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Ascoli's theorem for functions vanishing at infinity and selected applications

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Abstract

We give a new form of the Ascoli theorem for functions on $\mathbf{R}^N$ tending to some given closed subset Z of a complete metric space E at infinity. For instance, when E is a normed space and $Z = \{0\}$, the usual uniform decay requirement is replaced by the assumption that the 0 function is the only continuous function produced by some limiting process. This formulation, which has significant practical value in concrete applications, is described in its general form, but with emphasis on the case when Z is totally disconnected. Variants in Sobolev spaces and the properness of nonlinear ordinary differential operators are discussed.

Keywords: Ascoli theorem; Sobolev space; Properness

1. Introduction

Let E be a complete metric space with distance d and let $C_b(\mathbf{R}^N; E)$ be the (metric) space of E-valued bounded continuous functions on $\mathbf{R}^N$, equipped with the distance $d_\infty(u, v) := \sup_{x\in\mathbf{R}^N} d(u(x), v(x))$. Given a nonempty subset $Z \subset E$, we denote by $C_Z(\mathbf{R}^N; E)$ the closed subspace of $C_b(\mathbf{R}^N; E)$ of those functions tending to Z at infinity:

$$C_Z(\mathbf{R}^N; E) = \Big\{u \in C_b(\mathbf{R}^N; E):\ \lim_{|x|\to\infty} d\big(u(x), Z\big) = 0\Big\}. \tag{1}$$

By collapsing Z to a point z ("zero"), the functions of $C_Z(\mathbf{R}^N; E)$ may be viewed as functions vanishing at infinity. In fact, when $Z = \{z\}$ is a single point and with $\mathbf{S}^{N+1} \simeq$

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$\mathbf{R}^N \cup \{\infty\}$ being the unit sphere of $\mathbf{R}^{N+1}$, the space $C_{\{z\}}(\mathbf{R}^N; E)$ is isometrically isomorphic to the closed subspace of $C(\mathbf{S}^{N+1}; E)$ of those functions u such that $u(\infty) = z$. Therefore, Ascoli's theorem for $C(\mathbf{S}^{N+1}; E)$ has an immediate corollary for $C_{\{z\}}(\mathbf{R}^N; E)$, given below for future reference.

Theorem 1 (Ascoli; classical form). *Let E be a complete metric space and $z \in E$ be a given point. A subset $\mathcal{H} \subset C_{\{z\}}(\mathbf{R}^N; E)$ is relatively compact if and only if*

(a) *for every $x \in \mathbf{R}^N$, the set $\mathcal{H}(x)$ is relatively compact in E,*
(b) *$\mathcal{H}$ is equicontinuous,*
(c) *$\mathcal{H}$ tends uniformly to z at infinity, i.e., $d(u(x), z)$ can be made arbitrarily small, uniformly in $u \in \mathcal{H}$, for $|x|$ large enough.*

Condition (c) of Theorem 1 merely reflects the equicontinuity of $\mathcal{H}$ at $\infty \in \mathbf{S}^{N+1}$. In practice, checking condition (c) requires having some knowledge of the collective pointwise asymptotic behavior of the members of $\mathcal{H}$, which is not always directly accessible.

This paper elaborates on a version of Theorem 1, given in Theorem 2, in which condition (c) is replaced by the requirement that the only function $\tilde{u} \in C_b(\mathbf{R}^N; E)$ produced by some pointwise limiting process is the constant function $\tilde{u} = z$. While slight modifications of (a) and (b) are also needed, the net result remains a necessary and sufficient condition for relative compactness in $C_{\{z\}}(\mathbf{R}^N; E)$.

Although its proof is technically simple, this form of Ascoli's theorem has proved to have a considerable practical value, because it relies on a condition about *continuous functions* $\tilde{u}$. Whatever additional property these functions inherit from being involved in a given problem may be instrumental in showing that, indeed, $\tilde{u} = z$, as required by the theorem. In contrast, condition (c) of Theorem 1, which amounts to $\lim_{n\to\infty} d(u_n(x_n), z) = 0$ for all sequences $(u_n) \subset \mathcal{H}$ and $(x_n) \subset \mathbf{R}^N$ with $|x_n| \to \infty$, leaves no limiting mathematical object to examine in the light of problem-dependent features.

For instance, in many concrete applications, it is possible to characterize $\tilde{u}$ above as a solution of some known equation, thereby reducing the compactness question to showing that this equation has no solution other than $u = z$ (i.e., no nontrivial solution when $z = 0$). This is useful, directly or in a more subtle way, to establish the properness of several types of operators in various functional frameworks: Elliptic operators on $\mathbf{R}^N$, systems of ODEs on the line or half-line, convolution operators, etc. In such problems, other technical aspects incorporated to Theorem 2 are needed to consider, say, problems with N-periodic rather than constant, coefficients.

Whether Theorem 2 can be generalized when $\mathcal{H} \subset C_Z(\mathbf{R}^N; E)$ and Z is a nonempty closed subset of E depends upon the size of Z from a topological point of view: If Z is compact and totally disconnected, the answer is positive (Corollary 6), which, incidentally, yields a useful generalization of Theorem 2 for $C_{\{z\}}(\mathbf{R}^N; E)$ (Corollary 7). Otherwise, only a weaker form is true (Theorem 5), which gives a relative compactness criterion in the compact-open topology of $C_Z(\mathbf{R}^N; E)$. Still, this is not trivial since the uniform convergence on compact subsets alone ensures only that the limit points are in $C_b(\mathbf{R}^N; E)$, not $C_Z(\mathbf{R}^N; E)$.

Generalizations when $\mathbf{R}^N$ is replaced by a locally compact topological group are not investigated, but variants in Sobolev spaces are discussed in Section 4 when $E = \mathbf{R}^M$, or more generally a reflexive Banach space, and $Z = \{0\}$. When $mp > N$, these variants yield a simple characterization of the bounded subsets of $W^{m,p}(\mathbf{R}^N)$ which are relatively compact in $C_{\{0\}}(\mathbf{R}^N) := C_{\{0\}}(\mathbf{R}^N; \mathbf{R})$. In particular, this characterization shows that, for bounded subsets of $W^{m,p}(\mathbf{R}^N)$ with $mp > N$, relative compactness in $C_{\{0\}}(\mathbf{R}^N)$ is equivalent to relative compactness in $L^q(\mathbf{R}^N)$ for any $q \in (p, \infty)$ and weaker than relative compactness in $L^p(\mathbf{R}^N)$ (Corollary 10).

In a different direction, the results of Section 4 also provide an important first step in establishing the compactness of some subsets in Sobolev spaces, as exemplified by the proof of Theorem 14. Indeed, in spite of its resemblance with Ascoli's theorem, it does not appear that the classical criterion for compactness in $L^p(\mathbf{R}^N)$ can be reformulated in an equally convenient way.

The line of argument for the proof of Theorem 2 was first introduced in Rabier and Stuart [16], to investigate the Fredholmness and properness of nonlinear second order elliptic operators in $W^{2,p}(\mathbf{R}^N)$, $p > N$. However, that work does not make a connection with a general, problem-independent, compactness property in Sobolev spaces, let alone with the more remote theorem of Ascoli. This paper is the result of an attempt to identify the principles really involved in the procedure of [16].

In [19], Secchi and Stuart used the approach of [16], this time in $W^{1,2}(\mathbf{R}; \mathbf{R}^{2M})$, to obtain basic functional properties for the proof of the bifurcation of homoclinic solutions in nonlinear Hamiltonian systems. As an application of Theorems 2 and 9, we revisit and expand the properness results of [19] (Section 5). The example of ODE systems on the whole line is simpler to describe and was chosen here for precisely that reason, but, as already mentioned, there are numerous other applications in the same spirit.

For $R > 0$, $B_R \subset \mathbf{R}^N$ is the open ball with center 0 and radius R and $\widetilde{B}_R$ the complement of $\overline{B}_R$ in $\mathbf{R}^N$. Given $\xi \in \mathbf{R}^N$, we call τ_ξ the translation operator $\tau_\xi u := u(\xi + \cdot)$, where u is any function defined on $\mathbf{R}^N$. We shall also need the concept of δ-net in $\mathbf{R}^N$ ($\delta \geqslant 0$). This is simply a subset $S \subset \mathbf{R}^N$ such that $\operatorname{dist}(x, S) \leqslant \delta$ for every $x \in \mathbf{R}^N$. For instance, $S = \mathbf{R}^N$ is a δ-net for every $\delta \geqslant 0$ while $S = \mathbf{Z}^N$ is a δ-net if $\delta \geqslant \sqrt{N}/2$.

2. Relative compactness in $C_{\{z\}}(\mathbf{R}^N; E)$

As in the Introduction, E is a metric space with distance d, the point $z \in E$ is chosen once and for all and d_∞ denotes the corresponding distance on $C_b(\mathbf{R}^N; E) \supset C_{\{z\}}(\mathbf{R}^N; E)$.

Theorem 2 (Ascoli; new form). *Let E be a metric space, $z \in E$ be a given point and let $S \subset \mathbf{R}^N$ be any chosen δ-net. For a subset $\mathcal{H} \subset C_{\{z\}}(\mathbf{R}^N; E)$, the following statements are equivalent*:

(i) *$\mathcal{H}$ is relatively compact in $C_{\{z\}}(\mathbf{R}^N; E)$.*
(ii) *$\mathcal{H}(\mathbf{R}^N)$ is relatively compact in E, $\mathcal{H}$ is uniformly equicontinuous and if $\tilde{u} \in C_b(\mathbf{R}^N; E)$[1] and there are sequences $(u_n) \subset \mathcal{H}$ and $(\xi_n) \subset S$ with $\lim_{n\to\infty} |\xi_n| = \infty$ such that $\tilde{u}_n := \tau_{\xi_n} u_n \to \tilde{u}$ pointwise on $\mathbf{R}^N$, then $\tilde{u} = z$.*

[1] It is not enough to assume that $\tilde{u} \in C_{\{z\}}(\mathbf{R}^N; E)$.

Proof. (i) $\Rightarrow$ (ii) We begin with the relative compactness of $\mathcal{H}(\mathbf{R}^N)$ in E. As in the Introduction, identify $C_{\{z\}}(\mathbf{R}^N; E)$ with a closed subspace of $C(\mathbf{S}^{N+1}; E)$, so that $\mathcal{H}$ is relatively compact in $C(\mathbf{S}^{N+1}; E)$. Since the evaluation map $e(x,u) := u(x)$ is continuous from $\mathbf{S}^{N+1} \times C(\mathbf{S}^{N+1}; E)$ to E and $\mathbf{S}^{N+1}$ is compact, it follows that $\mathcal{H}(\mathbf{S}^{N+1}) = e(\mathbf{S}^{N+1} \times \mathcal{H})$ is relatively compact in E, so that $\mathcal{H}(\mathbf{R}^N) \subset \mathcal{H}(\mathbf{S}^{N+1})$ is relatively compact in E.

Next, the equicontinuity of $\mathcal{H}$ on $\mathbf{S}^{N+1}$ implies its uniform equicontinuity on $\mathbf{S}^{N+1}$ since $\mathbf{S}^{N+1}$ is compact. It is readily checked that the stereographic projection transforms a ball with radius $r > 0$ in $\mathbf{S}^{N+1}$ into a subset of $\mathbf{R}^N$ containing a ball with radius $r' > 0$ depending only upon r, which shows that $\mathcal{H}$ is uniformly equicontinuous on $\mathbf{R}^N$.

Lastly, with $\tilde{u}$, $(u_n) \subset \mathcal{H}$ and $(\xi_n) \subset S$ as in part (ii), we turn to the proof that $\tilde{u} = z$. Since $\mathcal{H}$ is relatively compact in $C_{\{z\}}(\mathbf{R}^N; E)$, there are $u \in C_{\{z\}}(\mathbf{R}^N; E)$ and a subsequence (u_{n_k}) such that $d_\infty(u_{n_k}, u) \to 0$. Thus, $d_\infty(\tilde{u}_{n_k}, \tau_{\xi_{n_k}} u) \to 0$ since translations do not change d_∞. Clearly, $\tau_{\xi_{n_k}} u \to z$ pointwise on $\mathbf{R}^N$ since $\lim_{n\to\infty} |\xi_n| = \infty$ and u tends to z at infinity. Since also $\tilde{u}_{n_k} \to \tilde{u}$ pointwise on $\mathbf{R}^N$ by hypothesis, it follows that $\tilde{u} = z$.

(ii) $\Rightarrow$ (i) It suffices to show that if $(u_n) \subset \mathcal{H}$ and if $(x_n) \subset \mathbf{R}^N$ satisfies $\lim_{n\to\infty} |x_n| = \infty$, then $\lim_{n\to\infty} d(u_n(x_n), z) = 0$. Indeed, if so, the conclusion follows from Theorem 1 since a straightforward contradiction argument shows that condition (c) of that theorem holds (and stronger variants of (a) and (b) are assumed in (ii)).

By contradiction, assume that there are $(u_n) \subset \mathcal{H}$ and $(x_n) \subset \mathbf{R}^N$ with $\lim_{n\to\infty} |x_n| = \infty$ such that $d(u_n(x_n), z)$ does not tend to 0. After replacing (u_n) and (x_n) by subsequences, we may assume that there is $\varepsilon > 0$ such that $d(u_n(x_n), z) \geqslant \varepsilon$ for all indices n. By definition of a δ-net, let $\xi_n \in S$ and $y_n \in \overline{B}_\delta$ be such that $x_n = \xi_n + y_n$, so that

$$d\big(\tilde{u}_n(y_n), z\big) \geqslant \varepsilon, \quad \forall n \in \mathbf{N}, \tag{2}$$

where $\tilde{u}_n := \tau_{\xi_n} u_n$. Let (y_{n_k}) be a subsequence such that $y_{n_k} \to y \in \overline{B}_\delta$.

Since $(u_{n_k}) \subset \mathcal{H}$ and $\mathcal{H}$ is uniformly equicontinuous, the sequence $(\tilde{u}_{n_k})$ is equicontinuous. Furthermore, $(\tilde{u}_{n_k}(x)) \subset \mathcal{H}(\mathbf{R}^N)$ for every $x \in \mathbf{R}^N$ and $\mathcal{H}(\mathbf{R}^N)$ is relatively compact by hypothesis, so that $(\tilde{u}_{n_k}(x))$ is relatively compact in E. It thus follows from the Arens–Myers generalization of Ascoli's theorem in the compact-open topology [3,11] that there are $\tilde{u} \in C(\mathbf{R}^N; E)$ and a subsequence $(\tilde{u}_{n_{k_\ell}})$ such that $\tilde{u}_{n_{k_\ell}} \to \tilde{u}$ uniformly on the compact subsets of $\mathbf{R}^N$. Also, $\tilde{u}(x) \in \overline{\mathcal{H}(\mathbf{R}^N)}$ for every $x \in \mathbf{R}^N$, so that $\tilde{u} \in C_b(\mathbf{R}^N; E)$ and hence $\tilde{u} = z$ from the assumptions made in (ii).

On the other hand, since $y_{n_k} \to y \in \overline{B}_\delta$, it follows from (2) and the uniform convergence of $(\tilde{u}_{n_{k_\ell}})$ to $\tilde{u}$ on $\overline{B}_\delta$ that $d(\tilde{u}(y), z) \geqslant \varepsilon$, which contradicts $\tilde{u} = z$. $\square$

Remark 3. If the set $\mathcal{H}$ is a sequence (u_n), the above proof shows that it suffices to consider the sequence (u_n) in part (ii) of Theorem 2, rather than every sequence $(u_{n(k)})$. This can also be seen by a contradiction argument. (The issue is not entirely trivial because of the arbitrary shifts involved in condition (ii).)

To see how condition (ii) breaks down in simple cases when $\mathcal{H}$ is not relatively compact, let $u \in C_{\{0\}}(\mathbf{R}; \mathbf{R}) \backslash \{0\}$ be a given function with compact support and let $\mathcal{H} = (\tau_n u)$. Here, $\mathcal{H}(\mathbf{R}) = u(\mathbf{R})$ is compact in $\mathbf{R}$ and $\mathcal{H}$ is uniformly equicontinuous, but if $\xi_n = -n$, then

$\tau_{-n}\tau_n u = u$ is not pointwise convergent to 0, so that condition (ii) with $S = \mathbf{R}^N$ and $\delta = 0$ does not hold. It is only slightly less trivial to show that condition (ii) also fails when S is any other δ-net.

Theorem 2 is still true when $\mathbf{R}^N$ is replaced by a closed convex cone K or even by more general unbounded closed subsets $K \subset \mathbf{R}^N$, invariant under some set of translations $\mathfrak{T}$ and hence having some "periodic" structure. For instance, $K = K_0 + \mathfrak{T}$ where $\mathfrak{T}$ is any (unbounded) subset of $\mathbf{Z}^N$. A δ-net $S \subset K$ can be obtained in the form $S = S_0 + \mathfrak{T}$ where S_0 is some δ-net in K_0, possibly a single point if K_0 is bounded (and then δ is the diameter of K_0). This includes cylinders $\overline{\omega} \times [0, \infty)$ where ω is a bounded open subset of $\mathbf{R}^{N-1}$: Just take $K_0 = \overline{\omega} \times [0, 1]$ and $\mathfrak{T} = \{0\} \times \mathbf{N}$.

3. Relative compactness in $C_Z(\mathbf{R}^N; E)$

This section is devoted to a partial extension of Theorem 2 when the singleton $\{z\}$ is replaced by a nonempty closed subset $Z \subset E$, which yields a genuine extension if also Z is compact and totally disconnected. Some preliminary discussion is needed.

Let E be a complete metric space and $Z \subset E$ be a nonempty subset. We denote by E/Z the set of equivalence classes for the relation

$$a \sim b \quad \Leftrightarrow \quad a = b \text{ or } a \in Z,\ b \in Z \tag{3}$$

and equip E/Z with the quotient topology, that is, $U \subset E/Z$ is open if and only if $\pi^{-1}(U)$ is open in E, where $\pi : E \to E/Z$ is the projection. In general, E/Z is not a metric space, even if Z is closed in E (a simple counterexample when E/Z is not first countable is given in Kelley [9, p. 104]). However, if E is compact, the following lemma, whose proof is given for completeness, is essentially a special case of a well-known result [9, p. 149].

Lemma 4. *If E is compact and Z is closed in E, then E/Z is a compact metric space. Furthermore, if $U \subset E/Z$ is an open neighborhood of*[2] *$\pi(Z)$ in E/Z, then $\pi^{-1}(U)$ contains some ε-neighborhood $W_\varepsilon := \{a \in E\colon\ d(a, Z) < \varepsilon\}$ of Z in E $(\varepsilon > 0)$.*

Proof. That E/Z is compact follows from the continuity of π. We begin with the "furthermore" part. Let $U \subset E/Z$ be an open neighborhood of $\pi(Z)$ in E/Z. By the continuity of π, $\pi^{-1}(U)$ is an open subset of E containing Z. Cover Z by finitely many open balls $B(b_i, \varepsilon_i) \subset \pi^{-1}(U)$, $b_i \in Z$, and let $\varepsilon > 0$ be a Lebesgue number for the covering. Then, $W_\varepsilon = \bigcup_{b \in Z} B(b, \varepsilon) \subset \pi^{-1}(U)$.

To prove the metrizability of E/Z, we rely on Urysohn's metrization theorem (see [9]). It suffices to show that points are closed in E/Z and that E/Z is second countable.

That points are closed follows at once from the remark that Z is closed in E and π is a bijection of $E \backslash Z$ onto $(E/Z) \backslash \pi(Z)$. A countable basis for the topology of E/Z is obtained as follows: Since E is compact, it is separable and hence $E \backslash Z$ is open in E and separable. Let then (V_n) be a countable basis for the topology of $E \backslash Z$ and, for $m \in \mathbf{N}$, set

[2] In this statement and elsewhere, we implicitly identify the singleton $\pi(Z)$ with the unique point in it.

$W_m := \{a \in E\colon d(a, Z) < 1/m\}$, an open subset of E containing Z. Note that $\pi(V_n)$ and $\pi(W_m)$ are open in E/Z since $\pi^{-1}(\pi(V_n)) = V_n$ and $\pi^{-1}(\pi(W_m)) = W_m$.

If now $U \subset E/Z$ is open, then either $\pi(Z) \notin U$ or $\pi(Z) \in U$. In the first case, $\pi^{-1}(U)$ is contained in $E\backslash Z$ and hence is the union of some of the V_n. In the second, $\pi^{-1}(U)$ contains Z, so that $W_m \subset \pi^{-1}(U)$ for some m by the first part of the proof. It follows that $\pi^{-1}(U) = W_m \cup (\pi^{-1}(U)\backslash Z)$. Since $\pi^{-1}(U)\backslash Z$ is open in $E\backslash Z$, $\pi^{-1}(U)$ is the union of W_m and some of the V_n. This shows that $(\pi(V_n)) \cup (\pi(W_m))$ is a basis for the topology of E/Z.

Theorem 5. *Let E be a complete metric space and $Z \subset E$ be a nonempty closed subset. Let $S \subset \mathbf{R}^N$ be some δ-net and let $\mathcal{H} \subset C_Z(\mathbf{R}^N; E)$ satisfy the following conditions*:

(i) *$\mathcal{H}$ is uniformly equicontinuous,*
(ii) *$\mathcal{H}(\mathbf{R}^N)$ is relatively compact in E,*
(iii) *If $\tilde{u} \in C_b(\mathbf{R}^N; E)$ and there are sequences $(u_n) \subset \mathcal{H}$ and $(\xi_n) \subset S$ with $\lim_{n\to\infty} |\xi_n| = \infty$ such that $\tilde{u}_n := \tau_{\xi_n} u_n \to \tilde{u}$ pointwise on $\mathbf{R}^N$, then $\tilde{u}(\mathbf{R}^N) \subset Z$.*

Then, $\mathcal{H}$ is relatively compact in $C_Z(\mathbf{R}^N; E)$ for the compact-open topology. Furthermore, the following (stronger) property holds: Every sequence $(u_n) \subset \mathcal{H}$ contains a subsequence (u_{n_k}) converging uniformly to some $u \in C_Z(\mathbf{R}^N; E)$ on the compact subsets of $\mathbf{R}^N$ and tending uniformly to Z at infinity (i.e., for every $\varepsilon > 0$, there are $k_0 \in \mathbf{N}$ and $R > 0$ such that $d(u_{n_k}(x), Z) < \varepsilon$ whenever $k > k_0$ and $|x| > R$).

Proof. There is no loss of generality in replacing E by $\overline{\mathcal{H}(\mathbf{R}^N)}$ and Z by $Z \cap \overline{\mathcal{H}(\mathbf{R}^N)}$ and hence, by (ii), we may assume that E and Z are compact. If so, E/Z is a (compact) metric space by Lemma 4. We now check that (ii) $\Rightarrow$ (i) in Theorem 2 can be used with $\pi \circ \mathcal{H} \subset C_{\pi(Z)}(\mathbf{R}^N; E/Z)$, where of course $\pi \circ \mathcal{H} := \{\pi \circ u\colon u \in \mathcal{H}\}$.

First, to see that the inclusion $\pi \circ \mathcal{H} \subset C_{\pi(Z)}(\mathbf{R}^N; E/Z)$ holds, let $u \in \mathcal{H}$ be given and let U be an open neighborhood of $\pi(Z)$ in E/Z. By Lemma 4, $\pi^{-1}(U)$ contains $W_\varepsilon := \{a \in E\colon d(a, Z) < \varepsilon\}$ for some $\varepsilon > 0$, and $u(x) \in W_\varepsilon$ for $|x|$ large enough since $u \in C_Z(\mathbf{R}^N; E)$. Thus, $\pi \circ u(x) \in U$ for $|x|$ large enough, which means that $\pi \circ u(x) \to \pi(Z)$ as $|x| \to \infty$ and hence that $\pi \circ u \in C_{\pi(Z)}(\mathbf{R}^N; E/Z)$.

It follows from (ii) and the continuity of π that $\pi \circ \mathcal{H}(\mathbf{R}^N)$ is relatively compact in E/Z. Next, since E and E/Z are compact metric spaces, π is uniformly continuous. Together with (i), this yields that $\pi \circ \mathcal{H}$ is uniformly equicontinuous. Lastly, let $(\pi \circ u_n) \subset \pi \circ \mathcal{H}$ and $(\xi_n) \subset S$ be sequences such that $\lim_{n\to\infty} |\xi_n| = \infty$ and that $\tilde{v}_n := \tau_{\xi_n}(\pi \circ u_n)$ tends pointwise to $\tilde{v} \in C_b(\mathbf{R}^N; E/Z)$, so that $\tilde{v}(x) = \lim_{n\to\infty} \pi \circ u_n(\xi_n + x)$ for every $x \in \mathbf{R}^N$.

We claim that $\tilde{v} = \pi \circ \tilde{u}$, where $\tilde{u} \in C_b(\mathbf{R}^N; E)$. Indeed, as in the proof of Theorem 2, it follows from (i) that $(\tilde{u}_n) := (\tau_{\xi_n} u_n)$ is equicontinuous and then, by (ii) and the Arens–Meyers version of Ascoli's theorem, there are $\tilde{u} \in C(\mathbf{R}^N; E)$ and a subsequence $(\tilde{u}_{n_k})$ such that $\tilde{u}_{n_k} \to \tilde{u}$ uniformly on the compact subsets of $\mathbf{R}^N$. That $\tilde{u} \in C_b(\mathbf{R}^N; E)$ follows from (ii) and from $\tilde{u}(\mathbf{R}^N) \subset \overline{\mathcal{H}(\mathbf{R}^N)}$, while $\pi \circ \tilde{u}_{n_k} \to \pi \circ \tilde{u}$ pointwise by the continuity of π. Thus, $\tilde{v} = \pi \circ \tilde{u}$, as claimed. But then, $\tilde{v} = \pi(Z)$ since, by (iii), $\tilde{u}(x) \in Z$ for every $x \in \mathbf{R}^N$.

From the above and Theorem 2, it follows that $\pi \circ \mathcal{H}$ is relatively compact in $C_{\pi(Z)}(\mathbf{R}^N; E/Z)$. We now show that this implies that $\mathcal{H}$ is relatively compact in $C_Z(\mathbf{R}^N; E)$ (and not merely $C_b(\mathbf{R}^N; E)$) for the compact-open topology. By the local compactness of $\mathbf{R}^N$, the compact-open topology is metrizable since E is metric and the problem reduces to showing that every sequence $(u_n) \subset \mathcal{H}$ has a subsequence (u_{n_k}) converging to some $u \in C_Z(\mathbf{R}^N; E)$, uniformly on the compact subsets of $\mathbf{R}^N$.

Once again by (i), (ii) and the Arens–Myers–Ascoli theorem, there are $u \in C_b(\mathbf{R}^N; E)$ and (u_{n_k}) such that $u_{n_k} \to u$ uniformly on the compact subsets of $\mathbf{R}^N$. The only issue is to show that $\lim_{|x|\to\infty} d(u(x), Z) = 0$. Since, from the above, $\pi \circ \mathcal{H}$ is relatively compact in $C_{\pi(Z)}(\mathbf{R}^N; E/Z)$, we may and will assume with no loss of generality that $\pi \circ u_{n_k}$ tends to $\pi \circ u$ in $C_{\pi(Z)}(\mathbf{R}^N; E/Z)$.

Let $\varepsilon > 0$ be given. With $W_\varepsilon := \{a \in E\colon\ d(a, Z) < \varepsilon\}$, we have $\pi^{-1}(\pi(W_\varepsilon)) = W_\varepsilon$ and hence that $\pi(W_\varepsilon)$ is an open neighborhood of $\pi(Z)$ in E/Z. By (c) of Theorem 1 for $\pi \circ \mathcal{H}$, there are $k_0 \in \mathbf{N}$ and $R > 0$ such that $\pi \circ u_{n_k}(x) \in \pi(W_\varepsilon)$ if $k > k_0$ and $|x| > R$. Hence, $u_{n_k}(x) \in \pi^{-1}(\pi(W_\varepsilon)) = W_\varepsilon$, i.e., $d(u_{n_k}(x), Z) < \varepsilon$, for $k > k_0$ and $|x| > R$. With $x \in \mathbf{R}^N$ now fixed such that $|x| > R$ and by letting $k \to \infty$, it follows from the convergence of $u_{n_k}(x)$ to $u(x)$ in E that $d(u(x), Z) \leqslant \varepsilon$, which is the desired property since $\varepsilon > 0$ is arbitrary. That (u_{n_k}) tends uniformly to Z at infinity is contained in the statement above that $u_{n_k}(x) \in W_\varepsilon$ for $k > k_0$ and $|x| > R$. □

While the conditions (i) and (ii) of Theorem 5 yield the relative compactness of $\mathcal{H}$ in $C_b(\mathbf{R}^N; E)$ for the compact-open topology, (iii) is needed to ensure that the limits of convergent sequences tend to Z at infinity. On the other hand, Theorem 5 does not imply the existence of a subsequence converging uniformly on $\mathbf{R}^N$, even if Z is compact. Indeed, for large enough k and $|x|$, both $u_{n_k}(x)$ and $u(x)$ must be close to Z, but not necessarily to the same point of Z. Therefore, Theorem 5 gives a result stronger than convergence on the compact subsets of $\mathbf{R}^N$ but weaker than uniform convergence on $\mathbf{R}^N$. Unless $Z = \{z\}$ is a singleton, for then Theorem 5 implies condition (c) of Theorem 1 for (u_{n_k}) and hence is equivalent to Theorem 2. As it turns out, $Z = \{z\}$ is not the only case when uniform convergence on $\mathbf{R}^N$ is true.

Recall that a topological space Z is said to be *totally disconnected* if, given $a, b \in Z$ with $a \neq b$, there are disjoint open (and hence closed) neighborhoods V_a and V_b of a and b, respectively, such that $V_a \cup V_b = Z$. Examples include discrete sets, convergent sequences and their limit in Hausdorff spaces, Cantor sets, etc. If Z is compact metric, then V_a and V_b are compact subsets of Z, whence $d(V_a, V_b) > 0$. In particular, if Z is a compact subset of a metric space E, there are disjoint open neighborhoods U_a and U_b of a and b in E such that $Z \subset U_a \cup U_b$ (just let U_a and U_b be ε-neighborhoods of V_a and V_b in E, respectively, with $\varepsilon < d(V_a, V_b)/2$).

Corollary 6. *Let E be a complete metric space and $Z \subset E$ be a nonempty, compact*[3] *and totally disconnected subset. Let $S \subset \mathbf{R}^N$ be any chosen δ-net. For a subset $\mathcal{H} \subset C_Z(\mathbf{R}^N; E)$, the following statements are equivalent*:

[3] From the given proof, (ii) ⇒ (i) remains true if Z is closed; the same thing is true of (i) ⇒ (ii) if E is locally compact.

(i) *$\mathcal{H}$ is relatively compact in $C_Z(\mathbf{R}^N; E)$.*
(ii) *$\mathcal{H}(\mathbf{R}^N)$ is relatively compact in E, $\mathcal{H}$ is uniformly equicontinuous and, if $\tilde{u} \in C_b(\mathbf{R}^N; E)$ and there are sequences $(u_n) \subset \mathcal{H}$ and $(\xi_n) \subset S$ with $\lim_{n\to\infty} |\xi_n| = \infty$ such that $\tilde{u}_n := \tau_{\xi_n} u_n \to \tilde{u}$ pointwise on $\mathbf{R}^N$, then $\tilde{u}(\mathbf{R}^N) \subset Z$.*[4]

Proof. (i) $\Rightarrow$ (ii) We begin with the remark that, if $N \geqslant 2$, every $u \in C_Z(\mathbf{R}^N; E)$ has a well defined limit $a \in Z$ at infinity. Indeed, otherwise, there are sequences $(x_n) \subset \mathbf{R}^N$ and $(y_n) \subset \mathbf{R}^N$ with $\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} |y_n| = \infty$ such that $(u(x_n))$ and $(u(y_n))$ tend to two distinct points a and b of Z. Since Z is compact and totally disconnected, let U_a and U_b be disjoint open neighborhoods of a and b in E, respectively, such that $Z \subset U_a \cup U_b$. From Lemma 4, $u(\widetilde{B}_R) \subset U_a \cup U_b$ for $R > 0$ large enough and, since $\widetilde{B}_R$ is connected and $U_a \cap U_b = \emptyset$, it follows that either $u(\widetilde{B}_R) \subset U_a$ or $u(\widetilde{B}_R) \subset U_b$. In both cases a contradiction arises with the fact that $x_n, y_n \in \widetilde{B}_R$ for n large enough while $u(x_n) \in U_a$ and $u(y_n) \in U_b$.

We continue the proof assuming $N \geqslant 2$. Let $(u_n) \subset \mathcal{H}$ and $(x_n) \subset \mathbf{R}^N$ be arbitrary sequences. Since $\mathcal{H}$ is relatively compact in $C_Z(\mathbf{R}^N; E)$, there are subsequences (u_{n_k}) tending uniformly to $u \in C_Z(\mathbf{R}^N; E)$ on $\mathbf{R}^N$ and (x_{n_k}) such that either $x_{n_k} \to x_0$ in $\mathbf{R}^N$ or $|x_{n_k}| \to \infty$. In the first case, $(u_{n_k}(x_{n_k}))$ tends to $u(x_0)$ and in the second, $(u_{n_k}(x_{n_k}))$ tends to $a \in Z$, where $a := \lim_{|x|\to\infty} u(x)$, whose existence was established at the beginning of the proof. Thus, $\mathcal{H}(\mathbf{R}^N)$ is relatively compact in E.

To show that $\mathcal{H}$ is uniformly equicontinuous, we argue by contradiction, thereby assuming that there are $\varepsilon > 0$ and sequences $(u_n) \subset \mathcal{H}$, $(x_n) \subset \mathbf{R}^N$ and $(y_n) \subset \mathbf{R}^N$ such that $|x_n - y_n| \to 0$ but $d(u_n(x_n), u_n(y_n)) \geqslant \varepsilon$. After passing to a subsequence, we may assume that (u_n) tends uniformly to $u \in C_Z(\mathbf{R}^N; E)$ on $\mathbf{R}^N$ and either that $x_n \to x_0$ in $\mathbf{R}^N$, whence $y_n \to x_0$, or that $|x_n| \to \infty$, whence $|y_n| \to \infty$. In both cases, $(u_n(x_n))$ and $(u_n(y_n))$ have the same limit, namely, $u(x_0)$ in the first case and $a := \lim_{|x|\to\infty} u(x) \in Z$ in the second. Thus, $d(u_n(x_n), u_n(y_n)) \to 0$, in contradiction with $d(u_n(x_n), u_n(y_n)) \geqslant \varepsilon$.

Lastly, if $\tilde{u} \in C_b(\mathbf{R}^N; E)$ and there are sequences $(u_n) \subset \mathcal{H}$ and $(\xi_n) \subset S$ with $\lim_{n\to\infty} |\xi_n| = \infty$ such that $\tilde{u}_n := \tau_{\xi_n} u_n \to \tilde{u}$ pointwise on $\mathbf{R}^N$, then, after replacing (u_n) by a subsequence, we may assume that (u_n) tends uniformly to $u \in C_Z(\mathbf{R}^N; E)$ on $\mathbf{R}^N$. As a result, $\tilde{u}(x_0) = \lim_{n\to\infty} u_n(\xi_n + x_0) = a := \lim_{|x|\to\infty} u(x) \in Z$ irrespective of $x_0 \in \mathbf{R}^N$. Thus, $\tilde{u} = a$ and, in particular, $\tilde{u}(\mathbf{R}^N) \subset Z$.

This completes the proof of (i) $\Rightarrow$ (ii) when $N \geqslant 2$. If $N = 1$, the only modification is that, now, $u \in C_Z(\mathbf{R}; E)$ has well defined limits $a_\mp \in Z$ at $\mp\infty$. The same arguments as above can then be used, with the only extra step of considering limits at ∞ and $-\infty$ separately.

(ii) $\Rightarrow$ (i). We begin with the remark that, as in the proof of Theorem 5, it is not restrictive to assume that E is compact (by replacing E by $\overline{\mathcal{H}(\mathbf{R}^N)}$ and Z by $Z \cap \overline{\mathcal{H}(\mathbf{R}^N)}$; that Z is compact and totally disconnected is not affected by this operation).

It follows from Theorem 5 that every sequence $(u_n) \subset \mathcal{H}$ contains a subsequence (u_{n_k}) converging uniformly to $u \in C_Z(\mathbf{R}^N; E)$ on the compact subsets of $\mathbf{R}^N$, with the additional property that, for every $\varepsilon > 0$, there are $k_0 \in \mathbf{N}$ and $R > 0$ such that

$$\{k > k_0,\ |x| > R\} \quad \Rightarrow \quad d\big(u_{n_k}(x), Z\big) < \varepsilon. \tag{4}$$

[4] And $\tilde{u}$ is constant since the points of Z are its connected components.

Claim. *If $N \geqslant 2$, there are $a \in Z$ and a subsequence $(u_{n_{k_\ell}})$ tending uniformly to a at infinity.*

Choose a sequence $(x_k) \subset \mathbf{R}^N$ such that $|x_k| \to \infty$. By (4) and the compactness of Z, we obtain $a \in Z$ and subsequences $(u_{n_{k_\ell}})$ and (x_{k_ℓ}) such that $d(u_{n_{k_\ell}}(x_{k_\ell}), a) \to 0$. To simplify the notation, assume $d(u_{n_k}(x_k), a) \to 0$, with no prejudice to (4). By contradiction, if (u_{n_k}) does not tend to a uniformly at infinity, there are a subsequence $(u_{n_{k_\ell}})$ and a sequence $(y_\ell) \subset \mathbf{R}^N$ with $|y_\ell| \to \infty$ such that $d(u_{n_{k_\ell}}(y_\ell), a)$ is bounded away from 0. After extracting another subsequence and since $d(u_{n_{k_\ell}}(y_\ell), Z) \to 0$ by (4) and Z is compact, we may assume that there is $b \in Z$, $b \neq a$, such that $d(u_{n_{k_\ell}}(y_\ell), b) \to 0$.

Since Z is compact and totally disconnected, there are disjoint open neighborhoods U_a and U_b of a and b in E, respectively, such that $Z \subset U_a \cup U_b$. By (4) and Lemma 4, $u_{n_{k_\ell}}(\widetilde{B}_R) \subset U_a \cup U_b$ if $R > 0$ and ℓ are large enough. Since $N \geqslant 2$, $\widetilde{B}_R$ is connected and hence $u_{n_{k_\ell}}(\widetilde{B}_R) \subset U_a$ since $u_{n_{k_\ell}}(x_{k_\ell}) \in U_a$ for ℓ large enough. Evidently, a contradiction arises with the fact that $y_\ell \in \widetilde{B}_R$ and $u_{n_{k_\ell}}(y_\ell) \in U_b$ for large ℓ. Thus, (u_{n_k}) tends to a uniformly at infinity. Since (u_{n_k}) stands for a subsequence in this statement, we have obtained $(u_{n_{k_\ell}})$ with the property that, for every $\varepsilon > 0$, there are $\ell_0 > 0$ and $R > 0$ such that

$$\big\{\ell > \ell_0,\ |x| > R\big\} \quad \Rightarrow \quad d\big(u_{n_{k_\ell}}(x), a\big) < \varepsilon. \tag{5}$$

Since (u_{n_k}) tends to u pointwise, it follows, by letting $\ell \to \infty$ in (5), that $d(u(x), a) \leqslant \varepsilon$ if $|x| > R$. But then, $d(u_{n_{k_\ell}}(x), u(x)) < 2\varepsilon$ if $|x| > R$ and $\ell > \ell_0$. Since (u_{n_k}) tends uniformly to u on $\overline{B}_R$, we infer that $d(u_{n_{k_\ell}}(x), u(x)) < 2\varepsilon$ if $|x| \leqslant R$ and ℓ is large enough, whence $d(u_{n_{k_\ell}}(x), u(x)) < 2\varepsilon$ for all $x \in \mathbf{R}^N$ and ℓ large enough. This shows that $(u_{n_{k_\ell}})$ tends uniformly to u on $\mathbf{R}^N$, which completes the proof when $N \geqslant 2$.

If $N = 1$, the above procedure yields, in place of (5), two points $a_\mp \in Z$ such that $\{\ell > \ell_0,\ x > R\} \Rightarrow d(u_{n_{k_\ell}}(x), a_+) < \varepsilon$ and that $\{\ell > \ell_0, x < -R\} \Rightarrow d(u_{n_{k_\ell}}(x), a_-) < \varepsilon$. The proof can then be completed by the same argument as in the case $N \geqslant 2$. □

As a corollary, we obtain a generalization of Theorem 2, in which the condition "$\tilde{u} = z$" in part (ii) is relaxed.

Corollary 7. *Let E be a complete metric space, $z \in E$ be a given point and let $S \subset \mathbf{R}^N$ be any chosen δ-net. For a subset $\mathcal{H} \subset C_{\{z\}}(\mathbf{R}^N; E)$, the following statements are equivalent*:

- (i) *$\mathcal{H}$ is relatively compact in $C_{\{z\}}(\mathbf{R}^N; E)$.*
- (ii) *$\mathcal{H}(\mathbf{R}^N)$ is relatively compact in E, $\mathcal{H}$ is uniformly equicontinuous and there is a compact and totally disconnected subset $Z \subset E$ with the following property*: *If $\tilde{u} \in C_b(\mathbf{R}^N; E)$ and there are sequences $(u_n) \subset \mathcal{H}$ and $(\xi_n) \subset S$ with $\lim_{n\to\infty} |\xi_n| = \infty$ such that $\tilde{u}_n := \tau_{\xi_n} u_n \to \tilde{u}$ pointwise on $\mathbf{R}^N$, then $\tilde{u}(\mathbf{R}^N) \subset Z$.*

Proof. Observe that $z \in Z$ in (ii) (choose $u_n = u \in \mathcal{H}$) and that $C_{\{z\}}(\mathbf{R}^N; E)$ is closed in $C_Z(\mathbf{R}^N; E)$, so that the relative compactness of $\mathcal{H}$ in $C_{\{z\}}(\mathbf{R}^N; E)$ is equivalent to its relative compactness in $C_Z(\mathbf{R}^N; E)$. Then, use Corollary 6. □

From Theorem 2, $Z = \{z\}$ works in (ii) of Corollary 7. That other sets Z may replace $\{z\}$ is useful when only $\tilde{u}(\mathbf{R}^N) \subset Z$ with a set Z larger than $\{z\}$ can be established a priori. For a concrete application, see Theorem 13 and subsequent examples.

4. Application to Sobolev spaces

If $m \in \mathbf{N}$, $p \in [1, \infty)$ and $mp > N$, then $W^{m,p}(\mathbf{R}^N)$ embeds in $C_{\{0\}}(\mathbf{R}^N) := C_{\{0\}}(\mathbf{R}^N; \mathbf{R})$, but the embedding is not compact. Equivalently, the unit ball of $W^{m,p}(\mathbf{R}^N)$ is not relatively compact in $C_{\{0\}}(\mathbf{R}^N)$. Thus, the question arises to characterize the bounded subsets of $W^{m,p}(\mathbf{R}^N)$ which are relatively compact in $C_{\{0\}}(\mathbf{R}^N)$. A simple answer will be derived from Theorem 2.

By arguing componentwise, the results of this section remain valid as stated when $W^{m,p}(\mathbf{R}^N)$ is replaced by $W^{m,p}(\mathbf{R}^N; \mathbf{R}^M)$ and will be used in this form in the next section. With appropriate modifications, they can also be generalized to $W^{m,p}(\mathbf{R}^N; E)$ where E is a reflexive Banach space, but since a convenient reference for all the needed properties of the spaces $W^{m,p}(\mathbf{R}^N; E)$ seems to be lacking, this case is only discussed in the final comments. When $E = \mathbf{R}$, see Adams [1].

Part (iii) of Lemma 8 below uses the well-known and easily checked fact that $W^{m,\infty}(\mathbf{R}^N)$ is isomorphic to a weak* closed subspace of $(L^\infty(\mathbf{R}^N))^{N^m+\cdots+1}$. Thus, $W^{m,\infty}(\mathbf{R}^N)$ can be equipped with the weak* topology of $(L^\infty(\mathbf{R}^N))^{N^m+\cdots+1}$ and the closed unit ball of $W^{m,\infty}(\mathbf{R}^N)$ is compact for this weak* topology.

Lemma 8. *Let $m \in \mathbf{N}$ and $p \in [1, \infty]$ be such that $mp > N$ and let $\mathcal{H} \subset W^{m,p}(\mathbf{R}^N)$ be a bounded subset. The following properties hold*:

- (i) *$\mathcal{H}$ is uniformly equicontinuous and $\mathcal{H}(\mathbf{R}^N)$ is relatively compact.*
- (ii) *If $p \in (1, \infty)$, a sequence $(u_n) \subset \mathcal{H}$ has a pointwise limit u if and only if $u \in W^{m,p}(\mathbf{R}^N)$ and $u_n \overset{w}{\rightharpoonup} u$ in $W^{m,p}(\mathbf{R}^N)$.*
- (iii) *If $p = \infty$, a sequence $(u_n) \subset \mathcal{H}$ has a pointwise limit u if and only if $u \in W^{m,\infty}(\mathbf{R}^N)$ and $u_n \overset{w*}{\rightharpoonup} u$ in $W^{m,\infty}(\mathbf{R}^N)$.*

Proof. (i) Since $mp > N$, there is $\sigma \in (0, 1]$ such that $W^{m,p}(\mathbf{R}^N) \hookrightarrow C^{0,\sigma}(\mathbf{R}^N)$, so that $|u(x) - u(y)| \leqslant M\|u\|_{m,p,\mathbf{R}^N}|x - y|^\sigma$ for all $u \in W^{m,p}(\mathbf{R}^N)$ and all $x, y \in \mathbf{R}^N$, where $M > 0$ is independent of x, y and u. This shows that $\mathcal{H}$ is uniformly equicontinuous. That $\mathcal{H}(\mathbf{R}^N)$ is relatively compact follows from the boundedness of $\mathcal{H}$ in $C_{\{0\}}(\mathbf{R}^N)$.

(ii) If $(u_n) \subset W^{m,p}(\mathbf{R}^N)$ and $u_n \overset{w}{\rightharpoonup} u$ in $W^{m,p}(\mathbf{R}^N)$, then, given $R > 0$, $u_{n_k} \overset{w}{\rightharpoonup} u$ in $C(\overline{B}_R)$ since the embedding $W^{m,p}(\mathbf{R}^N) \hookrightarrow C(\overline{B}_R)$ is continuous. In particular, $u_n \to u$ pointwise on $\overline{B}_R$ (hence on $\mathbf{R}^N$) since the point evaluations are continuous on $C(\overline{B}_R)$.[5]

Conversely, suppose that $(u_n) \subset \mathcal{H}$ has a pointwise limit u. Since $p \in (1, \infty)$, the space $W^{m,p}(\mathbf{R}^N)$ is reflexive and hence there are $v \in W^{m,p}(\mathbf{R}^N)$ and a subsequence (u_{n_k}) such

[5] We purposely ignored the fact that the embedding $W^{m,p}(\mathbf{R}^N) \hookrightarrow C(\overline{B}_R)$ is compact, since this is no longer true in infinite dimensional vector-valued generalizations.

that $u_{n_k} \overset{w}{\rightharpoonup} v$ in $W^{m,p}(\mathbf{R}^N)$. From the above, $u_{n_k} \to v$ pointwise on $\mathbf{R}^N$. This shows that $u = v \in W^{m,p}(\mathbf{R}^N)$ and that u is the only cluster point of (u_n) in the weak topology of $W^{m,p}(\mathbf{R}^N)$, so that $u_n \overset{w}{\rightharpoonup} u$ in $W^{m,p}(\mathbf{R}^N)$.

(iii) Modify the proof of (ii) as follows: First, if $(u_n) \subset W^{m,\infty}(\mathbf{R}^N)$ and $u_n \overset{w*}{\rightharpoonup} u$ in $W^{m,\infty}(\mathbf{R}^N)$, then $u_{n_k} \to u$ uniformly on $\overline{B}_R$ and hence pointwise on $\mathbf{R}^N$ by the compactness of the embedding $W^{m,\infty}(\mathbf{R}^N) \hookrightarrow C(\overline{B}_R)$. For the converse part, use the fact that a bounded sequence in $W^{m,\infty}(\mathbf{R}^N)$ has a weak* convergent subsequence. □

Theorem 9. *Let $m \in \mathbf{N}$ and $p \in (1,\infty)$ be such that $mp > N$ and let $S \subset \mathbf{R}^N$ be any chosen δ-net. For a bounded subset $\mathcal{H} \subset W^{m,p}(\mathbf{R}^N)$, the following statements are equivalent*:

(i) *$\mathcal{H}$ is relatively compact in $C_{\{0\}}(\mathbf{R}^N)$.*
(ii) *If $\tilde{u} \in W^{m,p}(\mathbf{R}^N)$ and there are sequences $(u_n) \subset \mathcal{H}$ and $(\xi_n) \subset S$ with $\lim_{n\to\infty} |\xi_n| = \infty$ such that $\tilde{u}_n := \tau_{\xi_n} u_n \overset{w}{\rightharpoonup} \tilde{u}$ in $W^{m,p}(\mathbf{R}^N)$, then $\tilde{u} = 0$.*

Proof. This follows readily from Lemma 8(i) and (ii) and Theorem 2 with $E = \mathbf{R}$ and $z = 0$. □

If $\mathcal{H}$ is a bounded subset of $W^{m,p}(\mathbf{R}^N)$ with $mp > N$ and $\mathcal{H}$ is relatively compact in $L^q(\mathbf{R}^N)$ for some $q \geqslant p$, then it is trivial that $\mathcal{H}$ is also relatively compact in $L^r(\mathbf{R}^N)$ for every $r \in [q,\infty)$ (use the boundedness of $\mathcal{H}$ in $C_{\{0\}}(\mathbf{R}^N)$). It is less trivial that this remains true for $r = \infty$:

Corollary 10. *Let $m \in \mathbf{N}$ and $p \in (1,\infty)$ be such that $mp > N$ and let $\mathcal{H} \subset W^{m,p}(\mathbf{R}^N)$ be a bounded subset. If $\mathcal{H}$ is relatively compact in $L^q(\mathbf{R}^N)$ for some $q \in [p,\infty)$, then $\mathcal{H}$ is also relatively compact in $C_{\{0\}}(\mathbf{R}^N)$.*[6]

Proof. We use (ii) $\Rightarrow$ (i) in Theorem 9 with $S = \mathbf{R}^N$. Let then $\tilde{u} \in W^{m,p}(\mathbf{R}^N)$ be such that there are sequences $(u_n) \subset \mathcal{H}$ and $(\xi_n) \subset \mathbf{R}^N$ with $\lim_{n\to\infty} |\xi_n| = \infty$ and $\tilde{u}_n := \tau_{\xi_n} u_n \overset{w}{\rightharpoonup} \tilde{u}$ in $W^{m,p}(\mathbf{R}^N)$. Evidently, $\tilde{u}_n \overset{w}{\rightharpoonup} \tilde{u}$ in $L^q(\mathbf{R}^N)$. On the other hand, let $u \in L^q(\mathbf{R}^N)$ and (u_{n_k}) be such that $\|u_{n_k} - u\|_{0,q,\mathbf{R}^N} \to 0$, so that $\|\tilde{u}_{n_k} - \tau_{\xi_{n_k}} u\|_{0,q,\mathbf{R}^N} \to 0$ by translation invariance. It is straightforward to check that $\tau_{\xi_{n_k}} u \overset{w}{\rightharpoonup} 0$ in $L^q(\mathbf{R}^N)$, so that $\tilde{u}_{n_k} \overset{w}{\rightharpoonup} 0$ in $L^q(\mathbf{R}^N)$ and hence $\tilde{u} = 0$. □

For instance, it follows from Corollary 10 and Lions' embedding theorem [10] that the embedding $W^{1,p}_{\text{radial}}(\mathbf{R}^N) \hookrightarrow C_{\{0\}}(\mathbf{R}^N)$ is compact if $p > N$.

Corollary 6 is relevant in the following variant of Theorem 9 when $p = \infty$. The proof follows at once from Corollary 6 and Lemma 8(i) and (iii).

Theorem 11. *Let $Z \subset \mathbf{R}$ be a totally disconnected compact subset and let $S \subset \mathbf{R}^N$ be any chosen δ-net. For a bounded subset $\mathcal{H} \subset W^{m,\infty}(\mathbf{R}^N) \cap C_Z(\mathbf{R}^N)$, $m \in \mathbf{N}$, the following statements are equivalent*:

[6] It is readily checked that the converse is true if $q \in (p,\infty)$, but not if $q = p$.

(i) *$\mathcal{H}$ is relatively compact in $C_Z(\mathbf{R}^N)$.*
(ii) *If $\tilde{u} \in W^{m,\infty}(\mathbf{R}^N)$ and there are sequences $(u_n) \subset \mathcal{H}$ and $(\xi_n) \subset S$ with $\lim_{n\to\infty} |\xi_n| = \infty$ such that $\tilde{u}_n := \tau_{\xi_n} u_n \overset{w*}{\rightharpoonup} \tilde{u}$ in $W^{m,\infty}(\mathbf{R}^N)$, then $\tilde{u}(\mathbf{R}^N) \subset Z$.*

By Corollary 7, Theorem 11(ii) also characterizes the relatively compact subsets $\mathcal{H}$ of $W^{m,\infty}(\mathbf{R}^N) \cap C_{\{z\}}(\mathbf{R}^N)$ for every $z \in Z$.

Theorems 9 and 11 still hold if $\mathbf{R}^N$ is replaced by an unbounded open subset $\Omega \subset \mathbf{R}^N$ with Lipschitz continuous boundary (so that $W^{m,p}(\Omega) \hookrightarrow C_{\{0\}}(\overline{\Omega})$), provided that $K = \overline{\Omega}$ satisfies the conditions described at the end of Section 2. More generally, when a continuous (linear or not) extension operator $\Lambda : W^{m,p}(\Omega) \to W^{m,p}(\mathbf{R}^N)$ is available, a subset $\mathcal{H}$ of $W^{m,p}(\Omega)$ is relatively compact in $C_{\{0\}}(\overline{\Omega})$ if and only if $\Lambda(\mathcal{H})$ is relatively compact in $C_{\{0\}}(\mathbf{R}^N)$, which reduces the problem to the case discussed above.

We now sketch the generalization of Theorem 9 when $\mathbf{R}$ is replaced by a Banach space E. The uniform equicontinuity in part (i) of Lemma 8 relies on the embedding $W^{m,p}(\mathbf{R}^N) \hookrightarrow C^{0,\sigma}(\mathbf{R}^N)$ for some $\sigma \in (0,1]$ when $mp > N$. This is proved by induction on m (starting with $m = 1$, $p > N$) by using the embedding $W^{1,p}(\mathbf{R}^N) \hookrightarrow L^q(\mathbf{R}^N)$ for $q \in [p, p/(N-p))$ if $p \in [1, N]$. The same procedure works with $W^{m,p}(\mathbf{R}^N; E)$: That $W^{1,p}(\mathbf{R}^N; E) \hookrightarrow C^{0,\sigma}(\mathbf{R}^N; E)$ when $p > N$ can be seen by the same proof as when $E = \mathbf{R}$ and the embedding $W^{1,p}(\mathbf{R}^N; E) \hookrightarrow L^q(\mathbf{R}^N; E)$ for $q \in [p, p^*)$ if $p \in [1, N]$ follows from $u \in W^{1,p}(\mathbf{R}^N; E) \Rightarrow \|u\| \in W^{1,p}(\mathbf{R}^N)$ [12, Theorem 1.1 and Corollary 1.1].

If E is reflexive and $p \in (1, \infty)$, then $L^p(\mathbf{R}^N; E)$ is reflexive (Edwards [5]) and hence $W^{m,p}(\mathbf{R}^N; E)$ is reflexive. As a result, part (ii) of Lemma 8 remains true with "pointwise limit" replaced by "pointwise weak limit". Therefore, it remains true as stated if $\mathcal{H}(\mathbf{R}^N)$ is relatively compact in E, for then a pointwise weak limit in E is also a pointwise limit in norm. It follows that if E is reflexive and $\mathcal{H}(\mathbf{R}^N)$ is relatively compact in E (which now must be assumed), Theorem 9 continues to hold with $W^{m,p}(\mathbf{R}^N)$ and $C_{\{0\}}(\mathbf{R}^N)$ replaced by $W^{m,p}(\mathbf{R}^N; E)$ and $C_{\{0\}}(\mathbf{R}^N; E)$, respectively.

Theorem 11 can also be generalized to the case when E is reflexive, but the proof of part (iii) of Lemma 8 does not go through since the embedding $W^{m,\infty}(\mathbf{R}^N; E) \hookrightarrow C(\overline{B}_R; E)$ is not compact in general and there are a few additional technicalities. We omit the details.

5. Application to the properness of ordinary differential operators

As a concrete application, we discuss the properness of a differential operator

$$u \mapsto \dot{u} - \mathbf{F}(u), \tag{6}$$

where $\dot{u} = \frac{du}{dt}$ and $\mathbf{F}$ is the Nemytskii operator associated with a continuous mapping $F : \mathbf{R}^M \to \mathbf{R}^M$, that is,

$$\mathbf{F}(u)(t) := F\big(u(t)\big), \tag{7}$$

for every function $u : \mathbf{R} \to \mathbf{R}^M$. When F also depends upon t, as will occasionally be assumed later, then

$$\mathbf{F}(u)(t) := F\big(t, u(t)\big). \tag{8}$$

It is understood that when $F(u)$ (respectively, $\mathbf{F}(u)$) is used, then $u \in \mathbf{R}^M$ (respectively, u is a function of t with values in $\mathbf{R}^M$).

There are two distinct aspects to properness: properness on the closed bounded subsets and existence of a priori bounds. The latter is not related to the theme of this paper and will not be addressed. Also, properness on the closed bounded subsets is not an issue for operators that can be written, or recast, after some suitable transformation, as compact perturbations of linear isomorphisms. However, differential operators on *unbounded* domains are generally not of this type in C^k, Hölder, or classical Sobolev spaces. In particular, this is true of $\frac{\mathrm{d}}{\mathrm{d}t} - \mathbf{F}$ on the real line.

Properness is especially important for Fredholm operators of index 0. Indeed, while the Leray–Schauder degree can only be used with compact perturbations of linear isomorphisms, many other degree theories have been worked out for various classes of *proper* Fredholm mappings of index 0 (see the discussion in [7]). For instance, the degree developed in [13], superseding the C^2 theory in [6], covers most other special cases and may be used in existence or bifurcation questions in much the same way as the Leray–Schauder degree. The discussion of such issues would take us too far afield, but they should be put in the direct perspective of the contents of this section.

We now turn to the properness properties of $\frac{\mathrm{d}}{\mathrm{d}t} - \mathbf{F}$. Our goal is to illustrate the use of the previous results while minimizing the extraneous difficulties as much as possible. Accordingly, we shall focus on the simpler problems and merely comment on some of the more general ones.

We denote by $Z = F^{-1}(0) \subset \mathbf{R}^M$ the zero set of F, assumed to be *nonempty*, and set

$$C_b^1(\mathbf{R}; \mathbf{R}^M) := \{u \in C^1(\mathbf{R}; \mathbf{R}^M)\colon\ u, \dot{u} \in C_b(\mathbf{R}; \mathbf{R}^M)\}, \tag{9}$$

equipped with the product metric, that is, with the $W^{1,\infty}(\mathbf{R}; \mathbf{R}^M)$ norm. We introduce the space

$$C_Z^1(\mathbf{R}; \mathbf{R}^M) := \{u \in C_b^1(\mathbf{R}; \mathbf{R}^M)\colon\ u \in C_Z(\mathbf{R}; \mathbf{R}^M),\ \dot{u} \in C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)\}, \tag{10}$$

a subspace of $C_b^1(\mathbf{R}; \mathbf{R}^M)$. In what follows, both spaces $C_Z(\mathbf{R}; \mathbf{R}^M)$ and $C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$ are equipped with the distance d_∞, here induced by the $L^\infty(\mathbf{R}; \mathbf{R}^M)$ norm.

It is obvious that $\frac{\mathrm{d}}{\mathrm{d}t}$ maps $C_Z^1(\mathbf{R}; \mathbf{R}^M)$ continuously into $C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$, but perhaps less obvious that $\mathbf{F}$ does the same thing. This and other preliminary items are collected in

Lemma 12. *The Nemytskii operator* $\mathbf{F}$ *has the following properties*:

(i) *It maps continuously* $C_Z(\mathbf{R}; \mathbf{R}^M)$ *into* $C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$. (*In particular,* $\frac{\mathrm{d}}{\mathrm{d}t} - \mathbf{F}$ *maps continuously* $C_Z^1(\mathbf{R}; \mathbf{R}^M)$ *into* $C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$.)
(ii) *It is sequentially weak* continuous from* $W^{1,\infty}(\mathbf{R}; \mathbf{R}^M)$ *into* $L^\infty(\mathbf{R}; \mathbf{R}^M)$.

Proof. (i) If $u \in C_Z(\mathbf{R}; \mathbf{R}^M)$, then u is bounded and hence $u(\mathbf{R}) \subset \overline{B}(0, r)$ (ball in $\mathbf{R}^M$) for some $r > 0$. Let $\varepsilon > 0$ be given. Since F is uniformly continuous on the compact set $Z \cap \overline{B}(0, 2r)$ and $F = 0$ on Z, there is $\delta > 0$ such that $|F(\xi)| < \varepsilon$ whenever $d(\xi, Z \cap \overline{B}(0, 2r)) < \delta$. With no loss of generality, assume that $\delta < r$. Since u tends to Z at infinity, it follows that $d(u(t), Z) < \delta$ for $|t|$ large enough. For any such t, let $z(t) \in Z$ be such that

$|u(t) - z(t)| < \delta < r$, so that $z(t) \in Z \cap \overline{B}(0, 2r)$. As a result, $d(u(t), Z \cap \overline{B}(0, 2r)) < \delta$ and, from the above, $|F(u(t))| < \varepsilon$. This shows that $\mathbf{F}(u) \in C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$.

For the continuity of $\mathbf{F}$, let $(u_n) \subset C_Z(\mathbf{R}; \mathbf{R}^M)$ be such that (u_n) tends to u in $C_Z(\mathbf{R}; \mathbf{R}^M)$. Then, there is $r > 0$ such that $u_n(t), u(t) \in \overline{B}(0, r)$ for all $t \in \mathbf{R}$ and all indices n. That $\mathbf{F}(u_n)$ tends to $\mathbf{F}(u)$ in $C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$ now follows from the uniform continuity of F on $\overline{B}(0, r)$ and the uniform convergence of (u_n) to u on $\mathbf{R}$.

(ii) If $(v_n) \subset W^{1,\infty}(\mathbf{R}; \mathbf{R}^M)$ and $v_n \overset{w*}{\rightharpoonup} v$ in $W^{1,\infty}(\mathbf{R}; \mathbf{R}^M)$, then, once again by Lemma 8(iii), (v_n) tends pointwise to v and hence $(\mathbf{F}(v_n))$ tends pointwise to $\mathbf{F}(v)$ on $\mathbf{R}$. Since $(\mathbf{F}(v_n))$ is bounded in $L^\infty(\mathbf{R}; \mathbf{R}^M)$, it follows (by dominated convergence) that $\mathbf{F}(v_n) \overset{w*}{\rightharpoonup} \mathbf{F}(v)$ in $L^\infty(\mathbf{R}; \mathbf{R}^M)$. □

Theorem 13. *Assume that $Z = F^{-1}(0)$ is nonempty and totally disconnected and that the only solutions $u \in C_b^1(\mathbf{R}; \mathbf{R}^M)$ of the equation $\dot{u} - \mathbf{F}(u) = 0$ are constant functions. Then*:

(i) *The operator $\frac{d}{dt} - \mathbf{F} : C_Z^1(\mathbf{R}; \mathbf{R}^M) \to C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$ is proper on the closed bounded subsets of $C_Z^1(\mathbf{R}; \mathbf{R}^M)$.*

(ii) *For every $z \in Z$, the operator $\frac{d}{dt} - \mathbf{F} : C_{\{z\}}^1(\mathbf{R}; \mathbf{R}^M) \to C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$ is proper on the closed bounded subsets of $C_{\{z\}}^1(\mathbf{R}; \mathbf{R}^M)$.*

Proof. (i) Since the continuity of $\frac{d}{dt} - \mathbf{F}$ was established in Lemma 12(i), we must only show that a bounded sequence $(u_n) \subset C_Z^1(\mathbf{R}; \mathbf{R}^M)$ such that $f_n := \dot{u}_n - \mathbf{F}(u_n) \to f$ in $C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$, has a norm-convergent subsequence in $C_Z^1(\mathbf{R}; \mathbf{R}^M)$. The main part of the proof consists in showing that (ii) ⇒ (i) in Corollary 6, which requires only the closedness of Z, can be used with $\mathcal{H} = (u_n)$.

Indeed, by Remark 3, this yields $u \in C_Z(\mathbf{R}; \mathbf{R}^M)$ and a subsequence (u_{n_k}) such that $u_{n_k} \to u$ in $C_Z(\mathbf{R}; \mathbf{R}^M)$. Then, $\mathbf{F}(u_{n_k}) \to \mathbf{F}(u)$ in $C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$ by the continuity of $\mathbf{F}$, whereas $u \in W^{1,\infty}(\mathbf{R}; \mathbf{R}^M)$ and $u_{n_k} \overset{w*}{\rightharpoonup} u$ in $W^{1,\infty}(\mathbf{R}; \mathbf{R}^M)$ by Lemma 8(iii). Thus, $\dot{u}_{n_k} = \mathbf{F}(u_{n_k}) + f_{n_k} \to \mathbf{F}(u) + f$ in $C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$ and $\mathbf{F}(u) + f = \dot{u}$ since $\dot{u}_{n_k} \overset{w*}{\rightharpoonup} \dot{u}$ in $L^\infty(\mathbf{R}; \mathbf{R}^M)$. This shows that $\dot{u} \in C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$ (so that $u \in C_Z^1(\mathbf{R}; \mathbf{R}^M)$) and that $\dot{u}_{n_k} \to \dot{u}$ in $C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$, whence $u_{n_k} \to u$ in $C_Z^1(\mathbf{R}; \mathbf{R}^M)$.

To complete the proof, we check that the conditions required in part (ii) of Corollary 6 hold with $\mathcal{H} = (u_n)$. Evidently, $Z = F^{-1}(0)$ is closed. Since (u_n) is bounded in $C_Z^1(\mathbf{R}; \mathbf{R}^M)$, i.e., in $W^{1,\infty}(\mathbf{R}; \mathbf{R}^M)$, Lemma 8(i) shows that $\mathcal{H}$ is uniformly equicontinuous and that $\mathcal{H}(\mathbf{R})$ is relatively compact in $\mathbf{R}^M$.

It remains to check the third condition for some δ-net $S \subset \mathbf{R}$. We simply choose $S = \mathbf{R}$ and $\delta = 0$. Let then $(\xi_n) \subset \mathbf{R}$ be a sequence such that $\lim_{n\to\infty} |\xi_n| = \infty$. Set $\tilde{u}_n := \tau_{\xi_n} u_n$, so that $(\tilde{u}_n)$ is bounded in $C_Z^1(\mathbf{R}; \mathbf{R}^M)$, i.e., in $W^{1,\infty}(\mathbf{R}; \mathbf{R}^M)$, and suppose that $(\tilde{u}_n)$ has a pointwise limit $\tilde{u} \in C_b(\mathbf{R}; \mathbf{R}^M)$. By Lemma 8(iii), $\tilde{u} \in W^{1,\infty}(\mathbf{R}; \mathbf{R}^M)$ and $\tilde{u}_n \overset{w*}{\rightharpoonup} \tilde{u}$ in $W^{1,\infty}(\mathbf{R}; \mathbf{R}^M)$, so that

$$\dot{\tilde{u}}_n \overset{w*}{\rightharpoonup} \dot{\tilde{u}} \quad \text{in } L^\infty\big(\mathbf{R}; \mathbf{R}^M\big). \tag{11}$$

From (11) and Lemma 12(ii),

$$\dot{\tilde{u}}_n - \mathbf{F}(\tilde{u}_n) \overset{w*}{\rightharpoonup} \dot{\tilde{u}} - \mathbf{F}(\tilde{u}) \quad \text{in } L^\infty(\mathbf{R}; \mathbf{R}^M). \tag{12}$$

On the other hand, since $f_n := \dot{u}_n - F(u_n) \to f$ in $C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$ and with $\tilde{f}_n := \tau_{\xi_n} f_n$, it is obvious that $\|\tilde{f}_n - \tau_{\xi_n} f\|_{\infty,\mathbf{R}^N} = \|f_n - f\|_{\infty,\mathbf{R}^N} \to 0$ and that $\tau_{\xi_n} f \overset{w*}{\rightharpoonup} 0$ in $L^\infty(\mathbf{R}; \mathbf{R}^M)$ (because f tends to 0 at infinity), whence $\tilde{f}_n \overset{w*}{\rightharpoonup} 0$ in $L^\infty(\mathbf{R}; \mathbf{R}^M)$. Since differentiation and translation commute, we have $\tilde{f}_n = \dot{\tilde{u}}_n - \mathbf{F}(\tilde{u}_n)$ and hence $\dot{\tilde{u}} - \mathbf{F}(\tilde{u}) = 0$ by (12). Since $\tilde{u} \in C_b(\mathbf{R}; \mathbf{R}^M)$ and F is continuous, it follows that $\tilde{u} \in C_b^1(\mathbf{R}; \mathbf{R}^M)$. Thus, $\tilde{u} = c$ since the equation $\dot{u} - \mathbf{F}(u) = 0$ has no other solution in $C_b^1(\mathbf{R}; \mathbf{R}^M)$ by hypothesis. But then, $c \in Z = F^{-1}(0)$ and hence $\tilde{u}(\mathbf{R}^N) \subset Z$. This completes the proof of (i).

(ii) This follows from (i) and the closedness of $C^1_{\{z\}}(\mathbf{R}; \mathbf{R}^M)$ in $C^1_Z(\mathbf{R}; \mathbf{R}^M)$ or, alternatively, by using Corollary 7 instead of Corollary 6 in the proof of (i) above. □

Since F maps $\mathbf{R}^M$ to $\mathbf{R}^M$, the assumption that $F^{-1}(0)$ is totally disconnected is little restrictive in practice. That $\dot{u} - \mathbf{F}(u) = 0$ has no nonconstant solution in $C_b^1(\mathbf{R}; \mathbf{R}^M)$ can be proved under various conditions, the simplest one being $F(u) \cdot u > 0$ for every $u \in \mathbf{R}^M \backslash \{0\}$, a case when $Z = \{0\}$. (If $u \in C^1(\mathbf{R}; \mathbf{R}^M) \backslash \{0\}$ and $\dot{u} - \mathbf{F}(u) = 0$, then $|u|^2$ is strictly increasing. But then, $F(u(t)) \cdot u(t)$ is bounded away from 0 by a positive constant α for $t \geqslant 0$, so that $\frac{\mathrm{d}|u|^2}{\mathrm{d}t} \geqslant 2\alpha$ in $[0, \infty)$ and u is not bounded.)

Other simple cases arise when $F = \nabla\Phi$ is a gradient and $Z = \{z\}$ is a singleton, or when $M = 1$ and $1/F$ is integrable. For example, $F(u) = |P(u)|^\alpha$ where $\alpha \in (0, 1)$ and P is a polynomial with $\deg P > \alpha^{-1}$ and simple real roots (and at least one such root to ensure $Z \neq \emptyset$). If so, $Z = P^{-1}(0)$ is finite.

A scalar second order example, thus corresponding to a first order 2×2 system, is

$$\ddot{v} - \mathbf{g}(v) = 0, \tag{13}$$

with $g \geqslant 0$ (vanishing at least at one point). Every solution is convex, and a bounded convex function on $\mathbf{R}$ is constant. Here, $Z = g^{-1}(0) \times \{0\}$ is totally disconnected if and only if $g^{-1}(0)$ is totally disconnected. In $\mathbf{R}^M$ now, if $G(v) \cdot v \geqslant 0$ for every $v \in \mathbf{R}^M$ and

$$\ddot{v} - \mathbf{G}(v) = 0, \tag{14}$$

then $|v|^2$ is convex, hence constant if v is bounded. If so, $\frac{\mathrm{d}^2|v|^2}{\mathrm{d}t^2} = 0$, that is, $\ddot{v} \cdot v + |\dot{v}|^2 = 0$, so that $\mathbf{G}(v) \cdot v + |\dot{v}|^2 = 0$ and hence $\dot{v} = 0$, i.e., v is constant.

Theorem 13 can be extended to the case when $F = F(t, u)$ is continuous, T-periodic in t and $Z := \{u \in \mathbf{R}^N\colon F(t, u) = 0, \ \forall t \in \mathbf{R}\} \neq \emptyset$. The arguments are similar, but now choosing $S = \{mT\colon m \in \mathbf{Z}\}$ instead of $S = \mathbf{R}$ since only the translations τ_{mT} commute with $\mathbf{F}$. This shows that the option of using a δ-net $S \neq \mathbf{R}$ is needed to handle some applications.

In Theorem 13, the condition that $Z = F^{-1}(0)$ is totally disconnected is nearly optimal, for $\frac{\mathrm{d}}{\mathrm{d}t} - \mathbf{F}$ is *not* proper on the closed bounded subsets of $C^1_Z(\mathbf{R}; \mathbf{R}^M)$ if Z contains a nontrivial C^1 curve. Indeed, it is easily seen that this yields the existence of $u \in C^1(\mathbf{R}; \mathbf{R}^M)$ such that $u(t) = z$ is constant for $|t| \geqslant 1$, $u(0) = z_0 \neq z$ and $u(\mathbf{R}) \subset Z$ (so that $u \in C^1_Z(\mathbf{R}; \mathbf{R}^M)$ and $\mathbf{F}(u) = 0$). Then, $u_n(t) := u(t/n)$ is bounded in $C^1_Z(\mathbf{R}; \mathbf{R}^M)$ and $\dot{u}_n - \mathbf{F}(u_n) = \dot{u}_n \to 0$ in $C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$. Yet, (u_n) has no convergent subsequence in

$C^1_Z(\mathbf{R};\mathbf{R}^M)$ because (u_n) converges pointwise to the constant function z_0 while $u_n(t) = z \neq z_0$ for $|t| \geqslant n$.

A counterexample to Theorem 13(ii) of a different nature is given by the linear second order problem

$$\ddot{v} + v = 0. \tag{15}$$

Given $\varphi \in C_0^\infty(\mathbf{R})$, it is readily checked that if $v_n(t) := \varphi(t/n)\sin t$, then $(v_n, \dot{v}_n)$ is bounded in $C^1_{\{0\}}(\mathbf{R};\mathbf{R}^2)$ and $\ddot{v}_n + v_n \to 0$ in $C_{\{0\}}(\mathbf{R})$, but $(v_n, \dot{v}_n)$ has no convergent subsequence in $C^1_{\{0\}}(\mathbf{R};\mathbf{R}^2)$. Here, $Z = \{0\}$ but of course $\ddot{v} + v = 0$ has bounded nonconstant solutions.

In Theorem 14 below, we prove a variant of Theorem 13 in the $W^{1,p} - L^p$ setting, based on Theorem 9. However, there are a few extra subtleties and different assumptions are involved. Once again, F is t-independent for simplicity but, in contrast to Theorem 13, no condition beyond $0 \in F^{-1}(0)$ is explicitly required of $F^{-1}(0)$.

Theorem 14. *Assume that $F \in C^1(\mathbf{R}^M;\mathbf{R}^M)$ and that $F(0) = 0$. If $p \in (1,\infty)$ and the operator $\frac{\mathrm{d}}{\mathrm{d}t} - \mathbf{F} : W^{1,p}(\mathbf{R};\mathbf{R}^M) \to L^p(\mathbf{R};\mathbf{R}^M)$ is Fredholm, then it is proper on the closed bounded subsets of $W^{1,p}(\mathbf{R};\mathbf{R}^M)$ if and only if the equation $\dot{u} - \mathbf{F}(u) = 0$ has no solution in $W^{1,p}(\mathbf{R};\mathbf{R}^M)\backslash\{0\}$.*

Proof. For the necessity, observe that, if $u \in W^{1,p}(\mathbf{R};\mathbf{R}^M)\backslash\{0\}$ and $\dot{u} - \mathbf{F}(u) = 0$, then $\tau_s u = u(s + \cdot) \in W^{1,p}(\mathbf{R};\mathbf{R}^M)\backslash\{0\}$ has the same norm as u, but $(\tau_s u)_{s\in\mathbf{R}}$ is certainly not relatively compact in $W^{1,p}(\mathbf{R};\mathbf{R}^M)$.

We now address the sufficiency. A repetition, with suitable modifications, of the proof of Theorem 13, shows that if $(u_n) \subset W^{1,p}(\mathbf{R};\mathbf{R}^M)$ is bounded and $f_n := \dot{u}_n - \mathbf{F}(u_n)$ is norm-convergent in $L^p(\mathbf{R};\mathbf{R}^M)$, then (u_n) is relatively compact in $C_{\{0\}}(\mathbf{R};\mathbf{R}^M)$. The modifications include using (the $\mathbf{R}^M$-valued variant of) Theorem 9 instead of Corollary 6 and showing that $\mathbf{F}$ maps $W^{1,p}(\mathbf{R};\mathbf{R}^M)$ into $L^p(\mathbf{R};\mathbf{R}^M)$ and is sequentially weakly continuous (see Remark 15 below).

In the remainder of the proof, we establish the relative compactness of (u_n) in $W^{1,p}(\mathbf{R};\mathbf{R}^M)$ rather than just $C_{\{0\}}(\mathbf{R};\mathbf{R}^M)$. After replacing (u_n) by a subsequence, it suffices to show that if $u_n \to u$ in $C_{\{0\}}(\mathbf{R};\mathbf{R}^M)$, then some subsequence (u_{n_k}) is norm-convergent in $W^{1,p}(\mathbf{R};\mathbf{R}^M)$. Since (u_n) is bounded in $W^{1,p}(\mathbf{R};\mathbf{R}^M)$, it is not restrictive to assume that $u_n \overset{w}{\rightharpoonup} u$ in $W^{1,p}(\mathbf{R};\mathbf{R}^M)$. For $v \in \mathbf{R}^M$, we have

$$F(v) = DF(0)v + G(v)v, \tag{16}$$

where $G(v) := \int_0^1 (DF(sv) - DF(0))\,\mathrm{d}s$, so that $G(0) = 0$ and $G : \mathbf{R}^M \to \mathcal{L}(\mathbf{R}^M)$ is continuous.

Claim. $\mathbf{G}(u_n)u_n \to \mathbf{G}(u)u$ *in $L^p(\mathbf{R};\mathbf{R}^M)$, where $\mathbf{G}$ is the Nemytskii operator associated with G.*

To see this, write $\mathbf{G}(u_n)u_n - \mathbf{G}(u)u = (\mathbf{G}(u_n) - \mathbf{G}(u))u_n + \mathbf{G}(u)(u_n - u)$. Since $u \in C_{\{0\}}(\mathbf{R};\mathbf{R}^M)$ and $G(0) = 0$, it follows that $\mathbf{G}(u) \in C_{\{0\}}(\mathbf{R};\mathcal{L}(\mathbf{R}^M))$ and the decay of $\mathbf{G}(u)$ at infinity implies that the multiplication by $\mathbf{G}(u)$ is a *compact* operator from $W^{1,p}(\mathbf{R};\mathbf{R}^M)$ to $L^p(\mathbf{R};\mathbf{R}^M)$.[7] As a result, since $u_n \overset{w}{\rightharpoonup} u$ in $W^{1,p}(\mathbf{R};\mathbf{R}^M)$, it follows that $\mathbf{G}(u)(u_n - u) \to 0$ in $L^p(\mathbf{R};\mathbf{R}^M)$.

Next, $\mathbf{G}(u_n) - \mathbf{G}(u) \to 0$ in $L^\infty(\mathbf{R};\mathcal{L}(\mathbf{R}^M))$ since $u_n \to u$ in $C_{\{0\}}(\mathbf{R};\mathbf{R}^M)$ and G is uniformly continuous on the compact subsets of $\mathbf{R}^M$. Since (u_n) is bounded in $W^{1,p}(\mathbf{R};\mathbf{R}^M)$ and hence in $L^p(\mathbf{R};\mathbf{R}^M)$, this shows that $(\mathbf{G}(u_n) - \mathbf{G}(u))u_n \to 0$ in $L^p(\mathbf{R};\mathbf{R}^M)$. This proves the claim.

By (16), $f_n := \dot{u}_n - \mathbf{F}(u_n) = \dot{u}_n - DF(0)u_n - \mathbf{G}(u_n)u_n$. From the above claim, the assumption that (f_n) is norm-convergent in $L^p(\mathbf{R};\mathbf{R}^M)$ thus implies that $\dot{u}_n - DF(0)u_n = f_n + \mathbf{G}(u_n)u_n$ is norm-convergent in $L^p(\mathbf{R};\mathbf{R}^M)$. But $\frac{d}{dt} - DF(0)$ is Fredholm from $W^{1,p}(\mathbf{R};\mathbf{R}^M)$ to $L^p(\mathbf{R};\mathbf{R}^M)$ by hypothesis, and *linear* Fredholm operators are proper on closed bounded subsets. This is Yood's criterion (see Deimling [4]), which states that properness on closed bounded subsets characterizes the linear semi-Fredholm operators of index $\nu < \infty$ (including $-\infty$). It follows that (u_n) does contain a norm-convergent subsequence in $W^{1,p}(\mathbf{R};\mathbf{R}^M)$ and the proof is complete. □

Remark 15. To see that $\mathbf{F}: W^{1,p}(\mathbf{R};\mathbf{R}^M) \to L^p(\mathbf{R};\mathbf{R}^M)$ is well defined and sequentially weakly continuous (used above), note first that if $v \in W^{1,p}(\mathbf{R};\mathbf{R}^M)$, then $DF(0)v \in L^p(\mathbf{R};\mathbf{R}^M)$ and $\mathbf{G}(v) \in L^\infty(\mathbf{R};\mathcal{L}(\mathbf{R}^M))$, so that, by (16), $\mathbf{F}(v) = DF(0)v + \mathbf{G}(v)v \in L^p(\mathbf{R};\mathbf{R}^M)$. Next, let $v_n \overset{w}{\rightharpoonup} v$ in $W^{1,p}(\mathbf{R};\mathbf{R}^M)$. Since $\mathbf{F}(v_n) = DF(0)v_n + \mathbf{G}(v_n)v_n$ and $DF(0)$ acts linearly and continuously from $W^{1,p}(\mathbf{R};\mathbf{R}^M)$ to $L^p(\mathbf{R};\mathbf{R}^M)$, it suffices to show that $\mathbf{G}(v_n)v_n \overset{w}{\rightharpoonup} \mathbf{G}(v)v$ in $L^p(\mathbf{R};\mathbf{R}^M)$. If $\varphi \in C_0^\infty(\mathbf{R};\mathbf{R}^M)$, it is clear that $\int_{\mathbf{R}} \mathbf{G}(v_n)v_n \cdot \varphi \to \int_{\mathbf{R}} \mathbf{G}(v)v \cdot \varphi$ since (v_n) tends to v uniformly on the compact subsets of $\mathbf{R}$. Thus, it remains only to check that $(\mathbf{G}(v_n)v_n)$ is bounded in $L^p(\mathbf{R};\mathbf{R}^M)$. This follows from the boundedness of (v_n) in $W^{1,p}(\mathbf{R};\mathbf{R}^M)$, hence in $C_{\{0\}}(\mathbf{R};\mathbf{R}^M)$ (so that $(\mathbf{G}(v_n))$ is bounded in $L^\infty(\mathbf{R};\mathcal{L}(\mathbf{R}^M)))$ and in $L^p(\mathbf{R};\mathbf{R}^M)$.

Like Theorem 13, Theorem 14 is still true when F is periodic in t and $F(\cdot,0) = 0$. That $\dot{u} - \mathbf{F}(u) = 0$ has no solution in $W^{1,p}(\mathbf{R};\mathbf{R}^M)\backslash\{0\}$ holds whenever there is no solution homoclinic to 0. For instance if $F = \nabla\Phi$ is a gradient (in particular, $M = 1$), irrespective of $F^{-1}(0)$.

Of course, no solution homoclinic to 0 exists if $DF(0)$ is positive or negative definite, which also ensures that $\frac{d}{dt} - \mathbf{F}$ is Fredholm (see below). The nonexistence issue has also been investigated, with a very different motivation, in more challenging problems that do not comply with general criteria; see Amick and McLeod [2] (traveling waves) or Hayashi [8] (neural networks), among others. On the other hand, the second order scalar equation

$$\ddot{v} - v + v^3 = 0, \tag{17}$$

[7] But $u \mapsto \mathbf{G}(u)u$ is *not* compact from $W^{1,p}(\mathbf{R};\mathbf{R}^M)$ to $L^p(\mathbf{R};\mathbf{R}^M)$; the proof that $\mathbf{G}(u_n)u_n \to \mathbf{G}(u)u$ in $L^p(\mathbf{R};\mathbf{R}^M)$ will use that u_n tends uniformly to u, which is not true for an arbitrary weakly convergent sequence in $W^{1,p}(\mathbf{R};\mathbf{R}^M)$. This is why the information provided by Theorem 9 is crucial.

(equivalent to a first order system) has the nonzero solution $v \in W^{2,p}(\mathbf{R})$ given by

$$v(t) = \frac{\sqrt{2}}{\cosh t} \tag{18}$$

and hence the corresponding operator is not proper on the closed bounded subsets of $W^{2,p}(\mathbf{R})$. Problems of this sort may have peculiar properties; see [14,15].

A sufficient condition for the Fredholmness of $\frac{\mathrm{d}}{\mathrm{d}t} - \mathbf{F}$, required in Theorem 14, is that $DF(0) \in \mathcal{L}(\mathbf{R}^M)$ has no imaginary eigenvalue. If so, the index is 0; see Sacker [18], or [17,19]. When F is C^1, the same spectral condition ensures that $\frac{\mathrm{d}}{\mathrm{d}t} - \mathbf{F}$ is Fredholm of index 0 between $C^1_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$ and $C_{\{0\}}(\mathbf{R}; \mathbf{R}^M)$. It is satisfied by the counterexample $\ddot{v} - v + v^3$ above, but not by $\ddot{v} + v$, discussed earlier.

The case when $F = F(t, u)$ is only "asymptotically" periodic is discussed in [19] when $p = 2$ and the system is Hamiltonian. Roughly speaking, asymptotic periodicity means that $F(t, u)$ looks like some limiting operator $F^{\infty}(t, u)$ or $F^{-\infty}(t, u)$ when $t \to \infty$ or $t \to -\infty$, respectively, where both F^{∞} and $F^{-\infty}$ are periodic in t, with possibly different periods. Theorems 13 and 14 can be extended to this case as well. What now matters is that the limiting equations $\dot{u} - \mathbf{F}^{\infty}(u) = 0$ and $\dot{u} - \mathbf{F}^{-\infty}(u) = 0$ have no nontrivial solutions.

In [17], Theorem 14 is used when $p = 2$ to prove properness for boundary value operators on $W^{1,p}(\mathbf{R}_+; \mathbf{R}^M)$ (half-line), which, together with a priori bounds, yields existence results by degree arguments. Interestingly, even for problems on the half-line, the useful criterion for properness remains that $\dot{u} - \mathbf{F}(u) = 0$ has no nontrivial solution on the *whole* line.

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