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Three new cut sets of fuzzy sets and new theories of fuzzy sets[☆]

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ABSTRACT

Three new cut sets are introduced from the view points of neighborhood and Q-neighborhood in fuzzy topology and their properties are discussed. By the use of these cut sets, new decomposition theorems, new representation theorems, new extension principles and new fuzzy linear mappings are obtained. Then inner project of fuzzy relations, generalized extension principle and new composition rule of fuzzy relations are given. In the end, we present axiomatic descriptions for different cut sets and show the three most intrinsic properties for each cut set.

1. Introduction

Let X be a set and $\mathcal{F}(X) = \{A|A : X \to [0, 1] \text{ is a mapping}\}$. For $A \in \mathcal{F}(X)$ and $\lambda \in [0, 1]$, $A_\lambda = \{x|x \in X, A(x) \geq \lambda\}$ and $A_{\underline{\lambda}} = \{x|x \in X, A(x) > \lambda\}$ are called λ-cut set and λ-strong cut set of fuzzy set A respectively [1]. The concept of cut sets plays an important role in fuzzy topology [2], fuzzy algebra [3,4], fuzzy measure [5–7] and fuzzy reasoning [8,9]. In fuzzy topology, a fuzzy point x_λ is said to be contained in a fuzzy set A or to belong to A, denoted by $x_\lambda \in A$, iff $A(x) \geq \lambda$. From the point of neighborhood, λ-cut set A_λ of A satisfies: $A_\lambda = \{x|x \in X, x_\lambda \in A\}$. Prof. Luo has even introduced the concept of strong neighborhood [10]. According to his view point, a fuzzy point x_λ $(0 < \lambda < 1)$ is said to strongly belong to A, denoted by $x_\lambda \underline{\in} A$, iff $A(x) > \lambda$. λ-strong cut set $A_{\underline{\lambda}}$ of A satisfies: $A_{\underline{\lambda}} = \{x|x \in X, x_\lambda \underline{\in} A\}$.

It is well known that Q-neighborhood plays an important role in fuzzy topology. According to [11], a fuzzy point x_λ is said to be (strong) quasi-coincident with A, denoted by $x_\lambda \underline{\in}_q A$, iff $\lambda + A(x) > \lambda$. Thus, a fuzzy point x_λ and a fuzzy set A have the following relations:

(1) x_λ belongs to A, denoted by $x_\lambda \in A$, iff $A(x) \geq \lambda$;
x_λ strongly belongs to A, denoted by $x_\lambda \underline{\in} A$, iff $A(x) > \lambda$;
(2) x_λ is strong quasi-coincident with A, denoted by $x_\lambda \underline{\in}_q A$, iff $\lambda + A(x) > 1$;
x_λ is quasi-coincident with A, denoted by $x_\lambda \in_q A$, iff $\lambda + A(x) \geq 1$;
(3) x_λ does not strongly belong to A, denoted by $x_\lambda \underline{\overline{\in}} A$, iff $A(x) \leq \lambda$;
x_λ does not belong to A, denoted by $x_\lambda \overline{\in} A$, iff $A(x) < \lambda$;
(4) x_λ is not strong quasi-coincident with A, denoted by $x_\lambda \in^q A$, iff $\lambda + A(x) \leq 1$; x_λ is not quasi-coincident with A, denoted by $x_\lambda \underline{\in}^q A$, iff $\lambda + A(x) < 1$;

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Based on the relations as above, we can define three new kinds of cut sets of fuzzy sets A.

(i) $A_\lambda = \{x|x \in X, x_\lambda \in A\}$ and $A_{\underline{\lambda}} = \{x|x \in X, x_\lambda \underline{\in} A\}$ are called λ-upper cut set and λ-strong upper cut set of fuzzy set A respectively.

(ii) $A^\lambda = \{x|x \in X, x_\lambda \overline{\in} A\} = \{x|x \in X, A(x) \le \lambda\}$ and $A^{\underline{\lambda}} = \{x|x \in X, x_\lambda \overline{\in} A\} = \{x|x \in X, A(x) < \lambda\}$ are called λ-lower cut set and λ-strong lower cut set of fuzzy set A respectively.

(iii) $A_{[\lambda]} = \{x|x \in X, x_\lambda \in_q A\} = \{x|x \in X, \lambda + A(x) \ge 1\}$ and $A_{\underline{[\lambda]}} = \{x|x \in X, x_\lambda \underline{\in}_q A\} = \{x|x \in X, \lambda + A(x) > 1\}$ are called λ-lower Q-cut set and λ-strong lower Q-cut set of fuzzy set A respectively.

(iv) $A^{[\lambda]} = \{x|x \in X, x_\lambda \in^q A\} = \{x|x \in X, \lambda + A(x) \le 1\}$ and $A^{\underline{[\lambda]}} = \{x|x \in X, x_\lambda \underline{\in}^q A\} = \{x|x \in X, \lambda + A(x) < 1\}$ are called λ-upper Q-cut set and λ-strong upper Q-cut set of fuzzy set A respectively.

In this paper,we shall discuss some properties of these cut sets in Section 2, and give new decomposition theorems, new representation theorems and new extension principles (in Sections 3–5 respectively). In Section 6, we shall explain extension principles by using theories of category. New fuzzy linear mappings are obtained in Section 7. In Section 8, we shall give the definition of inner project of fuzzy relation, generalized extension principle and new composition rule of fuzzy relations. In Section 9, we present axiomatic descriptions for different cut sets and show the three most intrinsic properties for each cut set.

2. Properties of cut set

Let $A, A^t (t \in T), B \in \mathcal{F}(X) = \{C|C : X \to [0,1] \text{ is a mapping}\}$, $\lambda, \lambda_1, \lambda_2, \alpha_t \in I = [0,1], (t \in T)$ and $a = \bigvee_{t\in T} \alpha_t, b = \bigwedge_{t\in T} \alpha_t$. Then the following properties are clear.

Property 2.1. (1) $(A \bigcup B)_\lambda = A_\lambda \bigcup B_\lambda$, $(A \bigcup B)_{\underline{\lambda}} = A_{\underline{\lambda}} \bigcup B_{\underline{\lambda}}$, $(A \bigcap B)_\lambda = A_\lambda \bigcap B_\lambda$, $(A \bigcap B)_{\underline{\lambda}} = A_{\underline{\lambda}} \bigcap B_{\underline{\lambda}}$;
(2) $\lambda_1 < \lambda_2 \Rightarrow A_{\lambda_1} \supseteq A_{\lambda_2}, A_{\underline{\lambda_1}} \supseteq A_{\underline{\lambda_2}}, A_{\lambda_1} \supseteq A_{\underline{\lambda_1}}, A_{\underline{\lambda_1}} \supseteq A_{\lambda_2}$;
(3) $(\bigcup_{t\in T} A^t)_\lambda \supseteq \bigcup_{t\in T} A^t_\lambda$, $(\bigcup_{t\in T} A^t)_{\underline{\lambda}} = \bigcup_{t\in T} A^t_{\underline{\lambda}}$, $(\bigcap_{t\in T} A^t)_\lambda = \bigcap_{t\in T} A^t_\lambda$, $(\bigcap_{t\in T} A^t)_{\underline{\lambda}} \subseteq \bigcap_{t\in T} A^t_{\underline{\lambda}}$;
(4) $(A^c)_\lambda = (A_{\underline{1-\lambda}})^c$, $(A^c)_{\underline{\lambda}} = (A_{1-\lambda})^c$;
(5) $A_a = \bigcap_{t\in T} A_{\alpha_t}, A_b \supseteq \bigcup_{t\in T} A_{\alpha_t}, A_{\underline{a}} \subseteq \bigcap_{t\in T} A_{\underline{\alpha_t}}, A_{\underline{b}} = \bigcup_{t\in T} A_{\underline{\alpha_t}}$.

Property 2.2. (1) $(A \bigcup B)^\lambda = A^\lambda \bigcap B^\lambda$, $(A \bigcup B)^{\underline{\lambda}} = A^{\underline{\lambda}} \bigcap B^{\underline{\lambda}}$, $(A \bigcap B)^\lambda = A^\lambda \bigcup B^\lambda$, $(A \bigcap B)^{\underline{\lambda}} = A^{\underline{\lambda}} \bigcup B^{\underline{\lambda}}$;
(2) $\lambda_1 < \lambda_2 \Rightarrow A^{\lambda_1} \subseteq A^{\lambda_2}, A^{\underline{\lambda_1}} \subseteq A^{\underline{\lambda_2}}, A^{\lambda_1} \supseteq A^{\underline{\lambda_1}}, A^{\lambda_1} \subseteq A^{\underline{\lambda_2}}$;
(3) $(\bigcup_{t\in T} A_t)^\lambda = \bigcap_{t\in T} A^\lambda_t$, $(\bigcup_{t\in T} A_t)^{\underline{\lambda}} \subseteq \bigcap_{t\in T} A^{\underline{\lambda}}_t$, $(\bigcap_{t\in T} A_t)^\lambda \supseteq \bigcup_{t\in T} A^\lambda_t$, $(\bigcap_{t\in T} A_t)^{\underline{\lambda}} = \bigcup_{t\in T} A^{\underline{\lambda}}_t$;
(4) $(A^c)^\lambda = (A^{\underline{1-\lambda}})^c$, $(A^c)^{\underline{\lambda}} = (A^{1-\lambda})^c$;
(5) $A^a \supseteq \bigcup_{t\in T} A^{\alpha_t}, A^b = \bigcap_{t\in T} A^{\alpha_t}, A^{\underline{a}} = \bigcup_{t\in T} A^{\underline{\alpha_t}}, A^{\underline{b}} \subseteq \bigcap_{t\in T} A^{\underline{\alpha_t}}$.

Property 2.3. (1) $(A \bigcup B)_{[\lambda]} = A_{[\lambda]} \bigcup B_{[\lambda]}$, $(A \bigcup B)_{\underline{[\lambda]}} = A_{\underline{[\lambda]}} \bigcup B_{\underline{[\lambda]}}$, $(A \bigcap B)_{[\lambda]} = A_{[\lambda]} \bigcap B_{[\lambda]}$, $(A \bigcap B)_{\underline{[\lambda]}} = A_{\underline{[\lambda]}} \bigcap B_{\underline{[\lambda]}}$;
(2) $\lambda_1 < \lambda_2 \Rightarrow A_{[\lambda_1]} \subseteq A_{[\lambda_2]}, A_{\underline{[\lambda_1]}} \subseteq A_{\underline{[\lambda_2]}}, A_{[\lambda_1]} \subseteq A_{\underline{[\lambda_2]}}, A_{\underline{[\lambda_1]}} \subseteq A_{[\lambda_2]}$;
(3) $(\bigcup_{t\in T} A^t)_{[\lambda]} \supseteq \bigcup_{t\in T} A^t_{[\lambda]}$, $(\bigcup_{t\in T} A^t)_{\underline{[\lambda]}} = \bigcup_{t\in T} A^t_{\underline{[\lambda]}}$, $(\bigcap_{t\in T} A^t)_{[\lambda]} = \bigcap_{t\in T} A^t_{[\lambda]}$, $(\bigcap_{t\in T} A^t)_{\underline{[\lambda]}} \subseteq \bigcap_{t\in T} A^t_{\underline{[\lambda]}}$;
(4) $(A^c)_{[\lambda]} = (A_{\underline{[1-\lambda]}})^c$, $(A^c)_{\underline{[\lambda]}} = (A_{[1-\lambda]})^c$;
(5) $A_{[a]} \supseteq \bigcup_{t\in T} A_{[\alpha_t]}, A_{[b]} = \bigcap_{t\in T} A_{[\alpha_t]}, A_{\underline{[a]}} = \bigcup_{t\in T} A_{\underline{[\alpha_t]}}, A_{\underline{[b]}} \subseteq \bigcap_{t\in T} A_{\underline{[\alpha_t]}}$.

Property 2.4. (1) $(A \bigcup B)^{[\lambda]} = A^{[\lambda]} \bigcap B^{[\lambda]}$, $(A \bigcup B)^{\underline{[\lambda]}} = A^{\underline{[\lambda]}} \bigcap B^{\underline{[\lambda]}}$, $(A \bigcap B)^{[\lambda]} = A^{[\lambda]} \bigcup B^{[\lambda]}$, $(A \bigcap B)^{\underline{[\lambda]}} = A^{\underline{[\lambda]}} \bigcup B^{\underline{[\lambda]}}$;
(2) $\lambda_1 < \lambda_2 \Rightarrow A^{[\lambda_1]} \supseteq A^{[\lambda_2]}, A^{\underline{[\lambda_1]}} \supseteq A^{\underline{[\lambda_2]}}, A^{[\lambda_1]} \supseteq A^{\underline{[\lambda_1]}}, A^{\underline{[\lambda_1]}} \supseteq A^{[\lambda_2]}$;
(3) $(\bigcup_{t\in T} A_t)^{[\lambda]} = \bigcap_{t\in T} A^{[\lambda]}_t$, $(\bigcup_{t\in T} A_t)^{\underline{[\lambda]}} \subseteq \bigcap_{t\in T} A^{\underline{[\lambda]}}_t$, $(\bigcap_{t\in T} A_t)^{[\lambda]} \supseteq \bigcup_{t\in T} A^{[\lambda]}_t$, $(\bigcap_{t\in T} A_t)^{\underline{[\lambda]}} = \bigcup_{t\in T} A^{\underline{[\lambda]}}_t$,
(4) $(A^c)^{[\lambda]} = (A^{\underline{[1-\lambda]}})^c$, $(A^c)^{\underline{[\lambda]}} = (A^{[1-\lambda]})^c$;
(5) $A^{[a]} = \bigcap_{t\in T} A^{[\alpha_t]}, A^{\underline{[a]}} \subseteq \bigcap_{t\in T} A^{\underline{[\alpha_t]}}, A^{[b]} \supseteq \bigcup_{t\in T} A^{[\alpha_t]}, A^{\underline{[b]}} = \bigcup_{t\in T} A^{\underline{[\alpha_t]}}$.

3. Decomposition theorems

Let $\mathcal{P}(X)$ be power set of set X and $I = [0,1]$. For $\lambda \in I$ and $B \in \mathcal{P}(X)$, we define $\lambda B, \lambda \cdot B, \lambda \circ B, \lambda \diamond B$ as fuzzy subsets of X respectively and

$$(\lambda B)(x) = \begin{cases} \lambda, & \text{if } x \in B \\ 0, & \text{if } x \overline{\in} B \end{cases}, \qquad (\lambda \cdot B)(x) = \begin{cases} \lambda, & \text{if } x \in B \\ 1, & \text{if } x \overline{\in} B \end{cases}$$

$$(\lambda \circ B)(x) = \begin{cases} 1, & \text{if } x \in B \\ \lambda, & \text{if } x \overline{\in} B \end{cases}, \qquad (\lambda \diamond B)(x) = \begin{cases} 0, & \text{if } x \in B \\ \lambda, & \text{if } x \overline{\in} B. \end{cases}$$

Clearly, we have the following decomposition theorem:

Theorem 3.1. (1) $A = \bigcup_{\lambda\in I} \lambda A_\lambda, A = \bigcap_{\lambda \in I} \lambda \circ A_\lambda, A^c = \bigcap_{\lambda\in I} \lambda^c \cdot A_\lambda, A^c = \bigcup_{\lambda\in I} \lambda^c \diamond A_\lambda$;
(2) $A = \bigcup_{\lambda\in I} \lambda A_{\underline{\lambda}}, A = \bigcap_{\lambda \in I} \lambda \circ A_{\underline{\lambda}}, A^c = \bigcap_{\lambda\in I} \lambda^c \cdot A_{\underline{\lambda}}, A^c = \bigcup_{\lambda\in I} \lambda^c \diamond A_{\underline{\lambda}}$;

(3) *Let* $H : I \to \mathcal{P}(X)$ *satisfying* $A_{\underline{\lambda}} \subseteq H(\lambda) \subseteq A_\lambda$ *for any* $\lambda \in I$. *Then*
(i) $A = \bigcup_{\lambda\in I} \lambda H(\lambda), A = \bigcap_{\lambda\in I} \lambda \circ H(\lambda), A^c = \bigcap_{\lambda\in I} \lambda^c \cdot H(\lambda), A^c = \bigcup_{\lambda\in I} \lambda^c \diamond H(\lambda)$;
(ii) $\lambda_1 < \lambda_2 \Rightarrow H(\lambda_1) \supseteq H(\lambda_2)$;
(iii) $A_\lambda = \bigcap_{\alpha<\lambda} H(\alpha), A_{\underline{\lambda}} = \bigcup_{\alpha>\lambda} H(\alpha)$.

Theorem 3.2. (1) $A = \bigcap_{\lambda\in I} \lambda \cdot A^\lambda, A = \bigcup_{\lambda\in I} \lambda \diamond A^\lambda, A^c = \bigcup_{\lambda\in I} \lambda^c A^\lambda, A^c = \bigcap_{\lambda\in I} \lambda^c \circ A^\lambda$;
(2) $A = \bigcap_{\lambda\in I} \lambda \cdot A^{\underline{\lambda}}, A = \bigcup_{\lambda\in I} \lambda \diamond A^{\underline{\lambda}}, A^c = \bigcup_{\lambda\in I} \lambda^c A^{\underline{\lambda}}, A^c = \bigcap_{\lambda\in I} \lambda^c \circ A^{\underline{\lambda}}$;
(3) *Let* $H : I \to \mathcal{P}(X)$ *satisfying* $A^{\underline{\lambda}} \subseteq H(\lambda) \subseteq A^\lambda$ *for any* $\lambda \in I$. *Then*
(i) $A = \bigcap_{\lambda\in I} \lambda \cdot H(\lambda), A = \bigcup_{\lambda\in I} \lambda \diamond H(\lambda), A^c = \bigcup_{\lambda\in I} \lambda^c H(\lambda), A^c = \bigcap_{\lambda\in I} \lambda^c \circ H(\lambda)$;
(ii) $\lambda_1 < \lambda_2 \Rightarrow H(\lambda_1) \subseteq H(\lambda_2)$;
(iii) $A^\lambda = \bigcap_{\alpha>\lambda} H(\alpha), A^{\underline{\lambda}} = \bigcup_{\alpha<\lambda} H(\alpha)$.

Theorem 3.3. (1) $A = \bigcup_{\lambda\in I} \lambda^c A_{[\lambda]}, A = \bigcap_{\lambda\in I} \lambda^c \circ A_{[\lambda]}, A^c = \bigcap_{\lambda\in I} \lambda \cdot A_{[\lambda]}, A^c = \bigcup_{\lambda\in I} \lambda \diamond A_{[\lambda]}$;
(2) $A = \bigcup_{\lambda\in I} \lambda^c A_{\underline{[\lambda]}}, A = \bigcap_{\lambda\in I} \lambda^c \circ A_{\underline{[\lambda]}}, A^c = \bigcap_{\lambda\in I} \lambda \cdot A_{\underline{[\lambda]}}, A^c = \bigcup_{\lambda\in I} \lambda \diamond A_{\underline{[\lambda]}}$;
(3) *Let* $H : I \to \mathcal{P}(X)$ *satisfying* $A_{\underline{[\lambda]}} \subseteq H(\lambda) \subseteq A_{[\lambda]}$ *for any* $\lambda \in I$. *Then*
(i) $A = \bigcup_{\lambda\in I} \lambda^c H(\lambda), A = \bigcap_{\lambda\in I} \lambda^c \circ H(\lambda), A^c = \bigcap_{\lambda\in I} \lambda \cdot H(\lambda), A^c = \bigcup_{\lambda\in I} \lambda \diamond H(\lambda)$;
(ii) $\lambda_1 < \lambda_2 \Rightarrow H(\lambda_1) \subseteq H(\lambda_2)$;
(iii) $A_{[\lambda]} = \bigcap_{\alpha>\lambda} H(\alpha), A_{\underline{[\lambda]}} = \bigcup_{\alpha<\lambda} H(\alpha)$.

Theorem 3.4. (1) $A = \bigcap_{\lambda\in I} \lambda^c \cdot A^{[\lambda]}, A = \bigcup_{\lambda\in I} \lambda^c \diamond A^{[\lambda]}, A^c = \bigcup_{\lambda\in I} \lambda A^{[\lambda]}, A^c = \bigcap_{\lambda\in I} \lambda \circ A^{[\lambda]}$;
(2) $A = \bigcap_{\lambda\in I} \lambda^c \cdot A^{\underline{[\lambda]}}, A = \bigcup_{\lambda\in I} \lambda^c \diamond A^{\underline{[\lambda]}}, A^c = \bigcup_{\lambda\in I} \lambda A^{\underline{[\lambda]}}, A^c = \bigcap_{\lambda\in I} \lambda \circ A^{\underline{[\lambda]}}$;
(3) *Let* $H : I \to \mathcal{P}(X)$ *satisfying* $A^{\underline{[\lambda]}} \subseteq H(\lambda) \subseteq A^{[\lambda]}$ *for any* $\lambda \in I$. *Then*
(i) $A = \bigcap_{\lambda\in I} \lambda^c \cdot H(\lambda), A = \bigcup_{\lambda\in I} \lambda^c \diamond H(\lambda), A^c = \bigcup_{\lambda\in I} \lambda H(\lambda), A^c = \bigcap_{\lambda\in I} \lambda \circ H(\lambda)$;
(ii) $\lambda_1 < \lambda_2 \Rightarrow H(\lambda_1) \supseteq H(\lambda_2)$;
(iii) $A^{[\lambda]} = \bigcap_{\alpha<\lambda} H(\alpha), A^{\underline{[\lambda]}} = \bigcup_{\alpha>\lambda} H(\alpha)$.

4. Representation theorems

Definition 4.1 (*[10,12]*). Let mapping $H : I \to \mathcal{P}(X)$ satisfy: $\lambda_1 < \lambda_2 \Rightarrow H(\lambda_1) \supseteq H(\lambda_2)$. Then H is called a set embedding over X.

For example, $H(\lambda) = A_\lambda$ ($H(\lambda) = A_{\underline{\lambda}}, H(\lambda) = A^{[\lambda]}, H(\lambda) = A^{\underline{[\lambda]}}$ respectively) is a set embedding over X.

Let $\mathcal{U}(X)$ be a set of all set embedding over X. In $\mathcal{U}(X)$, we define:

$$\bigcup_{\gamma\in\Gamma} H_\gamma : \left(\bigcup_{\gamma\in\Gamma} H_\gamma\right)(\lambda) = \bigcup_{\gamma\in\Gamma} H_\gamma(\lambda),$$

$$\bigcap_{\gamma\in\Gamma} H_\gamma : \left(\bigcap_{\gamma\in\Gamma} H_\gamma\right)(\lambda) = \bigcap_{\gamma\in\Gamma} H_\gamma(\lambda),$$

$$H^c : H^c(\lambda) = (H(1-\lambda))^c.$$

Then $(\mathcal{U}(X), \bigcup, \bigcap, c)$ is a De Morgan lattice.

Let $H \in \mathcal{U}(X)$ and $T_1(H) = \bigcup_{\lambda\in I} \lambda H(\lambda), T_2(H) = \bigcap_{\lambda\in I} \lambda \circ H(\lambda), T_3(H) = \bigcap_{\lambda\in I} \lambda^c \cdot H(\lambda), T_4(H) = \bigcup_{\lambda\in I} \lambda^c \diamond H(\lambda)$.

Theorem 4.1. *From above* $T_i : \mathcal{U}(X) \to \mathcal{F}(X), H \mapsto T_i(H)$ $(i = 1, 2)$, *we have*

(1) $(T_i(H))_{\underline{\alpha}} \subseteq H(\alpha) \subseteq (T_i(H))_\alpha$ *for any* $\alpha \in I$;
(2) $T_i(\bigcup_{\gamma\in\Gamma} H_\gamma) = \bigcup_{\gamma\in\Gamma} T_i(H_\gamma), T_i(\bigcap_{\gamma\in\Gamma} H_\gamma) = \bigcap_{\gamma\in\Gamma} T_i(H_\gamma), T_i(H^c) = (T_i(H))^c$.

Proof. When $i = 1$, please see [10].

When $i = 2$, $T_2(H) = \bigcap_{\lambda\in I} \lambda \circ H(\lambda)$, then $T_2(H)(x) = \bigwedge\{\lambda | x\overline{\in} H(\lambda)\}$ for any $x \in X$. Then

$x \in H(\alpha) \Rightarrow x \in H(\lambda)$ for any $\lambda \le \alpha \Rightarrow (x\overline{\in} H(\lambda) \Rightarrow \lambda > \alpha) \Rightarrow T_2(H)(x) \ge \alpha \Rightarrow x \in (T_2(H))_\alpha$. It follows that $H(\alpha) \subseteq (T_2(H))_\alpha$.

On the other hand, $x\overline{\in} H(\alpha) \Rightarrow T_2(H)(x) = \bigwedge\{\lambda | x\overline{\in} H(\lambda)\} \le \alpha \Rightarrow x\overline{\in} (T_2(H))_{\underline{\alpha}}$. It follows that $(T_2(H))_{\underline{\alpha}} \subseteq H(\alpha)$.

Therefore $(T_2(H))_{\underline{\alpha}} \subseteq H(\alpha) \subseteq (T_2(H))_\alpha$.

(2) Please see [10]. □

Theorem 4.2. *For above* $T_i : \mathcal{U}(X) \to \mathcal{F}(X), H \mapsto T_i(H)$ $(i = 3, 4)$, *we have*

(1) $(T_i(H))^{\underline{[\alpha]}} \subseteq H(\alpha) \subseteq (T_i(H))^{[\alpha]}$ *for any* $\alpha \in I$;
(2) $T_i(\bigcup_{\gamma\in\Gamma} H_\gamma) = \bigcap_{\gamma\in\Gamma} T_i(H_\gamma), T_i(\bigcap_{\gamma\in\Gamma} H_\gamma) = \bigcup_{\gamma\in\Gamma} T_i(H_\gamma), T_i(H^c) = (T_i(H))^c$.

Proof. When $i = 3$, $T_3(H) = \bigcap_{\lambda \in I} \lambda^c \cdot H(\lambda)$, then $T_3(H)(x) = \bigwedge\{\lambda^c | x \in H(\lambda)\}$ for any $x \in X$. Then

(1) $x \in H(\alpha) \Rightarrow T_3(H)(x) \le \alpha^c = 1 - \alpha \Rightarrow \alpha + T_3(H)(x) \le 1 \Rightarrow x \in (T_3(H))^{[\alpha]}$. It follows that $H(\alpha) \subseteq (T_3(H))^{[\alpha]}$.

On the other hand, $x \overline{\in} H(\alpha) \Rightarrow x \overline{\in} H(\lambda)$ for any $\lambda \ge \alpha \Rightarrow (x \in H(\lambda) \Rightarrow \lambda < \alpha) \Rightarrow (x \in H(\lambda) \Rightarrow \lambda^c > \alpha^c) \Rightarrow T_3(H)(x) \ge \alpha^c \Rightarrow \alpha + T_3(H)(x) \ge 1 \Rightarrow x \overline{\in} (T_3(H))^{\underline{[\alpha]}}$. It follows that $(T_3(H))^{\underline{[\alpha]}} \subseteq H(\alpha)$.

Therefore $(T_3(H))^{\underline{[\alpha]}} \subseteq H(\alpha) \subseteq (T_3(H))^{[\alpha]}$.

(2) By $\left(T_3(\bigcup_{\gamma \in \Gamma} H_\gamma)\right)^{\underline{[\lambda]}} = \bigcup_{\alpha > \lambda}(\bigcup_{\gamma \in \Gamma} H_\gamma)(\alpha) = \bigcup_{\alpha > \lambda} \bigcup_{\gamma \in \Gamma} H_\gamma(\alpha) = \bigcup_{\gamma \in \Gamma} \bigcup_{\alpha > \lambda} H_\gamma(\alpha) = \bigcup_{\gamma \in \Gamma} \left(T_3(H_\gamma)\right)^{\underline{[\lambda]}} = \left(\bigcap_{\gamma \in \Gamma} T_3(H_\gamma)\right)^{\underline{[\lambda]}}$, we have $T_3(\bigcup_{\gamma \in \Gamma} H_\gamma) = \bigcap_{\gamma \in \Gamma} T_3(H_\gamma)$.

By $\left(T_3(\bigcap_{\gamma \in \Gamma} H_\gamma)\right)^{[\lambda]} = \bigcap_{\alpha < \lambda}(\bigcap_{\gamma \in \Gamma} H_\gamma)(\alpha) = \bigcap_{\alpha < \lambda} \bigcap_{\gamma \in \Gamma} H_\gamma(\alpha) = \bigcap_{\gamma \in \Gamma} \bigcap_{\alpha < \lambda} H_\gamma(\alpha) = \bigcap_{\gamma \in \Gamma} \left(T_3(H_\gamma)\right)^{[\lambda]} = \left(\bigcup_{\gamma \in \Gamma} T_3(H_\gamma)\right)^{[\lambda]}$, we have $T_3(\bigcap_{\gamma \in \Gamma} H_\gamma) = \bigcup_{\gamma \in \Gamma} T_3(H_\gamma)$.

By $(T_3(H^c))^{\underline{[\lambda]}} = \bigcup_{\alpha > \lambda} H^c(\alpha) = \bigcup_{\alpha > \lambda}(H(1-\alpha))^c = \left(\bigcap_{1-\alpha < 1-\lambda} H(1-\alpha)\right)^c = \left((T_3(H))^{[1-\lambda]}\right)^c = (T_3(H)^c)^{\underline{[\lambda]}}$, we have $T_3(H^c) = T_3(H)^c$.

The proof of $i = 4$ is similar. □

Definition 4.2. Let mapping $H : I \to \mathcal{P}(X)$ satisfy: $\lambda_1 < \lambda_2 \Rightarrow H(\lambda_1) \subseteq H(\lambda_2)$. Then H is called an order set embedding over X.

For example, $H(\lambda) = A^\lambda$ ($H(\lambda) = A^{\underline{\lambda}}$, $H(\lambda) = A_{[\lambda]}$, $H(\lambda) = A_{\underline{[\lambda]}}$ respectively) is an order set embedding over X.

Let $\mathcal{V}(X)$ be a set of all order set embedding over X. In $\mathcal{V}(X)$, we define:

$$\bigcup_{\gamma \in \Gamma} H_\gamma : \left(\bigcup_{\gamma \in \Gamma} H_\gamma\right)(\lambda) = \bigcap_{\gamma \in \Gamma} H_\gamma(\lambda),$$

$$\bigcap_{\gamma \in \Gamma} H_\gamma : \left(\bigcap_{\gamma \in \Gamma} H_\gamma\right)(\lambda) = \bigcup_{\gamma \in \Gamma} H_\gamma(\lambda),$$

$$H^c : H^c(\lambda) = (H(1-\lambda))^c.$$

Then $(\mathcal{V}(X), \bigcup, \bigcap, c)$ is a De Morgan lattice.

Let $H \in \mathcal{V}(X)$ and $T_5(H) = \bigcap_{\lambda \in I} \lambda \cdot H(\lambda)$, $T_6(H) = \bigcup_{\lambda \in I} \lambda \diamond H(\lambda)$, $T_7(H) = \bigcup_{\lambda \in I} \lambda^c H(\lambda)$, $T_8(H) = \bigcap_{\lambda \in I} \lambda^c \circ H(\lambda)$.

Theorem 4.3. *From above $T_i : \mathcal{V}(X) \to \mathcal{F}(X)$, $H \mapsto T_i(H)$ $(i = 5, 6)$, we have*

(1) $(T_i(H))^{\underline{\alpha}} \subseteq H(\alpha) \subseteq (T_i(H))^\alpha$ *for any* $\alpha \in I$;
(2) $T_i(\bigcup_{\gamma \in \Gamma} H_\gamma) = \bigcup_{\gamma \in \Gamma} T_i(H_\gamma)$, $T_i(\bigcap_{\gamma \in \Gamma} H_\gamma) = \bigcap_{\gamma \in \Gamma} T_i(H_\gamma)$, $T_i(H^c) = (T_i(H))^c$.

Proof. When $i = 5$, $T_5(H) = \bigcap_{\lambda \in I} \lambda \cdot H(\lambda)$, then $T_5(H)(x) = \bigwedge\{\lambda | x \in H(\lambda)\}$ for any $x \in X$. Then

(1) $x \overline{\in} H(\alpha) \Rightarrow x \overline{\in} H(\lambda)$ for any $\lambda \le \alpha \Rightarrow (x \in H(\lambda) \Rightarrow \lambda > \alpha) \Rightarrow T_5(H)(x) \ge \alpha \Rightarrow x \overline{\in} (T_5(H))^{\underline{\alpha}}$. It follows that $H(\alpha) \supseteq (T_5(H))^{\underline{\alpha}}$.

On the other hand, $x \in H(\alpha) \Rightarrow T_5(H)(x) = \bigwedge\{\lambda | x \in H(\lambda)\} \le \alpha \Rightarrow x \in (T_5(H))^\alpha$. It follows that $(T_5(H))^\alpha \supseteq H(\alpha)$.

Therefore $(T_5(H))^{\underline{\alpha}} \subseteq H(\alpha) \subseteq (T_5(H))^\alpha$.

(2) By $\left(T_5(\bigcup_{\gamma \in \Gamma} H_\gamma)\right)^\lambda = \bigcap_{\alpha > \lambda}(\bigcup_{\gamma \in \Gamma} H_\gamma)(\alpha) = \bigcap_{\alpha > \lambda} \bigcap_{\gamma \in \Gamma} H_\gamma(\alpha) = \bigcap_{\gamma \in \Gamma} \bigcap_{\alpha > \lambda} H_\gamma(\alpha) = \bigcap_{\gamma \in \Gamma} \left(T_5(H_\gamma)\right)^\lambda = \left(\bigcup_{\gamma \in \Gamma} T_5(H_\gamma)\right)^\lambda$, we have $T_5(\bigcup_{\gamma \in \Gamma} H_\gamma) = \bigcup_{\gamma \in \Gamma} T_5(H_\gamma)$.

By $\left(T_5(\bigcap_{\gamma \in \Gamma} H_\gamma)\right)^{\underline{\lambda}} = \bigcup_{\alpha < \lambda}(\bigcap_{\gamma \in \Gamma} H_\gamma)(\alpha) = \bigcup_{\alpha < \lambda} \bigcup_{\gamma \in \Gamma} H_\gamma(\alpha) = \bigcup_{\gamma \in \Gamma} \bigcup_{\alpha < \lambda} H_\gamma(\alpha) = \bigcup_{\gamma \in \Gamma} \left(T_5(H_\gamma)\right)^{\underline{\lambda}} = \left(\bigcap_{\gamma \in \Gamma} T_5(H_\gamma)\right)^{\underline{\lambda}}$, we have $T_5(\bigcap_{\gamma \in \Gamma} H_\gamma) = \bigcap_{\gamma \in \Gamma} T_5(H_\gamma)$.

By $(T_5(H^c))^{\underline{\lambda}} = \bigcup_{\alpha < \lambda} H^c(\alpha) = \bigcup_{\alpha < \lambda} (H(1-\alpha))^c = \left(\bigcap_{1-\alpha > 1-\lambda} H(1-\alpha)\right)^c = \left((T_5(H))^{1-\lambda}\right)^c = (T_5(H)^c)^{\underline{\lambda}}$, we have $T_5(H^c) = T_5(H)^c$.

The proof of $i = 6$ is similar. □

Theorem 4.4. *Let $T_i : \mathcal{V}(X) \to \mathcal{F}(X)$, $H \mapsto T_i(H)$ $(i = 7, 8)$, we have*

(1) $(T_i(H))_{\underline{[\alpha]}} \subseteq H(\alpha) \subseteq (T_i(H))_{[\alpha]}$ *for any* $\alpha \in I$;
(2) $T_i(\bigcup_{\gamma \in \Gamma} H_\gamma) = \bigcap_{\gamma \in \Gamma} T_i(H_\gamma)$, $T_i(\bigcap_{\gamma \in \Gamma} H_\gamma) = \bigcup_{\gamma \in \Gamma} T_i(H_\gamma)$, $T_i(H^c) = (T_i(H))^c$.

Proof. When $i = 7$, $T_7(H) = \bigcup_{\lambda\in I} \lambda^c H(\lambda)$, then $T_7(H)(x) = \bigvee\{\lambda^c | x \in H(\lambda)\}$ for any $x \in X$. Then

(1) $x \in H(\alpha) \Rightarrow T_7(H)(x) \geq \alpha^c = 1 - \alpha \Rightarrow \alpha + T_7(H)(x) \geq 1 \Rightarrow x \in (T_7(H))_{[\alpha]}$. It follows that $H(\alpha) \subseteq (T_7(H))_{[\alpha]}$.

On the other hand, $x \overline{\in} H(\alpha) \Rightarrow x \overline{\in} H(\lambda)$ for any $\lambda \leq \alpha \Rightarrow (x \in H(\lambda) \Rightarrow \lambda > \alpha) \Rightarrow (x \in H(\lambda) \Rightarrow \lambda^c < \alpha^c) \Rightarrow T_7(H)(x) \leq \alpha^c \Rightarrow \alpha + T_7(H)(x) \leq 1 \Rightarrow x \overline{\in} (T_7(H))_{\underline{[\alpha]}}$. It follows that $(T_7(H))_{\underline{[\alpha]}} \subseteq H(\alpha)$.

Therefore $(T_7(H))_{\underline{[\alpha]}} \subseteq H(\alpha) \subseteq (T_7(H))_{[\alpha]}$.

(2) By $\left(T_7(\bigcup_{\gamma\in\Gamma} H_\gamma)\right)_{[\lambda]} = \bigcap_{\alpha>\lambda}(\bigcup_{\gamma\in\Gamma} H_\gamma)(\alpha) = \bigcap_{\alpha>\lambda}\bigcap_{\gamma\in\Gamma} H_\gamma(\alpha) = \bigcap_{\gamma\in\Gamma}\bigcap_{\alpha>\lambda} H_\gamma(\alpha) = \bigcap_{\gamma\in\Gamma}\left(T_7(H_\gamma)\right)_{[\lambda]} = \left(\bigcap_{\gamma\in\Gamma} T_7(H_\gamma)\right)_{[\lambda]}$, we have $T_7(\bigcup_{\gamma\in\Gamma} H_\gamma) = \bigcap_{\gamma\in\Gamma} T_7(H_\gamma)$.

By $\left(T_7(\bigcap_{\gamma\in\Gamma} H_\gamma)\right)_{\underline{[\lambda]}} = \bigcup_{\alpha<\lambda}(\bigcap_{\gamma\in\Gamma} H_\gamma)(\alpha) = \bigcup_{\alpha<\lambda}\bigcup_{\gamma\in\Gamma} H_\gamma(\alpha) = \bigcup_{\gamma\in\Gamma}\bigcup_{\alpha<\lambda} H_\gamma(\alpha) = \bigcup_{\gamma\in\Gamma}\left(T_7(H_\gamma)\right)_{\underline{[\lambda]}} = \left(\bigcup_{\gamma\in\Gamma} T_7(H_\gamma)\right)_{\underline{[\lambda]}}$, we have $T_7(\bigcap_{\gamma\in\Gamma} H_\gamma) = \bigcup_{\gamma\in\Gamma} T_7(H_\gamma)$.

By $(T_7(H^c))_{\underline{[\lambda]}} = \bigcup_{\alpha<\lambda} H^c(\alpha) = \bigcup_{\alpha<\lambda}(H(1-\alpha))^c = \left(\bigcap_{1-\alpha>1-\lambda} H(1-\alpha)\right)^c = \left((T_7(H))_{[1-\lambda]}\right)^c = (T_7(H)^c)_{\underline{[\lambda]}}$, we have $T_7(H^c) = \overline{T_7}(H)^c$.

The proof of $i = 8$ is similar. □

5. Extension principles

5.1. Extension principles of a single variable

Let $f : X \to Y$ be a mapping. Then f can be extended as four mappings from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$:

$$f_i : \mathcal{P}(X) \to \mathcal{P}(Y), A \mapsto f_i(A) \quad (i = 1, 2, 3, 4)$$

where

$$f_1(A) = \{f(a) | a \in A\} \triangleq f(A) \tag{1}$$

$$f_2(A) = \{f(a) | a \overline{\in} A\} \triangleq f(A^c) \tag{2}$$

$$f_3(A) = (f(A))^c \tag{3}$$

$$f_4(A) = (f(A^c))^c. \tag{4}$$

Theorem 5.1 ([10]). *Let $f : X \to Y$ be a mapping. If we put*

$$f_1 : \mathcal{F}(X) \to \mathcal{F}(Y), \qquad A \mapsto f_1(A) \triangleq \bigcup_{\lambda\in I} \lambda f(A_\lambda)$$

then $f_1(A)(y) = \bigvee_{f(x)=y} A(x)$ for any $y \in Y$.

$f_1(A)$ is denoted as $f(A)$, which is the extension principle of Zadeh [13] and is the extension of (1).

Clearly, we have the following extension principles:

Theorem 5.2. *Let $f : X \to Y$ be a mapping. If we write*

$$f_2 : \mathcal{F}(X) \to \mathcal{F}(Y), \qquad A \mapsto f_2(A) \triangleq \bigcup_{\lambda\in I} \lambda f(A^{[\lambda]})$$

then $f_2(A)(y) = \bigvee_{f(x)=y} A^c(x)$ for any $y \in Y$.

(Denoted by $f_2(A) = f_c(A)$, which is the extension of (2)).

Theorem 5.3. *Let $f : X \to Y$ be a mapping. If we write*

$$f_3 : \mathcal{F}(X) \to \mathcal{F}(Y), \qquad A \mapsto f_3(A) \triangleq \bigcup_{\lambda\in I} \lambda \cdot f(A_{[\lambda]})$$

then $f_3(A)(y) = \bigwedge_{f(x)=y} A^c(x)$ for any $y \in Y$.

(Denoted by $f_3(A) = f^c(A)$, which is the extension of (3)).

Theorem 5.4. *Let $f : X \to Y$ be a mapping. If we write*

$$f_4 : \mathcal{F}(X) \to \mathcal{F}(Y), \qquad A \mapsto f_4(A) \triangleq \bigcup_{\lambda\in I} \lambda \cdot f(A^{\lambda})$$

then $f_4(A)(y) = \bigwedge_{f(x)=y} A(x)$ for any $y \in Y$.

(Denoted by $f_4(A) = f_c^c(A)$, which is the extension of (4)).

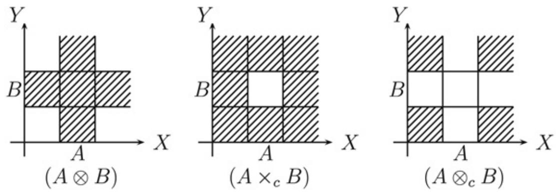

Fig. 1. Products of fuzzy sets.

5.2. Products of fuzzy sets

Definition 5.1. Let $A \in \mathcal{F}(X), B \in \mathcal{F}(Y)$.

(1) If $(A \times B)(x, y) = A(x) \wedge B(y), \forall x \in X, \forall y \in Y$, then $A \times B$ is called inner product A and B.

(2) If $(A \otimes B)(x, y) = A(x) \vee B(y), \forall x \in X, \forall y \in Y$, then $A \otimes B$ is called outer product of A and B.

(3) If $(A \times_c B)(x, y) = A^c(x) \vee B^c(y), \forall x \in X, \forall y \in Y$, then $A \times_c B$ is called inner complementary product of A and B.

(4) If $(A \otimes_c B)(x, y) = A^c(x) \wedge B^c(y), \forall x \in X, \forall y \in Y$, then $A \otimes_c B$ is called outer complementary product of A and B.

Remark 5.1. (1) $A \times B$, $A \otimes B$, $A \times_c B$ and $A \otimes_c B$ are fuzzy relations from X to Y. (2) When A and B are classical sets, $A \otimes B$, $A \times_c B$ and $A \otimes_c B$ can be explained by Fig. 1.

It is easy to prove the following result:

Theorem 5.5. (1) $A \times B = \bigcup_{\lambda \in I} \lambda(A_\lambda \times B_\lambda)$; (2) $A \otimes B = \bigcap_{\lambda \in I} \lambda \cdot (A^\lambda \times B^\lambda)$; (3) $A \times_c B = \bigcap_{\lambda \in I} \lambda \cdot (A_{[\lambda]} \times B_{[\lambda]})$; (4) $A \otimes_c B = \bigcup_{\lambda \in I} \lambda(A^{[\lambda]} \times B^{[\lambda]})$.

5.3. Extension principles of n-variables

Let $X = X_1 \times X_2 \times \cdots \times X_n$, $Y = Y_1 \times Y_2 \times \cdots \times Y_m$, and $f : X \to Y$ be a mapping, then we have

Theorem 5.6. *Let $f_i : \mathcal{F}(X_1) \times \mathcal{F}(X_2) \times \cdots \times \mathcal{F}(X_n) \to \mathcal{F}(Y)$, where for $A_i \in \mathcal{F}(X_i)(1 \leqslant i \leqslant n)$,*

$f_1(A_1, \ldots, A_n) = \bigcup_{\lambda \in I} \lambda f\left((A_1)_\lambda \times \cdots \times (A_n)_\lambda\right)$,

$f_2(A_1, \ldots, A_n) = \bigcup_{\lambda \in I} \lambda f\left(A_1^{[\lambda]} \times \cdots \times A_n^{[\lambda]}\right)$,

$f_3(A_1, \ldots, A_n) = \bigcap_{\lambda \in I} \lambda \cdot f\left(A_1^{\lambda} \times \cdots \times A_n^{\lambda}\right)$,

$f_4(A_1, \ldots, A_n) = \bigcap_{\lambda \in I} \lambda \cdot f\left((A_1)_{[\lambda]} \times \cdots \times (A_n)_{[\lambda]}\right)$,

$f_5(A_1, \ldots, A_n) = \bigcup_{\lambda \in I} \lambda f\left((A_1)_\lambda \otimes \cdots \otimes (A_n)_\lambda\right)$,

$f_6(A_1, \ldots, A_n) = \bigcup_{\lambda \in I} \lambda f\left(A_1^{[\lambda]} \otimes \cdots \otimes A_n^{[\lambda]}\right)$,

$f_7(A_1, \ldots, A_n) = \bigcap_{\lambda \in I} \lambda \cdot f\left(A_1^{\lambda} \otimes \cdots \otimes A_n^{\lambda}\right)$,

$f_8(A_1, \ldots, A_n) = \bigcap_{\lambda \in I} \lambda \cdot f\left((A_1)_{[\lambda]} \otimes \cdots \otimes (A_n)_{[\lambda]}\right)$.

Then

$f_1(A_1, \ldots, A_n)(y) = \bigvee_{f(x_1,\ldots,x_n)=y} \bigwedge_{i=1}^n A_i(x_i)$,

$f_2(A_1, \ldots, A_n)(y) = \bigvee_{f(x_1,\ldots,x_n)=y} \bigwedge_{i=1}^n A_i^c(x_i)$,

$f_3(A_1, \ldots, A_n)(y) = \bigwedge_{f(x_1,\ldots,x_n)=y} \bigvee_{i=1}^n A_i(x_i)$,

$f_4(A_1, \ldots, A_n)(y) = \bigwedge_{f(x_1,\ldots,x_n)=y} \bigvee_{i=1}^n A_i^c(x_i)$,

$f_5(A_1, \ldots, A_n)(y) = \bigvee_{f(x_1,\ldots,x_n)=y} \bigvee_{i=1}^n A_i(x_i)$,

$f_6(A_1, \ldots, A_n)(y) = \bigvee_{f(x_1,\ldots,x_n)=y} \bigvee_{i=1}^n A_i^c(x_i)$,

$f_7(A_1, \ldots, A_n)(y) = \bigwedge_{f(x_1,\ldots,x_n)=y} \bigwedge_{i=1}^n A_i(x_i)$,

$f_8(A_1, \ldots, A_n)(y) = \bigwedge_{f(x_1,\ldots,x_n)=y} \bigwedge_{i=1}^n A_i^c(x_i)$.

Theorem 5.6 is called the extension principles of n-variable.

6. Explanation of extension principles based on category theory

In [14], Wang explained extension principle of Zadeh by using the category theory. In this section, we will explain other extension principles by using the category theory.

Let $\mathbf{FSet}_i$ $(i = 1, 2, 3, 4)$ be a category, where its object is $(X, \mathcal{F}(X))$, and X is a set, i.e. $|\mathbf{FSet}_i| = \{(X, \mathcal{F}(X)) \mid X \text{ is a set}\}$; an i-morphism from object $(X, \mathcal{F}(X))$ to object $(Y, \mathcal{F}(Y))$ is a mapping $f_i : \mathcal{F}(X) \to \mathcal{F}(Y)$ and satisfies condition: there exists a mapping $f : X \to Y$ such that $f_1(A) = f(A), f_2(A) = f_c(A), f_3(A) = f^c(A)$ and $f_4(A) = f_c^c(A)$, i.e. $\mathbf{Mor}_i\left((X, \mathcal{F}(X)), (Y, \mathcal{F}(Y))\right) = \{f_i \mid f_i \text{ is an } i\text{-morphism from } (X, \mathcal{F}(X)) \text{ to } (Y, \mathcal{F}(Y))\}$.

i-**Identity**: identity from $(X, \mathcal{F}(X))$ to $(X, \mathcal{F}(X))$ is a mapping $\mathrm{Id}_i : \mathcal{F}(X) \to \mathcal{F}(X)$ satisfying that $\mathrm{Id}_1(A) = \mathrm{Id}_4(A) = A$ and $\mathrm{Id}_2(A) = \mathrm{Id}_3(A) = A^c$.

i-**Composition**: For $i = 1$, $(g_1 \vartriangle f_1)(A) \triangleq g(f(A)) = (g \circ f)(A)$, where $g \circ f$ is a composition of mappings f and g; for $i = 2$, $(g_2 \vartriangle f_2)(A) = (g \circ f)_c(A)$, for $i = 3$, $(g_3 \vartriangle f_3)(A) = (g \circ f)^c(A)$, for $i = 4$, $(g_4 \vartriangle f_4)(A) = (g \circ f)^c_c(A)$. Clearly, the compositions defined as above satisfies associativity. Thus **FSet** $_i$ is a category ($i = 1, 2, 3, 4$).

Theorem 6.1. *If **Set** is a category of classical sets, then category **FSet** $_i$ is an isomorphism with category **Set** ($i = 1, 2, 3, 4$).*

Proof. Let $i \in \{1, 2, 3, 4\}$ and

$$F_i : \mathbf{Set} \to \mathbf{FSet}_i, X \to (X, \mathcal{F}(X)), f \to F_i(f) = f_i.$$

Then F_i is an isomorphism functor ($i = 1, 2, 3, 4$). Therefore category $\mathbf{FSet}_i$ is an isomorphism with category **Set** ($i = 1, 2, 3, 4$). □

7. Fuzzy linear mappings

Definition 7.1. Let $T : \mathcal{F}(X) \to \mathcal{F}(Y)$, $\lambda_r \in I$ and $A_r \in \mathcal{F}(X) (r \in \Gamma)$.

(1) If $T(\bigcup_{r\in\Gamma} \lambda_r A_r) = \bigcup_{r\in\Gamma} \lambda_r T(A_r)$, then T is called a $(\bigvee, \bigwedge)$-fuzzy linear mapping.
(2) If $T(\bigcap_{r\in\Gamma} \lambda_r \circ A_r) = \bigcap_{r\in\Gamma} \lambda_r \circ T(A_r)$, then T is called a $(\bigwedge, \bigvee)$-fuzzy linear mapping.
(3) If $T(\bigcup_{r\in\Gamma} \lambda_r A_r) = \bigcap_{r\in\Gamma} \lambda_r^c \circ T(A_r)$, then T is called a $(\bigwedge, \bigvee^c)$-fuzzy linear mapping.
(4) If $T(\bigcap_{r\in\Gamma} \lambda_r \circ A_r) = \bigcup_{r\in\Gamma} \lambda_r^c T(A_r)$, then T is called a $(\bigvee, \bigwedge^c)$-fuzzy linear mapping.

Theorem 7.1. *Let $T : \mathcal{F}(X) \to \mathcal{F}(Y)$ be a $(\bigwedge, \bigvee)$-fuzzy linear mapping, then there is only a fuzzy relation R_T such that*

$$T(A)(y) \triangleq (A \odot R)(y) = \bigwedge_{x\in X} \left(A(x) \bigvee R_T(x, y)\right). \tag{5}$$

Conversely, for any fuzzy relation $R \in \mathcal{F}(X \times Y)$, there is only a $(\bigwedge, \bigvee)$-fuzzy linear mapping $T_R : \mathcal{F}(X) \to \mathcal{F}(Y)$ such that $T_R(A) = A \odot R$.

Proof. Let T be a $(\bigwedge, \bigvee)$-fuzzy linear mapping and let

$$f_T : X \to \mathcal{F}(Y), x \mapsto f_T(x) \triangleq T\left(\{x\}^c\right).$$

If we put $\lambda_x = A(x)$, $\forall x \in X$, then

$$A = \bigcap_{x\in X} \lambda_x \circ \{x\}^c.$$

Let $R_T(x, y) = f_T(x)(y)$, $\forall x \in X$, $\forall y \in Y$, then

$$T(A) = T\left(\bigcap_{x\in X} \lambda_x \circ \{x\}^c\right) = \bigcap_{x\in X} \lambda_x \circ f_T(x).$$

It follows that

$$T(A)(y) = \bigwedge_{x\in X} \left(A(x) \bigvee f_T(x)(y)\right) = \bigwedge_{x\in X} \left(A(x) \bigvee R_T(x, y)\right) = (A \odot R_T)(y).$$

Let R be another fuzzy relation satisfying (5), then

$$R_T(x', y') = T\left(\{x'\}^c\right)(y') = \bigwedge_{x\in X} \left(\{x'\}^c(x) \bigvee R(x, y')\right) = R(x', y'), \quad \forall (x', y') \in X \times Y.$$

So $R = R_T$, i.e. R_T satisfying (5) is unique.
Conversely, let $R \in \mathcal{F}(X \times Y)$ and

$$T_R : \mathcal{F}(X) \to \mathcal{F}(Y), \qquad A \mapsto T_R(A) = A \odot R.$$

Then

$$\begin{aligned} T_R\left(\bigcap_{r\in\Gamma} \lambda_r \circ A_r\right)(y) &= \bigwedge_{x\in X} \left(\left(\bigcap_{r\in\Gamma} \lambda_r \circ A_r\right)(x) \bigvee R(x, y)\right) \\ &= \bigwedge_{x\in X} \left(\bigwedge_{r\in\Gamma} \left(\lambda_r \bigvee A_r(x) \bigvee R(x, y)\right)\right) \end{aligned}$$

$$= \bigwedge_{r\in\Gamma}\bigwedge_{x\in X}\left(\lambda_r \bigvee A_r(x) \bigvee R(x,y)\right)$$

$$= \bigwedge_{r\in\Gamma}\left(\lambda_r \bigvee \left(\bigwedge_{x\in X}(A_r(x) \bigvee R(x,y))\right)\right)$$

$$= \bigcap_{r\in\Gamma}\lambda_r \circ T_R(A_r)(y).$$

Hence $T_R(\bigcap_{r\in\Gamma}\lambda_r \circ A_r) = \bigcap_{r\in\Gamma}\lambda_r \circ T_R(A_r)$, i.e., T_R is a $(\bigwedge, \bigvee)$-fuzzy linear mapping. □

Similarly, we have

Theorem 7.2. *If T is a $(\bigwedge, \bigvee^c)$-fuzzy mapping, then there is only a fuzzy relation R_T such that $T(A)(y) \triangleq (A^c \odot R_T)(y) = \bigwedge_{x\in X}\left(A^c(x) \bigvee R_T(x,y)\right)$.*

Conversely, for any fuzzy relation $R \in \mathcal{F}(X\times Y)$, there is only a $(\bigwedge, \bigvee^c)$-fuzzy linear mapping $T_R : \mathcal{F}(X) \to \mathcal{F}(Y)$ such that $T_R(A) = A^c \odot R$.

Theorem 7.3. *If T is a $(\bigvee, \bigwedge^c)$-fuzzy linear mapping, then there is only a fuzzy relation R_T such that $T(A)(y) \triangleq (A^c \circ R)(y) = \bigvee_{x\in X}\left(A^c(x) \bigwedge R_T(x,y)\right)$.*

Conversely, for any fuzzy relation $R \in \mathcal{F}(X\times Y)$, there is only a $(\bigvee, \bigwedge^c)$-fuzzy linear mapping $T_R : \mathcal{F}(X) \to \mathcal{F}(Y)$ such that $T_R(A) = A^c \circ R$.

Remark 7.1. Let $f : X \to Y$ be a mapping, and $T_i : \mathcal{F}(X) \to \mathcal{F}(Y)$ satisfy $T_1(A) = f(A)$, $T_2(A) = f_c^c(A)$, $T_3(A) = f^c(A)$ and $T_4(A) = f_c(A)$. Then T_1, T_2, T_3 and T_4 are $(\bigvee, \bigwedge)$-, $(\bigwedge, \bigvee)-$, $(\bigvee, \bigwedge^c)-$, $(\bigwedge, \bigvee^c)$-fuzzy linear respectively. Therefore Definition 7.1 is a generalization of Extension Principles.

8. Generalized extension principle

8.1. Inner project of fuzzy relation

Definition 8.1. Let $R \in \mathcal{F}(X\times Y)$ and $R^X \in \mathcal{F}(X)$ satisfying

$$(R^X)(x) = \bigwedge_{y\in Y} R(x,y).$$

Then R^X is called a inner project of R over X. If $R_X \in \mathcal{F}(X)$ satisfies

$$(R_X)(x) = \bigvee_{y\in Y} R(x,y)$$

then R_X is called a outer project of R over X.

In the same way, we can define R^Y and R_Y.

By the use of concepts of inner project and outer project, we can explain extension principle of a single variable as following:

Theorem 8.1. *If $f : X \to Y$ is a mapping, then*

(1) $f(A) = \left((A\times Y)\cap R_f\right)_Y$,
(2) $f_c(A) = \left((A^c\times Y)\cap R_f\right)_Y$,
(3) $f^c(A) = \left((A^c\times Y)\cup R_f^c\right)^Y$,
(4) $f_c^c(A) = \left((A\times Y)\cup R_f^c\right)^Y$,

where $A \in \mathcal{F}(X)$, $R_f = \{(x, f(x)) \mid x\in X\}$, R_f^c is complementary set of R_f in $X\times Y$.

8.2. Generalized extension principle

Definition 8.2. Let $f : X \to \mathcal{F}(Y)$ be a fuzzy mapping. If we put

$$f_\lambda(x) = (f(x))_\lambda,\ T_{f_\lambda}(A_\lambda) = \left((A_\lambda\times Y)\cup R_{f_\lambda}\right)^Y, \quad \forall\lambda\in I,$$

where $R_{f_\lambda}(x, y) = f_\lambda(x)(y)$ and

$$T_f : \mathcal{F}(X) \to \mathcal{F}(Y), \qquad T_f(A) = \bigcap_{\lambda \in I} \lambda \circ T_{f_\lambda}(A_\lambda),$$

then T_f is called a fuzzy inner mapping from X to Y induced by f.

Theorem 8.2. *If $f : X \to \mathcal{F}(Y)$ is a fuzzy mapping, $R_f(x, y) = f(x)(y)$ and T_f is a fuzzy inner mapping from X to Y induced by f, then $T_f(A) = \left((A \times Y) \cup R_f\right)^Y$, $\forall A \in \mathcal{F}(X)$.*

Proof. $T_f(A)(y) = \bigwedge_{\lambda \in I} \left(\lambda \vee T_{f_\lambda}(A_\lambda)(y)\right) = \bigwedge_{\lambda \in I} \left(\lambda \vee \left(\bigwedge_{x \in X} (A_\lambda(x) \vee f_\lambda(x)(y))\right)\right) = \bigwedge_{\lambda \in I} \bigwedge_{x \in X} \left(\lambda \bigvee A_\lambda(x) \bigvee f_\lambda(x)(y)\right) = \bigwedge_{x \in X} \bigwedge_{\lambda \in I} \left(\lambda \bigvee A_\lambda(x) \bigvee f_\lambda(x)(y)\right)$.

We only need to prove that

$$\bigwedge_{\lambda \in I} (\lambda \vee A_\lambda(x) \vee f_\lambda(x)(y)) = \left(\bigwedge_{\lambda \in I} (\lambda \vee A_\lambda(x))\right) \bigvee \left(\bigwedge_{\lambda \in I} (\lambda \vee f_\lambda(x)(y))\right). \tag{6}$$

Assume that there is a $\alpha \in I$ such that

$$\bigwedge_{\lambda \in I} (\lambda \vee A_\lambda(x) \vee f_\lambda(x)(y)) > \alpha > \left(\bigwedge_{\lambda \in I} (\lambda \vee A_\lambda(x))\right) \bigvee \left(\bigwedge_{\lambda \in I} (\lambda \vee f_\lambda(x)(y))\right).$$

Then $\bigwedge_{\lambda \in I} (\lambda \vee A_\lambda(x)) < \alpha$ and $\bigwedge_{\lambda \in I} (\lambda \vee f_\lambda(x)(y)) < \alpha$, so there exists $\lambda_1, \lambda_2 \in I$ such that $\lambda_1 \vee A_{\lambda_1}(x) < \alpha$ and $\lambda_2 \vee A_{\lambda_2}(y) < \alpha$.

We let $\lambda_1 \leqslant \lambda_2$, then it follows that $\lambda_2 \vee A_{\lambda_2}(x) \vee f_{\lambda_2}(x)(y) < \alpha$ from $A_{\lambda_1} \supseteq A_{\lambda_2}$, and consequently $\bigwedge_{\lambda \in I} (\lambda \vee A_\lambda(x) \vee f_\lambda(x)(y)) \leqslant \alpha$.

This is a contradiction. Therefore expression (6) holds. Thus

$$\begin{aligned} T_f(A)(y) &= \bigwedge_{x \in X} \left(\left(\bigwedge_{\lambda \in I} (\lambda \vee A_\lambda(x))\right) \vee \left(\bigwedge_{\lambda \in I} (\lambda \vee f_\lambda(x)(y))\right)\right) \\ &= \bigwedge_{x \in I} (A(x) \vee f(x)(y)) = \bigwedge_{x \in X} \left(A(x) \vee R_f(x, y)\right) = \left((A \times Y) \cup R_f\right)^Y. \quad \square \end{aligned}$$

Definition 8.3. Let $S \in \mathcal{F}(X \times Y)$, $R \in \mathcal{F}(Y \times X)$. Let

$$(S \odot R)(x, z) = \bigwedge_{y \in Y} (S(x, y) \vee R(y, z)), \quad \forall (x, z) \in X \times Z.$$

$S \odot R$ is called the inner composition of S and R.

By the use of generalized extension principle, we can explain inner composition of fuzzy relation S and R as follows.

Theorem 8.3. *Let $f : Y \to \mathcal{F}(Z)$ be a fuzzy mapping. If*

$$T_f : \mathcal{F}(X \times Y) \to \mathcal{F}(X \times Z)$$
$$S \mapsto T_f(S) = \left((S \times Z) \cup (X \times R_f)\right)^{X \times Z},$$

then $T_f(S) = S \odot R_f$.

The proof of Theorem 8.3 is obvious.

By the use of Theorem 8.3, inner composition of fuzzy relation $S \in \mathcal{F}(X \times Y)$ and $R \in \mathcal{F}(Y \times Z)$ can be divided into the following process.

(1) Extension: to make fuzzy sets of $X \times Y \times Z : S \times Z$ and $X \times R$.
(2) Union: to make fuzzy sets of $X \times Y \times Z : (S \times Z) \cup (X \times R)$.
(3) Inner project: $S \odot R = ((S \times Z) \cup (X \times R))^{X \times Z} \in \mathcal{F}(X \times Z)$.

9. The axiomatic descriptions for different cut sets

From Properties 2.1–2.4, we have known that each cut set of fuzzy sets has the similar properties. In this section, we shall give a general definition of cut set of a fuzzy set. We shall present the axiomatic descriptions for different cut sets and show the three most intrinsic properties for each cut set.

Definition 9.1. Let $f : [0,1] \times \mathcal{F}(X) \to \mathcal{P}(X)$ be a mapping. Then $f(\lambda, A)$ is called a f-cut set of fuzzy set A.

Theorem 9.1. *If the mapping $f : [0,1] \times \mathcal{F}(X) \to \mathcal{P}(X)$ satisfies the following conditions:*

(1) $f(\lambda, \bigcup_{t\in T} A_t) = \bigcup_{t\in T} f(\lambda, A_t)$;
(2) *When $A \in \mathcal{P}(X)$ and $\lambda < 1$, we have $f(\lambda, A) = A$;*
(3) $f(\lambda, \alpha A) = \begin{cases} \emptyset, & \alpha \leqslant \lambda \\ f(\lambda, A), & \alpha > \lambda \end{cases}$, $\forall A \in \mathcal{F}(X), \alpha, \lambda \in [0,1]$.

Then $f(\lambda, A) = A_{\underline{\lambda}}$ for any $\lambda \in I$ and $A \in \mathcal{F}(X)$.

Proof. When $\lambda < 1, f(\lambda, A) = f(\lambda, \bigcup_{\alpha\in I} \alpha A_{\underline{\alpha}}) = \bigcup_{\alpha\in I} f(\lambda, \alpha A_{\underline{\alpha}}) = \bigcup_{\alpha>\lambda} f(\lambda, A_{\underline{\alpha}}) = \bigcup_{\alpha>\lambda} A_{\underline{\alpha}} = A_{\underline{\lambda}}$; When $\lambda = 1$, we let $\alpha = 1$, then $f(\lambda, A) = \emptyset = A_{\underline{1}}$.

Therefore, $f(\lambda, A) = A_{\underline{\lambda}}, \forall \lambda \in I, A \in \mathcal{F}(X)$.

Theorem 9.2. *If the mapping $f : [0,1] \times \mathcal{F}(X) \to \mathcal{P}(X)$ satisfies the following conditions:*

(1) $f(\lambda, \bigcap_{t\in T} A_t) = \bigcap_{t\in T} f(\lambda, A_t)$;
(2) *When $\lambda > 0$ and $A \in \mathcal{P}(X)$, we have $f(\lambda, A) = A$;*
(3) $f(\lambda, \alpha \circ A) = \begin{cases} X, & \alpha \geqslant \lambda \\ f(\lambda, A), & \alpha < \lambda \end{cases}$, $\forall A \in \mathcal{F}(X), \alpha, \lambda \in I$.

Then $f(\lambda, A) = A_\lambda$ for any $\lambda \in I$ and $A \in \mathcal{F}(X)$.

Proof. When $\lambda > 0, f(\lambda, A) = f(\lambda, \bigcap_{\alpha\in I} \alpha \circ A_\alpha) = \bigcap_{\alpha\in I} f(\lambda, \alpha \circ A_\alpha) = \bigcap_{\alpha<\lambda} f(\lambda, A_\alpha) = \bigcap_{\alpha<\lambda} A_\alpha = A_\lambda$; When $\lambda = 0$, we let $\alpha = 0$, then $f(\lambda, A) = f(\lambda, \alpha \circ A) = X = A_0$. □

Therefore, $f(\lambda, A) = A_\lambda, \forall \lambda \in I, A \in \mathcal{F}(X)$.

Theorem 9.3. *If the mapping $f : [0,1] \times \mathcal{F}(X) \to \mathcal{P}(X)$ satisfies the following conditions:*

(1) $f(\lambda, \bigcup_{t\in T} A_t) = \bigcup_{t\in T} f(\lambda, A_t)$;
(2) *When $\lambda > 0$ and $A \in \mathcal{P}(X)$, we have $f(\lambda, A) = A$;*
(3) $f(\lambda, \alpha A) = \begin{cases} \emptyset, & \alpha + \lambda \leqslant 1 \\ f(\lambda, A), & \alpha + \lambda > 1 \end{cases}$, $\forall A \in \mathcal{F}(X), \alpha, \lambda \in I$.

Then $f(\lambda, A) = A_{[\underline{\lambda}]}$ for any $\lambda \in I$ and $A \in \mathcal{F}(X)$.

Proof. When $\lambda > 0, f(\lambda, A) = f(\lambda, \bigcup_{\alpha\in I} \alpha^c A_{[\underline{\alpha}]}) = \bigcup_{\alpha\in I} f(\lambda, \alpha^c A_{[\underline{\alpha}]}) = \bigcup_{\alpha^c+\lambda>1} f(\lambda, A_{[\underline{\alpha}]}) = \bigcup_{\lambda>\alpha} A_{[\underline{\alpha}]} = A_{[\underline{\lambda}]}$; When $\lambda = 0$, we let $\alpha = 1$, then $f(\lambda, A) = \emptyset = A_{[\underline{0}]}$. □

Therefore, $f(\lambda, A) = A_{[\underline{\lambda}]}, \forall \lambda \in I, A \in \mathcal{F}(X)$.

Theorem 9.4. *If the mapping $f : [0,1] \times \mathcal{F}(X) \to \mathcal{P}(X)$ satisfies the following conditions:*

(1) $f(\lambda, \bigcap_{t\in T} A_t) = \bigcap_{t\in T} f(\lambda, A_t)$;
(2) *When $\lambda < 1$ and $A \in \mathcal{P}(X)$, we have $f(\lambda, A) = A$;*
(3) $f(\lambda, \alpha \circ A) = \begin{cases} X, & \lambda + \alpha \geqslant 1 \\ f(\lambda, A), & \lambda + \alpha < 1 \end{cases}$, $\forall A \in \mathcal{F}(X), \alpha, \lambda \in I$.

Then $f(\lambda, A) = A_{[\lambda]}$ for any $\lambda \in I$ and $A \in \mathcal{F}(X)$.

Proof. when $\lambda < 1, f(\lambda, A) = f(\lambda, \bigcap_{\alpha\in I} \alpha^c \circ A_{[\alpha]}) = \bigcap_{\alpha\in I} f(\lambda, \alpha^c \circ A_{[\alpha]}) = \bigcap_{\lambda+\alpha^c<1} f(\lambda, A_{[\alpha]}) = \bigcap_{\lambda<\alpha} A_{[\alpha]} = A_{[\lambda]}$; When $\lambda = 1$, we let $\alpha = 0$, then $f(\lambda, A) = f(\lambda, \alpha \circ A) = X = A_{[\lambda]}$. □

Therefore, $f(\lambda, A) = A_{[\lambda]}, \forall \lambda \in I, A \in \mathcal{F}(X)$.

Theorem 9.5. *If the mapping $f : [0,1] \times \mathcal{F}(X) \to \mathcal{P}(X)$ satisfies the following conditions:*

(1) $f(\lambda, \bigcap_{t\in T} A_t) = \bigcup_{t\in T} f(\lambda, A_t)$;
(2) *When $\lambda > 0$ and $A \in \mathcal{P}(X)$, we have $f(\lambda, A) = A^c$;*
(3) $f(\lambda, \alpha \cdot A) = \begin{cases} \emptyset, & \alpha \geqslant \lambda \\ f(\lambda, A^c), & \alpha < \lambda \end{cases}$, $\forall A \in \mathcal{F}(X), \alpha, \lambda \in I$.

Then $f(\lambda, A) = A^{\underline{\lambda}}$ for any $\lambda \in I$ and $A \in \mathcal{F}(X)$.

Proof. When $\lambda > 0, f(\lambda, A) = f(\lambda, \bigcap_{\alpha\in I} \alpha \cdot A^{\underline{\alpha}}) = \bigcup_{\alpha\in I} f(\lambda, \alpha \cdot A^{\underline{\alpha}}) = \bigcup_{\alpha<\lambda} f(\lambda, (A^{\underline{\alpha}})^c) = \bigcap_{\alpha<\lambda} A^{\underline{\alpha}} = A^{\underline{\lambda}}$; When $\lambda = 0$, we let $\alpha = 0$, then $f(\lambda, A) = f(\lambda, \alpha \cdot A^c) = \emptyset = A^{\underline{\lambda}}$. □

Therefore, $f(\lambda, A) = A^{\underline{\lambda}}, \forall \lambda \in I, A \in \mathcal{F}(X)$.

Theorem 9.6. *If the mapping $f : [0,1] \times \mathcal{F}(X) \to \mathcal{P}(X)$ satisfies the following conditions:*

(1) $f(\lambda, \bigcup_{t\in T} A_t) = \bigcap_{t\in T} f(\lambda, A_t)$;

(2) *When* $\lambda < 1$ *and* $A \in \mathcal{P}(X)$, *we have* $f(\lambda, A) = A^c$;
(3) $f(\lambda, \alpha \diamond A) = \begin{cases} X, & \alpha \leqslant \lambda \\ f(\lambda, A^c), & \alpha > \lambda \end{cases}, \forall A \in \mathcal{F}(X), \alpha, \lambda \in I.$

Then $f(\lambda, A) = A^{\lambda}$ *for any* $\lambda \in I$ *and* $A \in \mathcal{F}(X)$.

Proof. When $\lambda < 1, f(\lambda, A) = f(\lambda, \bigcup_{\alpha\in I} \alpha \diamond A^{\alpha}) = \bigcap_{\alpha\in I} f(\lambda, \alpha \diamond A^{\alpha}) = \bigcap_{\alpha>\lambda} f(\lambda, (A^{\alpha})^c) = \bigcap_{\alpha>\lambda} A^{\alpha} = A^{\lambda}$; When $\lambda = 1$, we let $\alpha = 1$, then $f(\lambda, A) = f(\lambda, 1 \diamond A^c) = X = A^1$. □

Therefore, $f(\lambda, A) = A^{\lambda}, \forall \lambda \in I, A \in \mathcal{F}(X)$.

Theorem 9.7. *If the mapping* $f : [0, 1] \times \mathcal{F}(X) \to \mathcal{P}(X)$ *satisfies the following conditions:*

(1) $f(\lambda, \bigcap_{t\in T} A_t) = \bigcup_{t\in T} f(\lambda, A_t)$;
(2) *When* $\lambda < 1$ *and* $A \in \mathcal{P}(X)$, *we have* $f(\lambda, A) = A^c$;
(3) $f(\lambda, \alpha \cdot A) = \begin{cases} \emptyset, & \lambda + \alpha \geqslant 1 \\ f(\lambda, A^c), & \lambda + \alpha < 1 \end{cases}, \forall A \in \mathcal{F}(X), \alpha, \lambda \in I.$

Then $f(\lambda, A) = A^{[\underline{\lambda}]}$ *for any* $\lambda \in I$ *and* $A \in \mathcal{F}(X)$.

Proof. When $\lambda < 1, f(\lambda, A) = f(\lambda, \bigcap_{\alpha\in I} \alpha^c \cdot A^{[\underline{\alpha}]}) = \bigcup_{\alpha\in I} f(\lambda, \alpha^c \cdot A^{[\underline{\alpha}]}) = \bigcup_{\lambda+\alpha^c<1} f(\lambda, (A^{[\underline{\alpha}]})^c) = \bigcup_{\lambda<\alpha} A^{[\underline{\alpha}]} = A^{[\underline{\lambda}]}$; When $\lambda = 1$, we let $\alpha = 0$, then $f(\lambda, A) = f(\lambda, 0 \cdot A^c) = \emptyset = A^{[\underline{\lambda}]}$. □

Therefore, $f(\lambda, A) = A^{[\underline{\lambda}]}, \forall \lambda \in I, A \in \mathcal{F}(X)$.

Theorem 9.8. *If the mapping* $f : [0, 1] \times \mathcal{F}(X) \to \mathcal{P}(X)$ *satisfies the following conditions:*

(1) $f(\lambda, \bigcup_{t\in T} A_t) = \bigcap_{t\in T} f(\lambda, A_t)$;
(2) *When* $\lambda > 0$ *and* $A \in \mathcal{P}(X)$, *we have* $f(\lambda, A) = A^c$;
(3) $f(\lambda, \alpha \diamond A) = \begin{cases} X, & \lambda + \alpha \leqslant 1 \\ f(\lambda, A^c), & \lambda + \alpha > 1 \end{cases}, \forall A \in \mathcal{F}(X), \alpha, \lambda \in I.$

Then $f(\lambda, A) = A^{[\underline{\lambda}]}$ *for any* $\lambda \in I$ *and* $A \in \mathcal{F}(X)$.

Proof. When $\lambda > 0, f(\lambda, A) = f(\lambda, \bigcup_{\alpha\in I} \alpha^c \diamond A^c) = \bigcap_{\alpha\in I} f(\lambda, \alpha^c \diamond A^c) = \bigcap_{\lambda+\alpha^c>1} f(\lambda, (A^{[\alpha]})^c) = \bigcap_{\lambda>\alpha} A^{[\alpha]} = A^{[\lambda]}$; When $\lambda = 0$, we let $\alpha = 1$, then $f(\lambda, A) = f(\lambda, 1 \diamond A^c) = X = A^{[0]}$.

Therefore, $f(\lambda, A) = A^{[\lambda]}, \forall \lambda \in I, A \in \mathcal{F}(X)$. □

10. Conclusions

In this paper, three new cut sets of fuzzy sets are presented and their properties are discussed. Based on those cut sets of fuzzy sets, new decomposition theorems, new representation theorems, new extension principles and new fuzzy linear mappings are established. These discussions extended the theories of fuzzy sets.

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