Transitive actions of finite abelian groups of sup-norm isometries

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Abstract

There is a long-standing conjecture of Nussbaum which asserts that every finite set in $\mathbb{R}^n$ on which a cyclic group of sup-norm isometries acts transitively contains at most $2^n$ points. The existing evidence supporting Nussbaum’s conjecture only uses abelian properties of the group. It has therefore been suggested that Nussbaum’s conjecture might hold more generally for abelian groups of sup-norm isometries. This paper provides evidence supporting this stronger conjecture. Among other results, we show that if $\Gamma$ is an abelian group of sup-norm isometries that acts transitively on a finite set $X$ in $\mathbb{R}^n$ and $\Gamma$ contains no anticlockwise additive chains, then $X$ has at most $2^n$ points.

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1. Introduction

Let $X$ be a compact set in $\mathbb{R}^n$ and let $\Gamma$ be a group of sup-norm isometries $g: X \to X$ that map $X$ onto itself. Martus [13] has shown that if $\Gamma$ acts transitively on $X$, that is, for each $x, y \in X$ there exists $g \in \Gamma$ such that $g(x) = y$, then $X$ is a finite set with at most $n!2^n$ elements. Moreover, the upper bound is sharp as the following example shows: let $x = (1, 2, \ldots, n) \in \mathbb{R}^n$ and $X$ be the orbit of $x$ under the symmetry group of the $n$-cube, so $X = \{Px \in \mathbb{R}^n : P \text{ is an } n \times n \text{ signed permutation matrix}\}$. In case the group $\Gamma$ is cyclic, Nussbaum [15] has conjectured that the optimal upper bound for the size of $X$ is $2^n$. An example with $2^n$ elements consists of the set of vertices of the standard unit cube $\{-1, 1\}^n$. Indeed, as every

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two distinct vertices have the same sup-norm distance to each other, any cyclic permutation of the vertices is an isometry. All partial results supporting Nussbaum’s conjecture only exploit abelian properties of the group and at present no examples are known with more than \(2^n\) elements. It has therefore been suggested that the following more general conjecture might be true.

**Conjecture A.** If \(X\) is a finite set in \(\mathbb{R}^n\) on which an abelian group of sup-norm isometries acts transitively, then \(X\) has at most \(2^n\) elements.

This paper provides evidence for this claim. In fact, we shall formulate several conjectures stronger than Conjecture A and prove one of them for abelian group actions, where the group does not contain an anticlockwise additive chain. In addition, we provide results that support the other conjectures. Before going into the details, we recall the known results and discuss some motivating background material.

It is known that Conjecture A is true for \(n = 1, 2,\) and 3. Lyons and Nussbaum [12] proved the case \(n = 3\) and gave additional evidence. At present the best general bound for the size of \(X\) in Conjecture A is \(\max \, 2^k \binom{n}{k} \leq C3^n / \sqrt{n},\) where \(C > 0\) is a constant (Lemmens and Scheutzow [11]). Weaker bounds have been obtained in [5,13,15,22]. It is also known that the size of \(X\) cannot be a prime larger than \(2^n\) (Lyons and Nussbaum [16]).

The main motivation for analysing transitive actions of groups of sup-norm isometries lies in the study of the dynamics of non-expansive maps. Weller [23] has shown that if \(f: M \rightarrow M,\) where \(M \subset \mathbb{R}^n\) is closed, is non-expansive with respect to the sup-norm, then every bounded orbit of \(f\) converges to a periodic orbit of \(f.\) It is easy to verify that the restriction of a sup-norm non-expansive map \(f: M \rightarrow M\) to one of its periodic orbits is a sup-norm isometry. Moreover, the iterates of the restriction of \(f\) to a periodic orbit form a cyclic group that acts transitively on \(X.\) Therefore Nussbaum’s conjecture is equivalent to the following assertion: The period of each periodic point of a sup-norm non-expansive map \(f: M \rightarrow M,\) where \(M \subset \mathbb{R}^n,\) does not exceed \(2^n.\) Sup-norm non-expansive maps arise in various areas in mathematics. A particularly interesting area is nonlinear Perron–Frobenius theory, in which one studies order-preserving homogeneous maps on cones in \(\mathbb{R}^n.\) Such maps are non-expansive with respect to Hilbert’s projective metric and Thompson’s part metric. It is known that if the cone is polyhedral, these maps have the same dynamical behaviour as sup-norm non-expansive maps [15]. Many results concerning the dynamics of non-expansive maps and nonlinear Perron–Frobenius theory can be found in [1,2,9–11,15,17–20,22,23].

Other motivation comes from metric geometry. A basic problem in metric geometry is to decide whether a finite metric space can be isometrically embedded into a given class of metric spaces. In particular, one can ask when a finite metric space can be isometrically embedded into a finite dimensional \(\ell_p\) space. This problem goes back to Fréchet [7], Menger [14], and Schoenberg [21]. More recent studies can be found in [4,6,24,25]. In this context Conjecture A asserts that if \((X, d)\) is a finite metric space on which an abelian group of \(d\)-isometries acts transitively and \(X\) can be isometrically embedded into \(\mathbb{R}^n\) under the sup-norm, then \(X\) has at most \(2^n\) points.

2. Statement of results and conjectures

In this section we state the results and formulate several conjectures that are stronger than Conjecture A. We begin however by recalling some basic definitions and preliminary facts. Let \((X, d)\) be a metric space. A map \(f: X \rightarrow X\) is called a \(d\)-isometry if \(d(f(x), f(y)) = d(x, y)\) for all \(x, y \in X.\) A group \(\Gamma'\) of \(d\)-isometries \(g: X \rightarrow X\) acts transitively on \(X\) if for each \(x, y \in X\)
there exists \( g \in \Gamma \) such that \( g(x) = y \). We denote the unit element in a group by \( e \). If \( 
abla \) is an abelian group action of isometries, then the following assertion is true \([12, \text{Lemma 2.3}]\).

**Lemma 2.1.** If \((X, d)\) is a metric space on which an abelian group \( \Gamma \) of \( d \)-isometries acts transitively, then for each \( g \in \Gamma \) there exists a constant \( d(g) \geq 0 \) such that \( d(g(x), x) = d(g) \) for all \( x \in X \).

Indeed, \( d(g(x), x) = d(f(g(x)), f(x)) = d(g(f(x)), f(x)) \) for all \( f \in \Gamma \), as \( \nabla \) is abelian. Hence the transitivity of \( \Gamma \) yields the assertion. Using Lemma 2.1 it is not hard to show that for each \( x \in X \) the map \( \varphi_x : \Gamma \to X \) given by \( \varphi_x(g) = g(x) \) for \( g \in \Gamma \) is bijective, so that \( |\Gamma| = |X| \) whenever the transitive group is abelian.

In this paper we only consider metric spaces \((X, d)\), where \( X \subset \mathbb{R}^n \) and the metric \( d_\infty \) is induced by the sup-norm: \( \|x\|_\infty = \max_i |x_i| \). For \( x \in \mathbb{R}^n \) and \( r \geq 0 \) we let \( B(x; r) \) denote the closed sup-norm ball around \( x \) with radius \( r \). For each \( 1 \leq i \leq n \) we have a partial ordering \( \leq_i \) on \( \mathbb{R}^n \) given by \( x \leq_i y \) if \( d_\infty(x, y) = y_i - x_i \). A sequence \( x^1, x^2, \ldots \) in \( \mathbb{R}^n \) is called an \( i \)-chain if \( x^k \leq_i x^{k+1} \) for all \( k \geq 1 \), or \( x^{k+1} \leq_i x^k \) for all \( k \geq 1 \). A finite sequence \( x^1, x^2, \ldots, x^m \) in \( \mathbb{R}^n \) is called an additive chain if

\[
d_\infty(x^1, x^m) = \sum_{k=1}^{m-1} d_\infty(x^k, x^{k+1}).
\]

Additive chains are preserved under isometries and play an important role in the analysis of transitive actions of abelian groups of sup-norm isometries. The length of an additive chain is the number of distinct points in the sequence. We note that if \( x^1, x^2, x^3 \) is an additive chain, then \( x^2 \) need not be on the straight line between \( x^1 \) and \( x^3 \), since there can be more than one geodesic between \( x^1 \) and \( x^3 \). Indeed, for \( x, z \in \mathbb{R}^n \) the metric interval, \( [x, z]_\infty = \{ y \in \mathbb{R}^n : x, y, z \) is an additive chain\} need not just consist of the line segment between \( x \) and \( z \). It is easy to verify that \( x^1, x^2, \ldots, x^m \) is an additive chain in \( \mathbb{R}^n \) if and only if it is an \( i \)-chain for some \( 1 \leq i \leq n \). Moreover, if \( x^1, x^2, \ldots, x^m \) is an additive chain in \( \mathbb{R}^n \) and \( x^1 \leq_i x^m \), then \( x^p \leq_i x^q \) for all \( 1 \leq p < q \leq m \). An additive chain is said to be saturated in \( X \subset \mathbb{R}^n \) if it is not a proper subsequence of any other additive chain in \( X \).

**Lemma 2.1** has the following useful consequence.

**Corollary 2.2** (\([12]\)). Suppose that \( X \) is a finite set in \( \mathbb{R}^n \) on which an abelian group \( \Gamma \) of sup-norm isometries acts transitively. If \( x^1, x^2, \ldots, x^m \) is an additive chain in \( X \) of length \( m \geq 2 \), and for \( 1 \leq k < m \) the map \( g_k \in \Gamma \) maps \( x^k \) to \( x^{k+1} \), then for each permutation \( \pi \) on \{1, \ldots, m - 1\} and for each \( x \in X \) the sequence

\[ x, g_{\pi(1)}(x), \ldots, g_{\pi(m-1)} \circ \cdots \circ g_{\pi(1)}(x) \]

is an additive chain of length \( m \).

**Proof.** Put \( g = g_{m-1} \circ \cdots \circ g_2 \circ g_1 \) and note that, since \( \Gamma \) is abelian, \( g = g_{\pi(m-1)} \circ \cdots \circ g_{\pi(2)} \circ g_{\pi(1)} \). For simplicity we write \( z^k = g_{\pi(k-1)} \circ \cdots \circ g_{\pi(1)}(x) \) for \( 2 \leq k \leq m \) and \( z^1 = x \). Using Lemma 2.1 we obtain the equalities:

\[
d_\infty(z^1, z^m) = d(g) = d_\infty(x^1, x^m) = \sum_{k=1}^{m-1} d_\infty(x^k, x^{k+1})
\]

\[
= \sum_{k=1}^{m-1} d(g_k) = \sum_{k=1}^{m-1} d(g_{\pi(k)}) = \sum_{k=1}^{m-1} d_\infty(z^k, z^{k+1}).
\]
Thus, \(z^1, z^2, \ldots, z^m\) is an additive chain of length \(m\), which completes the proof. \(\square\)

If \(\Gamma\) is an abelian group of sup-norm isometries acting transitively on a finite set \(X\) in \(\mathbb{R}^n\) and \(g_1, g_2, \ldots, g_m\) is a sequence in \(\Gamma\) such that

\[
d(g_m \circ \cdots \circ g_2 \circ g_1) = \sum_{i=1}^{m} d(g_i),
\]

then we call \(g_1, g_2, \ldots, g_m\) an additive chain in \(\Gamma\). Note that if \(g_1, g_2, \ldots, g_m\) is an additive chain in \(\Gamma\), then \(g_\pi(1), g_\pi(2), \ldots, g_\pi(m)\) is also an additive chain for each permutation \(\pi\) on \(\{1, \ldots, m\}\) by Corollary 2.2. Moreover, \(x^1, x^2, \ldots, x^{m+1}\) is an additive chain in \(X\) if and only if \(g_1, g_2, \ldots, g_m\), where \(g_i(x^i) = x^{i+1}\) for \(1 \leq i \leq m\), is an additive chain in \(\Gamma\).

Blokhuis and Wilbrink [5] proved the following useful result for additive chains.

**Lemma 2.3** ([5]). If \(X\) is a nonempty subset of \(\mathbb{R}^n\) and \(X\) has no additive chains of length \(m\), then \(|X| \leq (m - 1)^n\).

**Proof.** For each \(x \in X\) let \(h_i(x)\) be the length of the longest descending \(i\)-chain starting at \(x\). As \(X\) has no additive chains of length \(m\), \(h_i(x) < m\) for all \(x \in X\) and \(1 \leq i \leq n\). Now put \(h(x) = (h_1(x), \ldots, h_n(x))\) and note that \(h(x) \neq h(y)\) if \(x \neq y\), because \(d_{\infty}(x, y) = |x_i - y_i|\) for some \(i\), so that \(h_i(x) \neq h_i(y)\). As \(h(x) \in \{1, \ldots, m-1\}^n\) for each \(x \in X\), we conclude that \(|X| = |h(x); x \in X|\) \(\leq (m - 1)^n\). \(\square\)

In particular, we see that if a subset of \(\mathbb{R}^n\) has no additive chain of length 3 (that is no triangle equality), then it has at most \(2^n\) points. Therefore it would be nice if each finite set with a transitive abelian group of sup-norm isometries has no additive chains of length 3. Unfortunately this is not the case. For instance the set \(X = \{a, b, c, d\}\) in \(\mathbb{R}^2\), where \(a = (1, 0), b = (0, 1), c = (-1, 0),\) and \(d = (0, -1)\), has additive chains of length 3 and a transitive cyclic group of isometries given by \(\Gamma = \{P^k: 0 \leq k \leq 3\}\), where \(P(x_1, x_2) = (-x_2, x_1)\). We believe however that our sets are very close to having no additive chains of length 3. More precisely, we suspect the following conjecture to be true.

**Conjecture B.** If \(X\) is a finite set in \(\mathbb{R}^n\) on which an abelian group of sup-norm isometries acts transitively, then for each \(\varepsilon > 0\) and each \(x \in X\) there exists \(\tilde{x} \in B(x; \varepsilon)\) such that \(X' = \{\tilde{x}: x \in X\}\) has no additive chains of length 3.

In other words, Conjecture B says that we can perturb each point in \(X\) over an arbitrarily small distance so that the set of perturbed points \(X'\) has no additive chain of length 3. As \(|X| = |X'|\), when \(\varepsilon\) is small, Lemma 2.3 shows that Conjecture B implies Conjecture A. If \(X\) is a finite set in \(\mathbb{R}^n\) on which an abelian group of sup-norm isometries acts transitively and \(X\) satisfies the assertion in Conjecture B, then we say that \(X\) has a perturbation. In the example above a possible perturbation is: \(\tilde{a} = (1 + \varepsilon, \varepsilon), \tilde{b} = (-\varepsilon, 1 + \varepsilon), \tilde{c} = (-1 - \varepsilon, -\varepsilon),\) and \(\tilde{d} = (\varepsilon, -1 - \varepsilon)\).

It is clear that one cannot perturb a set \(X\) in \(\mathbb{R}^n\) over an arbitrary small distance to a set with no additive chains of length 3 if there exist \(x, y\) and \(z\) in \(X\) such that \(y\) is in the interior of the metric interval \([x, z]_\infty\). It turns out that \(X\) does not have such triples if \(X\) has an abelian group of sup-norm isometries acting transitively on it. In fact, one can use [8, Proposition 3.2] to show the following more general statement: If \(X\) is a finite set in \(\mathbb{R}^n\) on which an abelian group of sup-norm isometries acts transitively, then for each \(x \neq z\) in \(X\) the intersection of \(X\) with the relative interior of \([x, z]_\infty\) is empty.
We note that Conjecture B does not require $X'$ to have a transitive abelian group of sup-norm isometries. However, in all the examples we know we can perturb and keep the transitive group of isometries, which suggests the following conjecture.

**Conjecture C.** If $X$ is a finite set in $\mathbb{R}^n$ and on $X$ an abelian group of sup-norm isometries acts transitively, then for each $\epsilon > 0$ and each $x \in X$ there exists $\tilde{x} \in B(x; \epsilon)$ such that $X' = \{\tilde{x}: x \in X\}$ has no additive chains of length 3 and $I'' = \{g': X' \to X' | g \in I'\}$, where each $g'$ is defined by $g'(\tilde{x}) = g(x)$ for $\tilde{x} \in X'$, is an abelian group of sup-norm isometries that acts transitively on $X'$.

By definition $I''$ is an abelian group that acts transitively on $X'$. The point however is that each $g' \in I''$ is a sup-norm isometry. If $X$ is a finite set in $\mathbb{R}^n$ on which an abelian group of sup-norm isometries acts transitively and $X$ satisfies the assertion in Conjecture C, then we say that $X$ has a perturbation that preserves the group. In the example above, $X' = \{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}$ satisfies Conjecture C.

A simple way to perturb a set to a set that has no triangle equalities, but still has a transitive abelian group of sup-norm isometries, is to stretch the distance between every two points by a fixed amount $\delta > 0$. Indeed, we shall show in Lemma 3.2 that for sufficiently small $\delta > 0$, the perturbed set $X'$ cannot have any additive chains other than the ones in the original set $X$. Moreover, if $x, y, z$ is an additive chain in $X$, then

$$d_{\infty}(\tilde{x}, \tilde{z}) = d_{\infty}(x, z) + \delta < d_{\infty}(x, y) + d_{\infty}(y, z) + 2\delta = d_{\infty}(\tilde{x}, \tilde{y}) + d_{\infty}(\tilde{y}, \tilde{z}),$$

so that $\tilde{x}, \tilde{y}, \tilde{z}$ is not an additive chain in $X'$. It is also easy to verify that $I''$ given in Conjecture C is a transitive abelian group of sup-norm isometries on $X'$. As a matter of fact, for each $\tilde{x} \neq \tilde{y}$ in $X'$ we have that

$$d_{\infty}(g'(\tilde{x}), g'(\tilde{y})) = d_{\infty}(g(x), g(y)) + \delta = d_{\infty}(x, y) + \delta = d_{\infty}(\tilde{x}, \tilde{y}).$$

At present we do not know of any example for which we cannot stretch the distances. Therefore the following conjecture might be true.

**Conjecture D.** If $X$ is a finite set in $\mathbb{R}^n$ and on $X$ an abelian group of sup-norm isometries acts transitively, then for each $\epsilon > 0$ and each $x \in X$ there exists $\tilde{x} \in B(x; \epsilon)$ such that $X' = \{\tilde{x}: x \in X\}$ has the property that $d_{\infty}(\tilde{x}, \tilde{y}) = d_{\infty}(x, y) + 2\epsilon$ for all $x \neq y$ in $X$.

Although there are no known counter-examples to Conjecture D, we do not strongly believe it to be true. If $X$ is a finite set in $\mathbb{R}^n$ on which an abelian group of sup-norm isometries acts transitively and $X$ satisfies the assertion in Conjecture D, we say that $X$ can be stretched. In the example above, the perturbation $X' = \{a', b', c', d'\}$ actually satisfies Conjecture D.

The conjectures can be ordered according to their strength as follows:

Conjecture D $\Rightarrow$ Conjecture C $\Rightarrow$ Conjecture B $\Rightarrow$ Conjecture A.

This paper provides some evidence for these conjectures. In particular, we shall prove in Section 3 the following theorem.

**Theorem 2.4.** If $X$ is a finite set in $\mathbb{R}^n$ and on $X$ an abelian group of sup-norm isometries acts transitively, then for each $\epsilon > 0$ and each $x \in X$ there exists $\tilde{x} \in B(x; \epsilon)$ such that $d_{\infty}(\tilde{x}, \tilde{y}) = d_{\infty}(x, y) + \epsilon$ or $d_{\infty}(\tilde{x}, \tilde{y}) = d_{\infty}(x, y) + 2\epsilon$ for all $x \neq y$ in $X$. 
For $\varepsilon$ sufficiently small, the set $X'$ in Theorem 2.4 has no additive chains of length 4, and hence Theorem 2.4 almost yields Conjecture B. In addition, we prove the following assertion, which supports Conjecture D.

**Theorem 2.5.** If $X$ is a finite set in $\mathbb{R}^n$ on which an abelian group of sup-norm isometries acts transitively and there exists $k \geq 1$ such that each saturated additive chain in $X$ has length 2 or $2k$, then $X$ can be stretched.

In Section 4 we shall prove Conjecture C for a special class of transitive actions of abelian groups of sup-norm isometries. To state the result precisely it is convenient to introduce the following definition.

**Definition 2.6.** Let $X$ be a finite set in $\mathbb{R}^n$ on which an abelian group $\Gamma$ of sup-norm isometries acts transitively. An additive chain $f, g$ in $\Gamma \setminus \{e\}$ is said to be **clockwise** if $f^{-1}, g$ is not an additive chain. It is called an **anticlockwise** additive chain, otherwise.

We can show Conjecture C if $\Gamma$ does not contain any anticlockwise additive chains. In fact, we prove the following theorem in Section 4.

**Theorem 2.7.** If $X$ is a finite set in $\mathbb{R}^n$ on which an abelian group $\Gamma$ of sup-norm isometries acts transitively and if $\Gamma$ does not contain an anticlockwise additive chain, then $X$ has a perturbation that preserves the group.

There are many examples of abelian groups of sup-norm isometries that act transitively on a finite set in $\mathbb{R}^n$ and do not have any anticlockwise additive chains, particularly among those coming from periodic orbits of sup-norm non-expansive maps on $\mathbb{R}^n$. In fact, it seems harder to construct examples that do have anticlockwise additive chains. For instance, if $X = \{x^1, \ldots, x^6\} \subset \mathbb{R}^3$ is given by

\[
\begin{align*}
  x^1 &= (2, 1, 0), & x^2 &= (2, 2, 1), \\
  x^3 &= (1, 2, 2), & x^4 &= (0, 1, 2), \\
  x^5 &= (1, 0, 1), & x^6 &= (2, 0, 1),
\end{align*}
\]

then the cyclic group $\Gamma$ generated by $g: X \to X$, where $g(x^k) = x^{k+1 \mod 6}$, is a transitive group of sup-norm isometries on $X$ that only has clockwise additive chains. Unfortunately there also exist examples of transitive abelian groups of sup-norm isometries that have anticlockwise additive chains, e.g., the set $X = \{x^1, \ldots, x^8\} \subset \mathbb{R}^4$ given in the table below, where the abelian group is cyclic and generated by the map $g: X \to X$ given by $g(x^k) = x^{k+1 \mod 8}$.

\[
\begin{align*}
  x^1 &= (2, 1, 2, 1), & \tilde{x}^1 &= (2 + \varepsilon, 1 + \varepsilon, 2 + \varepsilon, 1 + \varepsilon) \\
  x^2 &= (1, 2, 1, 2), & \tilde{x}^2 &= (1 - \varepsilon, 2 + \varepsilon, 1 + \varepsilon, 2 + \varepsilon) \\
  x^3 &= (0, 1, 0, 1), & \tilde{x}^3 &= (-\varepsilon, 1 + \varepsilon, -\varepsilon, 1 + \varepsilon) \\
  x^4 &= (1, 0, 1, 0), & \tilde{x}^4 &= (1 - \varepsilon, -\varepsilon, 1 - \varepsilon, -\varepsilon) \\
  x^5 &= (2, 1, 0, 1), & \tilde{x}^5 &= (2 + \varepsilon, 1 - \varepsilon, -\varepsilon, 1 - \varepsilon) \\
  x^6 &= (1, 2, 1, 0), & \tilde{x}^6 &= (1 - \varepsilon, 2 + \varepsilon, 1 + \varepsilon, -\varepsilon) \\
  x^7 &= (0, 1, 2, 1), & \tilde{x}^7 &= (-\varepsilon, 1 - \varepsilon, 2 + \varepsilon, 1 - \varepsilon) \\
  x^8 &= (1, 0, 1, 2), & \tilde{x}^8 &= (1 - \varepsilon, -\varepsilon, 1 - \varepsilon, 2 + \varepsilon)
\end{align*}
\]
It is easy to verify that \( g, g^3 \) is an anticlockwise additive chain. Although this example is not covered by Theorem 2.7, the set \( X′ = \{\tilde{x}^1, \ldots, \tilde{x}^8\} \) in the table shows an ad hoc way of stretching the distances for this case (consistent with Conjecture D).

An obvious sufficient condition for a group not to have an anticlockwise additive chain is the following.

**Lemma 2.8.** If \( X \) is a finite set in \( \mathbb{R}^n \) with a transitive abelian group \( \Gamma \) of sup-norm isometries and each saturated additive chain in \( X \) is of the form \( x, g(x), g^2(x), \ldots, g^k(x) \) for some \( g \in \Gamma, k \geq 1 \) and \( x \in X \), then \( \Gamma \) contains no anticlockwise additive chain.

**Proof.** Suppose by way of contradiction that \( f, h \) is an anticlockwise additive chain in \( \Gamma \). Then there exist \( g \in \Gamma \setminus \{e\} \) and integers \( k, m \geq 1 \) such that \( f = g^k, h = g^m \) and \( d(h \circ f) = d(g^{k+m}) = (k+m)d(g) \). By assumption \( f, h \) is an additive chain, so that

\[
d(h \circ f^{-1}) = d(h) + d(f^{-1}) = d(h) + d(f) = d(h \circ f) = (k+m)d(g).
\]

But also \( d(h \circ f^{-1}) = d(g^{m-k}) = |m-k|d(g) \), which is clearly a contradiction. \( \square \)

A simple situation where \( \Gamma \) does have an anticlockwise additive chain occurs when there exist \( f \neq g \) in \( \Gamma \setminus \{e\} \) such that \( f, g \) is an additive chain and \( f^2 = e \) or \( g^2 = e \). However, under the special assumption that \( \Gamma \) contains at most one pair \( \{f, g\} \) such that \( f, g \) is an anticlockwise additive chain, and if, in addition, \( f^2 = e \) or \( g^2 = e \), we can show that the underlying set \( X \) admits a perturbation that preserves the group. As the proof of this result is quite involved and only a minor step forward from having no anticlockwise additive chains, we shall omit it.

The main idea behind the proof of Theorem 2.7 can be summarized as follows. For each \( g \in \Gamma \) we first define a sup-norm non-expansive map \( G : \mathbb{R}^n \to \mathbb{R}^n \) that extends \( g \). The Aronszajn–Panitchpakdi theorem [3] guarantees that such a non-expansive extension exists, and we shall use a specific one. Subsequently a subset \( Q_X \) of \( \mathbb{R}^n \) is identified such that \( X \subset Q_X \) and for each \( g \in \Gamma \) the restriction of \( G \) to \( Q_X \) is a sup-norm isometry. We shall see that for each \( \tilde{x} \in Q_X \) the set \( X′ = \{G(\tilde{x}) : g \in \Gamma\} \) has a transitive abelian group \( \Gamma′ \) of sup-norm isometries. Furthermore, if \( \tilde{x} \in Q_X \) is close to some \( x \in X \), then \( X′ \) has the same cardinality as \( X \) and \( X′ \) does not contain any additive chains other than ones in \( X \). If, in addition, \( \Gamma \) has a clockwise additive chain, we shall see that \( \tilde{x} \in Q_X \) can be chosen arbitrarily close to some \( x \in X \) and such that group \( \Gamma′ \) acting on \( X′ \) has fewer clockwise additive chains than \( \Gamma \). We can apply this result iteratively and end up after finitely many steps with a set \( X^* \) in \( \mathbb{R}^n \) arbitrary close to \( X \) that has a transitive abelian group of sup-norm isometries, but no additive chains of length 3.

**3. Perturbation to additive chains of length 3**

In this section we prove Theorems 2.4 and 2.5. A useful idea, which was introduced in [11], is the notion of an extreme pair. A pair of distinct points \( \{x, y\} \) in \( X \) is said to be an extreme pair in \( X \) if there exists no \( z \in X \) such that \( z, x, y \) or \( x, y, z \) is an additive chain of length 3. Extreme pairs have the following property.

**Lemma 3.1 ([11]).** If \( X \) is a finite set in \( \mathbb{R}^n \) on which an abelian group \( \Gamma \) of sup-norm isometries acts transitively, then \( \{x, y\} \) is an extreme pair in \( X \) if and only if there exists no \( z \in X \) such that \( x, y, z \) is an additive chain of length 3.

**Proof.** It is clear from the definition that if \( \{x, y\} \) is an extreme pair in \( X \), then there exists no \( z \in X \) such that \( x, y, z \) is an additive chain of length 3. To prove the other implication we assume
that \( \{x, y\} \) is not an extreme pair in \( X \). By definition there exists \( z \in X \) such that \( z, x, y \) or \( x, y, z \) is an additive chain of length 3. If \( x, y, z \) is an additive chain of length 3, we are done. If, on the other hand, \( z, x, y \) is an additive chain of length 3, then it follows from Corollary 2.2 that \( x, y, f(y) \) is an additive chain of length 3 if \( f \in \Gamma \) is chosen such that \( f(z) = x \), and the proof is completed. □

The next lemma of this section tells us that if we arbitrarily perturb each point in a finite set, then the set of perturbed points has no new additive chains, as long as the perturbations are sufficiently small.

**Lemma 3.2.** Let \( X \) be a finite set in \( \mathbb{R}^n \) and let

\[
\Delta_1 = \min\{\delta > 0 : \delta = d_\infty(x, y) - (x_i - y_i) \text{ for some } x, y \in X \text{ and } 1 \leq i \leq n\}.
\]

Suppose that for each \( x \in X \) a point \( \tilde{x} \in B(x, \varepsilon) \) is selected and \( \varepsilon < \Delta_1/4 \). If \( \tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^m \) is an additive chain, then \( x^1, x^2, \ldots, x^m \) is an additive chain in \( X \). Moreover, if \( \tilde{x} \preceq_i \tilde{y} \), then \( x \preceq_i y \).

**Proof.** As a sequence of points in \( \mathbb{R}^n \) is an additive chain if and only if it is an \( i \)-chain for some \( 1 \leq i \leq n \), it suffices to prove the second assertion: \( \tilde{x} \preceq_i \tilde{y} \) implies \( x \preceq_i y \). So suppose that \( \tilde{x} \preceq_i \tilde{y} \). Since \( \tilde{x} \in B(x, \varepsilon) \) and \( \tilde{y} \in B(y, \varepsilon) \) we have that

\[
d_\infty(x, y) \leq d_\infty(x, \tilde{x}) + d_\infty(\tilde{x}, \tilde{y}) + d_\infty(\tilde{y}, y) \leq 2\varepsilon + \tilde{y}_i - \tilde{x}_i
\]

\[
\leq 2\varepsilon + |\tilde{y}_i - y_i| + y_i - x_i + |x_i - \tilde{x}_i| \leq 4\varepsilon + y_i - x_i.
\]

But \( \varepsilon < \Delta_1/4 \), so that \( \Delta_1 > d_\infty(x, y) - (y_i - x_i) \). Consequently the definition of \( \Delta_1 \) implies that \( d_\infty(x, y) = y_i - x_i \). □

The proof of Theorem 2.4 is a variant of the proof of [11, Theorem 2.1].

**Proof of Theorem 2.4.** Let \( X \) be a finite set in \( \mathbb{R}^n \) on which an abelian group \( \Gamma \) of sup-norm isometries acts transitively. Take \( \varepsilon > 0 \) arbitrary. For each \( x \in X \) we define \( \tilde{x} \in B(x, \varepsilon) \) by

\[
\tilde{x}_i = \begin{cases} 
  x_i - \varepsilon & \text{if there exists } y \in X \text{ with } \{x, y\} \text{ an extreme pair in } X \text{ and } x \preceq_i y, \\
  x_i + \varepsilon & \text{if there exists } y \in X \text{ with } \{x, y\} \text{ an extreme pair in } X \text{ and } y \preceq_i x, \\
  x_i & \text{otherwise,}
\end{cases}
\]

for \( 1 \leq i \leq n \). Further put \( X' = \{\tilde{x} : x \in X\} \). Notice that \( \tilde{x} \) is well defined, because \( \tilde{x}_i = x_i - \varepsilon \) and \( \tilde{x}_i = x_i + \varepsilon \) imply that there exist \( y, z \in X \) such that \( \{x, y\} \) and \( \{x, z\} \) are extreme pairs in \( X \) with \( x \preceq_i y \) and \( z \preceq_i x \). But this implies that \( z, x, y \) is an additive chain of length 3, which contradicts the fact that \( \{x, y\} \) is an extreme pair in \( X \).

Now take \( \varepsilon > 0 \) arbitrary. We claim that for each \( x \neq y \) in \( X \),

\[
d_\infty(\tilde{x}, \tilde{y}) = d_\infty(x, y) + \varepsilon \quad \text{or} \quad d_\infty(\tilde{x}, \tilde{y}) = d_\infty(x, y) + 2\varepsilon. \tag{1}
\]

To prove (1) we note by construction \( d_\infty(\tilde{x}, \tilde{y}) = d_\infty(x, y) \pm r\varepsilon \), where \( r = 0, 1, \) or 2. Therefore it is sufficient to show that \( d_\infty(\tilde{x}, \tilde{y}) = d_\infty(x, y) \) for all \( x \neq y \) in \( X \). So suppose that \( x, y \in X \) and \( x \neq y \). Now let \( z^1, z^2, \ldots, z^m \) be an additive chain in \( X \), with \( z^1 = x \) and \( z^2 = y \), and suppose that \( m \) is maximal. We claim that \( \{z^1, z^m\} \) is an extreme pair in \( X \). Indeed, there exists \( u \in X \) such that \( z^1, z^m, u \) is an additive chain of length 3, as \( m \) is maximal. Therefore it follows from Lemma 3.1 that \( \{z^1, z^m\} \) is an extreme pair in \( X \).

Remark that \( z^1 \preceq_i z^m \) or \( z^m \preceq_i z^1 \) for some \( 1 \leq i \leq n \). In the first case \( \tilde{x}_i = x_i - \varepsilon \) by definition, and we now show that \( \tilde{y}_i \neq y_i - \varepsilon \). If \( \tilde{y}_i = y_i - \varepsilon \), then there exists \( u \in X \) such that
$\{y, u\}$ is an extreme pair in $X$ with $y \leq u$. As $z^1, z^2, \ldots, z^m$ is an additive chain and $z^1 \leq z_i \leq z^m$ we have that $z^1 \leq z^2 \leq \ldots \leq z^m$ is an $i$-chain, and hence $x \leq y$. This however implies that $x, y, u$ is an additive chain of length 3, which contradicts the fact that $\{y, u\}$ is an extreme pair in $X$. Thus, we conclude that $\tilde{y}_i > y_i - \varepsilon$. By using this inequality we see that

$$d_\infty(\tilde{x}, \tilde{y}) \geq \tilde{y}_i - \tilde{x}_i > y_i - \varepsilon - x_i + \varepsilon = d_\infty(x, y).$$

Likewise, if $z^m \leq z^1$, then $\tilde{y}_i \neq y_i + \varepsilon$. Indeed, if $\tilde{y}_i = y_i + \varepsilon$, then there exists $v \in X$ such that $\{y, v\}$ is an extreme pair in $X$ and $v \leq y$. Again, as $z^1, z^2, \ldots, z^m$ is an additive chain and $z^m \leq z^1$ it follows that $z^m \leq z_i \leq z^1$, so that $y \leq x$. This implies that $v, y, x$ is an additive chain of length 4, which contradicts the fact that $\{y, v\}$ is an extreme pair. Thus $\tilde{y}_i < y_i + \varepsilon$ in this case. From this inequality we deduce that

$$d_\infty(x, y) \leq \tilde{x}_i - \tilde{y}_i > x_i + \varepsilon - y_i - \varepsilon = d_\infty(x, y),$$

and this completes the proof of the theorem. □

Theorem 2.4 has the following consequence.

Corollary 3.3. If $X$ is a finite set in $\mathbb{R}^n$ on which an abelian group of sup-norm isometries acts transitively, then $|X| \leq 3^n$.

Proof. Let $\varepsilon > 0$ be as in Lemma 3.2 and $X'$ as in Theorem 2.4. By Lemma 2.3 it suffices to show that $X'$ has no additive chains of length 4. Suppose by way of contradiction that $\tilde{x}^1, \tilde{x}^2, \tilde{x}^3, \tilde{x}^4$ is an additive chain of length 4 in $X'$. Then

$$d_\infty(\tilde{x}^1, \tilde{x}^4) = \sum_{j=1}^{3} d_\infty(\tilde{x}^j, \tilde{x}^{j+1})$$

(2)

and $x^1, x^2, x^3, x^4$ is an additive chain in $X$ by Lemma 3.2, so that

$$d_\infty(x^1, x^4) = \sum_{j=1}^{3} d_\infty(x^j, x^{j+1}).$$

(3)

On the other hand,

$$d_\infty(\tilde{x}^1, \tilde{x}^4) \leq d_\infty(x^1, x^4) + 2\varepsilon \quad \text{and} \quad \sum_{j=1}^{3} d_\infty(\tilde{x}^j, \tilde{x}^{j+1}) \geq \sum_{j=1}^{3} d_\infty(x^j, x^{j+1}) + 3\varepsilon,$$

which together with (3) contradict Eq. (2). □

We note that the estimate in Corollary 3.3 is weaker than the upper bound, $\max_k 2^k \binom{n}{k}$, proved in [11]. To conclude this section we prove Theorem 2.5.

Proof of Theorem 2.5. For each $x \in X$ we define a point $\tilde{x} \in B(x; \varepsilon)$ in the following manner. If $x \in X$ is a member of a saturated additive chain $x^1, \ldots, x^{2m}$ of length $2m$ in $X$, where $d_\infty(x^1, x^{2m}) = x_1 - x^{2m}$, and $x = x^p$, then we define $\tilde{x}_i = x_i + \varepsilon$ if $1 \leq p \leq m$ and $\tilde{x}_i = x_i - \varepsilon$ if $m < p \leq 2m$. Note that by assumption $m = 1$ or $m = k$. For all other coordinates $1 \leq i \leq n$ we put $\tilde{x}_i = x_i$. We need to show that $\tilde{x}$ is well defined and that $d_\infty(\tilde{x}, \tilde{y}) = d_\infty(x, y) + 2\varepsilon$ for all $x \neq y$ in $X$.

Suppose by way of a contradiction that $\tilde{x}$ is not well defined. Then there exists $1 \leq i \leq n$ such that $\tilde{x}_i = x_i + \varepsilon$ and $\tilde{x}_i = x_i - \varepsilon$. Hence there exist saturated additive chains, say $y^1, \ldots, y^{2m_1}$
and \(z^1, \ldots, z^{2m_2}\) in \(X\) such that \(d_\infty(y^1, z^{2m_1}) = y^1 - y_i\), \(d_\infty(z^1, z^{2m_2}) = z_i^1 - z_i^{2m_2}\), \(x = y_p\), with \(1 \leq p \leq m_1\), and \(x = z^q\), with \(m_2 < q \leq 2m_2\). This implies that
\[
z^1, \ldots, z^{q-1}, x, y^{p+1}, \ldots, y^{2m_1}
\]
is an additive chain. If \(m_1 = m_2 = k\), then the additive chain in (4) has length \((q - 1) + 1 + (2m_1 - p) \geq 2k + 1\), which contradicts the assumption that every saturated has length 2 or 2k.

Next we show that \(d_\infty(\tilde{x}, \tilde{y}) = d_\infty(x, y) + 2\varepsilon\) for all \(x \neq y \in X\). We know that \(x, y\) is a subsequence of a saturated additive chain of length \(2m\), where \(m = 1\) or \(m = k\). In fact, it follows from Corollary 2.2 that \(x, y\) is a subsequence of an additive chain \(z^1, \ldots, z^{2m}\) in \(X\) such that \(x = z^p\) with \(1 \leq p \leq m\) and \(y = z^q\) with \(m < q \leq 2m\). By possibly interchanging the roles of \(x\) and \(y\) we may assume that \(d_\infty(z^1, z^{2m}) = z_i^1 - z_i^{2m}\). This implies that \(z^1, \ldots, z^{2m}\) is an \(i\)-chain and \(d_\infty(x, y) = x_i - y_i\). By definition \(\tilde{x}_i = x_i + \varepsilon\) and \(\tilde{y}_i = y_i - \varepsilon\), so that
\[
d_\infty(\tilde{x}, \tilde{y}) \geq x_i + \varepsilon - (y_i - \varepsilon) = d_\infty(x, y) + 2\varepsilon.
\]
As \(\tilde{x} \in B(x; \varepsilon)\) and \(\tilde{y} \in B(y; \varepsilon)\), \(d_\infty(\tilde{x}, \tilde{y}) \leq d_\infty(x, y) + 2\varepsilon\), and hence \(d_\infty(\tilde{x}, \tilde{y}) = d_\infty(x, y) + 2\varepsilon\). □

For a specific subclass of sup-norm non-expansive maps, called topical maps (see [11]), one can use similar arguments to show that each periodic orbit has a perturbation. To prove this result one has to combine the ideas in the proof of Theorem 2.4 with the proof of [11, Theorem 4.2].

4. A proof of Conjecture C in a special case

In this section we prove Theorem 2.7. Suppose that \(X\) is a finite set in \(\mathbb{R}^n\) with a transitive abelian group \(\Gamma\) of sup-norm isometries. For each \(g \in \Gamma\) we first define a sup-norm non-expansive extension \(G: \mathbb{R}^n \to \mathbb{R}^n\) to the whole of \(\mathbb{R}^n\) by
\[
G(z)_i = \min\{y_i : |y_i - g(x)_i| \leq d_\infty(z, x)\ 	ext{for all } x \in X\}
\]
for \(1 \leq i \leq n\) and \(z \in \mathbb{R}^n\). It is easy to verify that
\[
G(z)_i = \max_{x \in X}(g(x)_i - d_\infty(z, x)) \quad \text{for } 1 \leq i \leq n \text{ and } z \in \mathbb{R}^n.
\]
We now show that \(G: \mathbb{R}^n \to \mathbb{R}^n\) is a non-expansive extension of \(g\).

**Lemma 4.1.** The map \(G: \mathbb{R}^n \to \mathbb{R}^n\) as defined in (5) is a sup-norm non-expansive map that extends \(g\).

**Proof.** It is straightforward to verify from (5) that \(G\) is an extension of \(g\). Let \(y, z \in \mathbb{R}^n\) and suppose that \(d_\infty(G(y), G(z)) = G(y)_i - G(z)_i\). Let \(x \in X\) be such that \(G(y)_i = g(x)_i - d_\infty(y, x)\). Then
\[
d_\infty(G(y), G(z)) = G(y)_i - G(z)_i
\]
\[
\leq (g(x)_i - d_\infty(y, x)) - (g(x)_i - d_\infty(z, x))
\]
\[
\leq d_\infty(y, z),
\]
so that \(G\) is sup-norm non-expansive. □
From this lemma it follows that every periodic orbit of a sup-norm non-expansive map is also the orbit of a map $G: \mathbb{R}^n \rightarrow \mathbb{R}^n$, where each coordinate function $G(z)_i$ is a min–max combination of expressions of the type $a \pm z_j$ where $1 \leq j \leq n$ and $a \in \mathbb{R}$.

Next we identify a subset $Q_X$ of $\mathbb{R}^n$ such that for each $g \in \Gamma$ the restriction of $G$ to $Q_X$ is a sup-norm isometry. To do so, it is useful to introduce the following notions. Let $M$ be subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ and define

$$B_M(x) = \bigcap_{z \in M} B(z; d_\infty(x, z)).$$

**Definition 4.2.** A point $x \in \mathbb{R}^n$ is called a minimal point of $M$ if

1. $y \in B_M(x)$ implies that $y$ is on the boundary of $B(z; d_\infty(x, z))$ for all $z \in M$, or equivalently, $d_\infty(x, z) = d_\infty(y, z)$ for all $z \in M$, and
2. $x_i = \min\{y_i : y \in B_M(x)\}$ for all $1 \leq i \leq n$.

The set of all minimal points of $M$ is denoted by $Q_M$.

We will show that the restriction of $G$ to $Q_X$ is a sup-norm isometry, but before we do that we make the following elementary observation.

**Lemma 4.3.** If $M$ is a finite subset of $\mathbb{R}^n$, then for each $x \in \mathbb{R}^n$ the set $Q_M \cap B_M(x)$ is nonempty.

**Proof.** Label the elements of $M$ by $z^1, \ldots, z^k$ and put

$$A_0 = B_M(x) = \bigcap_{z \in M} B(z; d_\infty(x, z)).$$

As $A_0$ is a nonempty compact set, we can find a point $x^1 \in A_0$ that has minimal distance to $z^1$. Now set

$$A_1 = \left(\bigcap_{i=2}^{k} B(z^i; d_\infty(x^1, z^i))\right) \bigcap B(z^1; d_\infty(x^1, z^1))$$

and note that $A_1$ is a nonempty subset of $A_0$. Hence there exists $x^2 \in A_1$ that has minimal distance to $z^2$. Subsequently, let

$$A_2 = \left(\bigcap_{i=3}^{k} B(z^i; d_\infty(x^1, z^i))\right) \bigcap B(z^1; d_\infty(x^1, z^1)) \bigcap B(z^2; d_\infty(x^2, z^2))$$

and repeat the procedure until we have found elements $x^1, \ldots, x^k$ and a nonempty subset of $A_0$:

$$A_k = \bigcap_{i=1}^{k} B(z^i; d_\infty(x^i, z^i)).$$

Define $x^* \in \mathbb{R}^n$ by $x^*_i = \min\{y_i : y \in A_k\}$ for $1 \leq i \leq n$. By construction $x^*$ is a minimal point for $M$ and moreover $x^* \in A_k \subset A_0$, so that $x^* \in Q_M \cap B_M(x)$. □

Next we show that each $G$ is a sup-norm isometry that maps $Q_X$ onto itself.

**Proposition 4.4.** Let $X$ be a finite set in $\mathbb{R}^n$ on which an abelian group $\Gamma$ of sup-norm isometries acts transitively. Then for each $g \in \Gamma$ the restriction of the map $G: \mathbb{R}^n \rightarrow \mathbb{R}^n$
to \(Q_X\) is a sup-norm isometry that maps \(Q_X\) onto itself. Moreover, for each \(x' \in Q_X\) the set \(X' = \{G(x') : g \in \Gamma\}\) has a transitive abelian group \(\Gamma'\) of sup-norm isometries, given by \(\Gamma' = \{G : X' \to X' \mid g \in \Gamma\}\).

**Proof.** We first show that if \(z \in \mathbb{R}^n\) is a minimal point of \(X\), then \(G(z)\) is one too. Let \(g \in \Gamma\) and \(y \in B_X(G(z))\). As \(X\) is a finite set, there exists an integer \(p \geq 1\) such that \(g^p(x) = x\) for all \(x \in X\). Since \(G\) extends \(g\) and \(d_\infty(G(z), G(x)) \geq d_\infty(y, G(x))\) for all \(x \in X\), we have that

\[
\tag{6}
d_\infty(z, x) \geq d_\infty(G(z), G(x)) \geq d_\infty(y, G(x)) \geq d_\infty(G^{p-1}(y), x)
\]

for all \(x \in X\). Therefore \(G^{p-1}(y) \in B_X(z)\) and \(d_\infty(z, x) = d_\infty(G^{p-1}(y), x)\) for all \(x \in X\), as \(z\) is a minimal point of \(X\). Thus Eq. (6) implies that \(d_\infty(G(z), G(x)) = d_\infty(y, G(x))\) for all \(x \in X\), which proves the first property in Definition 4.2. We also find that \(d_\infty(z, x) = d_\infty(G(z), G(x))\) for all \(x \in X\). By using the definition of \(G\) we deduce

\[
G(z)_i = \min\{y_i : |y_i - G(x)_i| \leq d_\infty(z, x)\text{ for all } x \in X\}
\]

\[
= \min\left\{y_i : y \in \bigcap_{x \in X} B(G(x); d_\infty(z, x))\right\}
\]

\[
= \min\left\{y_i : y \in \bigcap_{x \in X} B(G(x); d_\infty(G(z), G(x)))\right\}
\]

\[
= \min\{y_i : y \in B_X(G(z))\}
\]

for \(1 \leq i \leq n\). Thus, \(G(z)\) is a minimal point of \(X\) and hence \(G\) maps \(Q_X\) into itself.

Next we show that \(\{G : Q_X \to Q_X \mid g \in \Gamma\}\) is an abelian group of sup-norm isometries. We observe first that if \(f, g \in \Gamma\) and \(h = f \circ g\), then \(H = F \circ G\), where \(F, G\) and \(H\) are the extensions of \(f, g\) and \(h\) respectively. Indeed,

\[
F(G(z))_i = \max_{x \in X} (f(x)_i - d_\infty(G(z), x)) = \max_{x \in X} (f(g(x))_i - d_\infty(G(z), G(x)))
\]

\[
= \max_{x \in X} (h(x)_i - d_\infty(z, x)) = H(z)_i
\]

for all \(1 \leq i \leq n\) and \(z \in Q_X\). As \(\Gamma\) is an abelian group, we have that \(F \circ G = G \circ F\) and \((F \circ G) \circ H = F \circ (G \circ H)\). Moreover, if \(z \in Q_X\) and \(e\) is the unit element in \(\Gamma\), then

\[
E(z)_i = \min\{y_i : |y_i - e(x)_i| \leq d_\infty(z, x)\text{ for all } x \in X\}
\]

\[
= \min\left\{y_i : y \in \bigcap_{x \in X} B(e(x); d_\infty(z, x))\right\}
\]

\[
= \min\left\{y_i : y \in \bigcap_{x \in X} B(x; d_\infty(z, x))\right\}
\]

\[
= z_i
\]

for all \(1 \leq i \leq n\), so that \(E : Q_X \to Q_X\) is the identity on \(Q_X\). From this it follows that each \(G\) has an inverse. We remark that there exists \(p \geq 1\) such that \(g^p(x) = x\) for all \(x \in X\). Therefore \(G^p(z) = E(z) = z\) for all \(z \in Q_X\) and hence each \(z \in Q_X\) is a periodic point of \(G\). Since \(G\) is non-expansive, this implies that \(G\) is a sup-norm isometry that maps \(Q_X\) onto itself and hence \(\{G : Q_X \to Q_X \mid g \in \Gamma\}\) is an abelian group of sup-norm isometries.

Now let \(X' \subseteq Q_X\) and \(X' = \{G(x') : g \in \Gamma\}\). Clearly \(G\) maps \(X'\) into itself for all \(g \in \Gamma\) and \(\Gamma' = \{G : X' \to X' \mid g \in \Gamma\}\) is an abelian group of sup-norm isometries that acts transitively on \(X'\). \(\square\)
It is not hard to prove that if we take \( x' \in Q_X \) sufficiently close to \( x \in X \), then \( |X'| = |X| \). Indeed, we have the following lemma.

**Lemma 4.5.** Let \( X \) be a finite set in \( \mathbb{R}^n \) on which an abelian group \( \Gamma \) of sup-norm isometries acts transitively and let \( \Delta_2 = \min_{g \in \Gamma \setminus \{e\}} d(g) \). If \( x' \in Q_X \) and \( x \in X \) are such that \( d_\infty(x, x') < \Delta_2/2 \), then for \( X' = \{ G(x') : g \in \Gamma \} \) we have that \( |X'| = |X| \).

**Proof.** Define \( \alpha: X \to X' \) by \( \alpha(g(x)) = G(x') \) for all \( g \in \Gamma \). The map \( \alpha \) is well defined by Lemma 2.1. As \( \Gamma \) acts transitively on \( X \), \( \alpha \) maps \( X \) onto \( X' \). But \( \alpha \) is also injective, because

\[
d_\infty(F(x'), G(x')) \geq d_\infty(F(x), G(x)) - d_\infty(F(x), F(x')) - d_\infty(G(x), G(x')) \\
\geq d_\infty(x, f^{-1}(g(x))) - \Delta_2 > 0,
\]

as \( F \) and \( G \) are isometries on \( Q_X \). Thus we find that \( |X| = |X'| \). \( \Box \)

To prove Theorem 2.7 it is convenient to associate with each finite set \( X \) in \( \mathbb{R}^n \) a directed coloured graph \( G_X \) without loops, but with possibly multiple arrows.

**Definition 4.6.** Let \( X \) be a finite set in \( \mathbb{R}^n \). Then the directed \( n \)-coloured graph \( G_X \) with multiple arrows is defined as follows: the vertex set of \( G_X \) is \( X \) and there exists an arrow from \( x \) to \( y \) in \( G_X \) with colour \( i \) if \( y \leq x \) and \( x \neq y \).

An important step in the proof of Theorem 2.7 is to show that if \( X \) is a finite set in \( \mathbb{R}^n \) on which an abelian group \( \Gamma \) of sup-norm isometries acts transitively and \( \Gamma \) contains a clockwise additive chain, then \( X \) has a perturbation \( X' \) that preserves the group and the graph \( G_X \) contains fewer arrows than the graph of \( G_X \).

**Proposition 4.7.** If \( X \) is a finite set in \( \mathbb{R}^n \) on which an abelian group \( \Gamma \) of sup-norm isometries acts transitively and \( \Gamma \) contains a clockwise additive chain, then for each \( \epsilon > 0 \) and \( x \in X \) there exists \( x' \in Q_X \cap B(x; \epsilon) \) such that the graph \( G_{X'} \) of \( X' = \{ G(x') : g \in \Gamma \} \) has fewer arrows than \( G_X \) and \( |X'| = |X| \).

**Proof.** Suppose \( \Gamma \) has a clockwise additive chain \( f, g \) and let \( x \in X \). Then, \( f(x), g(f(x)) \) is an additive chain in \( X \). Let \( \epsilon > 0 \) be such that \( \epsilon < \max\{\Delta_1/4, \Delta_2/2\} \), where \( \Delta_1 \) and \( \Delta_2 \) are as in Lemmas 3.2 and 4.5 respectively. There are two cases to consider: \( f = g \) and \( f \neq g \). We begin with the first one: \( f = g \). Let \( r \) be the largest integer such that \( x, f(x), \ldots, f^r(x) \) is an additive chain in \( X \). It follows from Corollary 2.2 that \( f^{-r+1}(x), \ldots, x, f(x) \) is also an additive chain in \( X \). Thus, \( f^{-r+1}(x), \ldots, x, f(x) \) is a \( k \)-chain in \( x \) for some \( 1 \leq k \leq n \). By changing the sign on the \( k \)-th coordinate of each point in \( X \), we may assume without loss of generality that \( r d(f) = d_\infty(f^{-r+1}(x), f(x)) = f(x)_k - f^{-r+1}(x)_k \). In particular, this implies that there is an arrow from \( f(x) \) to \( x \) with colour \( k \) in \( G_X \). Now select a point \( y \in [f(x), f^2(x)]_\infty \) such that

\[
d_\infty(y, f(x)) = \epsilon, \quad d_\infty(y, f^2(x)) = d(f) - \epsilon, \quad \text{and} \quad d_\infty(y, f^{-r+1}(x)) < rd(f) + \epsilon.
\]

Such a point \( y \) exists since \( d_\infty(f^{-r+1}(x), f^2(x)) < (r + 1)d(f) \) and \( r \) is maximal. It follows from Lemma 4.3 that we may assume that \( y \in Q_X \). Now put \( x' = F^{-1}(y) \) and let \( X' = \{ G(x') : g \in \Gamma \} \). As \( y \in B(f(x); \epsilon) \) and \( \epsilon < \Delta_2/2 \) we know that \( |X'| = |X| \) by Lemma 4.5. Since \( x' \in Q_X \) and \( F \) is an isometry on \( Q_X \), \( d_\infty(x, x') = \epsilon \) and \( d_\infty(x', f(x)) = d(f) - \epsilon \), so that \( d_\infty(x', y) = d(f) \). Indeed,

\[
d_\infty(x', y) \leq d_\infty(x', f(x)) + d_\infty(f(x), y) = d(f),
\]
and
\[ d_\infty(x', y) \geq d_\infty(x, y) - d_\infty(x, x') = d_\infty(x, y) - \varepsilon. \]

By using the inequality
\[ d_\infty(x, y) = d_\infty(f^{-1}(x), x') \geq d_\infty(f^{-1}(x), f(x)) - d_\infty(f(x), x') = d(f) + \varepsilon, \]
we see that \( d_\infty(x', y) = d(f) \). Furthermore we note that
\[ y_k - x'_k < f'^{-r+1}(x)_k + rd(f) + \varepsilon - (f'^{-r-1}(x)_k + (r - 1)d(f) + \varepsilon) = d(f). \]

Hence there is no arrow from \( y = F(x') \) to \( x' \) with colour \( k \) in \( G_{X'} \). As \( \varepsilon < \Delta_1/4 \), it follows from Lemma 3.2 that \( G_{X'} \) has fewer arrows than \( G_X \). This proves the proposition in the first case.

Now let us assume that \( x, f(x), g(f(x)) \) is an additive chain and \( f \neq g \). We may assume that \( x, f(x), f^2(x) \) is not an additive chain. Moreover, since \( f, g \) is a clockwise additive chain, we know that \( f^2(x), f(x), g(f(x)) \) is not an additive chain. Let \( r \) be the largest integer such that
\[ f(x), x, g^{-1}(x), \ldots, g^{-r}(x) \]
is an additive chain and suppose that it is a \( k \)-chain. By changing the sign on the \( k \)-th coordinate of each point in \( X \) we may assume without loss of generality that \( d_\infty(f(x), g^{-r}(x)) = f(x)_k - g^{-r}(x)_k \) and hence there is an arrow from \( f(x) \) to \( x \) with colour \( k \) in \( G_X \).

As \( f^2(x), f(x), g(f(x)) \) is not an additive chain, we can find \( y \in [f(x), f^2(x)]_\infty \cap [f(x), g(f(x))]_\infty \) such that
\[ d_\infty(y, f(x)) = \varepsilon, \quad d_\infty(y, f^2(x)) = d(f) - \varepsilon, \quad d_\infty(y, g(f(x))) = d(g) - \varepsilon, \quad \text{and} \quad d_\infty(y, g^{-r}(x)) < d_\infty(f(x), g^{-r}(x)) + \varepsilon. \]

The last inequality can be fulfilled as \( g(f(x)), f(x), x, g^{-1}(x), \ldots, g^{-r}(x) \) is not an additive chain. Indeed, if \( g(f(x)), f(x), x, g^{-1}(x), \ldots, g^{-r}(x) \) is an additive chain, then \( f(x), x, g^{-1}(x), \ldots, g^{-r}(x), g^{-(r+1)}(x) \) is also an additive chain, which contradicts the fact that \( r \) is maximal. By Lemma 4.3 we may assume that \( y \in Q_X \). Put \( x' = F^{-1}(y) \) and let \( X' = \{ G(x') : g \in \mathcal{F} \} \). As \( \varepsilon < \Delta_2/2 \) and \( y \in B(f(x); \varepsilon) \), we know by Lemma 4.5 that \( |X'| = |X| \).

We further note that \( x' \in Q_X \) and therefore \( d_\infty(x, x') = \varepsilon \) and \( d_\infty(x', f(x)) = d(f) - \varepsilon \). This implies that \( x' \in [x, f(x)]_\infty \), so that
\[ x'_k = g^{-r}(x)_k + d_\infty(x, g^{-r}(x)) + \varepsilon. \]

Therefore
\[ y_k - x'_k < g^{-r}(x)_k + d_\infty(f(x), g^{-r}(x)) + \varepsilon - g^{-r}(x)_k - d_\infty(x, g^{-r}(x)) - \varepsilon = d(f). \]

On the other hand,
\[ d_\infty(x', y) \geq d_\infty(x, g(f(x))) - d_\infty(x', x) - d_\infty(y, g(f(x))) = d(f) + d(g) - \varepsilon - (d(g) - \varepsilon) = d(f). \]

Thus we conclude that there is no arrow from \( y = F(x') \) to \( x' \) with colour \( k \) in \( G_{X'} \). As \( \varepsilon < \Delta_1/4 \), we find that \( G_{X'} \) contains fewer arrows than \( G_X \) by Lemma 3.2. This completes the proof of the proposition. \( \square \)

By using Proposition 4.7 we can now prove Theorem 2.7.
Proof of Theorem 2.7. Suppose by way of a contradiction that the collection of all sets in $\mathbb{R}^n$ of size $m$ that have a transitive abelian group of sup-norm isometries, no anticlockwise additive chains, and no perturbation that preserves the group is nonempty. Within this collection there is a set, say $X$, for which the graph $G_X$ has the minimum number of arrows. Denote its group by $\Gamma$. As $X$ has no perturbation that preserves the group, $\Gamma$ must have an additive chain. Otherwise we could take $X' = X$. By assumption the additive chain is clockwise. We claim that for each $\varepsilon > 0$ and for each $x \in X$ there exists $\tilde{x} \in B(x; \varepsilon)$ such that $X' = \{\tilde{x}: x \in X\}$ has a transitive abelian group $\Gamma' = \{g': X' \to X' | g \in \Gamma\}$, with the property that $g(\tilde{x}) = g(x)$ for all $\tilde{x} \in X'$, and $G_{X'}$ has fewer arrows than $G_X$. To achieve this we fix $x \in X$ and for each $z \in X$ we take $\tilde{z} = G(x')$, where $x' \in Q_X \cap B(x; \varepsilon)$ is as in Proposition 4.7 and $G$ is such that $g(x) = z$. We know that $|X'| = |X|$ and moreover the graph $G_{X'}$ contains fewer arrows than $G_X$. It follows from Proposition 4.4 that if we let $g' = G$ for each $g \in \Gamma$, then the group $\Gamma'$ is a transitive abelian group of sup-norm isometries, which has the desired property. Since $G_X$ has the minimal number of arrows, $X'$ has a perturbation $X''$ that preserves the group. But this is a contradiction, as $X''$ is also a perturbation for $X$ that preserves the group. \qed

If $\Gamma$ has an anticlockwise additive chain, it can happen that the set $Q_X$ is too small for finding a perturbation with fewer additive chains. Consider for instance the set $X = \{a, b, c, d\}$ in $\mathbb{R}^2$, where $a = (1, 0)$, $b = (0, 1)$, $c = (-1, 0)$, and $d = (0, -1)$. Then $\Gamma' = \{e, (ab)(cd), (ac)(bd), (ad)(bc)\}$ is an abelian group of sup-norm isometries that acts transitively on $X$ and $\Gamma'$ has an anticlockwise additive chain. In this example we find for each $x'$ in $Q_X$ (illustrated in Fig. 1) that the graph of $X' = \{G(x'): g \in \Gamma\}$ is the same as the graph of $X$, whenever $x'$ is sufficiently close to a point in $X$. This illustrates the limitations of our method. It therefore seems desirable to find other sup-norm non-expansive extensions of the isometries in the group that allow a larger set in which one can perturb the orbit. In general there are many ways to extend a sup-norm isometry non-expansively, but it seems hard to give an explicit recipe for generating such extensions, other than the one we have used here.

5. Final remarks

Instead of thinking about perturbations one can just consider the directed coloured graph $G_X$ associated with $X$. By using the height vectors in the proof of Lemma 2.3, it can be shown that the following conjecture implies Conjecture A.
Conjecture A’. If $X$ is a finite set in $\mathbb{R}^n$ on which an abelian group of sup-norm isometries acts transitively, then the directed coloured graph $G_X$ has a subgraph $H_X$ with vertex set $X$ such that every pair of distinct vertices in $H_X$ is connected by an arrow in $H_X$ and $H_X$ contains no mono-coloured directed path of length 2.

To see that Conjecture A’ implies Conjecture A, let $h_i(x)$ be the length the longest directed path in $H_X$ with colour $i$ starting at $x$. As $H_X$ contains no mono-coloured paths of length 2, the height vector $h(x) = (h_1(x), \ldots, h_n(x))$ is in $\{0, 1\}^n$ for each $x \in X$. Moreover, $h(x) \neq h(y)$ if $x \neq y$, as there exists at least one arrow between $x$ and $y$. Thus, Conjecture A’ implies that $X$ contains at most $2^n$ points. Moreover Conjecture B implies Conjecture A’. To see this we remark that it follows from Lemma 3.2 that the graph $G_X'$ associated with the set $X'$ in Conjecture B is a subgraph of $G_X$ as long as $\epsilon > 0$ is sufficiently small. Moreover, $G_X'$ satisfies Conjecture A’, since $X'$ has no additive chain of length 3. At present we do not know whether any of the conjectures that we have formulated are equivalent.

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References