these computations after giving the proof of the theorem. The key technical tool in this proof is the following familiar lemma, whose proof we sketch for completeness.

LEMMA. *Let σ be a product of nontrivial disjoint cycles $\sigma_1, ..., \sigma_k$ in S_n, no two of which have the same length. Then every element of S_n which commutes with σ is a product $\tau\rho$, where τ is a product of powers of the cycles σ_i, $1 \leqslant i \leqslant k$, and ρ is disjoint from σ (i.e., nothing is moved by both ρ and σ).*

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Proof (Sketch). Suppose τ commutes with σ. Suppose $1 \leqslant s \leqslant n$ and $\mathrm{Orbit}_\sigma(s) = \{\sigma^i(s) : i \in Z\}$ has $m > 1$ elements. Since $\tau^i\sigma^j(s) = \sigma^j\tau^i(s)$ for all i and j, we have

$$\tau^i(\mathrm{Orbit}_\sigma\, s) = \mathrm{Orbit}_\sigma(\tau^i(s)).$$

Since σ has only one orbit with m elements,

$$\tau^i(s) \in \mathrm{Orbit}_\sigma(\tau^i(s)) = \mathrm{Orbit}_\sigma(s),$$

so $\mathrm{Orbit}_\tau(s) \subset \mathrm{Orbit}_\sigma(s)$. Thus τ breaks up into a product of disjoint factors $\tau_1 \tau_2 \cdots \tau_{k+1}$, where for all $i \leqslant k$, τ_i moves only elements moved by σ_i and τ_{k+1} is disjoint from σ. But then τ_i and σ_i commute, so τ_i is a power of σ_i for all $i \leqslant k$ (a transitive cyclic subgroup of a permutation group is its own centralizer [2, 10.3.4 and 10.3.5]).

Proof of the Theorem. Let δ denote the cycle $(9, 10, ..., n)$, so δ is trivial if $n = 8$. In applying the above lemma it will be important to note that the length of δ cannot be 2, 3, 4, or 6.

There exists a cover of S_n, call it $\mathscr{A}$, by maximal abelian subgroups of S_n such that $|\mathscr{A}| = \beta_n$. Let $\mathscr{A}^*$ be the set of all elements of $\mathscr{A}$ which contain a product of disjoint cycles of the form $(abc)(de)\,\delta$. Let $\mathscr{B}$ denote the set of subgroups of S_n generated by four disjoint cycles a, b, c, d, where a, b, c have lengths 3, 3, 2, respectively, and $d = \delta$. Finally, let $\mathscr{C}$ denote the set of subgroups of S_n generated by three disjoint cycles a, b, c, where $c = \delta$ and a and b have lengths either 3 and 4 or 2 and 6, respectively.

CLAIM 1. $\mathscr{A} \backslash \mathscr{A}^* \supset \mathscr{C}$.

Proof. Suppose a and b are cycles, disjoint from each other and δ, of lengths either 2 and 6 or 3 and 4. There exists $H \in \mathscr{A}$ with $ab\delta \in H$. By the lemma, every element of H is a product of powers of a, b, and δ, so by maximality, H is exactly the set of such products. Hence H contains no elements of the form $(cde)(fg)\,\delta$. This shows $\mathscr{A} \backslash \mathscr{A}^* \supset \mathscr{C}$.

CLAIM 2. $\bigcup (\mathscr{B} \cup \mathscr{C}) \supset \bigcup \mathscr{A}^*$.

Proof. Suppose $a \in H \in \mathscr{A}^*$. We must show a is in some element of $\mathscr{B} \cup \mathscr{C}$. Without loss of generality we may assume that $b := (123)(45)\,\delta \in H$. By the lemma, a must have the form $(123)^i\,(45)^j\,\delta^k \rho$, where ρ is $(\ )$, (678), (687), (67), (68), or (78). If ρ is the identity or a 3-cycle, then clearly a is in an element of $\mathscr{B}$, and if ρ is a transposition, then clearly a is in an element of either $\mathscr{B}$ (if j is even) or $\mathscr{C}$ (if j is odd).

CLAIM 3. $|\mathscr{B}| \leqslant |\mathscr{A}^*|$.

Proof. $\mathscr{B}$ is easily checked to have 280 elements, and S_n is checked to have 4×280 elements of the form $ab\delta$, where a, b and δ are disjoint cycles and a and b have lengths 2 and 3, respectively. Every such element lies in a member of $\mathscr{A}^*$, and no member of $\mathscr{A}^*$ contains more than 4 of them since by the lemma, at most 4 such elements can commute with any one of them. (Details: there are exactly ten elements of the form $(efg)(hi)\delta$ which commute with, say, $(123)(45)\delta$, and one checks directly that no five of them pairwise commute.) It follows that $\mathscr{A}^*$ has at least 280 elements.

Let us now suppose that $\alpha_n = \beta_n$; we deduce a contradiction. Claims 1 and 2 imply that $(\mathscr{A} \backslash \mathscr{A}^*) \cup \mathscr{B}$ is a cover of S_n by abelian subgroups, while Claim 3 implies that this cover has β_n elements. Our hypothesis says there is a subset E of S_n with β_n elements, no two of which commute. Clearly each element of $(\mathscr{A} \backslash \mathscr{A}^*) \cup \mathscr{B}$ contains exactly one element of E. Let γ be the unique element of E in $H := \langle (123), (456), (78), \delta \rangle$ (note $H \in \mathscr{B}$). Then γ has the form $(123)^i (456)^j (78)^k \delta^m$. It is easy to see that 3 must divide either i or j; for example, if $i = j = 1$ then γ lies in the group $L := \langle (1\ 4\ 2\ 5\ 3\ 6), (78), \delta \rangle$ in $\mathscr{A} \backslash \mathscr{A}^*$ (Claim 1), contradicting that γ does not commute with $L \cap E$. Thus we may suppose without loss of generality that $j = 0$. Therefore by our lemma anything that commutes with $(123)(78)\delta$ will also commute with γ, so we may suppose without loss of generality that $\gamma = (123)(78)\delta$.

Now let $\gamma' = (123)^i (578)^j (46)^k \delta^m$ be the unique element of $E \cap H'$, where $H' := \langle (123), (578), (46), \delta \rangle$. Since γ and γ' do not commute, 3 does not divide j. Arguing as above we conclude that 3 divides i and that without loss of generality we may assume $\gamma' = (578)(46)\delta$. An analogous argument (replacing H and H' by H' and $\langle (578), (246), (13), \delta \rangle$, respectively) shows that we may assume that $(246)(13)\delta$ is in E. Continuing in this way we deduce that we may assume without loss of generality that $(135)(78)\delta$ and $(678)(24)\delta$ are in E (consider in turn the groups $\langle (246), (135), (78), \delta \rangle$ and $\langle (135), (678), (24), \delta \rangle$). An analogous chain of deductions starting again with the group H shows that we may assume without loss of generality that the permutations $(478)(56)\delta$, $(156)(23)\delta$, $(234)(78)\delta$, and $(678)(15)\delta$ are in E. But then E has a pair of commuting elements, namely, $(678)(24)\delta$ and $(678)(15)\delta$. This contradiction establishes the theorem.

Remark. A corollary of the above proof is that the sequence $(\beta_n - \alpha_n)_{n>0}$ is unbounded. Indeed, $\beta_n - \alpha_n$ is at least as great as the number of ways of picking a cycle in S_n of length $n-8$. (Roughly, for each choice of "δ" we obtain a subset F of $\mathscr{A}$ such that $\bigcup F$ does not contain a collection of $|F|$ pairwise noncommuting elements, and these subsets of $\mathscr{A}$ are disjoint.)

PROPOSITION. *$\beta_8 = 6994$ and $\beta_{10} = 483270$.*

Proof (Sketch). Consider the following eight partitions of the number 8:

$$8 = 8 = 7 + 1 = 6 + 2 = 5 + 3 = 5 + 2 + 1$$
$$= 4 + 3 + 1 = 4 + 2 + 2 = 3 + 3 + 2.$$

Let $\mathscr{D}$ denote the set of all subgroups of S_8 generated by s disjoint cycles of lengths $k_1, ..., k_s$, where $k_1 + \cdots + k_s$ is one of the above partitions of 8. One easily checks that $\mathscr{D}$ is a cover of S_8 with 6994 elements. Hence it suffices to show, using the notation of the proof of the theorem, that $|\mathscr{A}| \geqslant |\mathscr{D}|$. For any cover $\mathscr{E}$ of S_8 and any positive integers m and n, let $\mathscr{E}(m, n)$ denote the set of all elements of $\mathscr{E}$ containing a permutation of cycle structure m, n. Thus, for example, $\mathscr{A}^* = \mathscr{A}(3, 2)$ and $\mathscr{B} = \mathscr{D}(3, 2)$. Let $\mathscr{D}_0$ be the set of all elements of $\mathscr{D}$ associated with the first six partitions of 8 above. Then

$$\mathscr{D} = \mathscr{D}_0 \cup \mathscr{D}(4, 2) \cup \mathscr{D}(3, 2).$$

Arguing as in Claim 1 above we see that $\mathscr{A} \supset \mathscr{D}_0$. Claim 3 says $|\mathscr{D}(3, 2)| \leqslant |\mathscr{A}(3, 2)|$. A similar but easier argument shows that $|\mathscr{D}(4, 2)| \leqslant |\mathscr{A}(4, 2)|$. (There are 6×420 permutations of cycle type 4, 2 and every one of them commutes with exactly three others not including itself.) Note that the sets $\mathscr{D}_0$, $\mathscr{A}(4, 2)$ and $\mathscr{A}(3, 2)$ are pairwise disjoint. Therefore

$$|\mathscr{A}| \geqslant |\mathscr{D}_0| + |\mathscr{A}(4, 2)| + |\mathscr{A}(3, 2)|$$
$$\geqslant |\mathscr{D}_0| + |\mathscr{D}(4, 2)| + |\mathscr{D}(3, 2)| = |\mathscr{D}|,$$

as required. Thus $\beta_8 = 6994$.

A similar but easier argument yields the value of β_{10}. Consider the following 12 partitions of 10:

$$10 = 10 = 9 + 1 = 8 + 2 = 7 + 3 = 7 + 2 + 1$$
$$= 6 + 4 = 6 + 3 + 1 = 6 + 2 + 2$$
$$= 5 + 4 + 1 = 5 + 3 + 2 = 4 + 3 + 2 + 1$$
$$= 4 + 2 + 2 + 2.$$

Using notation analogous to that appearing above, we let $\mathscr{D}$ be the set of subgroups of S_{10} generated by s disjoint cycles of lengths $k_1, ..., k_s$, where $k_1 + \cdots + k_s$ is one of the above partitions of 10, and we let $\mathscr{A}$ be a cover of S_{10} by β_{10} maximal abelian subgroups. $\mathscr{D}$ is easily checked to be a cover of S_{10} by 483270 subgroups. Let $\mathscr{D}_0$ now denote the set of all elements of $\mathscr{D}$ arising from the first eleven of the partitions of 10 (all except

$4+2+2+2$). Using the lemma we deduce that $\mathscr{A} \supset \mathscr{D}_0$. (Argue as in the proof of Claim 1 above; for the partition $10=6+2+2$ consider the elements of $\mathscr{A}$ containing permutations of cycle type 6, 2.) $\mathscr{D}$ has 9450 elements associated with the last partition, $10=4+2+2+2$. There are 6×9450 permutations in S_{10} of cycle type 4, 2, 2. Each of these lies in some element of $\mathscr{A} \backslash \mathscr{D}_0$. At most six of the 12 elements of this cycle type which commute with any one of them can lie in an abelian subgroup of S_{10}. Thus $\mathscr{A} \backslash \mathscr{D}_0$ has at least 9450 elements. Thus $\beta_{10}=|\mathscr{A}|=|\mathscr{D}|=483270$.

PROBLEMS. The primary interest in studying the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ is perhaps the insight these examples give into the behavior of maximal sets of pairwise noncommuters and minimal covers by abelian subgroups for arbitrary finite groups. It is not clear to me that this motivation justifies further work in this area. Having made that disclaimer, let me confess to an unslaked curiosity about these sequences and mention three questions that seem particularly interesting.

(1) What is α_8?

(2) Does $\alpha_{10}=\beta_{10}$?

(3) Does the sequence (β_n/α_n: $n=1, 2, \ldots$) take on any value infinitely often? Indeed, is 1 the only value which is taken on more than once?

One can show that $6908 \leqslant \alpha_8 \leqslant 6988$; the lower limit comes from a computer assisted construction of a set of pairwise noncommuters and the upper limit from an elaboration of the proof of the theorem. Beyond these crude computations, the first question is untouched. The second has received some work but appears difficult. (α_{10} is known to within 0.4% [1, 5A].) An answer to the third question would nicely round out our picture of the sequence in question.

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