A Two-Grid Method for Solving Elliptic Problems with Inhomogeneous Boundary Conditions

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(Received and accepted February 1994)

Abstract—A Galerkin finite element method using a two-grid technique for solving elliptic problems with inhomogeneous essential boundary conditions is considered. At the boundary, the function is approximated by a projection of the given data on a finite dimensional space; in the interior of the domain, a two-grid method is used for solving the algebraic system. An analysis of the $L^2$-error is made and sample computations are provided.

Keywords—Elliptic problem, Inhomogeneous boundary conditions, Finite element, Two-grid method.

1. INTRODUCTION

Consider the inhomogeneous problem of the form

\begin{align*}
L_\Omega u &= f, \quad \text{in } \Omega, \\
L_\Gamma u &= g, \quad \text{on } \Gamma,
\end{align*}

where $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is bounded, $\Gamma = \partial \Omega$, $L_\Omega$ is an elliptic operator and equation (1.2) represents a general notation for the boundary conditions.

An approximating problem can be associated with the equations (1.1), (1.2):

\begin{align*}
L_{h,\Omega} u_{h,\Omega} &= f_h, \\
L_{h,\Gamma} u_{h,\Gamma} &= g_h.
\end{align*}

In [1], the following approach has been discussed. Equation (1.3) is obtained by means of a finite differences or finite elements method and equation (1.4) represents a set of boundary conditions obtained by different interpolation techniques. For solving (1.3), multigrid methods have been considered, which present the advantage of reduced computer work, especially in threedimensional problems [1,2].

Another approach for the discretization has been considered in [3]. Equation (1.3) is obtained using a Galerkin finite element approximation on the finite dimensional spaces $V_h \subset H^1(\Omega)$, while equation (1.4) represents the equation obtained if the boundary data is approximated using the $L^2(\Gamma)$-projection $P_h g$ of $g$ onto the restriction $S_h$ of $V_h$ to $\Gamma$. Under further assumptions, a

This research has been supported in part by a grant from the Mobil Corporation.
comparison made with the case when \( g_h \) represents an \( S_h \)-interpolant of \( g \) has shown \([3,4]\) that the error rate is the same in the two situations. However, the advantage of using the \( L^2(\Gamma) \)-projection is that the \( L^2 \)-error is minimum over the space \( S_h \).

The purpose of this paper is to present a method for solving the above problem on a polyhedral domain; the algorithm combines the method of using the \( L^2(\Gamma) \)-projection with a two-grid technique for solving the algebraic system. Under appropriate assumptions \([1]\), a similar algorithm using a multi-grid method can be implemented in the same manner. An error analysis is made for the \( L^2 \)-norm. Sample computations are provided for the Poisson equation with Dirichlet boundary conditions on a square in \( \mathbb{R}^2 \).

2. THE APPROXIMATING PROBLEM

Throughout this paper, \( C \) will denote a constant, taking different values in different cases; the Sobolev space of order \( r \) associated with the set \( R \) will be denoted by \( H^r(R) \), with the norm

\[
\|u\|_{r,R} = \left( \sum_{|\alpha| \leq r} \int_R |\partial^\alpha u|^2 \right)^{1/2}.
\]

Also, \( H^r_0(R) \) will denote the space of those functions in \( H^r(R) \) which vanish on the boundary.

An example of a simple elliptic operator will be considered, but the discussion in Sections 2 and 3 can be extended to more general problems. As a model problem, let (1.1) be the Poisson equation and (1.2) represent a Dirichlet inhomogeneous boundary condition, i.e.,

Given \( f \in H^{-2}(\Omega) \) and \( g \in H^{-1/2}(\Gamma) \), find \( u \in H^r(\Omega) \) such that

\[
\begin{align*}
-\Delta u &= f, \quad \text{in } \Omega, \\
u &= g, \quad \text{on } \Gamma,
\end{align*}
\]

where \( \Omega \in \mathbb{R}^n \) is a convex polyhedron.

Let \( V_h \) be a finite dimensional subspace of \( H^1(\Omega) \) corresponding to a grid on \( \Omega \) of generic mesh-spacing \( h \). Let \( V_{h,0} = \{ v_h \in V_h : v_h = 0 \text{ on } \Gamma \} \). Since \( \Omega \) is a polyhedron, \( V_{h,0} \subseteq H^1_0(\Omega) \). Let \( S_h \) be the restriction of \( V_h \) to the boundary, i.e., \( S_h = \{ \psi_h : \forall v_h \in V_h, \text{ with } \psi_h = v_h |_{\Gamma} \} \).

Let \( P_h g \) be the \( L^2(\Gamma) \)-projection of \( g \) onto \( S_h \). Using a standard Galerkin weak formulation of the problem (2.1)-(2.2), one can obtain the following approximate problem \([3]\).

Given \( f \in H^{-2}(\Omega) \) and \( g \in H^{-1/2}(\Gamma) \), find \( u_h \in V_h \) such that

\[
\begin{align*}
\int_{\Gamma} \nabla u_h \nabla v_h &= \int_{\Omega} f v_h, & \text{for all } v_h \in V_{h,0},
\end{align*}
\]

To give a more explicit formulation, assume that \( V_h = \text{span}[\Phi_1, \Phi_2, \ldots, \Phi_{N+M}] \), with \( \Phi_j \) chosen such that \( V_{h,0} = \text{span}[\Phi_1, \Phi_2, \ldots, \Phi_N] \) and \( S_h = \text{span}[\Phi_{N+1} |_{\Gamma}, \Phi_{N+2} |_{\Gamma}, \ldots, \Phi_{N+M} |_{\Gamma}] \). Let \( \Psi_j = \Phi_{N+j} |_{\Gamma} \), for \( j = 1, 2, \ldots, M \), so that \( S_h = \text{span}[\Psi_1, \Psi_2, \ldots, \Psi_M] \). Then \( P_h g \in S_h \) can be written uniquely as

\[
P_h g(x) = \sum_{j=1}^{M} g_j \Psi_j(x), \quad \text{for all } x \in \Gamma,
\]

where \( g_j (1 \leq j \leq M) \) are some generalized weights, satisfying the property

\[
\| g - P_h g \|_{0,\Gamma} = \inf_{\hat{g} \in S_h} \| g - \hat{g} \|_{0,\Gamma}.
\]

Also, \( u_h \in V_h \) can be written uniquely as

\[
u_h(x) = \sum_{j=1}^{N+M} u_j \Phi_j(x), \quad \text{for all } x \in \Omega.
\]
Consequently, since \( u_h |_\Gamma = P_h g \),

\[
u_h(x) = \sum_{j=1}^{N} u_j \Phi_j(x) + \sum_{j=1}^{M} g_j \Phi_{N+j}(x), \quad \text{for all } x \in \Omega. \tag{2.8}
\]

Now taking into account that \( \{\Phi_1, \Phi_2, \ldots, \Phi_N\} \) is a basis for \( V_{h,0} \), the approximating problem becomes the following:

Given \( f \in H^{2}(\Omega) \) and \( g \in H^{1/2}(\Gamma) \), find \( u = (u_j)_{1 \leq j \leq N+M} \in \mathbb{R}^{N+M} \), such that

\[
ug_j = gj, \quad \text{for } 1 < j < M \quad \text{(where } g_j \text{ are the weights in (2.6)) and}
\]

\[
\sum_{j=1}^{N} u_j \int_{\Omega} \nabla \Phi_j \nabla \phi_k + \sum_{j=1}^{M} g_j \int_{\Omega} \nabla \Phi_{N+j} \nabla \phi_k = \int_{\Omega} f \phi_k, \quad \text{for all } k = 1, \ldots, N, \tag{2.9}
\]

or, making the notation

\[
f_0(v^h) = \int_{\Omega} f v^h - \sum_{j=1}^{M} g_j \int_{\Omega} \nabla \Phi_{N+j} \nabla v^h, \quad \text{for all } v^h \in V_{h,0}, \tag{2.10}
\]

the problem becomes the following:

Solve

\[
L_h u_\Omega = f_\Omega, \quad \text{for } u_\Omega = (u_j)_{1 \leq j \leq N} \in \mathbb{R}^N, \tag{2.11}
\]

where \( L_h \) is a \( N \times N \) matrix with elements

\[
(L_h)_{jk} = \int_{\Omega} \nabla \Phi_j \nabla \Phi_k, \quad 1 \leq j, k \leq N, \tag{2.12}
\]

and \( f_\Omega \) is a \( N \times 1 \) vector with components

\[
(f_\Omega)_k = f_0(\Phi_k), \quad 1 \leq k \leq N. \tag{2.13}
\]

Notice that the matrix \( L_h \) is the stiffness matrix for the homogeneous problem associated with the given problem, and is a symmetric positive definite matrix.

The \( L^2(\Gamma) \)-projection of \( g \) onto \( S_h \), given by (2.5)-(2.6), can be obtained practically by solving the \( M \times M \) linear system: \( D u_r = g \), for \( u_r = (u_{N+j})_{1 \leq j \leq M} = (g_j)_{1 \leq j \leq M} \in \mathbb{R}^M \), where \( D_{jk} = \int_{\Gamma} \Psi_j \Psi_k, \) for \( 1 \leq j, k \leq M \) and \( (g)_k = \int_{\Gamma} g \Psi_k, \) for \( 1 \leq k \leq M \).

Thus, the approximating problem (2.3)-(2.4) reduces to solving the system

\[
L_h u_\Omega = f_\Omega, \tag{2.14}
\]

\[
D u_r = g, \tag{2.15}
\]

for \( u_\Omega \in \mathbb{R}^N \) and \( u_r \in \mathbb{R}^M \).

3. IMPLEMENTATION OF THE TWO-GRID ALGORITHM

Assume that a fine grid and a coarse one are associated with the domain \( \Omega \), of generic mesh spacings \( h \) and \( 2h \), respectively. The use of \( P_h g \) to approximate the solution on \( \Gamma \), combined with a variant of a two-grid algorithm indicated in [1] yield the following method.

Solve (2.15) for \( u_r \) (for example by direct solving: Gauss elimination).

Find \( f_\Omega \) using (2.13) and (2.10).

Let \( u^{(0)}_{h,\Omega} \) be given. (Make an initial guess of the solution \( u_h \) at the interior nodes, say \( u^{(0)}_{h,\Omega} = 0 \).
For $k = 1, 2, \ldots$, let 
\[ u_{h,\Omega}^{(k+1)} = M_h(\nu) u_{h,\Omega}^{(k)} + N_h f_\Omega, \tag{3.1} \]
where
\[ M_h(\nu) = (I - p L_{2h}^{-1} r L_h) S_h^k \tag{3.2} \]
is the two-grid iteration matrix, $S_h$ is the matrix corresponding to one presmoothing step, the integer $\nu$ indicates the number of presmoothing steps performed for one two-grid iteration, $p$ and $r$ are the prolongation (coarse-to-fine) and the restriction (fine-to-coarse), respectively, which are assumed to be adjoint ($r = p^*$. For the definition of $N_h$, see [1].

A stopping criterion can be
\[ ||u_{h}^{(k+1)} - u_{h}^{(k)}||_{s,\Omega} < \epsilon, \tag{3.3} \]
where $s$ is 0 or 1, $\epsilon$ is given, and for some integer $k$, $u_{h}^{(k)} \in V_h$ is defined by $u_{h}^{(k)} = P_h g$ on $\Gamma$ and $u_{h}^{(k)} = u_{h,\Omega}^{(k)}$ at the interior nodes. Then, after applying the algorithm until the stopping criterion is met, an approximate for $u_h$ is $u_{h}^{(k+1)}$.

A discussion of the convergence of the two-grid method (3.2) has been made in [1], with different types of iterative or semi-iterative techniques for the smoothing step (simple Jacobi, Gauss-Seidel with different kinds of orderings of the nodes, conjugate gradient, etc.). We concentrate only on error estimates for the approximate iterative solution $u_{h}^{(k)}$.

### 4. ERROR ESTIMATES

Assume that for the finite dimensional subspace $V_h$ of $H^1(\Omega)$, the following approximating property holds. There exists an $l \geq 1$ such that for any $v \in H^r(\Omega)$, $1 \leq r \leq l + 1$, there is a $v^h \in V_h$ with
\[ ||v - v^h||_{s,\Omega} \leq C h^{r-s} ||v||_{r,\Omega}. \tag{4.1} \]
Assume that $V_{h,0}$ and $S_h$ possess approximation properties similar to (4.1). For example, $V_h$ is the space of piecewise polynomials of degree $\leq l$ associated with the grid on $\Omega$. For the exact approximate solution $u_h$, we have [4] the following lemma.

**Lemma 1.** Assume that $f \in H^{-2}(\Omega)$, $g \in H^{-1/2}(\Gamma)$, and $\Omega$ is a bounded domain with polyhedral Lipschitz boundary. Then
\[ ||u_h - u||_{s,\Omega} \leq C h^{r-s}, \text{ for } s = 0, 1. \tag{4.2} \]

In case of convergence of the two-grid algorithm, it remains to give an estimate for the iterative error $||u_{h}^{(k)} - u||_{s,\Omega}$ or to specify the number of iterations required for this error to be of the same order as $||u_h - u||_{s,\Omega}$.

Let $\xi$ be the contraction number of the two-grid iteration with respect to the $L^2$-norm,
\[ \xi \geq |M_h(\nu)|_{0,h}, \text{ where } |M_h(\nu)|_{0,h} = \sup_{0 \neq v^h \in V_h} \frac{||M_h(\nu) v^h||_{0,\Omega}}{||v^h||_{0,\Omega}}. \]

Taking into account (3.1), the following can be derived [1]:
\[ ||u_{h}^{(k)} - u_{h}^{(0)}||_{0,\Omega} \leq \xi^k ||u_{h}^{(0)} - u_{h}||_{0,\Omega}. \tag{4.3} \]
Assume that the initial guess $u_{h}^{(0)}$ was made such that it vanishes at the interior nodes and is equal to $P_h g$ on $\Gamma$. Then, $u_{h}^{(0)} - u_h$ vanishes on $\Gamma$ and is equal to $u_h$ at the interior nodes, so
\[ ||u_{h}^{(0)} - u_{h}||_{0,\Omega} \leq ||u_h||_{0,\Omega}. \tag{4.4} \]
Assume that $\nu$ is chosen such that $\xi < 1$ (cf. [1, Theorem 7.1.2]). Then the following proposition holds.

**Proposition 2.** Under the assumptions of Lemma 1 and with the initial choice for $u^{(0)}_h$ specified above, the following inequality is true:

$$\left\| u^{(k)}_h - u \right\|_{0,\Omega} \leq C h^r + C \xi^k. \quad (4.5)$$

**Proof.** From (4.3), (4.4), and the triangle inequality

$$\left\| u_h \right\|_{0,\Omega} \leq \left\| u_h - u \right\|_{0,\Omega} + \left\| u \right\|_{0,\Omega},$$

one obtains

$$\left\| u^{(k)}_h - u_h \right\|_{0,\Omega} \leq C \xi^k h^r + C \xi^k, \quad (4.6)$$

and since $\xi < 1$,

$$\left\| u^{(k)}_h - u_h \right\|_{0,\Omega} \leq C h^r + C \xi^k. \quad (4.7)$$

Now, to obtain (4.5), combine (4.7), (4.2) for $s = 0$, and the triangle inequality

$$\left\| u^{(k)}_h - u \right\|_{0,\Omega} \leq \left\| u^{(k)}_h - u_h \right\|_{0,\Omega} + \left\| u_h - u \right\|_{0,\Omega}. \quad \Box$$

Proposition 2 shows that, in order to obtain the order $h^r$ for the error $\left\| u^{(k)}_h - u \right\|_{0,\Omega}$, a number of iterations $k$ must be performed, such that $\xi^k = O(h^r)$, which implies

$$k = O\left(\left(\frac{\log h^r}{\log \xi}\right)\right). \quad (4.8)$$

Thus, if the algorithm is applied for a given problem with the same $\nu$ and different mesh sizes ($h = h_1$ and $h = h_2$), the corresponding numbers $k_1$ and $k_2$ of two-grid iterations should satisfy

$$\frac{k_1}{k_2} \approx \frac{(\log h_1^r)/(\log \xi)}{(\log h_2^r)/(\log \xi)} = \frac{\log h_1}{\log h_2}. \quad (4.9)$$

Notice that the ratio $(k_1/k_2)$ does not depend on $\nu$ or $r$, but only on the mesh spacing $h$.

For some problems, $\xi$ has been calculated; for example, let the model problem (2.1)–(2.2) be considered on the unit square ($\Omega = (0, 1) \times (0, 1)$), with nine-point schemes used for discretization, restriction, and prolongation (see Figures 1–3). Then, for $\nu \geq 2$, the value of $\xi$ is [1]

$$\xi = \xi(\nu) = \frac{1}{2} \left(1 - \sqrt{\frac{\nu}{\nu + 1}}\right), \quad (4.10)$$

so that, if the method is applied for the same $h$ and different numbers of smoothing steps $\nu_1$ and $\nu_2$, the corresponding numbers $k_1$ and $k_2$ of two-grid iterations should satisfy

$$\frac{k_1}{k_2} \approx \frac{(\log h^r)/(\log \xi(\nu_1))}{(\log h^r)/(\log \xi(\nu_2))} = \frac{\log \xi(\nu_2)}{\log \xi(\nu_1)} = \frac{1 - \sqrt{\frac{\nu_1}{\nu_1 + 1}}}{1 - \sqrt{\frac{\nu_2}{\nu_2 + 1}}}. \quad (4.11)$$

This time, notice that the ratio $(k_1/k_2)$ does not depend on the mesh spacing $h$ or on $r$, but only on the number of smoothing steps. In Section 5, we illustrate (4.9) and (4.11) by numerical examples.
5. EXPERIMENTAL RESULTS

The computations presented are made for the same model problem (2.1)-(2.2), on $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$; a uniform criss-cross grid (Figure 1) of mesh spacing $h$ with nodes $x_j$ ($1 \leq j \leq N + M$) is associated with $\Omega$; the basis of $V_h$ consists of bilinears $\Phi_j$, which have the value 1 at the node $x_j$ and 0 at all the other nodes ($1 \leq j \leq N + M$); symmetric Gauss-Seidel iterations are used for the smoothing step; the restriction (fine-to-coarse) and the prolongation (coarse-to-fine) are made with nine-point rules (Figure 2 and 3). The exact solution is

$$u(x, y) = \rho^a \sin(a\theta),$$  \hspace{1cm} (5.1)

where $\rho^2 = x^2 + y^2$ and $\tan \theta = y/x$; the parameter $\alpha$ is controlling the smoothness of $u$. The stopping criterion was (3.3) with $s = 0$ and $\epsilon = 10^{-6} h^{a+1}$, i.e.,

$$\left\| u_h^{(k+1)} - u_h^{(k)} \right\|_{0, \Omega} < 10^{-6} h^{a+1}. \hspace{1cm} (5.2)$$

The results refer to $\alpha = 0.5$.

In Table 1, we compare the $L^2$-error $\left\| u_h^{(k)} - u \right\|_{0, \Omega}$ obtained after implementing the two-grid algorithm, with the $L^2$-errors obtained if direct solving or the symmetric Gauss-Seidel method (SGS) is used instead. For the approximate solution $u_h$, the $L^2$-errors agree with the estimate provided by (4.2). Table 1 shows that for the iterative approximate solution $u_h^{(k)}$, the error is of order $O(h^\alpha)$, as expected by imposing (4.8) for $k$. Notice that the condition (4.8) takes into account only the $L^2$-error $\left\| u_h^{(k)} - u \right\|_{0, \Omega}$, but not the $H^1$-error $\left\| u_h^{(k)} - u \right\|_{1, \Omega}$. However, Table 2
A Two-Grid Method

Table 1. Comparison of the $L^2$-errors for the exact solution (5.2) with $\alpha = 0.5$, using bilinear finite elements. The expected rate is: $2^{1.5} = 2.8284$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>Exact solve</th>
<th>(SGS) iterative method</th>
<th>Two-grid with $\nu$ (SGS) sweeps</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\nu = 2$</td>
<td></td>
</tr>
<tr>
<td>1/4</td>
<td>0.2848694(-1)</td>
<td>8693</td>
<td>8695</td>
<td>8695</td>
</tr>
<tr>
<td>1/8</td>
<td>0.1016501(-1)</td>
<td>6498</td>
<td>6501</td>
<td>6501</td>
</tr>
<tr>
<td>1/16</td>
<td>0.3603857(-2)</td>
<td>3780</td>
<td>3859</td>
<td>3859</td>
</tr>
<tr>
<td>1/6</td>
<td>0.1561018(-1)</td>
<td>1015</td>
<td>1017</td>
<td>1018</td>
</tr>
<tr>
<td>1/12</td>
<td>0.5544164(-2)</td>
<td>4101</td>
<td>4115</td>
<td>4166</td>
</tr>
</tbody>
</table>

Table 2. Comparison of the $H^1$-errors for the exact solution (5.2) with $\alpha = 0.5$, using bilinear finite elements. The expected rate is: $2^{0.5} = 1.4142$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>Exact solve</th>
<th>(SGS) iterative method</th>
<th>Two-grid with $\nu$ (SGS) sweeps</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\nu = 2$</td>
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<td>1/4</td>
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<tr>
<td>1/16</td>
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<tr>
<td>1/12</td>
<td>0.1139812</td>
<td>812</td>
<td>812</td>
<td>812</td>
</tr>
</tbody>
</table>

Table 3. The number of two-grid iterations and the rate obtained practically compared with the one expected in (4.9), when $\nu$ is fixed and $h$ is variable.

<table>
<thead>
<tr>
<th>$h$</th>
<th>Expected rate $k_h/k_{h_2}$</th>
<th>$\nu = 2$</th>
<th>$\nu = 3$</th>
<th>$\nu = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$K_h$ $K_{h_1}/K_{h_2}$</td>
<td>$K_h$</td>
<td>$K_{h_1}/K_{h_2}$</td>
<td>$K_h$ $K_{h_1}/K_{h_2}$</td>
</tr>
<tr>
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<td>6</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>0.500</td>
<td>6</td>
<td>5.545</td>
<td>5.555</td>
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<tr>
<td>1/16</td>
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<td>40</td>
<td>34</td>
<td>30</td>
</tr>
</tbody>
</table>

suggests that with the same $k$, the $H^1$-error behaves like $O(h^{-1})$, i.e., like $\|u_h - u\|_{1,\Omega}$ in (4.2), with $s = 1$. The proof of this is an open problem.

In Table 3, we illustrate the rate provided by (4.9) for different values of $\nu$ ($\nu = 2, 3, 4$); $k_h$ denotes the number of two-grid iterations expected from (4.8); $K_h$ denotes the number of two-grid iterations performed until the stopping criterion (5.2) is met, obtained practically.

In Table 4, equation (4.11) is illustrated for $\nu = 2, 3, 4$; notations similar to the ones above are used.
Table 4. The number of two-grid iterations and the rate obtained practically compared with the one expected in (4.11), when $h$ is fixed and $\nu$ is variable.

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>Expected rate $k_{\nu 1}/k_{\nu 2}$</th>
<th>$h = 1/4$</th>
<th>$h = 1/8$</th>
<th>$h = 1/16$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$K_\nu$</td>
<td>$K_{\nu 1}/K_{\nu 2}$</td>
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REFERENCES