Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings

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Abstract

In this paper, weak and strong convergence theorems are established for a three-step iterative scheme for asymptotically nonexpansive mappings in Banach spaces. Mann-type and Ishikawa -type iterations are included by the new iterative scheme. The results obtained in this paper extend and improve the recent ones announced by Xu and Noor, Glowinski and Le Tallec, Noor, Ishikawa, and many others.

Keywords: Asymptotically nonexpansive mapping; Completely continuous; Uniformly convex; Noor iterations

1. Introduction

Fixed-point iteration processes for asymptotically nonexpansive mapping in Banach spaces including Mann and Ishikawa iterations processes have been studied extensively by many authors; see [1–19,21,22]. Many of them are used widely to study the approximate solutions of the certain problems; see [4,6,11,12,21]. In 2000, Noor [12] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [4] applied three-step iterative schemes for

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finding the approximate solution of the elastoviscoplasticity problem, eigenvalue problem and liquid crystal theory.

Recently, Xu and Noor [23] introduced and studied a three-step scheme to approximate fixed point of asymptotically nonexpansive mappings in a Banach space, and Cho, Zhou and Guo [3] extended their schemes to the three-step iterative scheme with errors and gave weak and strong convergence theorems for asymptotically nonexpansive mapping in the intermediate sense. Inspired and motivated by these facts, a new class of three-step iterative scheme is introduced and studied in this paper. This scheme can be viewed as an extension for three-step and two-step iterative schemes of Glowinski and Le Tallec [4], Noor [12], Xu and Noor [23], and Ishikawa [8]. The scheme is defined as follows.

Let \( X \) be a normed space, \( C \) be a nonempty convex subset of \( X \), and \( T : C \to C \) be a given mapping. Then for a given \( x_1 \in C \), compute the sequence \( \{x_n\} \), \( \{y_n\} \) and \( \{z_n\} \) by the iterative scheme

\[
\begin{align*}
z_n &= a_n T^n x_n + (1 - a_n) x_n, \\
y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n) x_n, \\
x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n) x_n, \quad n \geq 1,
\end{align*}
\]

where \( \{a_n\} \), \( \{b_n\} \), \( \{c_n\} \), \( \{\alpha_n\} \), \( \{\beta_n\} \) are appropriate sequences in \([0, 1]\).

The iterative schemes (1.1) are called the modified Noor iterations. Noor iterations include the Mann–Ishikawa iterations as special cases. If \( c_n = \beta_n \equiv 0 \), then (1.1) reduces to Noor iterations defined by Xu and Noor [23]:

\[
\begin{align*}
z_n &= a_n T^n x_n + (1 - a_n) x_n, \\
y_n &= b_n T^n z_n + (1 - b_n) x_n, \\
x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \geq 1,
\end{align*}
\]

where \( \{a_n\} \), \( \{b_n\} \), \( \{\alpha_n\} \) are appropriate sequences in \([0, 1]\).

For \( a_n = c_n = \beta_n \equiv 0 \), then (1.1) reduces to the usual Ishikawa iterative scheme

\[
\begin{align*}
y_n &= b_n T^n x_n + (1 - b_n) x_n, \\
x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \geq 1,
\end{align*}
\]

where \( \{b_n\} \), \( \{\alpha_n\} \) are appropriate sequences in \([0, 1]\).

If \( a_n = b_n = c_n = \beta_n \equiv 0 \), then (1.1) reduces to the usual Mann iterative scheme

\[
\begin{align*}
x_{n+1} &= \alpha_n T^n x_n + (1 - \alpha_n) x_n, \quad n \geq 1,
\end{align*}
\]

where \( \{\alpha_n\} \) are appropriate sequences in \([0, 1]\).

The purpose of this paper is to establish several strong convergence theorems of the modified Noor iterations for completely continuous asymptotically nonexpansive mappings in a uniformly convex Banach space, and weak convergence theorems of the modified Noor iterations for asymptotically nonexpansive mappings in a uniformly convex Banach space with Opial’s condition. Our results extend and improve the corresponding ones announced by Xu and Noor [23], and others.

Now, we recall the well-known concepts and results.
Let $X$ be normed space and $C$ be a nonempty subset of $X$. A mapping $T : C \to C$ is said to be asymptotically nonexpansive on $C$ if there exists a sequence $\{k_n\}$, $k_n \geq 1$, with $\lim_{n \to \infty} k_n = 1$ such that

$$
\|T^n x - T^n y\| \leq k_n \|x - y\|
$$

for all $x, y \in C$ and each $n \geq 1$.

If $k_n \equiv 1$, then $T$ is known as a nonexpansive mapping. The mapping $T$ is called uniformly $L$-Lipschitzian if there exists a positive constant $L$ such that

$$
\|T^n x - T^n y\| \leq L \|x - y\|
$$

for all $x, y \in C$ and each $n \geq 1$.

It is easy to see that if $T$ is asymptotically nonexpansive, then it is uniformly $L$-Lipschitzian with the uniform Lipschitz constant $L = \sup \{k_n : n \geq 1\}$.

Recall that a Banach space $X$ is said to satisfy Opial’s condition [13] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that

$$
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|.
$$

In the sequel, the following lemmas are needed to prove our main results.

**Lemma 1.1** [18, Lemma 1]. Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$
a_{n+1} \leq (1 + \delta_n) a_n + b_n, \quad \forall n = 1, 2, \ldots
$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(i) $\lim_{n \to \infty} a_n$ exists.

(ii) $\lim_{n \to \infty} a_n = 0$ whenever $\liminf_{n \to \infty} a_n = 0$.

**Lemma 1.2** [21, Theorem 2]. Let $p > 1$, $r > 0$ be two fixed numbers. Then a Banach space $X$ is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that

$$
\|\lambda x + (1 - \lambda) y\| \leq \lambda \|x\|^p + (1 - \lambda) \|y\|^p - w_p(\lambda) g(\|x - y\|),
$$

for all $x, y \in B_r = \{x \in X : \|x\| \leq r\}$, $\lambda \in [0, 1]$, where

$$
w_p(\lambda) = \lambda (1 - \lambda)^p + p (1 - \lambda).
$$

**Lemma 1.3** [3, Lemma 1.4]. Let $X$ be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that

$$
\|\lambda x + \beta y + \gamma z\|^2 \leq \lambda \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \lambda \beta \gamma g(\|x - y\|),
$$

for all $x, y, z \in B_r$ and all $\lambda, \beta, \gamma \in [0, 1]$ with $\lambda + \beta + \gamma = 1$.

**Lemma 1.4** [3, Lemma 1.6]. Let $X$ be a uniformly convex Banach space, $C$ a nonempty closed convex subset of $X$, and $T : C \to C$ be an asymptotically nonexpansive mapping.
Then \(1 - T\) is demiclosed at 0, i.e., if \(x_n \to x\) weakly and \(x_n - Tx_n \to 0\) strongly, then \(x \in F(T)\), where \(F(T)\) is the set of fixed point of \(T\).

2. Main results

In this section, we prove weak and strong convergence theorems of the modified Noor iterations for asymptotically nonexpansive mapping in a Banach space. In order to prove our main results, the following lemmas are needed.

Lemma 2.1. If \(\{b_n\}\) and \(\{c_n\}\) are sequences in \([0, 1]\) such that \(\limsup_{n \to \infty} (b_n + c_n) < 1\) and \(\{k_n\}\) is a sequence of real numbers with \(k_n \geq 1\) for all \(n \geq 1\) and \(\lim_{n \to \infty} k_n = 1\), then there exist a positive integer \(N_1\) and \(\gamma \in (0, 1)\) such that \(c_n k_n < \gamma\) for all \(n \geq N_1\).

Proof. By \(\limsup_{n \to \infty} (b_n + c_n) < 1\), there exists a positive integer \(N_0\) and \(\eta \in (0, 1)\) such that

\[c_n \leq b_n + c_n < \eta, \quad \forall n \geq N_0.\]

Let \(\eta' \in (0, 1)\) with \(\eta' > \eta\). From \(\lim_{n \to \infty} k_n = 1\), there exists a positive integer \(N_1 \geq N_0\) such that

\[k_n - 1 < \frac{1}{\eta'} - 1, \quad \forall n \geq N_1,
\]

from which we have \(k_n < 1/\eta', \; \forall n \geq N_1\). Put \(\gamma = \eta/\eta'\), then we have \(c_n k_n < \gamma\) for all \(n \geq N_1\). \(\Box\)

The next lemma is crucial for proving the main theorems.

Lemma 2.2. Let \(X\) be a uniformly convex Banach space, and let \(C\) be a nonempty closed, bounded, and convex subset of \(X\). Let \(T\) be an asymptotically nonexpansive self-map of \(C\) with \(\{k_n\}\) satisfying \(k_n \geq 1\) and \(\sum_{n=1}^{\infty} (k_n - 1) < \infty\). Let \(\{\alpha_n\}, \{\beta_n\}, \{\alpha_n\}, \{b_n\}\) and \(\{c_n\}\) be real sequences in \([0, 1]\) such that \(b_n + c_n\) and \(\alpha_n + \beta_n\) are in \([0, 1]\) for all \(n \geq 1\). For a given \(x_1 \in C\), let \(\{x_n\}\), \(\{y_n\}\) and \(\{z_n\}\) be the sequences defined as in (1.1).

(i) If \(q\) is a fixed point of \(T\), then \(\lim_{n \to \infty} \|x_n - q\|\) exists.

(ii) If \(\liminf_{n \to \infty} \alpha_n > 0\) and \(0 < \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty} (b_n + c_n) < 1\), then \(\lim_{n \to \infty} \|T^n z_n - x_n\| = 0\).

(iii) \(0 \leq \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} (\alpha_n + \beta_n) < 1\), then \(\lim_{n \to \infty} \|T^n y_n - x_n\| = 0\).

(iv) \(0 \leq \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty} (b_n + c_n) < 1\) and

\[0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} (\alpha_n + \beta_n) < 1\], then \(\lim_{n \to \infty} \|T^n x_n - x_n\| = 0\).

Proof. From [5, Theorem 1], \(T\) has a fixed point \(x^* \in C\). Choose a number \(r > 0\) such that \(C \subseteq B_r\) and \(C - C \subseteq B_r\). By Lemma 1.2, there is a continuous, strictly increasing, and convex function \(g_1: [0, \infty) \to [0, \infty), g_1(0) = 0\) such that

\[
\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - w_2(\lambda) g_1(\|x - y\|) \tag{2.1}
\]
for all \( x, y \in B_r \), \( \lambda \in [0, 1] \), where \( w_2(\lambda) = \lambda(1 - \lambda)^2 + \lambda^2(1 - \lambda) \). It follows from (2.1) that
\[
\|z_n - x^*\|^2 = \|a_n(T^n x_n - x^*) + (1 - a_n)(x_n - x^*)\|^2
\]
\[
\leq a_n \|T^n x_n - x^*\|^2 + (1 - a_n)\|x_n - x^*\|^2 - w_2(a_n)g_1(\|T^n x_n - x_n\|)
\]
\[
\leq a_n k_n^2 \|x_n - x^*\|^2 + (1 - a_n)\|x_n - x^*\|^2
\]
\[
\leq \left(1 + a_n k_n^2 - a_n\right)\|x_n - x^*\|^2.
\]

By Lemma 1.3, there exists a continuous strictly increasing convex function \( g_2 : [0, \infty) \rightarrow [0, \infty) \), \( g_2(0) = 0 \), such that
\[
\|\lambda x + \beta y + \gamma z\|^2 \leq \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g_2(\|x - y\|),
\]
for all \( x, y, z \in B_r \), and all \( \lambda, \beta, \gamma \in [0, 1] \) with \( \lambda + \beta + \gamma = 1 \). It follows from (2.2) that
\[
\|y_n - x^*\|^2 = \|b_n(T^n z_n - x^*) + (1 - b_n - c_n)(x_n - x^*) + c_n(T^n x_n - x^*)\|^2
\]
\[
\leq b_n \|T^n z_n - x^*\|^2 + (1 - b_n - c_n)\|x_n - x^*\|^2 + c_n\|T^n x_n - x^*\|^2
\]
\[
- b_n(1 - b_n - c_n)g_2(\|T^n z_n - x_n\|)
\]
\[
\leq b_n k_n^2 \|z_n - x^*\|^2 + c_n k_n^2 \|x_n - x^*\|^2 + (1 - b_n - c_n)\|x_n - x^*\|^2
\]
and
\[
\|x_{n+1} - x^*\|^2 = \|\alpha_n(T^n y_n - x^*) + (1 - \alpha_n - \beta_n)(x_n - x^*) + \beta_n(T^n z_n - x^*)\|^2
\]
\[
\leq \alpha_n \|T^n y_n - x^*\|^2 + (1 - \alpha_n - \beta_n)\|x_n - x^*\|^2 + \beta_n\|T^n z_n - x^*\|^2
\]
\[
- \alpha_n(1 - \alpha_n - \beta_n)g_2(\|T^n y_n - x_n\|)
\]
\[
\leq \alpha_n k_n^2 \|y_n - x^*\|^2 + (1 - \alpha_n - \beta_n)\|x_n - x^*\|^2 + \beta_n k_n^2 \|z_n - x^*\|^2
\]
\[
- \alpha_n(1 - \alpha_n - \beta_n)g_2(\|T^n y_n - x_n\|)
\]
\[
= \|x_n - x^*\|^2 + (\alpha_n c_n k_n^4 + \alpha_n k_n^2(1 - b_n - c_n) - \alpha_n - \beta_n)\|x_n - x^*\|^2
\]
\[
+ (\alpha_n b_n k_n^4 + \beta_n k_n^2)\|z_n - x^*\|^2
\]
\[
- \alpha_n k_n^2 b_n(1 - b_n - c_n)g_2(\|T^n z_n - x_n\|)
\]
\[
- \alpha_n(1 - \alpha_n - \beta_n)g_2(\|T^n y_n - x_n\|)
\]
\[
\leq \|x_n - x^*\|^2 + (\alpha_n k_n^2 c_n (k_n^2 - 1)
\]
\[
+ \alpha_n (k_n^2 - 1) - \alpha_n k_n^2 b_n - \beta_n)\|x_n - x^*\|^2
\]
\[
+ (\alpha_n b_n k_n^4 + \beta_n k_n^2)(1 + a_n k_n^2 - a_n)\|x_n - x^*\|^2
\]
For all \( n \geq 1 \), it follows that
\[
\left\{ \begin{array}{l}
\|x_n - x^*\|^2 \\
\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + M(k_n^2 - 1)
\end{array} \right.
\]
for all \( n \geq 1 \). It follows that
\[
\begin{align*}
\alpha_n k_n^2 b_n (1 - b_n - c_n) g_2 (\|T^n z_n - x_n\|) \\
\alpha_n (1 - \alpha_n - \beta_n) g_2 (\|T^n y_n - x_n\|) \\
\end{align*}
\]
which implies by (2.3) that
\[
\begin{align*}
\alpha_n k_n^2 b_n (1 - b_n - c_n) g_2 (\|T^n z_n - x_n\|) \\
\alpha_n (1 - \alpha_n - \beta_n) g_2 (\|T^n y_n - x_n\|) \\
\end{align*}
\]
for all \( n \geq 1 \). Since \( \{k_n\} \) and \( C \) are bounded, there exists a constant \( M > 0 \) such that
\[
\left\{ \begin{array}{l}
\alpha_n k_n^2 c_n + \alpha_n + \alpha_n b_n k_n^2 + \beta_n + (\alpha_n b_n k_n^2 + \beta_n k_n^2) a_n \|x_n - x^*\|^2 \\
\end{array} \right.
\]
for all \( n \geq 1 \), which implies
\[
\alpha_n k_n^2 b_n (1 - b_n - c_n) g_2 (\|T^n z_n - x_n\|) \\
\alpha_n (1 - \alpha_n - \beta_n) g_2 (\|T^n y_n - x_n\|) \\
\end{align*}
\]
for all \( n \geq 1 \). Since \( \sum_{n=1}^{\infty} (k_n^2 - 1) < \infty \), it follows from Lemma 1.1 that
\[
\lim_{n \to \infty} \|x_n - q\| \text{ exists.}
\]
(i) If \( q \in F(T) \), by taking \( x^* = q \) in the inequality (2.3) we have
\[
\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 + M(k_n^2 - 1).
\]
Since \( \sum_{n=1}^{\infty} (k_n^2 - 1) < \infty \), it follows from Lemma 1.1 that
\[
\lim_{n \to \infty} \|x_n - q\| \text{ exists.}
\]
(ii) If \( \lim_{n \to \infty} \alpha_n > 0 \) and \( 0 < \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty} (b_n + c_n) < 1 \), then there exists a positive integer \( n_0 \) and \( \eta, \eta' \in (0, 1) \) such that
\[
0 < \eta < b_n, \quad 0 < \eta < \alpha_n \quad \text{and} \quad b_n + c_n < \eta' < 1 \quad \text{for all} \quad n \geq n_0.
\]
This implies by (2.3) that
\[
\eta^2 (1 - \eta') g_2 (\|T^n z_n - x_n\|) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + M(k_n^2 - 1)
\]
for all \( n \geq n_0 \). It follows from inequality (2.5) that for \( m \geq n_0 \),
\[ \sum_{n=n_0}^{m} g(z(T^n z_n - x_n)) \leq \frac{1}{\eta^2(1 - \eta')} \left( \sum_{n=n_0}^{m} (\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2) + M \sum_{n=n_0}^{m} (k_n^2 - 1) \right) \leq \frac{1}{\eta^2(1 - \eta')} \left( \|x_{n_0} - x^*\|^2 + M \sum_{n=n_0}^{m} (k_n^2 - 1) \right). \] (2.6)

Since \( 0 \leq t^2 - 1 \leq 2t(t - 1) \) for all \( t \geq 1 \), the assumption \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \) implies that \( \sum_{n=1}^{\infty} (k_n^2 - 1) < \infty \). Let \( m \to \infty \) in inequality (2.6) we get \( \sum_{n=n_0}^{m} g_2(\|T^n z_n - x_n\|) < \infty \), and therefore \( \lim_{m \to \infty} g_2(\|T^n z_n - x_n\|) = 0 \). Since \( g_2 \) is strictly increasing and continuous at 0 with \( g(0) = 0 \), it follows that \( \lim_{n \to \infty} \|T^n z_n - x_n\| = 0 \).

(iii) If \( 0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} (\alpha_n + \beta_n) < 1 \), then by using a similar method, together with inequality (2.4), it can be shown that \( \lim_{n \to \infty} \|T^n y_n - x_n\| = 0 \).

(iv) If \( 0 < \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty} (b_n + c_n) < 1 \) and \( 0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} (\alpha_n + \beta_n) < 1 \), by (ii) and (iii) we have

\[
\lim_{n \to \infty} \|T^n y_n - x_n\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|T^n z_n - x_n\| = 0.
\] (2.7)

From \( y_n = b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n) x_n \), we have

\[ \|y_n - x_n\| \leq b_n \|T^n z_n - x_n\| + c_n \|T^n x_n - x_n\|. \]

Thus

\[
\|T^n x_n - x_n\| \leq \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\|
\leq k_n \|x_n - y_n\| + \|T^n y_n - x_n\|
\leq k_n \left( b_n \|T^n z_n - x_n\| + c_n \|T^n x_n - x_n\| \right) + \|T^n y_n - x_n\|
= k_n b_n \|T^n z_n - x_n\| + c_n k_n \|T^n x_n - x_n\| + \|T^n y_n - x_n\|. \]

(2.8)

By Lemma 2.1, there exists positive integer \( n_1 \) and \( \gamma \in (0, 1) \) such that \( c_n k_n < \gamma \) for all \( n \geq n_1 \). This together with (2.8) implies that for \( n \geq n_1 \),

\[
(1 - \gamma) \|T^n x_n - x_n\| < (1 - c_n k_n) \|T^n x_n - x_n\|
\leq k_n b_n \|T^n z_n - x_n\| + \|T^n y_n - x_n\|.
\]

It follows from (2.7) that \( \lim_{n \to \infty} \|T^n x_n - x_n\| = 0 \). \( \square \)

**Theorem 2.3.** Let \( X \) be a uniformly convex Banach space, and \( C \) a nonempty closed, bounded and convex subset of \( X \). Let \( T \) be a completely continuous asymptotically nonexpansive self-map of \( C \) with \( \{k_n\} \) satisfying \( k_n \geq 1 \) and \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \). Let \( \{\alpha_n\}, \{\beta_n\}, \{\alpha_n\}, \{\beta_n\} \) be sequences of real numbers in \([0, 1]\) with \( b_n + c_n \in [0, 1] \) and \( \alpha_n + \beta_n \in [0, 1] \) for all \( n \geq 1 \), and

(i) \( 0 < \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty} (b_n + c_n) < 1 \), and

(ii) \( 0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} (\alpha_n + \beta_n) < 1 \).
Let \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) be the sequences defined by the modified Noor iterations (1.1). Then \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) converge strongly to a fixed point of \( T \).

**Proof.** By Lemma 2.2, we have
\[
\lim_{n \to \infty} \|T^n y_n - x_n\| = 0, \quad \lim_{n \to \infty} \|T^n z_n - x_n\| = 0, \quad \lim_{n \to \infty} \|T^n x_n - x_n\| = 0. \tag{2.9}
\]

Since \( x_{n+1} - x_n = \alpha_n (T^n y_n - x_n) + \beta_n (T^n z_n - x_n) \), we have
\[
\|x_{n+1} - T^n x_{n+1}\| \leq \|x_{n+1} - x_n\| + \|T^n x_{n+1} - T^n x_n\| + \|T^n x_n - x_n\| \\
\leq \|x_{n+1} - x_n\| + k_n \|x_{n+1} - x_n\| + \|T^n x_n - x_n\| \\
= (1 + k_n) \|x_{n+1} - x_n\| + \|T^n x_n - x_n\| \\
\leq (1 + k_n) \alpha_n \|T^n y_n - x_n\| + (1 + k_n) \beta_n \|T^n z_n - x_n\| \\
+ \|T^n x_n - x_n\|.
\]

This together with (2.9) implies that
\[
\|x_{n+1} - T^n x_{n+1}\| \to 0 \quad \text{(as } n \to \infty). \]

Thus
\[
\|x_{n+1} - T x_{n+1}\| \leq \|x_{n+1} - T^{n+1} x_{n+1}\| + \|T x_{n+1} - T^{n+1} x_{n+1}\| \\
\leq \|x_{n+1} - T^{n+1} x_{n+1}\| + k_1 \|x_{n+1} - T^n x_{n+1}\| \to 0,
\]

which implies
\[
\lim_{n \to \infty} \|T x_n - x_n\| = 0. \tag{2.10}
\]

Since \( T \) is completely continuous and \( \{x_n\} \subseteq C \) is bounded, there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \{T x_{n_k}\} \) converges. Therefore from (2.10), \( \{x_{n_k}\} \) converges. Let \( \lim_{k \to \infty} x_{n_k} = q \). By continuity of \( T \) and (2.10) we have that \( T q = q \), so \( q \) is a fixed point of \( T \). By Lemma 2.2(i), \( \lim_{n \to \infty} \|x_n - q\| \) exists. But \( \lim_{k \to \infty} \|x_{n_k} - q\| = 0 \). Thus \( \lim_{n \to \infty} \|x_n - q\| = 0 \).

Since
\[
\|y_n - x_n\| \leq b_n \|T^n z_n - x_n\| + c_n \|T^n x_n - x_n\| \to 0 \quad \text{as } n \to \infty, \quad \text{and}
\]
\[
\|z_n - x_n\| \leq a_n \|T^n x_n - x_n\| \to 0 \quad \text{as } n \to \infty,
\]

it follows that \( \lim_{n \to \infty} y_n = q \) and \( \lim_{n \to \infty} z_n = q \). \( \square \)

For \( c_n = \beta_n \equiv 0 \) in Theorem 2.3, we obtain the following result.

**Theorem 2.4** [23, Theorem 2.1]. Let \( X \) be a uniformly convex Banach space, and let \( C \) be a closed, bounded and convex subset of \( X \). Let \( T \) be a completely continuous asymptotically nonexpansive self-map of \( C \) with \( \{k_n\} \) satisfying \( k_n \geq 1 \) and \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \). Let \( \{a_n\}, \{b_n\}, \{a_n\} \) be real sequences in \([0, 1]\) satisfying
(i) \(0 < \lim \inf_{n \to \infty} b_n \leq \lim \sup_{n \to \infty} b_n < 1\), and
(ii) \(0 < \lim \inf_{n \to \infty} \alpha_n \leq \lim \sup_{n \to \infty} \alpha_n < 1\).

For a given \(x_1 \in C\), define
\[
z_n = a_n T^n x_n + (1 - a_n) x_n, \\
y_n = b_n T^n z_n + (1 - b_n) x_n, \\
x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \geq 1.\]

Then \(\{x_n\}, \{y_n\}, \{z_n\}\) converge strongly to a fixed point of \(T\).

When \(c_n = \beta_n = 0\) in Theorem 2.3, we can obtain Ishikawa-type convergence result which is a generalization of Theorem 3 in [14].

**Theorem 2.5.** Let \(X\) be a uniformly convex Banach space, and let \(C\) be a closed, bounded and convex subset of \(X\). Let \(T\) be a completely continuous asymptotically nonexpansive self-map of \(C\) with \(\{k_n\}\) satisfying \(k_n \geq 1\) and \(\sum_{n=1}^{\infty} (k_n - 1) < \infty\). Let \(\{b_n\}\) and \(\{\alpha_n\}\) be real sequences in \([0, 1]\) satisfying

(i) \(0 < \lim \inf_{n \to \infty} b_n \leq \lim \sup_{n \to \infty} b_n < 1\), and
(ii) \(0 < \lim \inf_{n \to \infty} \alpha_n \leq \lim \sup_{n \to \infty} \alpha_n < 1\).

For a given \(x_1 \in C\), define
\[
y_n = b_n T^n x_n + (1 - b_n) x_n, \\
x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \geq 1.\]

Then \(\{x_n\}\) and \(\{y_n\}\) converge strongly to a fixed point of \(T\).

For \(a_n = b_n = c_n = \beta_n = 0\), then Theorem 2.3 reduces to the following Mann-type convergence result, which is a generalization and refinement of [14, Theorem 2], [16, Theorem 1.5], and [17, Theorem 2.2].

**Theorem 2.6.** Let \(X\) be a uniformly convex Banach space, and let \(C\) be a closed, bounded and convex subset of \(X\). Let \(T\) be a completely continuous asymptotically nonexpansive self-map of \(C\) with \(\{k_n\}\) satisfying \(k_n \geq 1\) and \(\sum_{n=1}^{\infty} (k_n - 1) < \infty\). Let \(\{\alpha_n\}\) be a real sequence in \([0, 1]\) satisfying

\(0 < \lim \inf_{n \to \infty} \alpha_n \leq \lim \sup_{n \to \infty} \alpha_n < 1\).

For a given \(x_1 \in C\), define
\[
x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n) x_n, \quad n \geq 1.\]

Then \(\{x_n\}\) converges strongly to a fixed point of \(T\).
In the next result, we prove weak convergence for the modified Noor iterations (1.1) for asymptotically nonexpansive mapping in a uniformly convex Banach space satisfying Opial’s condition. To do this, we need a lemma.

**Lemma 2.7.** Let $X$ be a Banach space which satisfies Opial’s condition and let $\{x_n\}$ be a sequence in $X$. Let $u, v \in X$ be such that $\lim_{n \to \infty} \|x_n - u\|$ and $\lim_{n \to \infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to $u$ and $v$, respectively, then $u = v$.

**Proof.** Suppose that $u \neq v$. Then, by Opial’s condition, we have

\[
\lim_{n \to \infty} \|x_n - u\| = \lim_{k \to \infty} \|x_{n_k} - u\| < \lim_{k \to \infty} \|x_{n_k} - v\| = \lim_{n \to \infty} \|x_n - v\| < \lim_{k \to \infty} \|x_{m_k} - u\| = \lim_{n \to \infty} \|x_n - u\|,
\]

which is a contradiction. $\square$

**Theorem 2.8.** Let $X$ be a uniformly convex Banach space which satisfies Opial’s condition, and $C$ a nonempty closed, bounded and convex subset of $X$. Let $T$ be an asymptotically nonexpansive self-map of $C$ with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ be sequences of real numbers in $[0, 1]$ with $b_n + c_n \in [0, 1]$ and $\alpha_n + \beta_n \in [0, 1]$ for all $n \geq 1$, and

(i) $0 < \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty} (b_n + c_n) < 1$, and

(ii) $0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} (\alpha_n + \beta_n) < 1$.

Let $\{x_n\}$ be the sequence defined by the modified Noor iterations (1.1). Then $\{x_n\}$ converges weakly to a fixed point of $T$.

**Proof.** It follows from Lemma 2.2(iv) that $\lim_{n \to \infty} \|Tx_n - x_n\| = 0$. Since $X$ is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 1.4, we have $u \in F(T)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to $u$ and $v$, respectively. From Lemma 1.4, $u, v \in F(T)$. By Lemma 2.2(i), $\lim_{n \to \infty} \|x_n - u\|$ and $\lim_{n \to \infty} \|x_n - v\|$ exist. It follows from Lemma 2.7 that $u = v$. Therefore $\{x_n\}$ converges weakly to a fixed point of $T$. $\square$

When $c_n = \beta_n \equiv 0$ in Theorem 2.8, we obtain the following result.

**Corollary 2.9.** Let $X$ be a uniformly convex Banach space which satisfies Opial’s condition, and $C$ a nonempty closed, bounded and convex subset of $X$. Let $T$ be an asymptotically nonexpansive self-map of $C$ with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{\alpha_n\}$ be sequences of real numbers in $[0, 1]$ and


\(0 < \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty} b_n < 1, \) and
\(0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1.\)

Let \(\{x_n\}, \{y_n\} \) and \(\{z_n\} \) be the sequences defined by
\[
\begin{align*}
z_n &= a_n T^n x_n + (1 - a_n) x_n, \\
y_n &= b_n T^n z_n + (1 - b_n) x_n, \\
x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \geq 1.
\end{align*}
\]
Then \(\{x_n\}\) converges weakly to a fixed point of \(T.\)

When \(a_n = c_n = \beta_n \equiv 0\) in Theorem 2.8, we obtain Ishikawa-type weak convergence theorem as follows:

**Corollary 2.10.** Let \(X\) be a uniformly convex Banach space which satisfies Opial’s condition, and \(C\) a nonempty closed, bounded, and convex subset of \(X.\) Let \(T\) be an asymptotically nonexpansive self-map of \(C\) with \(\{k_n\}\) satisfying \(k_n \geq 1\) and \(\sum_{n=1}^{\infty} (k_n - 1) < \infty.\) Let \(\{b_n\}\) and \(\{\alpha_n\}\) be sequences of real numbers in \([0,1]\) such that
\(0 < \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty} b_n < 1, \) and
\(0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1.\)

Let \(\{x_n\}\) and \(\{y_n\}\) be the sequences defined by
\[
\begin{align*}
y_n &= b_n T^n x_n + (1 - b_n) x_n, \\
x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \geq 1.
\end{align*}
\]
Then \(\{x_n\}\) converges weakly to a fixed point of \(T.\)

When \(a_n = b_n = c_n = \beta_n \equiv 0\) in Theorem 2.8, we obtain Mann-type weak convergence theorem as follows:

**Corollary 2.11.** Let \(X\) be a uniformly convex Banach space which satisfies Opial’s condition, and \(C\) a nonempty closed, bounded and convex subset of \(X.\) Let \(T\) be an asymptotically nonexpansive self-map of \(C\) with \(\{k_n\}\) satisfying \(k_n \geq 1\) and \(\sum_{n=1}^{\infty} (k_n - 1) < \infty.\) Let \(\{\alpha_n\}\) be a sequence of real numbers in \([0,1]\) such that
\(0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1.\)

Let \(\{x_n\}\) be the sequence defined by
\[
\begin{align*}
x_{n+1} &= \alpha_n T^n x_n + (1 - \alpha_n) x_n, \quad n \geq 1.
\end{align*}
\]
Then \(\{x_n\}\) converges weakly to a fixed point of \(T.\)

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References