Parameter uniform numerical method for a weakly coupled system of second order singularly perturbed turning point problem with Robin boundary conditions

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Abstract

In this paper, a parameter uniform numerical method for a weakly coupled system of two second order singularly perturbed turning point problems with Robin boundary conditions is presented. It is assumed that both equations have a turning point at the same point. An appropriate piecewise uniform mesh is considered and a classical finite difference scheme is applied on this mesh. An error estimate is derived by using supremum norm and it is of order $O(N^{-1}\ln N)$. Numerical examples are given which validate theoretical results.

Keywords:
Singularly perturbed turning point problems; Boundary value problems; Finite difference scheme; Shishkin mesh; Parameter uniform; Robin boundary conditions;

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1. Introduction

The problems involving differential equations having non smooth solutions with singularities related to the boundary layers are called Singularly Perturbed Problems (SPPs). They arise in many branches of applied mathematics. Since the solutions of these equations have multiscale character, most of the classical numerical methods fail to give parameter uniform convergence. Therefore we have to construct some new methods which are parameter uniform. Many authors have constructed parameter uniform numerical methods for the past 40 years. For further reference one may refer [1–4] and [5].

One dimensional version of stationary convection-diffusion problems with a dominant convective term, speed field that changes its sign in the catch basin, geophysics and modeling thermal boundary layers in laminar flow are some areas where the Singularly Perturbed Turning Point Problems (SPTPPs) arise [6].

Since the coefficient of convection term of SPTPPs vanishes inside the domain, the analytical and numerical treatments of SPTPPs are more complicated than SPPs. When the perturbation parameter $\varepsilon$ is small the standard dis-

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cretization methods for solving SPTPPs are sometimes unstable and fail to give accurate results. To obtain accurate results and parameter uniform convergence, we apply classical finite difference scheme on Shishkin mesh.

In [7], the author derived estimate for the solution of SPTPPs. Wasow [6], O’ Malley [8] and Roos et.al [5] studied the qualitative aspects of these problems. Farrell [9] and Berger et al. [10] derived a general sufficient condition for a uniformly convergent scheme for the second order turning point problem. In [11], E. O’Riordan and J. Quinn, proposed a parameter uniform numerical method for SPTPP with an interior layer. In [12] and [13] the authors obtained a second order convergence for SPTPP. For more detail one may refer [14] and the references therein.

An asymptotic expansion of solution for the third order SPTPP was constructed by Jia-qi Mo et al. [15]. Parameter uniform numerical method for a third order SPTPPs is given in [17]. System of singularly perturbed turning point problems arise in spherical shells and shallow cap dimpling [18]. In [19], the author suggested an asymptotic numerical method for solving a perturbed nonlinear system with turning points that consists of replacing the continuous problem with a sequence of constant coefficient problems on abutting intervals.

2. Statement of the Problem

Motivated by the works of [16,19,20], we consider the following system of second order singularly perturbed boundary value problem with a turning point at \( x = 0 \):

Find \( \bar{u} = (u_1, u_2)^T \in Y = C^2(\Omega = [-1, 1]) \cap C^3(\Omega = (-1, 1)) \) such that

\[
\begin{align*}
\dot{L}_\bar{u}(x) & = \left[ L_1 \bar{u}(x) = \frac{e u_1'(x) + a_1(x)u_1'(x) + b_{11}(x)u_1(x) + b_{12}(x)u_2(x)}{a_1(x)u_1'(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x)} = f_1(x), \quad x \in \Omega, \\
L_2 \bar{u}(x) & = \frac{e u_2'(x) + a_2(x)u_2'(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x)}{a_2(x)u_2'(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x)} = f_2(x), \quad x \in \Omega,
\end{align*}
\]

with the boundary conditions

\[
\begin{align*}
B_{10}u_1(-1) & \equiv \beta_{10}u_1(-1) - \varepsilon \beta_{11}u_1'(-1) = A_1, \quad B_{11}u_1(1) \equiv \gamma_{11}u_1(1) + \varepsilon \gamma_{12}u_1'(1) = B_1, \\
B_{20}u_2(-1) & \equiv \beta_{20}u_2(-1) - \varepsilon \beta_{21}u_2'(-1) = A_2, \quad B_{21}u_2(1) \equiv \gamma_{21}u_2(1) + \varepsilon \gamma_{22}u_2'(1) = B_2,
\end{align*}
\]

and the assumptions

\[
\begin{align*}
b_{10} \geq 0, \quad b_{20} \geq 0, \quad b_{11} + b_{12} \leq b_1 < 0, \quad b_{21} + b_{22} \leq b_2 < 0, \\
|a_k(x)| \leq a_k > 0, \quad \text{for } 0 < |x| \leq 1, \quad a_0(x) = 0, \quad b_0(x) = 0, \\
\alpha_k + b_k < 0 \quad \text{and} \quad |\alpha_k(x)| \geq |a_k(0)|/2 \quad \forall x \in \bar{\Omega}, \quad \text{for } k = 1, 2.
\end{align*}
\]

where the functions \( a_1(x), a_2(x), b_{11}(x), b_{12}(x), b_{21}(x), b_{22}(x), f_1(x) \) and \( f_2(x) \) are sufficiently smooth on \( \bar{\Omega} \), \( 0 < \varepsilon \leq 1 \), \( \beta_{j0}, \beta_{j1} \geq 0, \beta_{j0} - \varepsilon \beta_{j1} \geq 0, \gamma_{j1}, \gamma_{j2} \geq 0, \quad j = 1, 2 \). The above system can be written in the vector form as

\[
\dot{L}_{\bar{u}}(x) = \left( L_1 \bar{u}(x) \right)^T = \left( e \frac{d}{dx} 0 \right) \bar{u}(x) + \left( \frac{a_1(x) \frac{d}{dx}}{a_2(x) \frac{d}{dx}} 0 \right) \bar{u}(x) + \left( \begin{array}{c}
b_{11} \\
b_{12} \\
b_{21} \\
b_{22}
\end{array} \right) \bar{u}(x) = \tilde{f}(x), \quad x \in \Omega,
\]

with the boundary conditions

\[
\begin{align*}
B_{10}u_1(-1) & \equiv \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad B_{20}u_2(-1) \equiv \begin{bmatrix} B_{11}u_1(1) \\ B_{21}u_2(1) \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},
\end{align*}
\]

where \( \tilde{f}(x) = (f_1(x), f_2(x))^T \). With the above assumptions (1c), the turning point problem (1a)-(1b) possesses a unique solution having twin boundary layers at \( x = 1 \) and \( x = -1 \) [10].

Throughout the paper, \( C, C_1 \) denote generic positive constants independent of the singular perturbation parameter \( \varepsilon \) and the discretization parameter \( N \) of the discrete problem. Let \( Y : D \to \mathbb{R} \). The appropriate norm for studying the convergence of numerical solution to the exact solution is the maximum norm \( \|w\|_D = \sup_{x\in D} |w(x)| \). In case of vectors \( \bar{w} = (w_1, w_2)^T \), we define \( \bar{w}(x) = (|w_1(x)|, |w_2(x)|)^T \) and \( \|\bar{w}\|_D = \max\{\|w_1\|_D, \|w_2\|_D\} \).

3. Maximum Principle and Stability Result

In this section, we state the maximum principle and stability result on the solution for the problem (1).

**Theorem 3.1. (Maximum Principle)** Let \( \tilde{w}(x) = (w_1(x), w_2(x))^T \in Y \) be any function satisfying \( L_1 \tilde{w} \leq 0, \quad L_2 \tilde{w} \leq 0, \quad B_{j0}w_j(-1) \geq 0, \quad B_{j1}w_j(1) \geq 0, \quad \text{for } j = 1, 2 \). Then \( \tilde{w}(x) \geq 0, \quad \forall x \in \Omega \).
An immediate consequence of the maximum principle is the following stability result.

**Lemma 3.2. (Stability Result)** If \( u_1, u_2 \in Y \), then for \( i = 1, 2 \)
\[
|u_i(x)| \leq C \max \{|B_{10}u_1(1)|, |B_{11}u_1(1)|, |B_{20}u_2(1)|, |B_{21}u_2(1)|, \|L_1u_1\|_{\infty, \Omega}, \|L_2u_2\|_{\infty, \Omega} \}, \forall x \in \bar{\Omega}.
\]

*Note:* Since the operators \( L_j \), \( j = 1, 2 \) satisfy the above maximum principle, the solution \( \bar{u}(x) \) of the BVP (1) is unique, if it exists.

### 4. Analytical Results

In this section we present some analytical results for the solution \( \bar{u}(x) \) and its derivatives. Hereafter we shall denote the subdomains of \( \bar{\Omega} = [-1, 1] \) as \( \Omega_1 = [-1, -\delta], \Omega_2 = [-\delta, \delta] \) and \( \Omega_3 = [\delta, 1], 0 < \delta \leq 1/2 \). The choice of \( \delta = 1/2 \) can be found in [10]. And \( |a_k(x)| \geq \alpha > 0 \) for \( \delta < |x| \leq 1 \).

The following lemma gives estimates for \( \bar{u}(x) \) and its derivatives in the intervals \( \Omega_1 \) and \( \Omega_2 \) which exclude the turning point \( x = 0 \).

**Lemma 4.1.** Let \( \bar{u}(x) = (u_1(x), u_2(x))^T \) be the solution of (1). Then for \( j = 1, 2 \)
\[
|u_j^{(k)}(x)| \leq C \max \{|f_j|, |\bar{u}|\}, \text{ for } k = 1, 2
\]
\[
|u_j^{(3)}(x)| \leq C \max \{|f_j|, |f_j'|, |\bar{u}|\}, \text{ for } k = 3
\]
\[
\forall x \in \Omega_1 \cup \Omega_3,
\]
where \( C \) depends on \( |a_1|, |a_2|, |b_{11}|, |b_{12}|, |b_{22}|, |a_1'|, |a_2'| \) and \( |a_1| \).

**Proof.** Using the technique adopted in [[3], pp. 44,45] the present lemma can be proved in the subdomain \( \Omega_1 \). In a similar way one can prove an analogous result in the subdomain \( \Omega_3 \). \( \square \)

Let us denote \( d_i = b_{i2}(0)/a_1'(0) \) for \( i = 1, 2 \). And also note that \( d_1, d_2 < 0 \) always. The following lemma gives estimates for \( \bar{u}(x) \) and its derivatives in the interval \( \Omega_2 \) which includes the turning point \( x = 0 \).

**Lemma 4.2.** Let \( \bar{u}(x) = (u_1(x), u_2(x))^T \) be the solution of (1). Then for \( k = 1, 2, 3 \)
\[
|u_j^{(k)}(x)| \leq C, \text{ for } j = 1, 2 \text{ and } \forall x \in \Omega_2,
\]
where \( C \) depends on \( |a_1|, |a_2|, |b_{11}|, |b_{12}|, |b_{21}|, |b_{22}|, |a_1'|, |a_2'|, d_1 \) and \( d_2 \).

**Proof.** We prove this lemma by adopting the technique as in Berger et. al. [10]. From the Mean Value Theorem and the assumptions in (1c), we have
\[
|a_k(x)| = |a_k(x) - a_k(0)| = |x| |a_k'(\xi)| \geq \frac{|x| |a_k'(0)|}{2} = \frac{|x|}{2d_k} \text{ for } k = 1, 2.
\]

Then by the previous lemma the bound for \( \bar{u}(x) \) and its derivatives at \( x = \pm 1/2 \) are found where \( C \) depends on \( |a_1|, |a_2|, |b_{11}|, |b_{12}|, |b_{21}|, |b_{22}|, |a_1'|, |a_2'|, d_1 \) and \( d_2 \). If equations (1a) are differentiated \( k \) times, one can find that the differential equation satisfied by \( \bar{z}(x) = (\bar{u})^{(k)}(x) \) is
\[
\bar{z}''(x) + a_1(x)\bar{z}'(x) + [b_{11}(x) + k(a_1'(x))]\bar{z}'(x) + b_{12}'(x)\bar{z}_2(x) = g_1(x)
\]
\[
\bar{z}''(x) + a_2(x)\bar{z}_2'(x) + [b_{22}(x) + k(a_2'(x))]\bar{z}'_2(x) + b_{21}'(x)\bar{z}_2(x) = g_2(x)
\]
where \( g \) depends on \( \bar{u}, \cdots, (\bar{u})^{(k-1)} \) and on the \( k \)th order derivatives of \( a_1, a_2, b_{11}, b_{12}, b_{21}, b_{22} \). Applying lemma 3.2 with \( b_{22} \) is replaced by \( b_{22} + k(a_2') \) for \( i = 1, 2 \), \( b_{12} \) is replaced by \( b_{12}' \) and \( b_{21} \) is replaced by \( b_{21}' \) respectively, we obtain the required result by using an inductive argument. \( \square \)

To obtain the sharper bounds of solution \( \bar{u}(x) \) and its derivatives we decompose the solution \( \bar{u}(x) \) into regular and singular components as, \( \bar{u}(x) = \bar{v}(x) + \bar{w}(x) \), where \( \bar{v}(x) = (v_1(x), v_2(x))^T \) and \( \bar{w}(x) = (w_1(x), w_2(x))^T \). The regular component \( \bar{v}(x) \) can be written in the form of \( \bar{v} = \bar{v}_0 + \epsilon \bar{v}_1 + \epsilon^2 \bar{v}_2 \), where \( \bar{v}_0 = (v_{01}, v_{02})^T, \bar{v}_1 = (v_{11}, v_{12})^T \) and
Thus the regular component $\bar{v}(x)$ is the solution of
\[
\bar{L}(\bar{v}) = \bar{f},
\]
\[
\begin{pmatrix}
B_{10}v_1(-1) \\
B_{20}v_2(-1)
\end{pmatrix}
= \begin{pmatrix}
B_{10}v_10(-1) + \varepsilon B_{10}v_11(-1) + \varepsilon^2 B_{10}v_{21}(-1) \\
B_{20}v_20(-1) + \varepsilon B_{20}v_12(-1) + \varepsilon^2 B_{20}v_{22}(-1)
\end{pmatrix}
\]
\[
\begin{pmatrix}
B_{11}v_1(1) \\
B_{21}v_2(1)
\end{pmatrix}
= \begin{pmatrix}
B_{11}v_11(1) + \varepsilon B_{11}v_12(1) + \varepsilon^2 B_{11}v_{21}(1) \\
B_{21}v_21(1) + \varepsilon B_{21}v_12(1) + \varepsilon^2 B_{21}v_{22}(1)
\end{pmatrix}
\]
and the singular component $\bar{w}(x)$ is the solution of
\[
\bar{L}(\bar{w}) = \bar{0},
\]
\[
\begin{pmatrix}
B_{10}w_1(-1) \\
B_{20}w_2(-1)
\end{pmatrix}
= \begin{pmatrix}
B_{10}v_11(-1) - B_{10}v_1(1) \\
B_{20}v_21(-1) - B_{20}v_2(1)
\end{pmatrix}.
\]
The following lemma provides the bound on the derivatives of the regular and singular components of the solution $\bar{u}(x)$.

**Lemma 4.3.** The smooth component $\bar{v}(x)$ and singular component $\bar{w}(x)$ and their derivatives satisfy the bounds for $k=0,1,2,3$, and $j=1,2$
\[
\|v_j^{(k)}(x)\| \leq C(1 + \varepsilon^{2-k}), \quad \forall \, x \in \Omega_1 \cup \Omega_3, \text{ and } |w_j^{(k)}(x)| \leq \begin{cases} \frac{C\varepsilon^{-k}\varepsilon^{(1+k)/\varepsilon}}{\gamma}, & x \in \Omega_1 \\ \frac{C\varepsilon^{-k}\varepsilon^{(1-k)/\varepsilon}}{\gamma}, & x \in \Omega_3 \end{cases}
\]
where $a_i(x) > 0$ for $x \in \Omega_1$ and $a_i(x) < 0$ for $x \in \Omega_3$.

**Proof.** Using appropriate barrier functions, applying Theorem 3.1 and adopting the method of proof used in [[3], p.44], the present lemma can be proved.

**Theorem 4.4.** The smooth component $\bar{v}(x)$ and singular component $\bar{w}(x)$ and their derivatives satisfy the bounds for $k=0,1,2,3$, and $j=1,2$
\[
\|v_j^{(k)}(x)\| \leq C(1 + \varepsilon^{2-k}), \quad \text{and } |w_j^{(k)}(x)| \leq C\varepsilon^{-k}(\varepsilon^{(1+k)/\varepsilon} + \varepsilon^{(1-k)/\varepsilon}), \quad \forall \, x \in \bar{\Omega}
\]

**Proof.** Lemma 4.2 guarantees that the solution of the SPTPP (1) and its derivatives are smooth in the domain $\Omega_2$. Hence, the proof is an immediate consequence of the above estimates on $\bar{v}^{(k)}(x)$ and $\bar{w}^{(k)}(x)$.

5. Discrete Problem

5.1. Mesh selection strategy

In this section, the system (1a)-(1b) is discretized using classical finite difference scheme on piecewise uniform meshes (Shishkin mesh). Consider the classical upwind scheme on a piecewise uniform mesh $\bar{\Omega}_N^\varepsilon$, $N \geq 4$ which is constructed by dividing the domain $\bar{\Omega}$ into three subintervals $\Omega_L = [-1,-1+\tau], \Omega_C = [-1+\tau,1-\tau]$ and $\Omega_R = [1-\tau,1]$ such that $\bar{\Omega} = \Omega_L \cup \Omega_C \cup \Omega_R$. The transition parameter $\tau$ is chosen to be $\min\left\{1, \frac{2\varepsilon \ln N}{\alpha}\right\}$.

5.2. Finite difference method for the problem (1a)-(1b)

The domain $\bar{\Omega}_N^\varepsilon$ is obtained by putting a uniform mesh with $N/4$ mesh elements in both $\Omega_L$ and $\Omega_R$ and a uniform mesh with $N/2$ elements in $\Omega_C$. The resulting fitted finite difference scheme is to find $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$ for $i = 0, 1, 2, \ldots, N$ such that for $x_i \in \bar{\Omega}_N^\varepsilon$.
\[
\begin{cases}
L_N \bar{U}(x_i) := L_1^n \bar{U}(x_i) := \varepsilon^2 U_1(x_i) + a_1(x_i)D^+ U_1(x_i) + b_1(x_i)U_1(x_i) + b_2(x_i)U_2(x_i) = f_1(x_i), & i = 1(1)N-1 \\
L_2^n \bar{U}(x_i) := \varepsilon^2 U_2(x_i) + a_2(x_i)D^+ U_2(x_i) + b_2(x_i)U_1(x_i) + b_2(x_i)U_2(x_i) = f_2(x_i), & i = 1(1)N-1
\end{cases}
\]
\[
\begin{cases}
B_{10}^N U_1(x_0) = \beta_{10} U_1(x_0) - \varepsilon \beta_{11} D^+ U_1(x_0) = A_1, \\
B_{11}^N U_1(x_N) = \gamma_{11} U_1(x_N) + \varepsilon \gamma_{12} D^- U_1(x_N) = B_1,
\end{cases}
\]
\[
\begin{cases}
B_{20}^N U_2(x_0) = \beta_{20} U_2(x_0) - \varepsilon \beta_{21} D^+ U_2(x_0) = A_2, \\
B_{21}^N U_2(x_N) = \gamma_{21} U_2(x_N) + \varepsilon \gamma_{22} D^- U_2(x_N) = B_2.
\end{cases}
\]
where \( D^+ U_j(x_i) = \frac{U_j(x_{i+1}) - U_j(x_i)}{x_{i+1} - x_i} \), \( D^- U_j(x_i) = \frac{U_j(x_i) - U_j(x_{i-1})}{x_i - x_{i-1}} \),

\[
\frac{\partial^2 U_j(x_i)}{(x_{i+1} - x_{i-1})/2} \text{ and } D^+ U_j(x_i) = \begin{cases} D^+ U_j(x_i) & \text{if } a_j(x_i) > 0 \\ D^- U_j(x_i) & \text{if } a_j(x_i) < 0 \end{cases}
\]

6. Numerical solution estimates

Analogous to the results stated for the continuous problem in Theorem 3.1 and Lemma 3.2 one can prove the following results.

**Theorem 6.1.** Let \( \Psi(x_i) = (\Psi_1(x_i), \Psi_2(x_i))^T \), be any mesh function satisfying \( B^{N_j}_j \Psi_j(x_0) \geq 0, B^{N_j}_j \Psi_j(x_N) \geq 0 \) for \( j = 1, 2 \), \( L_j \Psi_j(x_i) \leq 0, \forall i = 1(1)N - 1 \) and \( L_j \Psi_j(x_i) \leq 0, \forall i = 1(1)N - 1 \). Then \( \Psi(x_i) \geq 0, \forall x_i \in \Omega^N_2 \).

**Lemma 6.2.** Consider the scheme (6)-(7). If \( \bar{z}(x_i) = (z_1(x_i), z_2(x_i))^T \) is any mesh function then, for all \( x_i \in \Omega^N_2 \),

\[
|z_j(x_i)| \leq C \max \left| B_j^N z_1(x_0), |B_{j1}^N z_1(x_N)|, |B_{j2}^N z_2(x_0)|, |B_{j2}^N z_2(x_N)| \right|, \max_{1 \leq j \leq N - 1} |L_j^N \bar{z}(x_i)|, j = 1, 2.
\]

**Lemma 6.3.** The solution of the constant coefficient problem

\[
e \delta^2 \Phi_i + \omega D^+ \Phi_i = 0 \quad 1 \leq i \leq N - 1, \quad \text{where } \omega > 0, \quad \beta_1 \Phi_0 - \varepsilon \beta_2 D^+ \Phi_0 = 0, \quad \gamma_1 \Phi_N + \varepsilon \gamma_2 D^+ \Phi_N = 0
\]

on a uniform mesh or the Shishkin mesh \( \Omega^N_2 \) satisfies \( D^+ \Phi_i \leq 0, \forall 1 \leq i \leq N - 1. \)

**Proof.** Using the technique adopted in [21], the present lemma can be proved.

**Lemma 6.4.** The solution of the constant coefficient problem

\[
e \delta^2 \Phi_i - \omega D^- \Phi_i = 0 \quad 1 \leq i \leq N - 1, \quad \text{where } \omega > 0, \quad \beta_1 \Phi_0 - \varepsilon \beta_2 D^+ \Phi_0 = 0, \quad \gamma_1 \Phi_N + \varepsilon \gamma_2 D^+ \Phi_N = 1
\]

on a uniform mesh or the Shishkin mesh \( \Omega^N_2 \) satisfies \( D^- \Phi_i \geq 0, \forall 1 \leq i \leq N - 1. \)

**Proof.** Using the technique adopted in [21], the present lemma can be proved.

The discrete solution \( \tilde{U}(x_i) \) can be decomposed into the sum as \( \tilde{U}(x_i) = \tilde{V}(x_i) + \tilde{W}(x_i) \) where \( \tilde{V}(x_i) \) and \( \tilde{W}(x_i) \) are regular and singular components respectively defined as

\[
\begin{align*}
\tilde{V}(x_i) &= \tilde{V}(x_i), \quad i = 1, 2, \ldots, N - 1, \\
B_{10}^N V_1(x_0) &= B_{10}^N v_1(-1), \quad B_{20}^N V_2(x_0) = B_{20}^N v_2(-1), \\
B_{11}^N V_1(x_N) &= B_{11}^N v_1(1), \quad B_{21}^N V_2(x_N) = B_{21}^N v_2(1)
\end{align*}
\]

\[
\begin{align*}
\tilde{W}(x_i) &= \tilde{W}(x_i), \quad i = 1, 2, \ldots, N - 1, \\
B_{10}^N W_1(x_0) &= B_{10}^N w_1(-1), \quad B_{20}^N W_2(x_0) = B_{20}^N w_2(-1), \\
B_{11}^N W_1(x_N) &= B_{11}^N w_1(1), \quad B_{21}^N W_2(x_N) = B_{21}^N w_2(1)
\end{align*}
\]

The error in the numerical solution can be written in the form \( \tilde{U} - u(x_i) = (\tilde{V} - \bar{v})(x_i) + (\tilde{W} - \bar{w})(x_i) \).

**Lemma 6.5.** At each mesh point \( x_i \in \Omega^N_2 \), the error of the regular component satisfies the estimate

\[
|\bar{v}(x_i)| \leq \left( \frac{CN^{-1}}{CN^{-1}} \right).
\]
Proof. Adopting the method of proof used in [3], the present lemma can be proved.

Lemma 6.6. At each mesh point $x_i \in \tilde{\Omega}_C^N$ the error of the singular component satisfies the estimate

$$|W - \tilde{w}(x_i)| \leq \frac{CN^{-1} \ln N}{C^{-1} \ln N}.$$ 

Proof. We have

$$|B_{10}^N(W_1 - w_1)(x_0)| \leq C\beta_1 \varepsilon(x_{i+1} - x_i)|w_1^2| \leq CN^{-1} \ln N,$$

and

$$|B_{11}^N(W_1 - w_1)(x_i)| \leq C\gamma_1 \varepsilon(x_{i+1} - x_i)|w_1^2| \leq CN^{-1} \ln N.$$ 

Similarly, $B_{20}^N(W_2 - w_2)(x_0) \leq CN^{-1} \ln N$ and $B_{21}^N(W_2 - w_2)(x_N) \leq CN^{-1} \ln N$.

We consider first the case $\tau = 1/2$ and so $\varepsilon \leq C\ln N$ and $h = N^{-1}$. Using Theorem 4.4,

we obtain

$$|L_N(W - \tilde{w})(x_i)| \leq \frac{CE^{-2N^{-1}}(e^{-\alpha(x_i+1)/\epsilon} + e^{-\alpha(1-x_i)/\epsilon})}{CEF^{-2N^{-1}}(e^{-\alpha(x_i+1)/\epsilon} + e^{-\alpha(1-x_i)/\epsilon})}$$

Consider the mesh functions $\Psi^*(x_i) = (\Psi^*_1(x_i), \Psi^*_2(x_i))^T$ defined as,

$$\Psi_1^*(x_i) = \begin{cases} C \frac{\exp(-x_i)}{\gamma_1(x_i/\epsilon)} e^{-N^{-1}Y_1(x_i)} \pm (W - w)(x_i), & \text{for } 0 \leq i \leq N/4 \\ C \frac{\exp(-x_i)}{\gamma_1(x_i/\epsilon)} e^{-N^{-1}(1 - x_i)} \pm (W - w)(x_i), & \text{for } N/4 \leq i \leq N/2 \\ C \frac{\exp(-x_i)}{\gamma_1(x_i/\epsilon)} e^{-N^{-1}(1 + x_i)} \pm (W - w)(x_i), & \text{for } N/2 \leq i \leq 3N/4 \\ C \frac{\exp(-x_i)}{\gamma_1(x_i/\epsilon)} e^{-N^{-1}Y_2(x_i)} \pm (W - w)(x_i), & \text{for } N/4 \leq i \leq N \end{cases}$$

$$\Psi_2^*(x_i) = \begin{cases} C \frac{\exp(-x_i)}{\gamma_1(x_i/\epsilon)} e^{-N^{-1}Y_3(x_i)} \pm (W - w)(x_i), & \text{for } 0 \leq i \leq N/4 \\ C \frac{\exp(-x_i)}{\gamma_1(x_i/\epsilon)} e^{-N^{-1}(1 - x_i)} \pm (W - w)(x_i), & \text{for } N/4 \leq i \leq N/2 \\ C \frac{\exp(-x_i)}{\gamma_1(x_i/\epsilon)} e^{-N^{-1}(1 + x_i)} \pm (W - w)(x_i), & \text{for } N/2 \leq i \leq 3N/4 \\ C \frac{\exp(-x_i)}{\gamma_1(x_i/\epsilon)} e^{-N^{-1}Y_4(x_i)} \pm (W - w)(x_i), & \text{for } N/4 \leq i \leq N \end{cases}$$

where $\sigma$ is a constant with $0 < \sigma < \alpha$, $Y_1$ is the solution of the constant coefficient problem (8)-(9) and $Y_2$ is the solution of the constant coefficient problem (10)-(11) with $\omega = \sigma$, $\beta_1 = \min(\beta_1, \beta_2, \beta_0, \beta_0', \beta_0'')$, $\gamma_1 = \min(\gamma_1, \gamma_2)$, $\gamma_2 = \min(\gamma_1, \gamma_2)$. Let $Y_3(x_i) = Y_1(x_i)$ and $Y_4(x_i) = Y_2(x_i)$. Choose $C$ large enough such that $\tilde{L}_N^N(\Psi^*) \leq 0$, and also $B_{ji}^N \Psi^*_j(x_0) \geq 0$ and $B_{ji}^N \Psi^*_j(x_N) \geq 0$ for $j = 1, 2$.

Applying Theorem 6.1 to the mesh function $\Psi^*(x_i)$ we have,

$$|W - \tilde{w}(x_i)| \leq \frac{CN^{-1} \ln N}{C^{-1} \ln N}.$$

We now consider the case $\tau = 2\varepsilon \ln N$. In this case the mesh is piecewise uniform with the mesh spacing $4\tau/N$ in the subintervals $\Theta_L, \Theta_R$ and $2\tau/N$ in the subinterval $\Theta_C$. We give separate proofs for coarse and fine mesh subintervals.

The subinterval $\Theta_C$ has no boundary layer, both $W$ and $w$ are small, and by the triangle inequality we have

$$|W - \tilde{w}(x_i)| \leq |W(x_i)| + |\tilde{w}(x_i)|.$$ 

It suffices to bound $W(x_i)$ and $\tilde{w}(x_i)$ separately. Now we consider the subinterval $[-1 + \tau, 0]$ for our discussion since one can obtain a similar proof for the subinterval $[0, 1 - \tau]$.

Using Lemma 4.3 we have

$$|\tilde{w}(x_i)| \leq \begin{cases} \frac{CN^{-1}}{C^{-1}} & \text{for } x_i \in [-1 + \tau, 0] \\ \frac{CN^{-1}}{C^{-1}} & \text{for } x_i \in [0, 1 - \tau] \end{cases}.$$ 

The bound for $|W(x_i)|$ is established by considering the following mesh functions $\tilde{\Psi}^*(x_i) = \frac{CN^{-1}(1 - x_i)}{C^{-1} \ln N} \pm \tilde{W}(x_i)$, where $C$ is a constant chosen such that $B_{ji}^N \Psi^*_j(x_0) \geq 0$ and $B_{ji}^N \Psi^*_j(x_N) \geq 0$ for $j = 1, 2$. Also $\tilde{L}_N^N(\Psi^*)(x_i) \leq 0$. Thus, applying Theorem 6.1, we have $\Psi^*(x_i) \geq 0$ and

$$|\tilde{W}(x_i)| \leq \frac{CN^{-1}(1 - x_i)}{C^{-1} \ln N}.$$
Combining the equations (17) to (19) we have,

$$|(\tilde{W} - \tilde{w})(x_j)| \leq \frac{C N^{-1}}{CN^{-1}} \quad \forall \ x_j \in [-1 + \tau, 0].$$

(20)

It remains to prove the results for $x_j \in \Omega_L$ and $x_j \in \Omega_R$. Let $x_j \in \Omega_L$. For $i = 0$ there is nothing to prove. For $x_j \in \Omega_L$ the proof follows the same lines as for the case $\tau = 1/2$ except that we use the discrete maximum principle on $\Omega_L$ and the already established bound $|\tilde{W}(x_{N/4})| \leq \frac{C N^{-1}}{CN^{-1}}$.

We have in this case, $|L^N(\tilde{W} - \tilde{w})(x_j)| \leq \begin{cases} C \tau e^{-2N-1} e^{-\alpha(1+x_j)/\varepsilon} & \forall i, \ 0 \leq i \leq N/4. \end{cases}$

We introduce the mesh functions $\Psi^k = (\Psi_1(x_j), \Psi_2(x_j))^T$ defined as

$$\Psi^k_j(x_j) = \frac{C e^{\sigma \epsilon}}{\sigma(\varepsilon - \sigma)} \tau e^{-1} N^{-1} Y_j(x_j) + C' N^{-1} \pm (\tilde{W} - \tilde{w})(x_j), \ j = 1, 2.$$

where $Y_j(x_j)$ is the solution of of the problem $e\partial^2 Y_j(x_j) + \sigma D^+ Y_j(x_j) = 0, i = 1, 2, \cdots, N-1, \beta_1 Y_j(x_0) - e\beta_2 D^+ Y_j(x_0) = 1, \gamma_1 Y_j(x_{N/4}) + e\gamma_2 D^+ Y_j(x_{N/4}) = 0$ and $\sigma, \beta_1, \beta_2, \gamma_1, \gamma_2$ are defined as before. Thus, for all $i, \ 0 \leq i \leq N/4$, $Y_j(x_j) = \lambda^{N/4-i} \sigma(\gamma_2/\gamma_1), Y_j(x_0) \leq 0$. Let $Y_2(x_j) = Y_j(x_j)$. It is easy to see that $B^N_{\psi}(\Psi^k(x_0)) > 0$, $B^N_{\psi}(\Psi^k(x_{N/4})) \geq 0$, for $j = 2, 1$, and $L^N\Psi^k \leq \bar{\lambda}$ for $1 \leq i < N/4$. Then by the discrete maximum principle we conclude that $\Psi^k \geq 0$, $\forall x_j \in \Omega_L$. That is $|\tilde{W} - \tilde{w})(x_j)| \leq \frac{C N^{-1}}{CN^{-1}} \ln N$, $0 \leq i \leq N/4$.

Similarly the proof follows for $x_j \in \Omega_R$. Combining the estimates for the singular components in different regions, we obtain $|\tilde{W} - \tilde{w})(x_j)| \leq \frac{C N^{-1} \ln N}{CN^{-1} \ln N}$, $0 \leq i \leq N$ as required.

\begin{theorem}
Let $\bar{u}(x) = (u_1(x), u_2(x))^T$, for all $x \in \bar{Q}$ be the solution of ($1$) and let $\bar{U}(x) = (U_1(x), U_2(x))^T$, for all $x_j \in \bar{Q}^N$ be the numerical solution of problem (6)-(7). Then we have

$$\sup_{0 < \varepsilon \leq 1} ||U_1 - u_1||_{L^\varepsilon} \leq CN^{-1} \ln N \quad \text{and} \quad \sup_{0 < \varepsilon \leq 1} ||U_2 - u_2||_{L^\varepsilon} \leq CN^{-1} \ln N$$

Proof. Proof follows immediately, if one applies Lemma 6.5 and Lemma 6.6 to $\tilde{U} - \bar{u} = \tilde{V} - \bar{v} + \tilde{W} - \bar{w}$. \hfill \Box

\section{Numerical Results}

In this section, an example is given to illustrate the numerical method discussed in this paper. We use the double mesh principle to estimate the error and compute the rate of convergence in our computed solution.

Define the double mesh differences to be $D^N_{e,j} = \left\{ \max_{x_i \in \Omega^N} |U^N_{j}(x_i) - U^N_{j}(x_i)| \right\}$, for $j = 1, 2$ and $D^N_j = \max_{x_i \in \Omega^N} D^N_{e,j}$ where $U^N_{j}(x_i)$ and $U^N_{j}(x_i)$ respectively, denote the numerical solution obtained using N and 2N mesh intervals. Further, we calculate the parameter robust order of convergence as $p_j = \log_2 \left( \frac{D^N_j}{D^N_{e,j}} \right)$, for $j = 1, 2$. The following example has a turning point at $x = 1/2$. The maximum error and the order of convergence of the Example 7.1 is presented in Table 1.

\begin{example} Consider the following system of singularly perturbed turning point problem

$$e u_1''(x) - 7(2x - 1)u_1'(x) - 10u_1'(x) + 2u_2(x) = -e^{-x}, \quad x \in (0, 1)$$

$$e u_2''(x) - 3(2x - 1)u_2'(x) - 7u_2'(x) + 3u_1(x) = x + 5, \quad x \in (0, 1)$$

$$u_1(0) - e u_1'(0) = 1, \quad u_2(0) - e u_2'(0) = 0, \quad u_1(1) + e u_1'(1) = 1, \quad u_2(1) + e u_2'(1) = 1.$$

\end{example}
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References


Table 1. Values of $D_{1}^{N}$, $p_{1}^{N}$ and $D_{2}^{N}$, $p_{2}^{N}$ for the solution components $U_{1}$ and $U_{2}$ respectively for Example 7.1

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<th>Number of mesh points N</th>
<th>64</th>
<th>128</th>
<th>256</th>
<th>512</th>
<th>1024</th>
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<tr>
<td>$D_{1}^{N}$</td>
<td>1.4436</td>
<td>4.9968e-1</td>
<td>2.2473e-1</td>
<td>1.1222e-1</td>
<td>5.9357e-2</td>
</tr>
<tr>
<td>$p_{1}^{N}$</td>
<td>1.5306</td>
<td>1.1528</td>
<td>1.0019</td>
<td>9.1886e-1</td>
<td>-</td>
</tr>
<tr>
<td>$D_{2}^{N}$</td>
<td>7.9382e-1</td>
<td>3.1690e-1</td>
<td>1.5740e-1</td>
<td>8.2448e-2</td>
<td>4.4219e-2</td>
</tr>
<tr>
<td>$p_{2}^{N}$</td>
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<td>1.0096</td>
<td>9.3289e-1</td>
<td>8.9884e-1</td>
<td>-</td>
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