JOURNAL OF ALGEBRA 87, 222–246 (1984)

Defect Groups and Character Heights in Blocks of Solvable Groups. II

DAVID GLUCK

Department of Mathematics, University of Wisconsin–Madison, Madison, Wisconsin 53706

AND

THOMAS R. WOLF

Department of Mathematics, University of Wisconsin–Milwaukee, Milwaukee, Wisconsin 53201

Communicated by Walter Feit

Received August 11, 1982

1. INTRODUCTION

All groups considered are finite, p denotes a prime, and $\mathrm{Irr}(G)$ is the set of ordinary irreducible characters of G. For a p-block B of G, there is a conjugacy class of p-subgroups D of G that are called defect groups of B. If $|D| = p^d$ and $|P| = p^m$, where $P \in \mathrm{Syl}_p(G)$, then $p^{m-d} \mid \chi(1)$ whenever $\chi \in \mathrm{Irr}(G) \cap B$, and the height of χ is the largest integer h such that $p^{m-d+h} \mid \chi(1)$.

Brauer [1] conjectured that every $\chi \in \mathrm{Irr}(G) \cap B$ has height 0 if and only if D is abelian. Brauer and Feit [2] proved the result if $d \leqslant 2$, and Reynolds [14] proved the result when $D \trianglelefteq G$. Fong [4] proved one direction for p-solvable G. Namely, if G is p-solvable and D is abelian, then each $\chi \in B \cap \mathrm{Irr}(G)$ has height 0.

We prove the converse for solvable G. This extends the results of part I of this paper (Wolf [15]), where the converse direction is proven for solvable G, provided $p \geqslant 5$ or that certain hypotheses are met when $p \leqslant 3$. To prove our results, we use a "reduction" theorem of Fong that allows us to assume that $B \cap \mathrm{Irr}(G) = \mathrm{Irr}(G \mid \alpha)$ for some $\alpha \in \mathrm{Irr}(\mathbf{O}_{p'}(G))$ (we note that $\mathrm{Irr}(G \mid \alpha) = \{\chi \in \mathrm{Irr}(G) \mid [\chi, \alpha^G] \neq 0\}$ and $[\ ,\ ]$ is the usual inner product of characters). Our main result is

THEOREM A. *Suppose that $N \trianglelefteq G$, that G/N is solvable, that $\phi \in \mathrm{Irr}(N)$, and that $p \nmid (\chi(1)/\phi(1))$ for all $\chi \in \mathrm{Irr}(G \mid \phi)$. Then the Sylow-$p$-subgroups of G/N are abelian.*

0021-8693/84 $3.00

Proof. This is Corollary 5.2 below. ▮

Theorem A, with Fong's reduction theorem, gives an affirmative answer to Brauer's conjecture for solvable groups.

THEOREM B. *Let B be a p-block of a group G and let D be a defect group of B. Assume $G/\mathbb{O}_{p'}(G)$ is solvable. If every $\chi \in B \cap \mathrm{Irr}(G)$ has height 0, then D is abelian.*

Proof. We argue by induction on $|G : \mathbb{O}_{p'}(G)|$. Since B is a p-block of the p-solvable group G, Lemma 1A of [5] shows that there exists a p-block b of a group M such that b and B have isomorphic defect groups, such that there is a height-preserving bijection from $B \cap \mathrm{Irr}(G)$ onto $b \cap \mathrm{Irr}(M)$, and such that either

(a) $\mathbb{O}_{p'}(G) \leqslant M < G$, or

(b) $M/\mathbb{O}_{p'}(M) \cong G/\mathbb{O}_{p'}(G)$, $b \cap \mathrm{Irr}(M) = \mathrm{Irr}(M \mid \alpha)$ for some $\alpha \in \mathrm{Irr}(\mathbb{O}_{p'}(M))$ and the defect groups of b are Sylow-subgroups of M.

We may assume by the induction argument that $B \cap \mathrm{Irr}(G) = \mathrm{Irr}(G \mid \theta)$ for some $\theta \in \mathrm{Irr}(\mathbb{O}_{p'}(G))$ and that the defect groups are Sylow-subgroups of G. The hypotheses imply that $p \nmid \chi(1)$ for all $\chi \in \mathrm{Irr}(G \mid \theta)$. Theorem A implies that the Sylow-p-subgroups of $G/\mathbb{O}_{p'}(G)$, and G are abelian. Thus D is abelian. ▮

A natural question to ask is whether Theorems A and B can be generalized. For example, is the derived length of a defect group bounded by the maximum character height of the block? The answer is affirmative for solvable G.

THEOREM C. *Assume that $N \trianglelefteq G$, that G/N is solvable, and that $\phi \in \mathrm{Irr}(N)$. Suppose that e is an integer and $p^{e+1} \nmid (\chi(1)/\phi(1))$ for all $\chi \in \mathrm{Irr}(G \mid \phi)$. Then the derived length* d.l.(P/N) *of a Sylow-p-subgroup P/N of G/N is at most $2e+1$.*

THEOREM D. *Let D be a defect group of a p-block B of a group G and assume that $G/\mathbb{O}_{p'}(G)$ is solvable. If e is a nonnegative integer and each $\chi \in B \cap \mathrm{Irr}(G)$ has height at most e, then* d.l.$(D) \leqslant 2e+1$.

Theorem D follows from Theorem C in the same manner than Theorem B follows from Theorem A. Before proving Theorem C, we need Lemma 1.1, which is proved by Isaacs [11, Lemma 1.6] under the additional hypothesis that $I_Q(\theta) = \{x \in Q \mid \theta^x = \theta\}$ equals Q. The noninvariant case follows from Isaacs' result and an easy induction argument using Clifford's theorem [13, 6.11].

1.1. LEMMA. *Assume* $N \trianglelefteq Q$, *that* Q/N *is a* p-*group, and that* e *is a nonnegative integer. If* $\theta \in \mathrm{Irr}(N)$ *and* $p^{e+1} \nmid (\chi(1)/\theta(1))$ *for all* $\chi \in \mathrm{Irr}(Q \mid \theta)$, *then* $\mathrm{d.l.}(Q/N) \leqslant e + 1$.

Proof of Theorem C. We argue by induction on $|G : N|$. We may assume that $\mathbf{O}_{p'}(G/N) = 1$ and $\mathbf{O}^{p'}(G/N) = G/N$. Let $K/N = \mathbf{O}_p(G/N)$ and $L/N = \mathbf{O}_{pp'}(G/N)$. If $L = G$, then $K = G$ and the result follows from Lemma 1.1. Let M/L be a chief factor of G, so that M/L is a nontrivial abelian p-group.

Choose $\phi \in \mathrm{Irr}(K \mid \theta)$ and an integer f such that $p^f \mid (\phi(1)/\theta(1))$ and $p^{f+1} \nmid (\mu(1)/\theta(1))$ for any $\mu \in \mathrm{Irr}(K \mid \theta)$. Then $p^{e-f+1} \nmid (\tau(1)/\phi(1))$ for any $\tau \in \mathrm{Irr}(G \mid \phi)$. The induction argument yields that $\mathrm{d.l.}(P/K) \leqslant 2(e-f) + 1$, and Lemma 1.1 yields that $\mathrm{d.l.}(K/N) \leqslant f + 1$. Thus $\mathrm{d.l.}(P/N) \leqslant 2(e-f) + 1 + f + 1 = 2e + 1 + (1-f)$. Hence, we may assume that $f = 0$ and that K/N is abelian. Since $K/N = \mathbf{O}_p(G/N)$ and $\mathbf{O}_{p'}(G/N) = 1$, it follows by Lemma 1.2.3 of [8] that $K/N = \mathbf{C}_{G/N}(K/N)$. In particular, $\mathrm{d.l.}(P/N \cap M/N) = 2$.

Choose $\eta \in \mathrm{Irr}(M \mid \theta)$ and a nonnegative integer g such that $p^g \mid (\eta(1)/\theta(1))$ and $p^{g+1} \nmid (\beta(1)/\theta(1))$ for all $\beta \in \mathrm{Irr}(M \mid \theta)$. By Theorem A, $g \geqslant 1$. The induction argument yields that $\mathrm{d.l.}(PM/M) \leqslant 2(e-g) + 1$. Since $\mathrm{d.l.}(P/N \cap M/N) = 2$ and $g \geqslant 1$, we have that $\mathrm{d.l.}(P/N) \leqslant 2(e-g) + 1 + 2 = 2e + 1 + 2(1-g) \leqslant 2e + 1$. ∎

Theorem C extends one of the main results (Corollary 3.6) of Isaacs [11]. In fact, Isaacs obtains the same bound when θ is a "p-character" (i.e., $\theta(1)$ is a power of p and the order of the linear character $\det(\theta)$ is a p-power). In particular, setting $N = 1$, Isaacs showed that derived length of a Sylow-p-subgroup of a solvable group G is bounded as a function of the "p-parts" of the degrees of the irreducible characters of G.

The remainder of this paper is aimed at proving Theorem A. If $p \geqslant 5$, this theorem follows from Theorem 2.5 of Part 1 [15]. The proofs for $p = 3$ and $p = 2$ are in Sections 4 and 5. Sections 2 and 3 deal with a certain module action that arises in a minimal counterexample to Theorem A. Suppose that $|M : M'| = p$, that $p \nmid |M'|$, and M is solvable. Assume that V is a faithful, irreducible $\mathscr{F}(M)$-module for a finite field $\mathscr{F}$ and that $p \nmid |\mathbf{C}_M(v)|$ for all $v \in V$. This limits the structure of M. In Section 2, we show that M' is cyclic or $M \cong \mathrm{SL}(2, 3)$ if V is primitive. In Section 3, we look at the structure of M when V is imprimitive. Our results in Section 3 lean heavily on Huppert's classification of doubly transitive solvable groups.

2. PRIMITIVE MODULES

The main purpose of this section is to characterize certain primitive module actions (Theorem 2.3). Lemma 2.1 follows from Theorem 15.16 of [13].

2.1. LEMMA. *Let G be a Frobenius group with kernel N and complement H. Suppose that V is an $\mathscr{F}[G]$-module for a field $\mathscr{F}$ whose characteristic does not divide $|N|$. If $\mathbb{C}_V(N)=0$, then* $\dim(V)=|H|\dim(\mathbb{C}_V(H))$.

Let E be elementary abelian of order 8. We may choose $U \leqslant \mathrm{Aut}(E)$ such that U is nonabelian of order 21, and we let J be the semidirect product EU. By applying Sylow's theorem to $\mathrm{Aut}(E)$ we may conclude that J is unique up to isomorphism.

2.2. DEFINITION. Throughout this paper, we let J be the group defined above.

2.3. THEOREM. *Let G be a solvable group that acts faithfully and irreducibly on a vector space V over a finite field $\mathscr{F}$. Assume that $K \trianglelefteq G$, $|G:K|=p$, $p \nmid |K|$,* and $\mathbb{O}^{p'}(G)=G$. *Suppose that $p \,\big|\, |\mathbb{C}_G(x)|$, for all $x \in V$. If V_N is homogeneous for all $N \trianglelefteq G$, then*

(i) *K is cyclic; or*

(ii) *$K \cong Q_8$, $|V|=9$, and $p=3$.*

Proof. We will carry out the proof in a series of steps. We let $P \in \mathrm{Syl}_p(G)$. The hypotheses imply that $K=G'$ is the unique maximal normal subgroup of G.

Step 1. V_K is irreducible.

Let V_0 be an irreducible K-submodule of V and let $0 \neq x \in V_0$. The hypotheses imply that $P_0 \leqslant \mathbb{C}_G(x)$ for some $P_0 \in \mathrm{Syl}_p(G)$. Since $K \trianglelefteq G$, we have that $N_G(V_0) \geqslant KP_0 = G$ and $V_0 = V$.

Step 2. There is a unique maximal normal abelian subgroup Z of G. Furthermore, Z is cyclic and $Z=\mathbb{Z}(K)$.

The hypotheses imply that $K \neq 1$ and that any normal abelian $A \leqslant G$ is in fact contained in K. Since V is a faithful homogeneous A-module, we have that A is cyclic (see Theorem 3.2.3 of [7]). Since $\mathrm{Aut}(A)$ is abelian and $K=G'$, it follows that $A \leqslant \mathbb{Z}(K)$. This completes Step 2.

Step 3. We may assume that $K > Z$. Otherwise the conclusion of the theorem is satisfied.

Since V_N is homogeneous for all $N \trianglelefteq G$, every normal abelian subgroup of G is cyclic (see Theorem 3.2.3 of [7]). It is well known that this condition strictly limits the structure of G. The key step in [15, Part 1, Theorem 3.3] was Step 3 proving that V_N is homogeneous for all $N \trianglelefteq G$. Steps 4, 5, and 6 may be proved by repeating Steps 5–8, and 14 of [15, Part 1, Theorem 3.3]. (Alternatively, they follow immediately from Step 2 above and Lemma 2.3, Corollary 2.4, and Lemma 2.5 of [16].)

Step 4. Let E/Z be a chief factor of G, let $B = \mathbb{C}_G(E)$, and $C = \mathbb{C}_G(E/Z)$. Then

(i) $E \leqslant K$;

(ii) E/Z is elementary abelian of order q^{2n} for a prime q and integer n;

(iii) $q \,\|\, |Z|$;

(iv) $BE = C \leqslant K$ and $B \cap E = Z$;

(v) K/C is isomorphic to a subgroup of $\mathrm{Sp}(2n, q)$;

(vi) $C = K$ if and only if $|E/Z| = 4$.

Step 5. There exist $E = E_1, ..., E_m \leqslant G$ such that:

(i) E_i/Z is a chief factor of G for each i;

(ii) $\mathbb{C}_{G/Z}(M/Z) = M/Z$, where $M = E_1 \cdots E_m$; and

(iii) $M/Z = E_1/Z \times \cdots \times E_m/Z$.

Step 6. Let W be an irreducible Z-submodule of V. Then

(i) $|Z| \mid (|W| - 1)$; and

(ii) $|V| = |W|^{te}$ for some positive integers t and e with $e^2 = |M : Z|$.

Step 7. (i) $p \leqslant 3$;

(ii) $|\mathrm{Syl}_p(G)|\, |\mathbb{C}_V(P)| \geqslant |V|$;

(iii) $\log(|\mathrm{Syl}_p(G)|) \geqslant ((q-1)/2q) \log(|V|)$; and

(iv) $\log(|\mathrm{Syl}_p(G)|) \geqslant \log(|V|)/2$ if $p = 3$.

We may assume that $p \leqslant 3$, since otherwise Theorem 3.3 of Part 1 [15] yields the desired result. Since $\mathbb{C}_G(x)$ contains a Sylow-p-subgroup of G whenever $x \in V$, part (ii) follows from the conjugacy part of Sylow's theorem. Lemma 1.7 of Part 1 [15] applied to the action of EP on V yields that $|\mathbb{C}_V(P)| \leqslant |V|^j$, where $j = (q+1)/2q$, and that $|\mathbb{C}_V(P)| \leqslant |V|^{1/2}$ if $p \neq 2$. Parts (iii) and (iv) then follow from part (ii).

Step 8. Assume that $|E/Z| \neq 4$. Then

(i) If s is a prime divisor of $|\mathbb{F}(G/C)|$, then $s \mid (q^{2n} - 1)$;

(ii) $1 \neq \mathbb{F}(G/C) \leqslant K/C$;

(iii) $\mathbb{C}_{G/C}(\mathbb{F}(G/C)) \leqslant \mathbb{F}(G/C)$;

(iv) If $1 \neq S$ is a Sylow-subgroup of $\mathbb{F}(G/C)$ and if $\mathbb{C}_S(P) = 1$, then $\dim(\mathbb{C}_{E/Z}(P)) = 2n/p$;

(v) If $\mathbb{F}(G/C)$ is cyclic, then $\mathbb{F}(G/C) = K/C$ and $\dim(\mathbb{C}_{E/Z}(P)) = 2n/p$.

Since E/Z is a chief factor of G, E/Z is an irreducible G/C-module. Let $1 \neq S \in \mathrm{Syl}_s(\mathbb{F}(G/C))$ for a prime s. Then $S \trianglelefteq G$ and $\mathbb{C}_{E/Z}(S) = 1$. Part (i) follows from counting orbits and part (iv) may be obtained by applying Lemma 2.1 to the action of SP on E/Z. For any solvable group $X \neq 1$, $\mathbb{F}(X) \neq 1$, and $\mathbb{F}(X)$ contains its own centralizer. If $\mathbb{F}(G/C) \nleqslant K/C$, then $\mathbb{F}(G/C) = G/C$ as $\mathbb{O}^{p'}(G) = G$. This implies that $PC/C \leqslant \mathbb{Z}(G/C)$ and $C = K$, contradicting Step 4. Parts (ii) and (iii) follow.

For (v), assume that $\mathbb{F}(G/C)$ is cyclic, so that $\mathrm{Aut}(\mathbb{F}(G/C))$ is abelian. Then $K/C = (G/C)' \leqslant \mathbb{C}_{G/C}(\mathbb{F}(G/C)) = \mathbb{F}(G/C)$ and thus $K/C = \mathbb{F}(G/C)$. Since $O^{p'}(G) = G$ and K/C is cyclic, $\mathbb{C}_{K/C}(P) = 1$. Part (v) now follows from parts (ii) and (iv).

Step 9. If $q = 2$ and $2 \leqslant n \leqslant 8$, then

(i) $n = 6$ and $|K/C| \leqslant 2^{36}$; or

(ii) $n = 8$ and $7 \nmid |K/C|$.

Assume that $q = 2$ and $2 \leqslant n \leqslant 8$. Then $p = 3$ by Step 7. By Step 4, K/C is isomorphic to a subgroup of $\mathrm{Sp}(2n, 2)$. Suppose that $n = 7$. Since $|\mathrm{Sp}(14, 2)| = 3^9 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 31 \cdot 43 \cdot 127 \cdot 2^{49}$, Step 8(i) implies that $|\mathbb{F}(G/C)| \mid 43 \cdot 127$. Then $\mathbb{F}(G/C)$ is cyclic and Step 8(v) implies that $3 \mid 14$. Thus $n \neq 7$. Similarly, it can be shown that n is not 2 or 5. If $n = 4$, then $|\mathbb{F}(G/C)| \mid 5^2 \cdot 17$ as $|\mathrm{Sp}(8, 2)| = 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 2^{16}$. Since $P \nleqslant \mathbb{C}_G(\mathbb{F}(G/C))$ by Step 8(iii), we must have that $\mathbb{F}(G/C)$ has a Sylow-subgroup S of order 25 such that $\mathbb{C}_S(P) = 1$, whence Step 8(iv) yields a contradiction. Thus n is 3, 6, or 8. Assume that $n = 3$. Since $|\mathrm{Sp}(6, 2)| = 3^4 \cdot 5 \cdot 7 \cdot 2^9$, Step 8 yields that $|\mathbb{F}(G/C)| = |K/C| = 7$ and G/C is nonabelian of order 21. The Frobenius group of order 21 is embedded in $GL(3, 2)$ as the normalizer of a Sylow-7-subgroup, so that the natural and contragredient representations of $GL(3, 2)$ give distinct irreducible representations of G/C over $GF(2)$, and these are the only faithful irreducible representation of G/C over $GF(2)$. Thus E/Z is not an irreducible G/C-module and not a chief factor of G. Hence $n = 6$ or 8.

Assume that $n = 6$. Since $|\mathrm{Sp}(12, 2)| = 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 31 \cdot 2^{36}$, we must have $|\mathbb{F}(G/C)| \mid 5^3 \cdot 7^2 \cdot 13$. Since $\mathbb{F}(G/C) \geqslant \mathbb{C}_{K/C}(\mathbb{F}(G/C)$ by Step 8, since $|K/C| \mid |\mathrm{Sp}(12, 2)|$, and since $\mathrm{Aut}(\mathbb{F}(G/C))$ is the direct product of the automorphism groups of the Sylow-subgroups of $\mathbb{F}(G/C)$; it follows that $|K/C| \mid 5^3 \cdot 7^2 \cdot 13 \cdot 2^{14} \cdot 31$ and $|K/C| \leqslant 2^{36}$. We may assume that $n = 8$.

Since $|\mathrm{Sp}(16, 2)| = 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17^2 \cdot 31 \cdot 43 \cdot 127 \cdot 257 \cdot 2^{64}$, $\mathbb{F}(G/C)|| 5^4 \cdot 17^2 \cdot 127$. Then $7 \nmid |\mathrm{Aut}(T)|$ if T is a Sylow-subgroup of $\mathbb{F}(G/C)$. Since $\mathbb{F}(G/C)$ is nilpotent and $\mathbb{C}_{G/C}(\mathbb{F}(G/C)) \leqslant \mathbb{F}(G/C)$, we have that $7 \nmid |G/C|$. This completes Step 9.

Step 10. If $n=1$, then $q \neq 3$.

Otherwise, $p=2$ by Step 7. Since $p \nmid |K|$ and $\mathrm{Sp}(2,3)$ is a $\{2,3\}$-group, Step 8 ((i) and (ii)) yield a contradiction.

Step 11. If $p=3$, we may assume that

(i) G involves J, and

(ii) $7 \mid |K/M|$.

If $7 \nmid |K/M|$, it follows via Step 5 that G does not involve J. And if G does not involve J, then Theorem 3.3 of Part 1 [15] yields that $K \cong Q_8$ and $|V|=9$, as desired.

Step 12. (i) $p=2$,

(ii) $m=1$, and

(iii) q^n is 5, 7, 11, 3^2, 5^2, 3^3 or 3^4.

By parts (iii) and (iv) of Step 7, we have that $|\mathrm{Syl}_p(G)| \geqslant |V|^s$, where $s=\frac{1}{2}$ when $p=3$ and $s=(q-1)/2q \geqslant \frac{1}{3}$ when $p=2$.

Without loss of generality, we may choose an integer k such that $0 \leqslant k \leqslant m$ and $|E_i/Z|=4$ if and only if $i \leqslant k$. For each i, $|E_i/Z| = q_i^{2n_i}$ for a prime q_i and integer n_i. Let $C_0=K$ and define C_i the centralizer in C_{i-1} of E_i/Z (for $1 \leqslant i \leqslant m$). By Step 4(v, vi), we have that $C_i = K$ for $i \leqslant k$ and that C_{i-1}/C_i is isomorphic to a subgroup of $\mathrm{Sp}(2n_i, q_i)$. By Step 5, $C_m = M$. Since $|\mathrm{Sp}(2n,q)| < q^{2n^2+n}$ and $|\mathrm{Syl}_p(G)| < |K|$, it follows that

$$\log(|\mathrm{Syl}_p(G)|) \leqslant \log(|Z|) + 2k\log(2) + \sum_{i=k+1}^{m} (2n_i^2 + 3n_i)\log(q_i).$$

By Step 6, $|V| = |W|^{te}$, where $e^2 = |M:Z|$. Thus the first paragraph of this step yields that

$$\log(|Z|) + 2k\log(2) + \sum_{i=k+1}^{m} (2n_i^2+3n_i)\log(q_i) \geqslant st\left(2^k \prod_{i=k+1}^{m} q_i^{n_i}\right)\log(|W|). \tag{I}$$

By Steps 4(iii) and 6(i), $q_i \leqslant |Z| < |W|$ for all i. Hence

$$1 + 2k + \sum_{i=k+1}^{m} (2n_i^2+3n_i) > st \cdot 2^k \cdot \prod_{i=k+1}^{m} q_i^{n_i}. \tag{II}$$

We will first assume that $p=2$ and proceed to show that conclusions (ii) and (iii) of this step hold when $p=2$. Since $p \nmid |K|$, we have that $k=0$ and that each q_i is odd. Since $s \geqslant \frac{1}{3}$, inequality II yields that

$$1 + 2l^2 + 3l > 3^{l-1},$$

where $l = \sum_{i=1}^{m} n_i$. This last inequality yields that $l \leqslant 4$. If $n_1 = n_2 = 2$, then inequality II implies that $29 > q_1^2 q_2^2/3$. But then $q_1 = q_2 = 3$ and inequality I implies that $|Z| \cdot 3^{28} \geqslant |W|^{27}$, a contradiction since $3 \mid |Z|$ and $|Z| \mid (|W| - 1)$. The case $n_1 = n_2 = 2$ cannot occur. To show that $m = 1$, we may assume that $n_1 = 1$, since $l \leqslant 4$. But then $q_1 \geqslant 5$ by Step 10, and now inequality II yields that $1 + 5 + 2(l-1)^2 + 3(l-1) > 5 \cdot 3^{l-2}$. Hence $l \neq 4$. If $n_2 = 2$, then inequality II implies that $1 + 5 + 14 > q_1 q_2^2/3 \geqslant q_1 q_2$, whence $q_1 = 5$ and $q_2 = 3$. Then $|Z| 5^5 \cdot 3^{14} > |W|^{15}$ by inequality I. This is a contradiction, since $15 \mid |Z|$ and $|Z| < |W|$. Thus $n_2 = 1$ and Step 10 yields that $q_2 \geqslant 5$. Inequality II implies that $11 + 2(l-2)^2 + 3(l-2) \geqslant q_1 q_2 \cdot 3^{l-3} \geqslant 5^2 \cdot 3^{l-3}$. Then $l = 2$ and $q_1 = q_2 = 5$. Inequality I yields that $|Z| 5^{10} > |W|^{25/3}$ and that $5^{10} > |W|^{22/3}$, a contradiction as $|W| > 11$. Hence $m = 1$, if $p = 2$. Furthermore $s = (q-1)/2q$ and $1 + 2n^2 + 3n > (q-1) q^{n-1}/2 \geqslant 3^{n-1}$ by inequality II. This inequality and Step 10 imply that $n \leqslant 4$ and $q^n = 5, 7, 11, 3^2, 5^2, 3^3$, or 3^4. This step is completed for $p = 2$.

We may now assume that $p = 3$, $3 \nmid |K|$, $s = 1/2$, and

$$|\mathrm{Syl}_3(G)| > |V|^{1/2}. \tag{III}$$

If $l = \sum_{i=k+1}^{m} n_i$, then $1 + 2k + 2l^2 + 3l > 2^{(k+l-1)}$ by inequality II and thus $1 + 2(k+l)^2 + 3(k+l) > 2^{(k+l-1)}$. This implies that $k + l \leqslant 8$. Assume that $q_{k+1} = 2$. If $n_1 = 8$, then $m = 1$, $C = E = M$, and Step 9 implies that $7 \nmid |K/M|$, contradicting Step 11. Thus $n_1 \neq 8$. Then Step 9 applied to E_i/Z for $i \geqslant k$ yields that $n_{k+1} = 6$ and $q_i \geqslant 5$ for all $i > k+1$. Then inequality II yields that $81 + 2k + 2(l-6)^2 + 3(l-6) > 2^{5+k} \cdot 5^{l-6}$. Since $l + k \leqslant 8$ and $l \geqslant 6$, we must have that $l = 6$ and $k = 0, 1$. Steps 4, 5, and 9 yield that $M = \mathbb{C}_G(E_{k+1}/Z)$ and that $|K/M| \leqslant 2^{36}$. Then $|\mathrm{Syl}_p(G)| \leqslant 2^{48+2k} |Z| < 2^{48+2k} |W|$. Since $\log(|V|) = 64 \cdot 2^k \cdot t \log(W)$, inequality III yields that $(48 + 2k) \log(2) > (32 \cdot 2^k - 1) \log(|W|)$. In either case ($k = 0$ or $k = 1$), this inequality implies that $|W| < 3$, a contradiction as $|Z| < |W|$. Hence $q_{k+1} \neq 2$, and thus $q_i \geqslant 5$ for all $i > k$.

Now inequality II yields that $1 + 2k + 2l^2 + 3l > 2^{k-1} 5^l$. As $k + l \leqslant 8$, the only solutions occur when $l \leqslant 2$. Since $C_i = K$ for all $i \leqslant k$ and since $7 \nmid |\mathrm{Sp}(j, 5)|$ for $j = 2$ or 4, it follows from Steps 4 and 5 that $7 \nmid |K/M|$ if each $q_i \leqslant 5$. By Step 11, we may assume that $l = \sum_{i=k+1}^{m} n_i \geqslant 1$ and that $q_{k+1} \geqslant 7$. But then inequality II implies that $1 + 2k + 2l^2 + 3l > 2^{k-1} \cdot 7 \cdot 5^{l-1}$. This inequality has no solutions when $l = 2$. Thus $l = 1$ and inequality II yields that $6 + 2k > 2^{k-1} \cdot q_{k+1}$ and thus $q_{k+1} = 7$ or 11, as $k \leqslant 8$. By Steps 4 and 5, $E_i/Z \leqslant \mathbb{Z}(K/Z)$ for $i \leqslant k$ and K/M is isomorphic to a subgroup of $\mathrm{Sp}(2, q_{k+1})$. But $\mathrm{Sp}(2, q_{k+1})$ does not involve the Frobenius group of order 56. Thus G does not involve J, contradicting Step 11. This completes Step 12.

Step 13. Conclusion. We have that $m=1$, $M=E_1=E=C$, $|E/Z|=q^{2n}$, and K/E is isomorphic to a subgroup of $\mathrm{Sp}(2n,q)$. We also have that $p=2$, $2\nmid|K|$, and $\log(|V|)=tq^n\log(|W|)$. By Steps 6 and 7, we have that

$$\log(|\mathrm{Syl}_p(G)|)\geqslant tq^{n-1}((q-1)/2)\log(|W|). \tag{IV}$$

Suppose that $q^n=5^2$. Since $|\mathrm{Sp}(4,5)|=2^7\cdot 3^2\cdot 13\cdot 5^4$ and since $p=2$, we have by Step 8 that $|\mathbb{F}(G/E)|\mid 3^2\cdot 13$ and $\mathbb{C}_{G/E}(\mathbb{F}(G/E))\leqslant\mathbb{F}(G/E)$. Then $|K/E:\mathbb{F}(G/E)|\mid|\mathrm{Aut}(\mathbb{F}(G/E)|$. Since $|K/E|$ is odd and divides $|\mathrm{Sp}(4,5)|$, we have that $|K/E|\mid 3^2\cdot 13$. Then $|\mathrm{Syl}_2(G)|\leqslant|K|\leqslant 3^2\cdot 13\cdot 5^4\cdot|Z|<3^2\cdot 13\cdot 5^4\cdot|W|$ and inequality IV implies that $3^2\cdot 13\cdot 5^4>|W|^9$ and $|W|<11$. This is a contradiction, as $q\mid(|W|-1)$ by Steps 4 and 6. Thus $q^n\neq 5^2$. Similar arguments show that q^n is not 3^4 or 11.

Suppose that $q^n=5$. Since $\mathrm{Sp}(2,5)=2^3\cdot 3\cdot 5$ and $2\nmid|K|$, Step 8 yields that $\mathbb{F}(G/E)=K/E$ is cyclic of order 3 and that $\mathbb{C}_{E/Z}(P)=5$. Then $|\mathrm{Syl}_2(G)|=|K:\mathbb{C}_K(P)|\leqslant 3\cdot 5\cdot|Z|\leqslant 15\cdot|W|$ and inequality IV yields that $|W|\leqslant 15$. Since $5\mid|Z|$ and $|Z|\mid(|W|-1)$, we have that $|W|=11$ and $|Z|=5$. But then $|\mathrm{Syl}_2(G)|=|K:\mathbb{C}_K(P)|\leqslant 3\cdot 5^2$, contradicting inequality IV. Thus $q^n\neq 5$. Similarly, we may argue that q^n is not 7 or 3^3. Thus $q^n=3^2$ by Step 12.

Since $|\mathrm{Sp}(4,3)|=2^7\cdot 5\cdot 3^4$, it follows from Step 8 that $|\mathbb{F}(G/E)|=|K/E|=5$ and $|\mathbb{C}_{E/Z}(P)|=3^2$. Then $|\mathrm{Syl}_2(G)|=|K:\mathbb{C}_K(P)|<5\cdot 3^2\cdot|Z|<5\cdot 3^2\cdot|W|$ and inequality IV implies that $45>|W|^{2t}$. Thus $t=1$, $|W|=4$, and $|Z|=3$. If $P\leqslant\mathbb{C}_G(Z)$, then $|\mathrm{Syl}_2(G)|\leqslant 5\cdot 3^2$, contradicting inequality IV. Thus $P\nleqslant\mathbb{C}_G(Z)$ and Lemma 2.1 applied to ZP implies that $|\mathbb{C}_V(P)|=|V|^{1/2}$. Step 7 now implies that $|\mathrm{Syl}_2(G)|\geqslant|V|^{1/2}=4^{9/2}$, a contradiction as $|\mathrm{Syl}_2(G)|=|K:\mathbb{C}_K(P)|\leqslant 5\cdot 3^3$. The proof is complete. ∎

3. Imprimitive Modules

In Theorem 1 of [6], Gluck determines all solvable primitive permutation groups (G,Ω) in which every $\Delta\subseteq\Omega$ has a nontrivial stabilizer in G. In all cases, $|\Omega|\leqslant 9$. Lemma 3.1 is a consequence of this result. We let D_{2n} denote the dihedral group of order $2n$.

3.1. Lemma. *Let G be a solvable primitive permutation group on a finite set Ω. Suppose that $p\mid|G|$, but $p^2\nmid|G|$. Assume that whenever $\Delta\subseteq\Omega$, then* $\mathrm{Stab}_G(\Delta)=\{x\in G\mid\Delta^x=\Delta\}$ *contains a Sylow-p-subgroup of G. Then*

(a) $|\Omega|=3$, $p=2$, *and* $G\cong D_6$;

(b) $|\Omega|=5$, $p=2$, *and* $G\cong D_{10}$; *or*

(c) $|\Omega|=8$, $p=3$, *and* $G\cong J$.

Proof. Let M be a minimal normal subgroup of G and let I be the stabilizer in G of some $\alpha \in \Omega$. A standard argument shows that $MI = G$, $M \cap I = 1$, $M = \mathbb{C}_G(M)$, and M acts regularly on Ω. In particular, $|M| = |\Omega|$ and I contains a Sylow-p-subgroup of G. Since each $\Delta \subseteq \Omega$ has a nontrivial point stabilizer, since $p \,\big|\, |I|$ and $p^2 \nmid |G|$; Theorem 1 of [6] yields that

(i) $|\Omega| = 3$ and $|I| = 2$;

(ii) $|\Omega| = 4$ and $|I| = 3$ or 6;

(iii) $|\Omega| = 5$ and $|I| = 2$;

(iv) $|\Omega| = 7$ and $|I| = 6$; or

(v) $|\Omega| = 8$ and $|I| = 21$;

If $|\Omega| = 7$, then I is cyclic since I acts faithfully on M. In this case, each involution in G fixes 3 elements of $\Omega_3 = \{\Delta \subseteq \Omega \,\big|\, |\Delta| = 3\}$. Since $3\,|\mathrm{Syl}_2(G)| = 3 \cdot 7 < \binom{7}{3} = |\Omega_3|$, some element of Ω_3 is not fixed by an involution of G. Thus $p \neq 2$ and hence $p = 3$. But again we can show that there are elements of Ω_3 not fixed by any elements of order 3, a contradiction. Thus (iv) does not hold. We may similarly argue that $|\Omega| \neq 4$. Thus (i), (iii), or (v) holds. Since I acts faithfully on M, $G = MI$, and $|\Omega| = M$; the conclusion of this lemma easily follows. ∎

Part (a) of Lemma 3.2 is standard.

3.2. LEMMA. *Assume that G is a solvable group that acts faithfully and irreducibly on a vector space V over a field $\mathcal{F}$. Suppose that $C \leqslant G$ is maximal with respect to $C \trianglelefteq G$ and V_C is not homogeneous. Let $V_1, \ldots, V_n$ be the homogeneous components of V_C. Then*

(a) *G/C permutes the V_i faithfully and primitively.*

Assume further that $p \,\big|\, |G/C|$, that $p^2 \nmid |G/C|$, and $p \nmid |G : \mathbb{C}_G(x)|$ for all $x \in V$. Then

(b) *n is 3, 5, or 8 and (resp.) p is 2, 2, or 3;*

(c) *G/C is isomorphic (resp.) to D_6, D_{10}, or J;*

(d) *$C/\mathbb{C}_C(V_i)$ acts transitively on the nonidentity elements of V_i for each i.*

Proof. Let M/C be a chief factor of G. Since V_M is homogeneous, it follows from Clifford's theorem that M/C transitively permutes the V_i. Since M/C is an abelian chief factor of G, we have that M/C acts regularly on the V_i and $|M/C| = n$. Let $I = N_G(V_1)$, so that $MI = G$ and $M \cap I = C$. Let $D/C = \mathbb{C}_{G/C}(M/C) \geqslant M/C$ and let $B = D \cap I \trianglelefteq MI = G$. Then B fixes each V_i and V_B is not homogeneous. Then $B = C$ and $D = M$ and M/C is the unique minimal normal subgroup of G/C. Thus G/C acts faithfully on the

V_i. Since M/C is an abelian chief factor of G/C, I is a maximal subgroup of G. Thus G/C acts primitively on the V_i, proving (a).

Let $0 \neq y \in V_1$ and $0 \neq z \in V_2$. Some Sylow-p-subgroup P_1 of G centralizes $y + z$. Since G and P_1 permute the V_i, P_1 must leave the set $\{V_1, V_2\}$ invariant. A similar argument shows that each $\Delta \subseteq \{V_1, ..., V_n\}$ is stabilized by some Sylow-p-subgroup of G/C. Parts (b) and (c) now follow from Lemma 3.1.

We next show that C acts transitively on the nonidentity elements of V_1. Let x_1 and x_2 be distinct nonzero elements of V_1, let $0 \neq y \in V_2$ and $0 \neq z \in V_3$. Assume that $p = 3$ and choose, for each j, $P_j \in \mathrm{Syl}_3(G)$ such that $P_j \leqslant \mathbb{C}_G(x_j + y + z)$. Since each Sylow-3-subgroup of G/C fixes exactly two of the V_i, we may choose $t_j \in G$ such that $CP_j = C\langle t_{j_1} \rangle$ for each j and such that $x_j^{t_j} = y$, $y^{t_j} = z$, and $z^{t_j} = x_j$. Then $x_1^{t_1 t_2^{-1}} = x_2$. Since each of the 28 Sylow-3-subgroups of G/C stabilizes exactly two $\Delta \subseteq \{V_1, ..., V_8\}$ with $|\Delta| = 3$, counting yields that $\{V_1, V_2, V_3\}$ is fixed by exactly one Sylow-3-subgroup of G/C. It follows that $CP_1 = CP_2$ and that $t_1 t_2^{-1} \in C$. Thus C is transitive on $V_1^{\#}$ if $p = 3$. A similar argument works for $p = 2$ (choose $P_j \in \mathrm{Syl}_2(G)$ that centralize $x_j + y$). This completes the proof. ∎

We next mention a number theoretic result of Birkhoff and Vandiver (see Herstein [9, p. 362]).

3.3. Lemma. *Let q be a prime and n a positive integer. There exists a prime p such that $p \mid (q^n - 1)$ but p does not divide $q^m - 1$ for all $0 < m < n$, unless $q^n = 2^6$ or $n = 2$ and q is a Mersenne prime.*

Conclusion (d) in Lemma 3.2 puts some restrictions on the structures of C and G. Huppert [10] has classified the solvable groups H that act faithfully on a vector space V of order q^n and transitively permute the nonidentity elements. Unless q^n is one of six values, Huppert has shown that V may be identified with the additive group of $GF(q^n)$ in such a way that H is a subgroup of $T(q^n)$, the group of semilinear transformations $\{x \to ax^{\sigma} \mid a \in GF(q^n),\ \sigma$ a field automorphism of $GF(q^n)\}$ of V. In particular, H is metacyclic.

3.4. Lemma. *Assume that H is a solvable group acting on a vector space V with $|V| = q^n$ and $q = 2, 3$. Assume that H acts transitively on $V^{\#}$ and that $q^n \neq 3^2, 3^4$. Further assume that $|H|$ is odd if $|V| = 2^6$. Then*

(i) *$H/\mathbb{F}(H)$ and $\mathbb{F}(H)$ are cyclic, the order of $H/\mathbb{F}(H)$ divides n; and*

(ii) *there exists a prime $p > n$ and Sylow-p-subgroup P of H such that $P \leqslant \mathbb{F}(H) = \mathbb{C}_H(P)$.*

Proof. Since H is a solvable group acting transitively on $V^{\#}$ and since $q^n \neq 3^2$, 3^4, 5^2, 7^2, 11^2, or 23^2, it follows from [10, Main Proposition], as the semidirect product HV is a doubly transitive group, that V may be identified with the additive group of $GF(q^n)$ in such a way that $H \leqslant T(q^n)$. We let S be the subgroup $\{x \to ax \mid a \in GF(q^n)\}$ of $T(q^n)$, so that S is a cyclic normal subgroup of $T(q^n)$ with cyclic factor group of order n and $|S| = (q^n - 1)$. We choose p as in Lemma 3.3 if $q^n \neq 2^6$ and let $p = 7$ if $q^n = 2^6$. Since $q^{p-1} \equiv 1 \pmod{p}$, $p > n$. Thus $T(q^n)$ has a cyclic normal Sylow-p-subgroup P. Then $P \leqslant S \leqslant D$, where D is the centralizer of P in $T(q^n)$. If $p \nmid (q^m - 1)$ for all $0 < m < n$, then P is not centralized by any field automorphism of $GF(q^n)$ and then $D = S$. If $q^n = 64$, then $|T(q^n)/S| = 6$, $p = 7$, and $p \nmid (2^2 - 1)$. In this case, P is not centralized by an automorphism of $GF(2^6)$ of order 3. In any case D/S is a 2-group and $D = S$ if $q^n \neq 2^6$.

We let $F = H \cap S$, so that F and H/F are cyclic. Since H acts transitively on $V^{\#}$, since $|S| = q^n - 1$, and since $p \nmid n$, we have that $P \leqslant H \cap S = F$. Then $\mathbb{C}_H(P) = D \cap H = S \cap H = F$, as either $D = S$ or $|H|$ is odd. But $P \in \mathrm{Syl}_p(\mathbb{F}(H))$ and $\mathbb{F}(H) \leqslant \mathbb{C}_H(P) \leqslant F \leqslant \mathbb{F}(H)$. Thus $F = \mathbb{F}(H)$, completing the proof. ∎

4. The Prime 3

Here we prove Theorem A for the prime three. We first start with some known character theoretic results. Let $N \trianglelefteq K$, $\phi \in \mathrm{Irr}(N)$, and $\theta \in \mathrm{Irr}(K \mid \phi)$. The following are equivalent (Exercise 6.3 of [13]):

(i) $\theta_N = e\phi$ with $e^2 = |K : N|$;

(ii) $I_K(\phi) = K$ and θ vanishes on $K - N$; and

(iii) $I_K(\phi) = K$ and θ is the unique irreducible constituent of ϕ^K.

In this situation, we say that ϕ or θ is *fully ramified* with respect to K/N. The following is immediate from Theorem 2.7 of Isaacs [12].

4.1. Theorem. *Suppose that $N \trianglelefteq K$, K/N is abelian, and $\phi \in \mathrm{Irr}(N)$ with $I_K(\phi) = K$. Then there exists $N \leqslant H \leqslant K$ such that each $\tau \in \mathrm{Irr}(H \mid \phi)$ extends ϕ and is fully ramified with respect to K/H. Furthermore if N, $K \trianglelefteq G$, and $I_G(\phi) = G$, then $H \trianglelefteq G$.*

In Theorem 4.1, H/N is the radical of a bilinear form defined on K/N. If ϕ is faithful and linear, the bilinear form can be taken to be the usual commutator map and $H = \mathbb{Z}(K)$. Lemma 4.2 is known.

4.2. Lemma. *Suppose that $N \trianglelefteq K$, K/N is abelian, and $\theta \in \mathrm{Irr}(K)$ is fully ramified with respect to K/N. Then*

(a) *$K/N \cong B \times B$ for some abelian group B; and*

(b) *if K/N is an abelian p-group, if $N, K \trianglelefteq G$, if $I_G(\theta) = G$, if $D/N = C_{G/N}(K/N)$, and if G/D is an abelian q-group for a prime $q \neq p$, then* $\operatorname{rank}(G/D) \leqslant \operatorname{rank}(K/N)/2$, (*where the rank of an abelian p-group P is* $\dim(\Omega_1(P))$).

Proof. Part (a) is Lemma 2 of [3]. We prove (b) by induction on $|K:N|$. Choose $D \leqslant H < G$, G/H cyclic, and $C/N = \mathbb{C}_{K/N}(H/D) \neq 1$. Since $C/N = \mathbb{C}_{K/N}(Q)$ for a Sylow-q-subgroup Q of H, Exercise 13.12 of [13] yields that θ is fully ramified with respect to K/C. Then the irreducible constituent of θ_N is fully ramified with respect to C/N, and so $\operatorname{rank}(C/N) \geqslant 2$. Since $C \trianglelefteq H$ and H/D acts faithfully on K/C, induction yields that $\operatorname{rank}(K/C) \geqslant 2\operatorname{rank}(H/D)$. By Fitting's lemma, $\operatorname{rank}(K/N) = \operatorname{rank}(C/N) + \operatorname{rank}(K/C)$. Thus $\operatorname{rank}(K/N) \geqslant 2\operatorname{rank}(H/D) + 2 \geqslant 2\operatorname{rank}(G/D)$. ∎

Lemma 4.3 is useful in Theorems 4.4 and 5.1. It is immediate from Theorem 13.31 and Exercise 13.10 of [13].

4.3. Lemma. *Assume that $N \leqslant K \trianglelefteq G$, $N \trianglelefteq G$, $(|K/N|, |G/K|) = 1$, and that G/K or K/N is solvable. Let $\phi \in \operatorname{Irr}(N)$ be invariant in G. Then*

(a) *there exists $\sigma \in \operatorname{Irr}(K \mid \phi)$ invariant in G; and*

(b) *σ is unique if $\mathbb{C}_{K/N}(S/N) = 1$ for a complement S/N of K/N in G/N.*

If $N \trianglelefteq G$ and $\phi \in \operatorname{Irr}(N)$ extends to $\chi \in \operatorname{Irr}(G)$, then $\beta \to \beta\chi$ is a bijection from $\operatorname{Irr}(G|N)$ onto $\operatorname{Irr}(G|\phi)$. A sufficient condition for ϕ to extend to G is that $I_G(\phi) = G$ and G/N has cyclic Sylow-subgroups. These known facts are summarized in Lemma 2.1 of part 1 [15] and will often be used without reference.

4.4. Theorem. *Suppose that Z is a normal (not necessarily central) subgroup of G, that G/Z is solvable, and that $\lambda \in \operatorname{Irr}(Z)$. If $3 \nmid (\chi(1)/\lambda(1))$ for all $\chi \in \operatorname{Irr}(G|\lambda)$, then G/Z has an abelian Sylow-3-subgroup.*

Proof. The proof will be by induction on $|G:Z|$ and will be done in a series of steps.

Step 1. We may assume that there exist $Z \leqslant N \leqslant K \trianglelefteq G$ such that

(a) N/Z is a chief factor of G and $\mathbb{C}_{G/Z}(N/Z) = N/Z$;

(b) $G/Z = \mathbb{O}^{3'}(G/Z)$;

(c) N/Z is a 3-group, $|G:K| = 3$, $K > N$, and $3 \nmid |K:N|$.

If $Z < H \triangle G$ and if $\theta \in \operatorname{Irr}(H \mid \lambda)$, then $3 \nmid (\theta(1)/\lambda(1))$ and $3 \nmid (\chi(1)/\theta(1))$ for all $\chi \in \operatorname{Irr}(G \mid \theta)$. Induction implies that G/H and H/Z have abelian Sylow-3-subgroups. In particular, we may assume that $\mathbb{O}_{3'}(G/Z) = 1$ and $\mathbb{O}^{3'}(G/Z) = G/Z$. We let $N/Z = \mathbb{O}_3(G/Z)$, so that $Z < N \trianglelefteq G$. We must have that $\mathrm{Irr}(N/Z)$ consists entirely of extensions of λ. Then each irreducible character of N/Z is linear. Thus N/Z is abelian and $N < G$. By Lemma 1.2.3 of [8], $N/Z = \mathbb{C}_{G/Z}(N/Z)$. Let K be a maximal normal subgroup of G, so that $|G : K| = 3$ and $K > N$. Since K/Z has an abelian Sylow-3-subgroup and $N/Z = \mathbb{C}_{G/Z}(N/Z)$, $3 \nmid |K : N|$.

We need just show that N/Z is a chief factor of G. We may choose $Z \leqslant L < N$ such that N/L is a chief factor of G and $\mathbb{C}_{N/Z}(K/N) \leqslant L/Z$. Since $3 \nmid |K/N|$, we have that K/N does not centralize N/L. If $Z < L$, the induction argument yields that G/L has an abelian Sylow-3-subgroup. Since $\mathbb{O}^3(G/L) = G/L$, we then have that G/N and hence K/N centralize N/L, a contradiction. This completes Step 1.

Step 2. Let $V = \mathrm{Irr}(N/Z)$. Then V is an elementary abelian 3-group and a faithful irreducible G/N-module.

Since N/Z is an elementary abelian 3-group, so is V. Since N/Z is abelian and since G/N acts faithfully on N/Z, G/N acts faithfully on V (see Theorem 6.32 of [13]). By Exercise 2.7 of [13], the map $A \to \{\lambda \in V \mid A \leqslant \ker(\lambda)\}$ is a bijection from the set of subgroups of N/Z onto the set of subgroups of V. Since the map is G-invariant and N/Z is a chief factor of G, V is an irreducible G/N-module.

Step 3. We may assume that

(a) $I_G(\lambda) = G$;

(b) λ is linear and faithful and $Z \leqslant Z(G)$;

(c) $3 \nmid |Z|$; and

(d) there is a unique G-invariant extension $\lambda^* \in \mathrm{Irr}(N)$ of λ. Also $\mathbb{O}_3(N) \leqslant \ker(\lambda^*)$.

Since $\mu \to \mu^G$ is a bijection from $\mathrm{Irr}(I_G(\lambda) \mid \lambda)$ onto $\mathrm{Irr}(G \mid \lambda)$, we have that $I_G(\lambda)$ must contain a Sylow-3-subgroup of G and that $3 \nmid (\mu(1)/\lambda(1))$ for all $\mu \in \mathrm{Irr}(I_G(\lambda) \mid \lambda)$. Hence we may assume that $I_G(\lambda) = G$. By applying a character triple isomorphism (see Chap. 11 of [13]), we may assume that λ is linear.

Since $3 \nmid \eta(1)$ for any $\eta \in \mathrm{Irr}(N \mid \lambda)$ by the hypotheses of this theorem and since N/Z is a 3-group, each $\eta \in \mathrm{Irr}(N \mid \lambda)$ extends λ. Since $3 \nmid |K/N|$ and $\mathbb{C}_{K/N}(N/Z) = 1$, it follows from Lemma 4.3 that there is a unique K-invariant extension $\lambda_0 \in \mathrm{Irr}(N \mid \lambda)$ of λ. The hypotheses imply that $3 \nmid |G : I_G(\lambda_0)|$, so that λ_0 is invariant in G. Since λ_0 is linear, there is a unique factorization $\lambda_0 = \lambda_1 \cdot \lambda_2$, where $o(\lambda_2) = |N : \ker(\lambda_2)|$ is a power of 3 and $(o(\lambda_1), 3) = 1$. We note that $\lambda = (\lambda_1)_Z \cdot (\lambda_2)_Z$ is also such a factorization of λ. Since λ_0 is invariant in G, so are λ_1 and λ_2. Since $3 \nmid |K/N|$, λ_2 extends to K (see Corollary 6.27 of [13]). Since a Sylow-3-subgroup of G/N is cyclic, it now follows that there is an extension $\beta \in \mathrm{Irr}(G)$ of λ_2. Then $\chi \to \beta^{-1}\chi$ is a bijection from $\mathrm{Irr}(G \mid \lambda)$ onto $\mathrm{Irr}(G \mid (\lambda_1)_Z)$. It involves no loss of generality to assume that $\beta = 1$ and $\lambda = (\lambda_1)_Z$. We may also assume that λ is faithful. Hence $3 \nmid |Z|$. Since λ is linear, faithful, and invariant in G, $Z \leqslant Z(G)$. This proves (a), (b), and (c).

Now $N = Z \times \mathbf{O}_3(N)$. We let λ^* be the unique extension λ^* of λ to N with $\mathbf{O}_3(N) \leqslant \ker(\lambda^*)$. Then $I_G(\lambda^*) = G$. By Lemma 4.3, λ^* is the unique K-invariant extension of λ to N. This yields part (d).

Step 4. For each $\beta \in V$, we have that $3 \nmid |G : I_G(\beta)|$.

The hypotheses imply that $3 \nmid |G : I_G(\eta)|$ for all $\eta \in \mathrm{Irr}(N \mid \lambda)$. Since $\beta \to \beta\lambda^*$ is a bijection from V onto $\mathrm{Irr}(N \mid \lambda)$ and since $I_G(\lambda^*) = G^*$, we have that $I_G(\beta) = I_G(\beta\lambda^*)$ for each $\beta \in V$ and thus that $3 \nmid |G : I_G(\beta)|$ for each $\beta \in V$. This proves Step 4.

Step 5. There exist $C, L \trianglelefteq G$ with $N \leqslant C \leqslant L \leqslant K$ such that

(a) $G/C \cong J$;

(b) $V = V_1 \oplus V_2 \oplus \cdots \oplus V_8$, where the V_i are irreducible C-modules and C/N_i acts transitively on $V_i^{\#}$ for each i, where $N_i = \mathbf{C}_C(V_i)$;

(c) G/C primitevely permutes the V_i;

(d) $|L/C| = 8$ and L/C acts regularly on the V_i.

First assume that K/N is cyclic or isomorphic to Q_8. As $|G : K| = 3$, λ^* extends to $\phi \in \mathrm{Irr}(G)$ (see Lemma 2.1 and Corollary 2.3 of Part 1 [15]). Since $K/N = (G/N)' > 1$, there exists $\delta \in \mathrm{Irr}(G/N)$ with $\delta(1) = 3$. But then $3 \mid \delta\phi(1)$ and $\delta\phi \in \mathrm{Irr}(G \mid \lambda)$, a contradiction. Hence K/N is not cyclic or isomorphic to Q_8. By Steps 2 and 4, Theorem 2.3, and Lemma 3.2, there exists $N \leqslant C \trianglelefteq G$ such that (a), (b), and (c) are satisfied. We prove (d) by letting L/C be the minimal normal subgroup of G/C, and we note that $L \leqslant K$ since $K/Z = (G/Z)'$ is the unique maximal normal subgroup of G/Z.

Step 6. (a) Assume that $N \leqslant M \trianglelefteq G$, that $\theta \in \mathrm{Irr}(M \mid \lambda)$ and that there exists $M \leqslant M_1 \trianglelefteq I_G(\theta)$ with $I_G(\theta)/M_1$ nonabelian of order 21. Then $7 \mid |M_1 : M|$;

(b) if $T/N \in \mathrm{Syl}_7(C/N)$ is normal in G and if $\mu \in \mathrm{Irr}(T \mid \lambda)$, then $7 \mid |G : I_G(\mu)|$;

(c) $|V_1| = 3^n$ for an integer $n \geqslant 6$.

To prove (a), assume that $7 \nmid |M_1 : M|$. Since $|I_G(\theta)/M_1| = 21$ and $(21, |M_1/M|) = 1$, it follows from Lemma 4.3 that there exists $\alpha \in \mathrm{Irr}(M_1 \mid \theta)$ with α invariant in $I_G(\theta)$. But then α extends to $\eta \in \mathrm{Irr}(I_G(\theta) \mid \theta)$. Since $I_G(\theta)/M_1$ is nonabelian of order 21, there exists $\delta \in \mathrm{Irr}(I_G(\theta)/M_1)$ with $\delta(1) = 3$. Then $\delta\eta \in \mathrm{Irr}(I_G(\theta) \mid \theta)$ and $(\delta\eta)^G \in \mathrm{Irr}(G \mid \theta) \subseteq \mathrm{Irr}(G \mid \lambda)$, a contradiction as $3 \mid (\delta\eta)^G(1)$. This proves (a).

To prove (b), assume that $7 \nmid |G : I_G(\mu)|$. Since $3 \nmid |G : I_G(\mu)|$, we have that $LI_G(\mu) = G$ and $I_G(\mu)/L \cap I_G(\mu)$ is nonabelian of order 21. This contradicts part (a), as $7 \nmid |L : T|$.

We have an integer n such that $|V_i| = 3^n$ for each i. If $n < 6$, then $7 \nmid |\mathrm{Aut}(V_1)|$ and $7 \nmid |C/N_1|$. Since G permutes the N_i and $\bigcap N_i = N$, we have that $N/N \in \mathrm{Syl}_7(C/N)$. Part (b) implies that λ^* is not invariant in G, a contradiction. This completes Step 6.

Step 7. Let S/N be the Fitting subgroup of C/N. Then

(a) S/N and C/S are abelian;

(b) $S/S \cap N_i$ is cyclic and acts fixed-point-freely on V_i for each i (i.e., $\mathbb{C}_S(\alpha) = S \cap N_i$ for $1 \neq \alpha \in V_i$);

(c) each prime divisor of C/S divides n; and

(d) there is a prime $p_0 > n$ and a Sylow-p_0-subgroup P_0/N of C/N such that $1 \neq P_0/N \leqslant S/N$ and $\mathbb{C}_{C/N}(P_0/N) = S/N$.

Since C/N_i acts transitively on $V_i^{\#}$ for each i (Step 5(b)) and since $|V_i| \geqslant 3^6$, it follows from Lemma 3.4 that if $S_i/N_i = \mathbb{F}(C/N_i)$, then we have that S_i/N_i and C/S_i are cyclic and $|C/S_i| \mid n$. Since $S_i/N_i \trianglelefteq C/N_i$ is cyclic and V_i is a faithful irreducible C/N_i-module, we have that S_i/N_i acts fixed-point-freely on V_i. To prove (a), (b), and (c), we need just show $S = \bigcap S_i$. Since $\bigcap S_i/N$ is a normal abelian subgroup of C/N, $\bigcap S_i \leqslant S$. But $SN_i \trianglelefteq C$ and SN_i/N_i is nilpotent. Hence $S \leqslant S_i$ for each i and $S = \bigcap S_i$.

To prove (d), we choose p_0 as in Lemma 3.4 applied to C/N_1 acting on V_1. Then $p_0 > n$ and $p_0 \nmid |C/S|$ by part (c). Let P_0 be the Sylow-p_0-subgroup of C/N. Then N_1P_0/N_1 is the Sylow-p_0-subgroup of C/N_1 and thus $\mathbb{C}_{C/N_1}(P_0) = S_1/N_1$ by Lemma 3.4. Since $P_0/N \trianglelefteq G/N$, since $\bigcap S_i = S$ and G permutes the S_i, we have that $\mathbb{C}_{C/N}(P_0) = S/N$.

Step 8. (a) If $N \leqslant A \trianglelefteq G$ with $A \leqslant C$ and $C/A \leqslant \mathbb{Z}(G/A)$, then $C = A$; and

(b) if $1 \neq R/S$ is a Sylow-subgroup of C/S, then $C/S = \mathbb{C}_{G/S}(R/S)$.

To prove (a), we may assume that $|C/A|$ is prime. If $(|C/A|, |G/C|) = 1$, then $G/A = C/A \times J_1$, where $J_1 \cong G/C \cong J$, a contradiction as $\mathbb{O}^{3'}(G/N) = G/N$. If $|C/A| = 7$, then $L/A = C/A \times B/A$, where $B/A \trianglelefteq G/A$ has order 8. Then $|G/B| = 3 \cdot 7^2$ and by Fitting's lemma $K/B = \mathbb{C}_{K/B}(t_0) \times [K/B, t_0]$, where $t_0 \in G/B$ has order 3. Then $\langle t_0 \rangle \cdot [K/B, t_0] \trianglelefteq G/B$ as $\mathbb{C}_{K/B}(t_0) \neq 1$. This is a contradiction as $\mathbb{O}^{3'}(G/B) = G/B$. We assume that $|C/A| = 2$. If L/A is abelian, we may apply Fitting's lemma to write $L/A = \mathbb{C}_{L/A}(G/L) \times [L/A, G]$ and $|\mathbb{C}_{L/A}(G/L)| = 2$. But then $(G/A)/[L/A, G]$ has normal Hall-subgroups of order 2 and index 2, a contradiction. We must have that L/A is nonabelian. Since L/C is a chief factor of G, we must have that $C/A = \mathbb{Z}(L/A)$, a contradiction as no class 2-group of order 16 has a center of order 2 (which can easily be shown by Theorem 4.1). This proves (a).

To prove (b), assume that $1 \neq R/S$ is a Sylow-subgroup of C/S and $\mathbb{C}_{G/S}(R/S) > C/S$. Then $L/S \leqslant \mathbb{C}_{G/S}(R/S)$ since L/C is the unique minimal normal subgroup of G/C. Since L transitively permutes the $R \cap S_i$ and $\bigcap S_i = S$, we have that $S = R \cap S_i$ for each i and R/S is cyclic. Then $R/S \leqslant \mathbb{Z}(K/S)$ and, by part (a), $K/S = \mathbb{C}_{G/S}(R/S)$.

Let $G_1 = \mathbb{N}_G(V_1)$, so that $|G_1/C| = 21$. Let $D = \mathbb{C}_G(V_1)$, so that $D \leqslant G_1$ and $D \cap C = N_1$. Consequently $R \cap DS_1 = R \cap (D \cap C) S_1 = R \cap N_1 S_1 = R \cap S_1 = S$. Thus the natural projection of G_1/S onto G_1/DS_1 carries R/S isomorphically onto RDS_1/DS_1. Since G_1/D is isomorphic to a subgroup of the semilinear group $T(3^n)$, it follows that any Sylow-3-subgroup of G_1 centralizes $R(DS_1)/DS_1$ and hence must centralize R/S. This implies that $R/S \leqslant \mathbb{Z}(G/S)$. Part (a) then yields $R = S$, a contradiction, completing Step 8.

Step 9. Suppose that $F/N \trianglelefteq G/N$ and $F \leqslant S$. If $\mathbb{C}_{G/N}(F/N) \nleqslant C/N$, then F/N is cyclic and $F/N \leqslant \mathbb{Z}(K/N)$.

Let $D/N = \mathbb{C}_{G/N}(F/N)$ and assume that $D \nleqslant C$. Since L/C is the minimal normal subgroup of G/C, $L \leqslant DC$. Since L/C transitively permutes the N_i, DC transitively permutes the $F \cap N_i$. But C fixes each N_i and D centralizes F/N. Thus $F \cap N_1 = \cdots = F \cap N_8$. Since $\bigcap N_i = N$, since S_i/N_i is cyclic, and since $F \leqslant S$; we have that F/N is cyclic. Since $\mathrm{Aut}(F/N)$ is abelian, $K/N = (G/N)' \leqslant D/N$. This completes Step 9.

Step 10. Suppose that $P/N \in \mathrm{Syl}_p(C/N)$ for a prime p that does not divide $|L/S|$. Assume that $N \leqslant W \leqslant P$ such that W/N is a chief factor of G. If $|W/N| \geqslant p^7$, then λ^* is fully ramified with respect to P/N.

We let $W_i = W \cap N_i$ for each i, let $W_{23} = W \cap N_2 \cap N_3$, etc. Since $W/W_i \cong WN_i/N_i$ is cyclic, since L/C permutes the W_i, and since $|W/N| \geqslant p^7$, we have that $W/N \nleqslant \mathbb{Z}(L/N)$ and that $W/N = [W/N, L]$. Since $p \nmid |L/S|$, we may write $P/N = Q/N \times Y/N$ via Fitting's lemma where $Y/N = \mathbb{C}_{P/N}(L)$ and $Q/N = [P/N, L] \geqslant W/N$. We let $D/N = \Omega_1(Q/N) \geqslant W/N$. Since $S/S \cap N_i$ is cyclic and $\bigcap N_i = N$, we have that $7 \leqslant \mathrm{rank}(W/N) \leqslant \mathrm{rank}(D/N) = \mathrm{rank}(Q/N) \leqslant \mathrm{rank}(P/N) \leqslant 8$. If $W < D$, then $D/W \trianglelefteq G/W$ is cyclic and $D/W \leqslant \mathbb{Z}((G/W)') = K/W$, a contradiction as $p \nmid |L/S|$ and $\mathbb{C}_{Q/N}(L/S) = 1$. Thus $W/N = D/N = \Omega_1(Q/N)$ is an irreducible G/S-module. It follows that Q/N is homocyclic and $\Omega_{j+1}(Q/N)/\Omega_j(Q/N)$ is an irreducible G/S-module of order 1 or $|W|$ for each j.

We may write $N = Z \times U$ where $U = \mathbb{O}_3(N)$ (see Step 3). For $a \in Q/U$, define $\phi_a \in \mathrm{Hom}(Y/N, N/U)$ by $\phi_a(y) = [y, a]$. Since $N/U \leqslant \mathbb{Z}(Y/U)$, we have that ϕ_a is well defined. Thus $a \to \phi_a$ defines a 1–1 homomorphism from $(Q/U)/\mathbb{C}_{Q/U}(Y/U)$ into $\mathrm{Hom}(Y/N, N/U)$, where multiplication in $\mathrm{Hom}(Y/N, N/U)$ is defined pointwise. Since Y/N and N/U are cyclic, so are

$\mathrm{Hom}(Y/N, N/U)$ and $(Q/U)/\mathbb{C}_{Q/U}(Y/U)$. Since $N/U \leqslant \mathbb{C}_{Q/U}(Y/U)$, since $\mathbb{C}_{Q/U}(Y/U)$ is G-invariant, and $(Q/U)/\mathbb{C}_{Q/U}(Y/U)$ is cyclic, it follows from the last paragraph that $\mathbb{C}_{Q/U}(Y/U) = Q/U$. Since $Y/U \leqslant \mathbb{Z}(P/U)$ and $\lambda^* \in \mathrm{Irr}(N/U)$, there exists a P-invariant extension $\mu \in \mathrm{Irr}(Y \mid \lambda^*)$.

By Theorem 4.1, there exists $H \trianglelefteq G$ such that $N \leqslant H \leqslant P$ and that each $\gamma \in \mathrm{Irr}(H \nmid \lambda^*)$ extends λ^* and is fully ramified with respect to P/H. If $H = N$, this step is complete. We may assume that $H > N$. Since any $\delta \in \mathrm{Irr}(P \mid \lambda^*)$ vanishes off H and since $\mu \in \mathrm{Irr}(Y \mid \lambda^*)$ is P-invariant and linear, we must have that $Y \leqslant H$. Since $W/N = \Omega_1(Q/N)$ is a chief factor of G, WY/Y is the unique minimal normal subgroup of G/Y contained in P/Y. To prove that $W \leqslant H$, we may assume that $H = Y$. By Lemma 4.2, $P/Y \cong A \times A$ for some abelian group A. Hence $Q/N \cong P/Y$ has even rank and $W/N = \Omega_1(Q/N)$ has even rank. We must then have that $\mathrm{rank}(\Omega_1(P/Z)) = \mathrm{rank}(W) = 8$. Hence $H = Y = N$, a contradiction. Thus $W \leqslant H$.

We have that each $\gamma \in \mathrm{Irr}(H \mid \lambda^*)$ extends λ^* and is fully ramified with respect to P/H. In particular, each such γ is invariant in P. Since $(|G_{12} : P|, |P : N|) = 1$, it follows from Lemma 4.3 that there exists $\gamma^* \in \mathrm{Irr}(H \mid \lambda^*)$ invariant in G_{12} (note that G_{12} denotes the stabilizer in G of $\{V_1, V_2\}$, so that G_{12}/C is cyclic of order 6). Let $t \in G_{12}/N$ have order 3. We may assume that t permutes both $\{V_3, V_4, V_5\}$ and $\{V_6, V_7, V_8\}$ non-trivially.

We next show that there exist linear characters $\rho \in \mathrm{Irr}(W|\lambda^*)$ and $\rho_0 \in \mathrm{Irr}(W_{12}|\lambda^*)$ such that ρ extends ρ_0, that $3 \nmid o(\rho_0)$, and ρ_0 is not invariant under any Sylow-3-subgroup of G_{12}/N (note $W_{12} \trianglelefteq G_{12}$). Since W/W_i is cyclic for each i and $|W/N| \geqslant 3^7$, we have that W_{12345} has rank at least two. Letting $X_j = W_{12345j}$ for $6 \leqslant j \leqslant 8$, we have that X_6, X_7, and X_8 are distinct and permuted nontrivially by t. Since G_{12}/C is cyclic of order 6, each 3-element of G_{12}/N permutes X_6, X_7, and X_8 nontrivially. Let $\eta \in \mathrm{Irr}(W_{12345}/X_6)$ be faithful. Then η is not invariant under any Sylow-3-subgroup of G_{12}. Let $\tau \in \mathrm{Irr}(W|\eta)$. Then τ is linear and $N \leqslant \ker(\tau)$. We let $\rho = \tau \cdot (\gamma^*_W)$ and let ρ_0 be the restriction of ρ to W_{12}. In particular, ρ and ρ_0 are linear. Since $3 \nmid |Z| \, |W/N|$, we have that $3 \nmid o(\rho_0)$. If γ_0 is the restriction of γ^* to W_{12345}, then ρ extends $\eta\gamma$. Since γ^* is invariant in G, it follows that neither $(\eta\gamma_0)$ nor ρ_0 is invariant under a Sylow-3-subgroup of G_{12}/N. We have shown what we stated at the beginning of this paragraph.

Let $\alpha_j \in V_j$ be nonprincipal characters for $j = 1, 2$ and let $\beta_0 = (\alpha_1, \alpha_2, 1, 1, 1, 1, 1, 1) \in V = \mathrm{Irr}(N/Z)$. Since W_{12} centralizes V_1 and V_2, since β_0 is linear with $o(\beta_0) = 3$, and since $3 \nmid |W_{12}/N|$, there is a unique extension $\beta \in \mathrm{Irr}(W_{12}|\beta_0)$ such that $o(\beta) = 3$. Since $\beta_N = (\alpha_1, \alpha_2, 1, \ldots, 1)$ and W/W_i acts fixed-point-freely on V_i for each i by Step 5, it follows that $I_W(\beta) = W_{12}$. Thus $\beta^W \in \mathrm{Irr}(W)$ and β^W restricted to W_{12} is $\beta_1 + \cdots + \beta_t$, where $\beta_1, \ldots, \beta_t \in \mathrm{Irr}(W_{12})$ are the distinct conjugates of β. Since $\beta^W \rho \in \mathrm{Irr}(W|\lambda^*)$ and $W \trianglelefteq G$, the hypotheses of the theorem imply that $\beta^W \rho$ is left invariant by some $s \in G/N$ of order 3. Since $3 \nmid |W/N|$, s must fix some irreducible constituent of $(\beta^W\rho)_N$ by Theorem 13.27 of [13]. Since each irreducible constituent of $(\beta^W\rho)_N$ has the form $\lambda^*(\sigma_1, \sigma_2, 1,..., 1)$ for nonprincipal σ_i $(i=1,2)$, we have that $s \in G_{12}$. Since $W_{12} \trianglelefteq G_{12}$, s must fix an irreducible constituent of $(\beta^W\rho)$ restricted to W_{12}, by Theorem 13.27 of [13]. It is easy to see that $\beta^W\rho$ restricted to W_{12} is $\beta_1\rho_0 + \cdots + \beta_l\rho_0$ (e.g., see Exercise 5.3 of [13]). Then s fixes $\beta_j\rho_0$ for some j. Since β_j and ρ_0 are linear, $o(\beta_j) = 3$ and $3 \nmid o(\rho_0)$, s must fix both β_j and ρ_0. This contradicts the last paragraph and completes this step.

Step 11. We may assume that $7 \nmid |C/S|$.

Assume that $7 \mid |C/S|$. By Step 7, $n \geqslant 7$ and $p_0 > 8$. Steps 7 and 9 yield that $\mathbb{C}_{G/N}(P_0/N) = S/N$ and $p_0 \nmid |G/S|$. Then $\Omega_1(P_0/N)$ is a faithful and completely reducible G/S-module. A Sylow-7-subgroup H/S of G/S is nonabelian by Step 8(b). Thus we may choose a chief factor W/N of G/N such that $W \leqslant P_0$ and $H/\mathbb{C}_H(W/N)$ is nonabelian. Thus rank$(W/N) \geqslant 7$ and Step 10 implies that λ^* is fully ramified with respect to P_0/N. Since $S/S \cap N_i$ is cyclic for each i and $\bigcap N_i = N$, rank$(P_0/N) \leqslant 8$. By Lemma 4.2, rank$(H_1/S) \leqslant 4$, where $H_1 = H \cap C$. In particular, rank$(\Omega_1(H_1/S)) \leqslant 4$.

By Step 8(b), L/C is a 2-group acting faithfully on $\Omega_1(H_1/S)$. But L/C is the unique minimal normal subgroup of G/C. Hence we may find a chief factor H_2/S of G/S such that $H_2/S \leqslant \Omega_1(H_1/S)$ and that G/C acts faithfully on H_2/S. Since K/C is a Frobenius group of order 56, Lemma 2.1 yields that rank$(H_2/S) \geqslant 7$, a contradiction. This completes Step 11.

Step 12. Let $T/N \in \mathrm{Syl}_7(C/N)$. Then

(a) T/N is cyclic; or

(b) λ^* is fully ramified with respect to T/N.

By Step 11, $T \leqslant S$. By Step 9, we may assume that $\mathbb{C}_{G/N}(T/N) \leqslant C/N$. First assume that $\mathbb{C}_{G/N}(T/N) = C/N$. We may choose a chief factor W/N of G/N such that $W/N \leqslant [T/N, L/C]$. Since K/C acts faithfully on W/N and is a Frobenius group of order 56, it follows from Lemma 2.1 that rank$(W/N) \geqslant 7$. In this case, Step 10 implies (b) above. We may assume that $\mathbb{C}_{G/N}(T/N) < C/N$.

For each i, we have that TN_i/N_i is the cyclic Sylow-7-subgroup of C/N_i and is contained in S_i/N_i. We let $D_i/N_i = \mathbb{C}_{C/N_i}(TN_i/N_i)$ and set $D = D_1 \cap \cdots \cap D_8$. Then $[D, T] \leqslant \bigcap N_i = N$ and it follows that $D/N = \mathbb{C}_{C/N}(T/N) = \mathbb{C}_{G/N}(T/N)$. Since TN_i/N_i is cyclic and $3 \nmid |C/N|$, we have that $|C/D_i| \leqslant 2$ for each i. Since $D < C$ and L/C transitively permutes the D_i, we have that $|C/D_i| = 2$ for each i. Also C/D and L/D are 2-groups. If L/D is abelian, then $D_1/D = \cdots = D_8/D$ as L/C transitively permutes the D_i. In this case $D_1 = D$ and $|C/D| = 2$. But then $C/D \leqslant Z(G/D)$, contradicting Step 8. Hence L/D is nonabelian.

Since L/D acts faithfully on T/N and $(|L/D|, |T/N|) = 1$, we have that $\Omega_1(T/N)$ is a faithful and completely reducible L/D-module. We may write $\Omega_1(T/N) = A/N \times B/N$ where A/N and B/N are L/D-modules with $(L/D)' \leqslant \mathbb{C}_{L/D}(A/N)$ and such that $(L/D)/\mathbb{C}_{L/D}(Y)$ is nonabelian if $1 \neq Y$ is an irreducible L/D-submodule of B/N. Then $B \trianglelefteq G$ and $N \neq B$ as L/D is nonabelian. We let W/N be a chief factor of G/N with $W \leqslant B$. In particular, $(L/D)/\mathbb{C}_{L/D}(W/N)$ is nonabelian.

Write $W/N = Y_1 \oplus \cdots \oplus Y_j$ where the Y_i are homogeneous components of W/N viewed as an C/D module. Then $j = |G : I_G(Y_1)|$. Assume that $K \leqslant I_G(Y_1)$. Then $L \leqslant I_G(Y_i)$ for all i. Since all the $L/\mathbb{C}_C(Y_i)$ are isomorphic and since L/D is a subdirect product of the $L/\mathbb{C}_C(Y_i)$, each $L/\mathbb{C}_C(Y_i)$ is nonabelian. Since C/D is elementary abelian, we have that $|\mathbb{C}/\mathbb{C}_C(Y_1)| = 2$. But $\mathbb{C}_C(Y_1)$ and $\mathbb{Z}(L/\mathbb{C}_C(Y_1))$ are invariant in K. Thus $C/\mathbb{C}(X_1) = \mathbb{Z}(L/\mathbb{C}_C(Y_1))$. Hence $L/\mathbb{C}_C(Y_1)$ has order 16, class 2, and a center of order 2, which is impossible (see, e.g., Theorem 4.1). Hence $K \nleqslant I_G(Y_1)$. Thus $7 | j$ or $2 | j$. Since $C \leqslant I_G(Y_1)$, since $j = |G : I_G(Y_1)|$ and since G/C has no subgroup of index 2, 4, or 6; we have that $j \geqslant 7$. Hence $\operatorname{rank}(W/N) \geqslant 7$ and Step 10 gives the desired conclusion of this step.

Step 13. Conclusion. Let $T_1/N \in \operatorname{Syl}_7(G/N)$, so that $|T_1 : T| = 7$. By Step 6(b), no $\mu \in \operatorname{Irr}(T | \lambda)$ is invariant in T_1. If λ^* is fully ramified with respect to T/N, then $(\lambda^*)^T$ has a unique irreducible constituent ϕ. Since $I_G(\lambda^*) = G$, we must have $I_G(\phi) = G$, a contradiction. By Step 12, T/N is cyclic. Hence $K/N = (G/N)' \leqslant \mathbb{C}_{G/N}(T/N)$ and T_1/N is abelian. If λ^* extends to $\gamma \in \operatorname{Irr}(T_1)$, then $\gamma_T \in \operatorname{Irr}(T | \lambda)$ is invariant in T_1, a contradiction. Thus T_1/N is not cyclic and $T_1/N = T/N \times T_0/N$ with $|T_0/N| = 7$ and $T_0 \nleqslant C$. We may assume that T_0 permutes the V_i with orbits $\{V_1\}$ and $\{V_2, ..., V_8\}$. Since $|T_0/N| = 7$, we may choose $1 \neq \beta_i \in V_i$ for $2 \leqslant i \leqslant 8$ such that T_0 permutes the β_i and $T_0 \leqslant I_G(\beta)$, where $\beta = (1, \beta_2, ..., \beta_8)$. Let $I = I_G(\beta) = I_G(\lambda^* \beta)$, so that $I \leqslant G_1 = I_G(V_1)$. By Step 4, $3 \nmid |G : I|$. Hence $I/C \cap I$ is nonabelian of order 21. Since $T/N \leqslant \mathbb{Z}(L/N)$, we have that $T \cap N_1 = \cdots = T \cap N_8 = N$ and thus T/N acts fixed-point-freely on each V_i by Step 7. Thus $I_T(\beta) = 1$ and $7 \nmid |(C \cap I)/N|$. This contradicts Step 6(a). The proof of the theorem is complete. ∎

5. The Prime Two

Theorem 5.1 proves Theorem A when the prime concerned is 2.

5.1. Theorem. *Suppose that Z is a normal (not necessarily central) subgroup of G, that G/Z is solvable, and that $\lambda \in \operatorname{Irr}(Z)$. If $2 \nmid (\chi(1)/\lambda(1))$ for all $\chi \in \operatorname{Irr}(G | \lambda)$, then G/Z has an abelian Sylow-2-subgroup.*

Proof. We argue by induction on $|G : Z|$ and the proof will be in a series of steps. Steps 1–9 are analogous to the corresponding Steps 1–9 of Theorem 4.4, and the almost identical proofs are omitted.

Step 1. We may assume that there exist $Z \leqslant N \leqslant K \trianglelefteq G$ such that

(a) N/Z is a chief factor of G and $\mathbb{C}_{G/Z}(N/Z) = N/Z$;

(b) $G/Z = \mathbb{O}^{2'}(G/Z)$; and

(c) N/Z is a 2-group, $|G : K| = 2$, and $2 \nmid |K/N|$.

Step 2. Let $V = \mathrm{Irr}(N/Z)$. Then V is an elementary abelian 2-group and a faithful irreducible G/N-module.

Step 3. We may assume that

(a) $I_G(\lambda) = G$;

(b) λ is linear and faithful and $Z \leqslant \mathbb{Z}(G)$;

(c) $2 \nmid |Z|$; and

(d) there is a unique G-invariant extension $\lambda^* \in \mathrm{Irr}(N)$ of λ.

Step 4. For each $\beta \in V$, we have that $2 \nmid |G : I_G(\beta)|$.

Step 5. There exists $C \trianglelefteq G$ with $N \leqslant C < K$ such that

(a) $G/C \cong D_{2q}$, the dihedral group for $q = 3$ or 5;

(b) $V = V_1 \oplus V_2 \oplus \cdots \oplus V_q$, where the V_i are irreducible C-modules and C/N_i acts transitively on $V_i^{\#}$ for each i, where $N_i = \mathbb{C}_C(V_i)$; and

(c) G/C primitively permutes the V_i.

Step 6. Assume that $N \leqslant M \trianglelefteq G$, that $\theta \in \mathrm{Irr}(M \mid \lambda)$, and that there is $M \leqslant M_1 \trianglelefteq I_G(\theta)$ with $I_G(\theta)/M_1 \cong D_{2q}$. Then $q \,\big|\, |M_1 : M|$.

Step 7. Let S/N be the Fitting subgroup of C/N. Then

(a) S/N and C/N are abelian;

(b) $S/S \cap N_i$ is cyclic and acts fixed-point-freely on V_i for each i (i.e., $\mathbb{C}_S(\alpha_i) = N_i$ for $1 \neq \alpha_i \in V_i$);

(c) each prime divisor of C/S divides n, where n is defined by $|V_1| = 2^n$; and

(d) there is a prime $p_0 > n$ and Sylow-p_0-subgroup P_0/N of C/N such that $1 \neq P_0/N \leqslant S/N$ and $\mathbb{C}_{C/N}(P_0/N) = S/N$.

Step 8. If $1 \neq R/S$ is a Sylow-subgroup of C/S, then $C/S = \mathbb{C}_{G/S}(R/S)$.

Step 9. Suppose that $F/N \trianglelefteq G/N$ and $F \leqslant S$. If $\mathbb{C}_{G/N}(F/N) \nleqslant C/N$, then F/N is cyclic and $F/N \leqslant \mathbb{Z}(K/N)$.

Step 10. Assume that $P/N \in \mathrm{Syl}_p(S/N)$ for a prime p that does not divide $|G/S|$. Then λ^* is fully ramified with respect to P/N.

Let $D/N = [P/N, G/N]$. Since $p \nmid |G:P|$, we have that $G/D = P/D \times M/D$ for a Hall-p'-subgroup M/D of G/D. Since $\mathbb{O}^{2'}(G/N) = G/N$, we have that $M = G$ and $[P/N, G/N] = P/N$. Since P/N is abelian, Fitting's lemma implies that $\mathbb{C}_{P/N}(G/N) = 1$.

By Theorem 4.1, we may choose $N \leqslant H \leqslant P$ with $H \trianglelefteq G$ such that each $\eta \in \mathrm{Irr}(H|\lambda^*)$ extends λ^* and is fully ramified with respect to P/H. By Lemma 4.3, there is some $\phi \in \mathrm{Irr}(H \mid \lambda^*)$ such that $I_G(\phi)$ contains a Hall-p'-subgroup of G. Since ϕ is fully ramified with respect to P/H, ϕ is invariant in P and $I_G(\phi) = G$. The hypotheses imply that $2 \nmid |G : I_G(\eta)|$ for any $\eta \in \mathrm{Irr}(H \mid \lambda^*)$. Since $\delta \to \delta\phi$ is a bijection from $\mathrm{Irr}(H/N)$ onto $\mathrm{Irr}(H \mid \lambda^*)$ and since $I_G(\phi) = G$, we have that $2 \nmid |G : I_G(\delta)|$ for all $\delta \in \mathrm{Irr}(H/N)$. If $t \in G/N$ is an involution and $\delta_0 \in \mathrm{Irr}(H/N)$ is inverted by t, then some involutions $s \in G/N$ fixes δ_0 and st inverts δ_0. Since $st \in K$ and since $|K/N|$ and $|\mathrm{Irr}(H/N)|$ are odd, we have that $\delta_0 = 1_H$. Hence t inverts no nonprincipal $\lambda \in \mathrm{Irr}(H/N)$ and $G/N = \mathbb{O}^{2'}(G/N)$ acts trivially on $\mathrm{Irr}(H/N)$. But $G/\mathbb{C}_G(H/N)$ acts faithfully on $\mathrm{Irr}(H/N)$ (see Theorem 6.32 of [13]). Thus $H/N \leqslant \mathbb{Z}(G/N)$. Thus $H = N$ by the first paragraph, and hence λ^* is fully ramified with respect to P/N. This completes Step 10.

Step 11. $C = S$.

We may assume that $C > S$. Since $|C/S|$ is odd, $n \neq 1, 2, 4$ by Step 7(c). Since $p_0 \,\big|\, |S|$, we have that $p_0 \,\big|\, |S_i/N_i|$ and $p_0 \mid (2^n - 1)$. Since $p_0 > n$, $p_0 \nmid |G/C|$. By Steps 7 and 9, $p_0 \nmid |G/S|$ and $S/N = \mathbb{C}_{G/N}(P_0/N)$. Then $\Omega_1(P_0/N)$ is a faithful and completely reducible G/S-module. Since $C > S$, we have K/S is nonabelian by Step 8. We may choose an irreducible K/S-module $Y \leqslant \Omega_1(P_0/N)$ such that $K/\mathbb{C}_C(Y)$ is nonabelian. Write $Y = Y_1 \oplus \cdots \oplus Y_l$ where the Y_j are the distinct homogeneous components of Y viewed as a C/S-module and $l = 1$ or q. If $l = 1$, then $C/\mathbb{C}_C(Y) = C/\mathbb{C}_C(Y_1) \leqslant \mathbb{Z}(K/\mathbb{C}_C(Y_1))$ as C/S is abelian. But then $K/\mathbb{C}_V(Y)$ is abelian, a contradiction. Thus $l \geqslant q$ and $\mathrm{rank}(P_0/N) \geqslant \mathrm{rank}(Y) \geqslant q$. But since $S/S \cap N_i$ is cyclic, $\mathrm{rank}(P_0/N) = q$. Since $p_0 \nmid |G/S|$, it follows from Step 10 and Lemma 4.2 that $\mathrm{rank}(P_0/N)$ is even, a contradiction as $q = 3, 5$. We may assume that $C = S$.

Step 12. Conclusion. Let $X_0 = \{(\beta_1, ..., \beta_q) \mid 1 \neq \beta_i \in V_i \text{ for each } i\}$ and let $\beta \in X_0$. Since $C = S$, we have by Step 7 that C/N_i acts fixed-point-freely on V_i for each i and thus $I_C(\beta) = N$. Since $G/C \cong D_{2q}$, $I_G(\beta)/N$ is isomorphic to a subgroup by D_{2q}. By Steps 4 and 6, $2 \,\big|\, |I_G(\beta)/N|$ and $I_G(\beta)/N = I_G(\lambda^*\beta)/N \ncong D_{2q}$. Hence $|I_G(\beta)/N| = 2$ and β is fixed by exactly one Sylow-2-subgroup of G/N. Choose an involution $t \in G/N$ such that $t \in G_1/N = N_G(V_1)/N$. Then V_1 is the unique V_i fixed by t and t fixes exactly $|\mathbb{C}_{V_i}(t)^{\#}|$

$(2^n-1)^{(q-1)/2}$ elements of X_0. Since $|X_0|=(2^n-1)^q$ and $\beta\in X_0$ is fixed by exactly one involution of G/N, we have that

$$|\mathrm{Syl}_2(G/N)|\,|\mathbb{C}_{V_1}(t)^{\#}|\,(2^n-1)^{(q-1)/2}=(2^n-1)^q. \tag{V}$$

Let $\beta_0=(1,\beta_2,...,\beta_q)$ with $1\neq\beta_i\in V_i$ for $2\leqslant i\leqslant q$. Since C/N_i is cyclic and acts fixed-point-freely on V_i and since $\bigcap N_i=1$, we have that $I_C(\beta_0)/N=N_2\cap\cdots\cap N_q/N$ is cyclic. But $I_G(\beta_0)\leqslant G_1$, so that $I_G(\lambda^*\beta_0)/N=I_G(\beta_0)/N$ has a cyclic normal subgroup of odd order and index 2. Hence $\lambda^*\beta_0$ extends to $I_G(\beta_0)$. The hypotheses imply that each $\chi\in\mathrm{Irr}(I_G(\lambda^*\beta_0)\,|\,\lambda^*\beta_0)$ has odd degree. Thus $2\nmid\mu(1)$ for all $\mu\in\mathrm{Irr}(I_G(\beta_0)/N)$ and $I_G(\beta_0)/N$ is cyclic. Thus β_0 is fixed by a unique involution of G/N. Let $X_1=\{(\beta_1,...,\beta_q)\,|\,\beta_i\in V_i$ and exactly one $\beta_j=1\}$. Each element of X_1 is fixed by exactly one involution and t fixes exactly $(2^n-1)^{(q-1)/2}$ elements of X_1. Thus

$$|\mathrm{Syl}_2(G/N)|\,(2^n-1)^{(q-1)/2}=q(2^n-1)^{q-1}. \tag{VI}$$

Equations (V) and (VI) yield that

$$q\,|\mathbb{C}_{V_1}(t)^{\#}|=2^n-1.$$

In particular, t does not centralize V_1. Since C/N_1 is cyclic, there is a dihedral group $H\langle t\rangle$ that is a subgroup of G_1/N_1 such that H/N_1 acts fixed-point-freely on V_1. By Lemma 2.1, $|\mathbb{C}_{V_1}(t)|=2^{n/2}$. Thus $q=2^{n/2}+1$. We now have that $q=3$, and $n=2$ or that $q=5$ and $n=4$.

Assume that $q=5$ and $n=4$. Let $\beta_0=(1,\beta_2,...,\beta_5)$ be as above. Since $I_G(\beta_0)\leqslant G_1$, we may choose β_0 so that $t\in I_G(\beta_0)/N$. We have that $I_G(\beta_0)/N$ is cyclic and t centralizes $I_C(\beta_0)/N=N_2\cap\cdots\cap N_5$, which is isomorphic to a factor group of C/N_1 centralized by t. But C/N_1 is cyclic of order 15 and the Sylow-5-subgroup A/N_1 of C/N_1 is not centralized by t as A/N_1 acts irreducibly on V_1 and $\mathbb{C}_{V_1}(t)$ is a nontrivial proper submodule of V_1. Hence $5\nmid|N_2\cap\cdots\cap N_5|$. It follows from Step 10 that the Sylow-3-subgroup F/N of S/N has even rank and thus $\mathrm{rank}(F/N)\leqslant 4$. Since $N_1\cap\cdots\cap N_5=N$ and G permutes the N_i, it is routine to see that $F\cap N_2\cap\cdots\cap N_5=N$. Since S is a $\{3,5\}$-group, we have that $N_2\cap\cdots\cap N_5=N$. If $1\neq\alpha_i\in V_i$ for $3\leqslant i\leqslant 5$, then $I_C(1,1,\alpha_3,\alpha_4\alpha_5)=N_3\cap N_4\cap N_5$ is isomorphic to a factor group of C/N_2 and is cyclic. We can now argue as in the last paragraph that each $\alpha\in X_2=\{(\alpha_1,...,\alpha_5)\,|$ exactly two $\alpha_j=1\}$ is fixed by a unique Sylow-2-subgroup of G/N. But t fixes $|\mathbb{C}_{V_1}(t)^{\#}|\cdot 2\cdot(2^n-1)=3\cdot 2\cdot 15$ elements of X_2 and $|X_2|=10\cdot 15^3$. Thus $3\cdot 2\cdot 15\,|\mathrm{Syl}_2(G/N)|=10\cdot 15^3$ and $|\mathrm{Syl}_2(G/N)|=3\cdot 5^3$. This contradicts Eq. (VI). Hence $q=3$ and $n=2$.

We have that $|\mathrm{Syl}_2(G/N)|=3^2$ by Eq. (VI). Since $|C/N_i|=3$, C/N is an elementary abelian 3-group. Choose an involution $s\in G_2/N$. Then st does not fix any V_i and the dihedral subgroup $\langle s,t\rangle$ of G/N has order 6 or 18. If

$o(st) = 3$, we may choose an $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in V$ with $1 \neq \alpha_i \in V_i$ such that (st) fixes α. This is a contradiction, as we have shown that $|\mathbb{C}_{G/N}(\alpha)| = 2$. Thus the dihedral group $\langle s, t\rangle$ has order 18 and contains all 9 involutions of G/N. Since $\mathbb{O}^{2'}(G/N) = G/N$, we have $G/N = \langle s, t\rangle$ and K/N is cyclic. Thus λ^* extends to G. In particular, there is a G-invariant extension $\phi \in \operatorname{Irr}(C \mid \lambda^*)$. This contradicts Step 6. The proof of Theorem 5.1 is complete. ∎

We next summarize results of Sections 4 and 5 and of Section 2 of part 1 [15] to derive Theorem A.

5.2. Corollary. *Let Z be a normal (not necessarily central) subgroup of G. Assume that G/Z is solvable and that $\lambda \in \operatorname{Irr}(Z)$. If $p \nmid (\chi(1)/\lambda(1))$ for all $\chi \in \operatorname{Irr}(G \mid \lambda)$, then the Sylow-p-subgroups of G/Z are abelian.*

Proof. Since $p \nmid |G : I_G(\lambda)|$, we may assume $G = I_G(\lambda)$. By a character triple isomorphism (see Chap. 11 of [13]), we may assume that λ is linear and $p \nmid \chi(1)$ for all $\chi \in \operatorname{Irr}(G \mid \lambda)$. The result now follows from Theorems 4.4 and 5.1 and from Theorem 2.5 of Part 1 [15].

Our techniques can extend Corollary 5.2 to a set π of primes and to Hall-π-subgroups. If the hypothesis "$p \nmid (\chi(1)/\lambda(1))$ for all $\chi \in \operatorname{Irr}(G \mid \lambda)$" is replaced by "$(\chi(1)/\theta(1))$ is a π'-number for all $\chi \in \operatorname{Irr}(G \mid \lambda)$," then we may conclude that the Hall-π-subgroups of G/Z are abelian. We omit the proof, which is very similar to that of Theorem 4.4.

Acknowledgments

We wish to thank I. M. Isaacs for fruitful discussion and suggestions.

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