Projective limits of group rings

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Abstract


A finite group $G$ may be written as a projective limit of certain quotients $G_i$. Denote by $\Gamma$ the corresponding projective limit of the integral group rings $\mathbb{Z}G_i$. The basic topic of the paper is the question whether $\Gamma$ may be a replacement of $\mathbb{Z}G$. In particular, this is studied in connection with the isomorphism problem of integral group rings and with the conjecture of Zassenhaus that different group bases of $\mathbb{Z}G$ are conjugate within $\mathbb{Q}G$.

Using such projective limits, a Čech style cohomology set yields obstructions for these conjectures to be true, if $G$ is soluble. This is used to construct two non-isomorphic groups as projective limits such that the projective limits of the corresponding group rings are semi-locally isomorphic.

On the other hand, it is shown that for special classes of groups certain $p$-versions of the Zassenhaus conjecture hold. These $p$-versions are weaker than the conjecture but still provide a strong positive answer to the Isomorphism problem. In particular, such $p$-versions hold when $G$ has a nilpotent commutator subgroup or when $G$ is a Frobenius or a 2-Frobenius group.

1. Introduction

In a series of papers [1–3] Karl Gruenberg and the second author have characterized those finite soluble groups $G$, whose augmentation ideal of the integral group ring $\mathbb{Z}G$ decomposes. Combining that description with results in this paper we can make the following conclusion:

Assume that the augmentation ideal of $\mathbb{Z}G$ decomposes. If $\mathbb{Z}G = \mathbb{Z}H$, then $G \cong H$.

This result should be regarded as an answer to one of the basic questions of the integral representation theory of finite groups:

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Which properties of the finite group are reflected by its integral representations, and hence by its integral group ring?

We shall deal in this article with the Isomorphism problem, the Zassenhaus conjecture and certain $p$-versions of the Zassenhaus conjecture from a conceptual point of view, by defining a Čech style cohomology set $\mathcal{H}$, which yields obstructions for these conjectures to be true.

For this we try to describe a finite group $G$ as projective limit with respect to certain families of normal subgroups $\{N_i\}_{1 \leq i \leq n}$; i.e.

$$G = \{(g_1, \ldots, g_n) \mid g_i \in G, g_i = g_i \text{ in } G/(N_i \cdot N_j)\}. \tag{1}$$

In Section 2 we show in Lemma 2.3 that this can be done provided $\bigcap N_i = 1$ and for every $p \in \pi(G)^2$ there is an index $i$ with $(p, |N_i|) = 1$. This applies in particular, if $N_i = O_{p_i}(G)$, where $\{p_i\} = \pi(G)$.

The basic question mentioned above should also be seen as the comparison between the category of groups and the category of rings. The behaviour of projective limits is different in both cases. If $G$ is the projective limit of the quotients $G_i$, then the projective limit $\Gamma_G = \Gamma_G(\{G_i\})$ of the group rings $\mathbb{Z}G_i$ does not coincide with $\mathbb{Z}G$. It is in general a relatively small proper quotient of $\mathbb{Z}G$. However, it nevertheless reflects many properties of the integral group ring and one topic of this article is to demonstrate that this projective limit seems to be an interesting substitute for the integral group ring. For further properties of this projective limit we refer to [8].

Let us first recall the problems we shall consider:

**Problem 1.1.** Let us assume that $\mathbb{Z}G = \mathbb{Z}H$ or that $\Gamma_G = \Gamma_H$ as augmented algebras.

1. The Isomorphism problem asks whether there exists an isomorphism $\rho : G \to H$.
2. The $p$-version of the Zassenhaus conjecture asks whether in addition such a $\rho = \rho_p$ can be chosen in such a way that its restriction to a Sylow $p$-subgroup $P$ is given by conjugation with an element $a_p \in \mathbb{Q}G$, $a_p \in \mathbb{Q} \otimes \mathbb{Z} \Gamma_G$ resp.
3. We shall say that the $p$-version of the Zassenhaus conjecture holds simultaneously for all $p$, if there exists an isomorphism $\rho : G \to H$ such that its restriction to every Sylow subgroup $S$ is given by conjugation with an element $a_S \subset \mathbb{Q}G$, $a_S \subset \mathbb{Q} \otimes \mathbb{Z} \Gamma_G$ resp., where $a_S$ will in general depend on $S$.

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$^1$ This idea goes back to the collaboration of L.L. Scott with the second author cf. [9, 10].
$^2$ For the finite group $G$ we denote by $\pi(G)$ the set of rational prime divisors of $|G|$.
$^3$ $O_p$ is the largest normal subgroup of order prime to $p$.
$^4$ This is defined in analogy to formula (1).
$^5$ $\Gamma_G$ is in a natural way augmented, and it is well known that assuming the above equality as augmented algebras is no restriction.
$^6$ This version was considered by the first author in [5, 5.13].
The Zassenhaus conjecture asks whether the above element \( a_p \) can be chosen to be independent of \( p \) for all primes \( p \), i.e., whether \( G \) and \( H \) are conjugate in \( QG, Q \otimes Z \Gamma_G \) resp.

The main result of this paper gives an explicit description of \( H \) in terms of \( G \) provided \( ZG = ZH \) as augmented algebras, in case \( G \) is soluble. Before we can state it, we have to introduce some more notation:

We show that \( G \) is the projective limit of the groups \( G_i = G/O_{p_i}(G) \), and we put \( G_{ij} = G/(O_{p_i} \cdot O_{p_j}) \). Denote the projective limit of the group rings \( ZG_i \) by \( \Gamma_{G_i}(C_i) \). Assume that we are given a ‘cocycle’ \( \rho = (\rho_{ij}) \) (cf. Definition 3.3) of conjugacy class preserving automorphisms \( \rho_{ij} \) of \( G_{ij} \). Then

\[
G(\rho) = \left\{ (g_i) \in \prod_{1 \leq i \leq n} G_i \mid \rho_{ij}(g_i \cdot G_{ij}) = (g_i \cdot G_{ij}) \right\}
\]

is a group. In Section 3 we elaborate on the question of when \( G \) and \( G(\rho) \) are isomorphic. One of the main results, proved in Section 4, is then as follows.

**Theorem 1.2.** Assume that \( \Gamma_G(C_0) = \Gamma_H(C_0) \) as augmented algebras and that \( G \) is soluble, then the Zassenhaus conjecture holds for the group rings \( ZG_i \) (cf. [12]), and there exists a cocycle \( \rho \) such that \( H = G(\rho) \).

Note that the hypothesis of Theorem 1.2 is in particular satisfied, if \( ZG = ZH \). So the result shows that then \( H = G(\rho) \), which shows in particular, that \( G \) and \( H \) share many properties.

In the last section the above theorem is used to construct two non-isomorphic groups \( G \) and \( H \) as projective limits such that the projective limits of the corresponding group rings are semi locally isomorphic. We do not know, whether or not for these groups \( ZG = ZH \).

This construction is based on a special class of groups, where we have necessary and sufficient conditions for when

\[
Z_{\pi(G)} \otimes Z \Gamma_G(C_0) = Z_{\pi(H)} \otimes Z \Gamma_H(C_0).
\]

Let \( H \) be a soluble group and consider it as projective limit with respect to \( \{ M_i \}_{1 < i < n} \), where \( \{ M_i \}_{1 < i < n} \) is contained in \( C_0 \) (cf. Remark 2.4(2)). Assume that \( H_0 = H_{ij} \) is the same for all pairs \( \{ i, j \}, i \neq j \). For the kernels \( K_i = \text{Ker}(H_i \rightarrow H_0) \) we require that \( K_i \) is a Sylow \( p_i \)-subgroup of \( H_i \). Then \( Z_{\pi(H)} \otimes Z \Gamma_H(C_0) = Z_{\pi(H)} \otimes Z \Gamma_H(C_0) \) if, and only if, \( G = H(\rho) \) for a cocycle \( \rho \). Moreover, \( H = H(\rho) \) if, and only if, \( \rho \) is a coboundary (cf. Proposition 6.1).

The importance of the \( p \)-version of the Zassenhaus conjecture lies in the fact

\[7\] This means that \( \rho_n = 1 \) and \( \rho_{ij} = \rho_{ij}^{-1} \).
that it holds for $\mathbb{Z}G$, if it holds for $\Gamma_G$, cf. Proposition 4.3. Note that this is definitely false for the Zassenhaus conjecture.

This is used in Section 4 to prove the $p$-version of the Zassenhaus conjecture for the following classes of groups:

**Theorem 1.3.** Assume that

1. $G$ has a nilpotent normal subgroup $N$ such that $G/N$ is nilpotent,
2. for each quotient $X$ of $G/N$ the group $\text{Aut}_p(X)$ consists of inner automorphisms only.

Then the $p$-version of the Zassenhaus conjecture holds simultaneously for all $p$ for $\Gamma_G$ and for $\mathbb{Z}G$.

An immediate consequence is the following:

**Corollary 1.4.** Assume that $G/F(G)$ is abelian. Then the $p$-version of the Zassenhaus conjecture holds simultaneously for all $p$ for $\Gamma_G$ and for $\mathbb{Z}G$.

The next result covers the class of soluble groups with decomposable integral augmentation ideal.

**Theorem 1.5.** If $G$ is a Frobenius group or a 2-Frobenius group, then the $p$-version of the Zassenhaus conjecture holds.

2. Projective limits of groups

Let $G$ be a finite group and let $\mathcal{N} = \{N_i | 1 \leq i \leq n\}$ be a family of normal subgroups. We let $\phi_i : G \to G/N_i = G_i$ be the natural reduction. $\mathcal{P}_n$ is the powerset of $\{1, \ldots, n\}$; it is partially ordered by inclusion. For $S \subseteq \mathcal{P}_n$ we set $G_S = G/(\prod_{i \in S} N_i)$ and $\phi_S : G \to G_S$ denotes the natural projection. For $S \subseteq T$ we have a corresponding induced homomorphism $\phi_{S,T} : G_S \to G_T$. Then the set $\{G_S, \phi_{S,T} | S \subseteq \mathcal{P}_n\}$ is a projective system, and we can form the projective limit

$$\hat{G} = \lim_{\to} \text{proj}_{S \subseteq \mathcal{P}_n}(G_S, \phi_{S,T})$$

$$= \{(g_S)_S = \phi_S(g_S) \in G_S, \phi_{S,T}(g_S) = \phi_{S,T}(g_{S'}) \text{ for } S, S' \subseteq T\}.$$

The special structure of the index set simplifies the situation considerably. For the

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*The group $\text{Aut}_p(X)$ consists of those automorphisms $\rho$ of the group $X$ such that for every prime $p$ and every $p$-power element $x \in X$ the elements $x$ and $\rho(x)$ are conjugate in $X$.

*F(G), the Fitting subgroup of $G$, is the largest nilpotent normal subgroup of $G$.

*A group $G$ is called a 2-Frobenius group, if it has a normal series $1 < N < T < G$ such that $T$ is a Frobenius group with Frobenius kernel $N$ and $G/N$ is a Frobenius group with Frobenius kernel $T/N$. The group $T$ is called the lower, the group $G/N$ is called the upper Frobenius group of $G$.

*Here $\prod$ denotes the product of subgroups inside $G$. 
sake of simplicity we shall write $G_i$ for $G_{i(1)}$ and $G_i$ for $G_{i(i)}$. Similar abbreviations we use for maps; in particular, we write $\phi_i$ for $\phi_{i(i)}$. Then $\phi_i \neq \phi_j$, but $G_{ij} = G_i$, and $G_i = G_i$.

**Claim 2.1.** Let

$$\hat{G}_i = \{(g_i)_{1 \leq i \leq n} \mid g_i \in G_i, \phi_{ij}(g_i) = \phi_{ji}(g_i)\};$$

then $\hat{G} = \hat{G}_i$.

**Proof.** Because of the universal property of the projective limit we get a natural map $\alpha: \hat{G}_i \to \hat{G}$,

$$\{(g_i)_{1 \leq i \leq n} \mid g_i \in G_i\} \to \{(g_S)_{s \in \pi_n} \mid g_S = \phi_{i,s}(g_i)\},$$

which is surely an isomorphism. $\square$

In the sequel we shall identify the groups $\hat{G}$ and $\hat{G}_i$. Since the elements in $G$ satisfy the above relations of the projective limit, there is a unique map

$$\gamma: G \to \hat{G}, \quad g \mapsto (g_i = g \cdot N_i)_{1 \leq i \leq n},$$

which has kernel $\bigcap_{1 \leq i \leq n} N_i$. Thus $\gamma$ is injective if, and only if, $\bigcap_{1 \leq i \leq n} N_i = 1$. In general it is not so easy to determine when the map $\gamma$ is surjective. This is surely the case, if $n = 2$, since then $G$ is the pullback of $G/N_1$ and $G/N_2$ over $G/(N_1 \cdot N_2)$.

**Example 2.2.** Let $G = \langle a, b \rangle$ be Klein’s four group, and let

$$N_1 = \langle a \rangle, \quad N_2 = \langle b \rangle \quad \text{and} \quad N_3 = \langle a \cdot b \rangle.$$ 

Then the injection from $G$ into the projective limit of this system is not surjective.

There is however a situation, which is relevant to the Isomorphism problem, in which $\gamma$ is surjective:

**Lemma 2.3.** Let $G$ be a finite group, and let $\{N_i\}_{1 \leq i \leq n}$ be a family of normal subgroups of $G$. Assume that

1. $\bigcap_{1 \leq i \leq n} N_i = \{1\}$,
2. for every rational prime divisor $p$ of $|G|$ there is at least one index $i = i(p)$ such that $(p, |N_{i(p)}|) = 1$.

Then $G$ is the projective limit of $\{G/N_i\}_{1 \leq i \leq n}$; i.e. $\gamma$ is an isomorphism.

**Remark 2.4.** (1) A family of normal subgroups $\{N_i\}$ satisfying the two conditions of the above Lemma 2.3 will be denoted by $N^0$. 


(2) The above conditions are satisfied for example for $G$ a finite group, if $C_0 = \{ N_i = O_{p_i}(G) \}_{1 \leq i \leq n}$, where $\{ p_i \}_{1 \leq i \leq n}$ runs over all prime divisors of $|G|$. Here we take $N_{i(p_i)} = N_i$. In the special case, if $N^{(0)} \subseteq C_0$, we write simply $C_0$.

(3) The above result holds more generally for $G$ a periodic group (i.e., every element in $G$ has finite order—for example a locally finite group) and a countable family of normal subgroups.

**Proof.** Consider the subgroup $G_0$ of the direct product $\prod_{i=1}^n G_i$ defined by

$$G_0 = \{(g_1N_1, \ldots, g_iN_i, \ldots, g_nN_n) \mid g_i \equiv g_j \bmod N_i \cdot N_j \}.$$  

(2)

By Claim 2.1 we may identify $\hat{G} = \lim_{\to} \text{proj}_{i \in \mathbb{N}}(G_i, \phi_i)$ with $G_0$. The image of $\gamma : G \to \hat{G}$ consists of $\{(gN_1, \ldots, gN_n) \mid g \in G\}$. We shall use induction on $k$ to show that $\gamma$ is surjective, by proving that an element $x = (g_1N_i, \ldots, g_nN_n) \in G_0$ lies in $\text{Im}(\gamma)$ provided in $x$ there are $n - k$ cosets with the same representative.

If $k = 0$, then obviously $x \in \text{Im}(\gamma)$. Assume that the statement is true for every element in $G_0$ with $n - (k - 1)$ identical representatives. If necessary after renumbering, we may assume that we are given the element—note that $k > 0$—

$$x = (g_1N_i, \ldots, g_kN_k, g_{k+1}N_{k+1}, \ldots, g_{n}N_n) .$$

Because $\gamma$ is a group homomorphism, we can assume—applying the case $k = 0$—that

$$x = (g_1N_i, \ldots, g_kN_k, N_{k+1}, \ldots, N_n) .$$

Moreover, for the same reason we may assume that $x$ has order $p^m$, for a rational prime number $p$. So in particular all components of $x$ have this order. We now have to distinguish two cases:

1. If there is an index $k + 1 \leq j \leq n$ such that $\left( p, \frac{|N_j|}{|N_i|} \right) = 1$, then we argue as follows, to show that $g_i \in N_i$: By the definition of $G_0$ we get that $g_i \in N_i \cdot N_j$. If we consider the natural homomorphism $\rho : N_i \cdot N_j \to (N_j \cdot N_i)/N_i = N_j/(N_j \cap N_i)$, then $\rho(g_i) = 1$, since $g_i$ is a $p$-element and $N_j$ has order prime to $p$. Thus $g_i \in \ker(\rho) = N_i$ and we are done by induction.

2. We may thus assume that $p$ divides $|N_i|$ for all $k + 1 \leq j \leq n$. Now we invoke the essential hypothesis (2) in Lemma 2.3: Then there must exist an index $1 \leq i \leq k$ such that $\left( p, \frac{|N_i|}{|N_j|} \right) = 1$. But then a similar argument as above shows that $g_i \in N_j$ for every $k + 1 \leq j \leq n$, and we are reduced to the case $k - 1$. In fact, our element $x$ can then be written as

$$x = (g_1N_i, \ldots, N_i, \ldots, g_kN_k, N_{k+1}, \ldots, N_n) .$$

\[ \square \]

\[ ^{12} \text{Note that this is a set, and so if } N_i = N_j, \text{ then it occurs only once.} \]
The projective limit is easily handled, if all the groups $G_{ij}$ coincide. This will be a case of special importance in our considerations on the Isomorphism problem.

**Claim 2.5.** Assume that $G$ is the projective limit of the groups $G_i = G/N_i$, and that for each pair $(i, j)$ with $i \neq j$ the groups $G_{ij} = G/(N_i \cdot N_j)$ coincide. Then the projective limit consists of\( \{(g_i)_{1 \leq i \leq n} \mid g_i \in G_i, \ \phi_{ij}^G(g_i) = \phi_{ij}^G(g_j), \ 2 \leq i \leq n\}\). A family of isomorphisms $\sigma_i : H_i \to G_i$, $1 \leq i \leq n$, gives rise to an isomorphism\( \lim_{\text{proj}}(H_i, \phi_{ij}^H) \to G \) if, and only if, $\sigma_i = \sigma_j \mod G_{ij}$; i.e. $\phi_{ij}^G \cdot \sigma_i = \phi_{ji}^G \cdot \sigma_j$.

**Proof.** Since the groups $G_{ij}$ are the same for all $(i, j)$, this means, that $N_i \cdot N_j = N_0$ is the same for all $(i, j)$. But then the projective limit consists of\( \{(g_i \cdot N_i)_{1 \leq i \leq n} \mid g_i = g_j \mod N_0\}\). The condition of the claim says $g_i = g_j \mod N_0$ for all $2 \leq i \leq n$. The statement now follows from the fact that $\mod N_0$ is an equivalence relation.

The statement about homomorphisms follows with similar arguments. \(\square\)

### 3. Čech-cohomology

These results were essentially noted by L.L. Scott in collaboration with the second author (cf. [10]).

We assume that the finite group $G$ is a projective limit of the groups $G_i = G/N_i$, $1 \leq i \leq n$, and we use the notation of Section 2. The maps

$$\phi_{S,T} : G_S \to G_T \quad \text{for } S \subseteq T$$

induce augmented homomorphisms

$$\phi_{S,T} : \mathbb{Z}G_S \to \mathbb{Z}G_T. \quad 13$$

Though $G$ is the projective limit of $\{G_S, \phi_{S,T}\}$, the group ring $\mathbb{Z}G$ is by no means the projective limit of $\{\mathbb{Z}G_S, \phi_{S,T}\}$. Let $\mathcal{N} = \{N_i\}$. As a matter of fact

$$\Gamma_c(\mathcal{N}) = \lim_{\text{proj}}_{S \in \mathcal{P}}(\mathbb{Z}G_S, \phi_{S,T}) \quad 14$$

is in general rationally a proper quotient of $\mathbb{Z}G$. The induced map $\phi : \mathbb{Z}G \to \Gamma_c(\mathcal{N})$ has kernel

$$\text{Ker}(\phi) = \bigcap_{1 \leq i \leq n} I(G, N_i), \quad (4)$$

where $I(G, N_i)$ is the kernel of the natural map $\mathbb{Z}G \to \mathbb{Z}G_i$. As for groups one shows

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13 It should not cause any confusion that we use the same name for the group homomorphism and the map induced on the group ring.

14 We use the same notation as for groups.
Let $H$ be a finite group with family of normal subgroups $\mathcal{M} = \{M_i\}$. Then the projective limits $\Gamma_G(N)$ and $\Gamma_H(M)$ are equal provided the following hold:

1. $|M_i| = |N_i|$, 
2. $\Gamma_G(\{N_i\}) = \Gamma_H(\{M_i\})$ as projective limits; i.e. we also have an equality $\mathbb{Z}G/N_i = \mathbb{Z}H/M_i$ compatible with the maps $\phi_{ij}^H$ and $\phi_{ij}^G$.

Note that it need not be the case that $\mathbb{Z}G = \mathbb{Z}H$.

**Definition 3.1.** Let $G$ be a finite soluble group. Then we have seen above that $G$ is the projective limit of any set $\mathcal{O}$. In this case we shall write $\Gamma_G(\mathcal{O})$ for the projective limit of the group rings $\mathbb{Z} G/O_i$ for $O_i \in \mathcal{O}$.

It should be noted that $G$ injects into $\Gamma_G(\mathcal{O})$, since $G$ is the projective limit of the groups $G/N_i$. Moreover, in the natural map $\phi: \mathbb{Z} G \to \Gamma_G(\mathcal{O})$ the kernel of $\phi$, $\text{Ker}(\phi) = \bigcap_{1 \leq i \leq n} I(G, N_i)$ is a characteristic submodule, since $O_p(G)$ is a characteristic subgroup of $G$.

We now come to the definition of various Čech style cohomology sets.

**Definition 3.2.** For the finite group $G$ we denote by $\text{Aut}(G)$ its group of automorphisms. $\text{Aut}_p(G)$ denotes the subgroup consisting of those automorphisms $\gamma \in \text{Aut}(G)$ such that for every $p$ and for every $p$-power element $g \in G$ (i.e. it has order a power of $p$) the elements $g$ and $\gamma(g)$ are conjugate in $G$; $\text{Aut}_p(G)$ is the one with $\gamma(g)$ and $g$ conjugate for every $g \in G$.

**Definition 3.3.** (1) Let $G$ be the projective limit with respect to normal subgroups $\{N_i\}_{1 \leq i \leq n}$ (cf. Section 2). We require that the normal subgroups $N_i$ are characteristic. We write $G_i = G/N_i$ and $G_{ij} = G/(N_i \cdot N_j)$ with natural homomorphisms $\phi_i: G \to G_i$ and $\phi_{ij}: G_i \to G_{ij}$.

We use the notation $G$ for $G$, if we want to stress that we view $G$ as a projective limit. We define the cocycles $Z(G, \text{Aut}_*(G)) = \{(\rho_{ij})_{1 \leq i, j \leq n} \mid \rho_{ij} \in \text{Aut}_*(G_{ij}), \rho_{ii} = \text{id}, \rho_{ij}^{-1} = \rho_{ji}\}$, where $\text{Aut}_*(-)$ stands for $\text{Aut}(-)$ or $\text{Aut}_v(-)$ or for $\text{Aut}_p(-)$.

Note that this is in general not a group with multiplication componentwise—for this one needs $\sigma_{ij} \cdot \rho_{ij} = \rho_{ij} \cdot \sigma_{ij}$ if $(\sigma_{ij})$ denotes another cocycle.

(2) We next define an equivalence relation on $Z(G, \text{Aut}_*(G))$: $\rho_i \in \text{Aut}_*(G_i)$ induces an automorphism $\bar{\rho}_i$ of $\text{Aut}_*(G_{ij})$. (One should write $\bar{\rho}_i$, but our notation is more suggestive for the cocycle setup.) Now

\[\text{Instead of requiring that } N_i \text{ is characteristic, it is often enough to require that } N_i \text{ is } *-\text{invariant; i.e. invariant with respect to } \text{Aut}_v(G). \text{ Note that every normal subgroup of } G \text{ is both } \text{Aut}_v(G) \text{ and } \text{Aut}_v(G) \text{ invariant.}\]
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$(\rho_{ij}) = (\sigma_{ij})$ if, and only if, $\tilde{\rho}_i \cdot \rho_{ij} \cdot \tilde{\rho}_{ij}^{-1} = \sigma_{ij}$,

for $\rho_i \in \text{Aut}_*(G_i)$, $\rho_j \in \text{Aut}_*(G_j)$, resp. This is easily seen to be an equivalence relation. With $(\rho_{ij})$ also the family $(\tilde{\rho}_i \cdot \rho_{ij} \cdot \rho_{ij}^{-1})$ is a cocycle for $\rho_i \in \text{Aut}_*(G_i)$, $1 \leq i \leq n$.

The equivalence classes form a pointed set, denoted by $\hat{H}(G, \text{Aut}_*(G))$. The class of the identity is the point and consists of the coboundaries

$$B(G, \text{Aut}_*(G))$$

which is easily seen to be a subset of the cocycles. This is a Čech style cohomology set\(^{16}\), and if we consider maps of such sets, then these are morphisms in the category of pointed sets.

The importance of this cohomology is apparent, if one deals with the question of isomorphisms of projective limits. Let $G$ be the projective limit induced from $\{G_i\}_{1 \leq i \leq n}$ and let $H$ be the projective limit of $\{H_i = G/M_i\}_{1 \leq i \leq n}$. Assume that we are given isomorphisms

$$\sigma_i : G_i \to H_i.$$  \hspace{1cm} (6)

**Definition 3.4.** With the notation in (6) we define

$$\text{Iso}_*(\sigma_i, G, H) = \{\tau : G \to H \mid \tau_i = \rho_i^{-1} \cdot \sigma_i : G_i \to H_i \text{ for } \rho_i \in \text{Aut}_*(H_i)\}.$$  

Similarly $\text{Iso}_*(\sigma, G, H)$ is defined for $\sigma : G \to H$.

Suppose that $\sigma_i$ induces an isomorphism from $G_{ij}$ to $H_{ij}$ for $1 \leq i, j \leq n$. As before we denote the induced isomorphism by $\tilde{\sigma}_i$. The obstruction, to when $G$ and $H$ are isomorphic lies in the cocycle

$$\sigma_{ij} = \tilde{\sigma}_i \cdot \tilde{\sigma}_j^{-1} : H_{ij} \to H_{ij}, \quad 1 \leq i, j \leq n,$$

$$\quad (\sigma_{ij}) \in Z(H, \text{Aut}_*(H)).$$  \hspace{1cm} (7)

In fact, we have the following:

**Lemma 3.5.** Let $\sigma_{ij}$ be defined as in (7). Then there exists $\tau \in \text{Iso}_*(\sigma_i, G, H)$ if, and only if, the cocycle $(\sigma_{ij})$ lies in $B(H, \text{Aut}_*(H))$; i.e. there exist $\rho_i \in \text{Aut}_*(H_i)$ with $\sigma_{ij} = \tilde{\rho}_i \cdot \tilde{\rho}_{ij}^{-1}$.

\(^{16}\) Our projective limit should be compared to the covering of a topological space by $n$ open sets.
Proof. Let us assume that \((\sigma_{ij})\) is a coboundary; i.e. there exist \(\rho_i \in \text{Aut}_*(H_i)\) with \(\sigma_{ij} = \rho_i \cdot \rho_j^{-1}\). We consider the family
\[
(\tau_i = \bar{\rho}_i^{-1} \cdot \sigma_i : G_i \to H_i)_{1 \leq i \leq n}.
\]
Then
\[
\tau_i \cdot \tau_j^{-1} = \bar{\rho}_i^{-1} \cdot \sigma_i \cdot \sigma_j^{-1} \cdot \bar{\rho}_j = \bar{\rho}_i^{-1} \cdot \bar{\rho}_i \cdot \bar{\rho}_j^{-1} \cdot \bar{\rho}_j = 1.
\]
Thus \((\tau_i)_{1 \leq i \leq n}\) induces an isomorphism from \(G\) to \(H\),
\[
\tau : (g_1, \ldots, g_n) \mapsto (\tau_1(g_1), \ldots, \tau_n(g_n)).
\]
Conversely, assume that \(\tau \in \text{Iso}_*((\sigma_i), G, H)\) is given. This induces isomorphisms \(\tau_i \in \text{Iso}_*((\sigma_i), G_i, H_i)\). We then define \(\rho_i = \tau_i \cdot \sigma_i^{-1}\), and hence \(\rho_i \in \text{Aut}_*(H_i)\). A direct calculation then shows that \(\sigma_{ij} = \rho_i \cdot \rho_j^{-1}\), and \((\sigma_{ij})\) is a coboundary in \(B(H, \text{Aut}_*(H))\). \(\square\)

The special case when all \(G_{ij}\) coincide needs some attention. For this we have to modify our cohomology set to adopt it to this situation.

Definition 3.6. Let \(G\) be a projective limit and assume, that \(G_{ij} = G_{ji}\) is the same for all \(i \neq j\). We define the cocycles
\[
Z \diamond (G, \text{Aut}_*(G)) = \{(\rho_{ij})_{1 \leq i, j \leq n} \mid \rho_{ii} \in \text{Aut}_*(G_0), \rho_{ii} = \text{id}, \rho_{ij} \cdot \rho_{jk} = \rho_{ik}\}. \tag{17}
\]
This is in general not a group. The equivalence relation and the coboundaries \(B \diamond (G, \text{Aut}_*(G))\) are defined as in Definition 3.3. The corresponding cohomology set is then denoted by
\[
\tilde{H} \diamond (G, \text{Aut}_*(G)).
\]
In the above situation, a cocycle is uniquely determined by \(\{\rho_{ii} \mid 2 \leq i \leq n\}\); accordingly, the condition for a coboundary is that there are \(\rho_i \in \text{Aut}_*(G_i)\) with \(\bar{\rho}_i \cdot \rho_i^{-1} = \rho_{ii}, 2 \leq i \leq n\).

Lemma 3.7. Assume that \(M_i \cdot M_j = M_n\) is the same for all \(i, j\) with \(i \neq j\). Let \(\sigma_{ij}\) be defined as in (7). Then there exists \(\sigma \in \text{Iso}_*((\sigma_i), G, H)\) if, and only if, there exist \(\rho_i \in \text{Aut}_*(H_i)\) with \(\sigma_{ii} = \bar{\rho}_i\) for \(2 \leq i \leq n\); i.e. if the maps \(\sigma_{ii}\) lift to \(\rho_i\) in \(\text{Aut}_*(H_i)\) for \(2 \leq i \leq n\).

\(^{17}\) Note that this only makes sense in case all of the groups \(G_{ij}\) are the same. These cocycles are thus a proper subset of \(Z(G, \text{Aut}_*(G))\), but \(B \diamond (G, \text{Aut}_*(G)) = B(G, \text{Aut}_*(G))\), so that \(\tilde{H} \diamond (G, \text{Aut}_*(G))\) has fewer elements than \(\tilde{H}(G, \text{Aut}_*(G))\).
Proof. We consider the family \((\tau_i = \rho_i^{-1} \cdot \sigma_i : G_i \to H_i), \ 2 \leq i \leq n\) and \(\tau_i = \sigma_i\). Then \(\tau_i \cdot \tau_j^{-1} = \sigma_i \cdot \sigma_j^{-1} \cdot \rho_j = 1\) for \(2 \leq j \leq n\). Thus \((\tau_i)\) induces an isomorphism in \(\text{Iso}_s((\sigma_i), G, H)\) by Claim 2.5. 

The importance of \(\hat{H}(G, \text{Aut}_s(G))\) lies also in the construction of the various modifications of projective limits.

Lemma 3.8. Let \(G = \lim.\text{proj}(G_s)\) be a projective limit.
(1) Given a cocycle \(\rho = (\rho_{ij}) \in Z(G, \text{Aut}_s(G)).\) Then

\[
G(\rho) = \left\{ (g_i) \in \prod_{1 \leq i \leq n} G_i \mid \rho_{ij} \cdot \phi_j(g_j) = \phi_j(g_i) \right\}
\]

is a group.

(2) \(G(\rho) = G\) with an isomorphism in \(\text{Iso}_s((\text{id}_G), G(\rho), G)\) as ‘projective limit’ if, and only if, \(\rho \in B(G, \text{Aut}_s(G))\) is a coboundary.

Proof. (1) The conditions \(\rho_{ij} = \text{id}\) and \(\rho_{ij} = \rho_{ij}^{-1}\) make the definition of \(H(\rho)\) well defined. Since \(\rho_{ij}\) and \(\phi_{ij}\) are group homomorphisms, it is easily seen that \(H(\rho)\) is a group.

(2) If \(\rho_{ij} = \tilde{\rho}_i \tilde{\rho}_j^{-1}\), then we get an isomorphism

\[
\sigma : G(\rho) \to G, \quad (g_i)_{1 \leq i \leq n} \to (\rho_i^{-1}(g_i))_{1 \leq i \leq n} ;
\]

in fact, \((g_i)_{1 \leq i \leq n} \in H(\rho)\) means that \(\rho_{ij}(g_j) = g_i\) for all \(1 \leq i \leq n\); i.e. \(\rho_i \cdot \rho_j^{-1}(g_j) = g_i\) and so \(\rho_j^{-1}(g_j) = \rho_i^{-1}(g_i)\) for all \(i\), this means that \((\rho_i^{-1}(g_i))_{1 \leq i \leq n} \in G\).

Similarly one shows that an isomorphism

\[
\sigma : G(\rho) \to G, \sigma \in \text{Iso}_s((\text{id}_G), G(\rho), G), \quad (g_i)_{1 \leq i \leq n} \to (\sigma_i^{-1}g_i),
\]

implies that \(\rho_{ij} = \sigma_i \sigma_j^{-1}\) is a coboundary in the corresponding cohomology. 

4. The Zassenhaus conjecture

We shall turn our attention to the Isomorphism problem, the \(p\)-version of the Zassenhaus conjecture and the conjecture itself, as explained in the introduction (cf. Problem 1.1).

Remark 4.1. Though the Zassenhaus conjecture is not true in general (cf. [10]), there is not yet known a counterexample to the \(p\)-version of the Zassenhaus conjecture.

\(^\star\) This is to be interpreted as \((\phi_j \cdot \rho_j)(\phi_i \cdot \rho_i)^{-1}\).
It is often useful to rephrase the Zassenhaus conjecture and its $p$-version as follows.

**Lemma 4.2.** Assume that $\mathbb{Z}G = \mathbb{Z}H$ as augmented algebras. Then the class sum correspondence [11] implies that one has for the class sums $K_g$ of $G$ and $K_h$ of $H$ resp. a correspondence $K_{\beta(g)} = K_g$ for a bijection $\beta : G \to H$.

(1) The Zassenhaus conjecture is true for $\mathbb{Z}G$ if, and only if, the above map $\beta$ can be chosen to be an isomorphism of groups.

(2) The $p$-version of the Zassenhaus conjecture holds if and only if there is an isomorphism $\rho = \rho_p : G \to H$ of groups such that for all $p$-power elements $g \in G$ one has $K_{\rho(g)} = K_{\rho_p(g)}$.

The importance of the $p$-version of the Zassenhaus conjecture is shown in the next fundamental result.

**Proposition 4.3.** Assume that $\mathbb{Z}G = \mathbb{Z}H$ as augmented algebras. Let $N^0$ be a family of subgroups for $G$ as defined in Remark 2.4. Then there exists a family $M^0$ of normal subgroups for $H$ such that $\Gamma_G(N^0) = \Gamma_H(M^0)$ as projective limits.

Then the $p$-version of the Zassenhaus conjecture holds for the pair of group bases $G$ and $H$ of $\mathbb{Z}G$ if and only if it holds for the images $G^*$ and $H^*$ in $\Gamma_G(N^0)$ under the natural map $\phi : \mathbb{Z}G \to \Gamma_G(N^0)$.

**Remark 4.4.** (1) The hypotheses of the proposition are surely satisfied if $N_i = O_{p_i}(G)$ (cf. Note 4.7).

(2) There are however other instances, where the hypotheses are satisfied, if for example $N_1 = O_{p_1}(G)$ and $N_2 = O_{p_2}(G)$.

(3) The corresponding statement for the Zassenhaus conjecture is false (cf. Example 4.16).

(4) It is not likely that the corresponding statement holds for the Isomorphism problem.

**Proof.** The normal subgroup correspondence gives a set of subgroups $M$ of $H$ which qualifies as $M^0$ [11], such that $\Gamma_G(N^0) = \Gamma_H(M^0)$ as projective limits. We note that an isomorphism $\rho : G \to H$ induces an isomorphism $\rho^* : G^* \to H^*$ via $\phi$ and conversely, every isomorphism between $G^*$ and $H^*$ arises in this way, since $G$ is isomorphic to $G^*$ via $\phi$.

If the $p$-version of the Zassenhaus conjecture holds for $\mathbb{Z}G$, then it obviously holds for $\Gamma_G(N^0)$.

Conversely, let $p$ be a fixed prime and assume that the $p$-version of the Zassenhaus conjecture holds for $\Gamma_G(N^0)$. Then there is an isomorphism $\rho : G \to H$, such that for each $p$-power element $g \in G$ we have $\phi(K_{\rho(g)}) = \phi(K_g) = \phi(K_{\rho_p(g)})$, with the notation used in Lemma 4.2. According to the above lemma we have to show that $K_{\rho(g)} = K_{\rho_p(g)}$. The proof will thus be finished, if we can show the following:

$^*$ $O_p(G)$ is the largest normal subgroup of $G$ of order a power of $p$. 
Claim 4.5. Let $K_x$ and $K_y$ be two class sums in $\mathbb{Z}G$ such that $\phi(K_x) = \phi(K_y)$, where $x$ and $y$ are $p$-power elements in $G$. Then $K_x = K_y$ in $\mathbb{Z}G$.

Proof. We now use our hypotheses that for $p$ there does exist a normal subgroup $N = N(p)$ of $G$ such that $(|N|, p) = 1$. Since $\phi(K_x) = \phi(K_y)$, we conclude that $K_x - K_y$ in $\mathbb{Z}G/N$. However, then $K_x = K_y$ in $\mathbb{Z}G$ by [11, p. 119, Lemma 1.3]. □

This also finishes the proof of the proposition. □

Let us recall from [12, 13] the most far-reaching result concerning the Isomorphism problem.

Theorem 4.6. Assume that for some prime $p$ the group $G$ has a normal $p$-subgroup $N$ such that $C_G(N)$ is contained in $N$. This condition is in particular satisfied in case $G$ is soluble and $O_p(G) = 1$. If $\mathbb{Z}G = \mathbb{Z}H$, then the Zassenhaus conjecture holds, i.e. there exists $a \in \mathbb{Q}G$—$a$ is even $p$-adically a unit in $\mathbb{Z}_pG$—such that $a \cdot G \\ a^{-1} = H$. □

Note 4.7. Let the groups $G = \text{proj.lim}(G/N_i)$ and $H = \text{proj.lim}(G/M_i)$ be given as projective limits. Assume that $\Gamma_G(\mathcal{N}^0) = \Gamma_H(\mathcal{M}^0)$, as projective limits. We shall next discuss the connections between the various situations around the Isomorphism problem for the quotients $\mathbb{Z}G/N_i$ and $\Gamma_G(\mathcal{N}^0)$.

(1) If the Isomorphism problem has a positive answer for the various rings $\mathbb{Z}G_i$, i.e. there exists an isomorphism $\sigma_i : G_i \rightarrow H_i$, then we get induced automorphisms

$$ \sigma_{ij} = \tilde{\sigma}_i \cdot \tilde{\sigma}_j^{-1} : H_i \rightarrow H_j \quad \text{with} \quad \sigma = (\sigma_{ij}) \in Z(H, \text{Aut}(H)). $$

(2) If the $p$-version of the Zassenhaus conjecture has a positive answer for the various rings $\mathbb{Z}G_i$, then in a similar way we obtain a cocycle $\sigma = \sigma_{ij}$ in $Z(H, \text{Aut}_p(H))$.

(3) If the Zassenhaus conjecture has a positive answer for the various rings $\mathbb{Z}G_i$, then in a similar way we obtain a cocycle $\sigma = \sigma_{ij}$ in $Z(H, \text{Aut}_p(H))$.

If we now apply Lemma 3.5, then we obtain the following result, which probably is also known to L.L. Scott.

Theorem 4.8. As in Note 4.7 we assume $\Gamma_G(\mathcal{N}^0) = \Gamma_H(\mathcal{M}^0)$.

(1) The groups $G$ and $H$ are isomorphic if, and only if, the cocycle from part (1) of the above note lies in $B(H, \text{Aut}(H))$.

(2) Under the assumption of part (2) above, the $p$-version of the Zassenhaus conjecture is true for

$$ \Gamma_G(\mathcal{N}^0) = \Gamma_H(\mathcal{M}^0) \quad \text{if, and only if,} \quad (\sigma_{ij}) \in B(H, \text{Aut}_p(H)). $$
Even more is true using Proposition 4.3, the \( p \)\text{-version} of the Zassenhaus conjecture holds for the pair \( G \) and \( H \) of group bases in \( \mathbb{Z}G \).

(3) Under the assumption of part (3) above, the Zassenhaus conjecture holds for 
\[ \Gamma_G(N^0) = \Gamma_H(M^0) \] if, and only if, \( (\sigma_{ij}) \in B(H, \text{Aut}_p(H)) \).

An immediate consequence is the following, since by Theorem 4.6 for \( \mathcal{N}^0 = \emptyset \) the assumption of the third part in Note 4.7 is satisfied.

**Corollary 4.9.** Assume that \( G \) is soluble and that \( \mathbb{Z}G = \mathbb{Z}H \) as augmented algebras. If the cocycle from Note 4.7(2) lies in \( B(H, \text{Aut}_p(H)) \), then the \( p \)-version of the Zassenhaus conjecture holds for \( \mathbb{Z}G \).

**Corollary 4.10.** Assume that \( G \) is a soluble group.

1. Let \( \Gamma_G(C) = \Gamma_H(C) \) and assume that the groups \( H_{ij} \) are abelian, then the Zassenhaus conjecture holds for \( \Gamma_G(C) \).

2. If \( \mathbb{Z}G = \mathbb{Z}H \), and if the groups \( H_{ij} \) are abelian, then the \( p \)-version of the Zassenhaus conjecture holds for \( \mathbb{Z}G \).

In order to apply these results to special classes of groups we have to discuss questions of when automorphisms of quotient groups can be lifted.

**Claim 4.11.** Let \( \rho \in \text{Aut}(G) \), and let \( M \) be a \( G \)-module. For an extension
\[
\mathcal{E}: \quad 1 \rightarrow M \rightarrow E \rightarrow G \rightarrow 1
\]
representing the zero element in \( H^2(G, M) \) the automorphism \( \rho \) extends to an automorphism \( \rho_0 \) of \( E \) if, and only if, the \( G \)-modules \( M \) and \( ^\rho M \) are isomorphic—where \( ^\rho M \) is \( M \) but the \( G \)-action is twisted by \( \rho \).

**Proof.** Let \( \phi : ^\rho M \rightarrow M \) be a \( G \)-homomorphism. If we first form the pullback of \( \mathcal{E} \) along \( \rho \) to get \( \mathcal{E}_{\rho} \) and then the pushout along \( \phi \), then we get the commutative diagram with exact rows and columns:

\[
\begin{array}{ccc}
1 & \rightarrow & M & \rightarrow & E & \rightarrow & G & \rightarrow & 1 \\
\downarrow & & \downarrow \rho_1 & & \downarrow \rho & & \downarrow & & \\
1 & \rightarrow & ^\rho M & \rightarrow & E_1 & \rightarrow & G & \rightarrow & 1 \\
\downarrow \phi & & \downarrow \rho_2 & & \downarrow & & \downarrow & & \\
1 & \rightarrow & M & \rightarrow & E & \rightarrow & G & \rightarrow & 1 \\
\end{array}
\]

The composition \( \rho_2 \cdot \rho_1 : E \rightarrow E \) is thus an automorphism of \( E \) extending \( \rho \). The converse is obvious.

We would like to stress that the pullback along \( \rho \) gives always rise to an isomorphism of groups but this is in general not an automorphism.
In general, there is no hope that the modules $M$ and $\gamma M$ are isomorphic. As a matter of fact, the automorphism $\rho$ induces an auto equivalence of the category of $\mathbb{Z}G$-modules. This equivalence is trivial if, and only if, $\rho$ induces an inner automorphism on $\mathbb{Z}G$.

The following result was essentially noted by L.L. Scott and the second author (cf. also [15]).

**Lemma 4.12.** Let $\gamma \in \text{Aut}_r(G)$ (i.e. $g$ and $\gamma(g)$ for every $g \in G$ are conjugate) and let

$$\mathcal{E}: 1 \to M \to E \to G \to 1$$

represent the zero element in $H^2(G, M)$ for a finite $G$-module $M = \sum_{1 \leq i \leq n} M_i$, where $M_i$ are the various $p_i$-primary components of $M$. If $M_i$ is a characteristic section in a finitely generated projective $\mathbb{Z}_p G$-module—i.e. there is a finitely generated projective $\mathbb{Z}_p G$-module $P$ and characteristic submodules $L_1 \subset L_2$ such that $M_i \cong L_2/L_1$—then $\gamma$ extends to an automorphism $\gamma_0$ of $\mathcal{E}$, in particular it extends to a group automorphism of $E$.

**Remark 4.13.** (1) The above conditions are satisfied for a semi-simple finite $G$-module $M$. In fact, such a module decomposes into a direct sum of simple modules, and they are the radical quotients of $p$-adic indecomposable projective modules.

(2) The above conditions are satisfied, if the modules $M_i$ have order prime to $|G|$. In fact, we can assume that $M = M_i$ is an indecomposable $\mathbb{Z}_p G$-module. Since $p$ does not divide $|G|$, the ring $\mathbb{Z}_p G$ is a direct sum of matrix rings $(R_i)_{n_i}$, where $R_i$ is an unramified extension of $\mathbb{Z}_p$. Since $M$ is indecomposable, it is a module for $(R_i)_{n_i}$ for some $i$. Then $M \cong L/p^n \cdot L$ for the projective indecomposable $(R_i)_{n_i}$-module $L$. This shows that $M$ is the epimorphic image of a projective module modulo a characteristic ideal.

**Proof.** Since $\mathcal{E}$ is the direct sum of the extensions with kernel $M_i$, we may assume that $M$ is an abelian $p$-group. We shall show, that for a projective $\mathbb{Z}_p G$-module $P$ the twisted module $\gamma P$ is $\mathbb{Z}_p G$-isomorphic to $P$. But $\gamma$ induces a central automorphism on $\mathbb{Z}_p G$ and thus is given by conjugation with a unit $a \in \mathbb{Q}G$, and so, $\mathbb{Q}P \cong \gamma \mathbb{Q}P$. We now invoke Swan's theorem [14], which implies $P \cong \gamma P$. Since such an isomorphism preserves characteristic sections, the modules $M$ and $\gamma M$ are isomorphic. The statement then follows from Claim 4.11. $\square$

**Theorem 4.14.** Assume that $G$ is a soluble group and $\Gamma_G(\mathcal{N}^0) = \Gamma_M(\mathcal{M}^0)$. Assume that the groups $H_{ij} = H_0$ are the same for all pairs $(i, j)$, and that the groups $H_i$ are split extensions

$$\mathcal{E}_i: 1 \to L_i \to H_i \to H_0 \to 1$$
with finite $H_0$-modules $L_i = \sum L_i^k$, where $L_i^k$ are the various $p_k$-primary components. Assume that $L_i^k$ is a characteristic section in a finitely generated projective $\hat{\mathbb{Z}}_{p_k} H_0$-module and furthermore that the Zassenhaus conjecture holds for the group rings $\mathbb{Z} G_i = \mathbb{Z} H_i$, then $G = H$, and even the $p$-version of the Zassenhaus conjecture holds, provided $\mathbb{Z} G = \mathbb{Z} H$.

**Proof.** We number the groups $H_i$ in such a way that for the given prime $p$ the kernel of the map $H \to H_i$ has order prime to $p$. Since the Zassenhaus conjecture holds for $\mathbb{Z} G_i = \mathbb{Z} H_i$, there are central automorphisms $\sigma_i : \mathbb{Z} G_i \to \mathbb{Z} H_i$ with $\sigma_i(G_i) = H_i$. These induce in the usual way an element $\sigma = (\sigma_i) = \sigma_i \bar{\sigma}_i^{-1} : H_0 \to H_0$, which is determined by $\{\sigma_i\}$ for $2 \leq i \leq n$. By Lemma 4.12 these maps can be lifted to automorphisms $\rho_i : H_i \to H_i$ for $2 \leq i \leq n$; we put $\rho_1 = 1$. We now consider the isomorphisms $\tau_i = \rho_i \sigma_i : G_i \to H_i$. Then the cocycle associated to this map is generated by

$$(\rho_i \sigma_i) \cdot (\rho_i \bar{\sigma}_i)^{-1} = \sigma_i \cdot \rho_i^{-1},$$

and hence gives rise to an isomorphism $\rho$ from $G$ to $H$; cf. Lemma 3.7. Since $\rho_1 = 1$, we conclude that $\rho$ induces $\sigma_1$, a central automorphism on $\mathbb{Z} G_1$. Now if we assume $\mathbb{Z} G = \mathbb{Z} H$ then we can apply Claim 4.5 and get the desired result. \square

The hypotheses of the last theorem are satisfied in particular, if $M_i$ are semi-simple $G_i$-modules or $\text{char}(M_i)$ is prime to $|G_i|$.

In case there are only two normal subgroups, we get the following result, since then the hypothesis $H_i = H_0$ is automatic.

**Corollary 4.15.** Assume that $G$ is a pullback

$$
\begin{array}{ccc}
G_1 & \longrightarrow & G_0 \\
\uparrow & & \uparrow \\
G & \longrightarrow & G_2
\end{array}
$$

with $G_i = G/N_i$.

(1) If the Zassenhaus conjecture holds for $\mathbb{Z} G_i$, then the $p$-version of the Zassenhaus conjecture is true for $\mathbb{Z} G$, in particular, the Isomorphism problem has a positive answer for $\mathbb{Z} G$, provided for every central isomorphism $\gamma$ of $G_i$, there exist $\rho_1 \in \text{Aut}(G_i)$ such that $\rho_1 \cdot \rho_2^{-1} = \gamma$. This latter condition is satisfied, in case $M_i = \text{Ker}(G_i \to G_0)$ is abelian and semi-simple as $\mathbb{Z} G_0$-module or the characteristic of $M_i$ is prime to $|G|$.\textsuperscript{20}

(2) Assume that the $p$-version of the Zassenhaus conjecture holds for $\mathbb{Z} G_i$, and assume that every $p$-central automorphism $\gamma$ of $G_0$; i.e. $\gamma \in \text{Aut}_p(G_0)$, can be written as

$$\rho_1 \cdot \rho_2^{-1} = \gamma \text{ for } \rho_i \in \text{Aut}(G_i).$$

\textsuperscript{20}It suffices to assume that $M_i$ satisfies the hypothesis of $M_i$ in Theorem 4.14.
Then the $p$-version of the Zassenhaus conjecture holds for $\mathbb{Z}G$; in particular, the Isomorphism problem has a positive answer for $\mathbb{Z}G$. □

**Example 4.16.** In the paper [10] an example of a group ring $\mathbb{Z}G$ and an augmented group basis $H$ was given such that for these two group bases $G$ and $H$ the Zassenhaus conjecture is not valid. However, in the projective limit with respect to $\{O_{p_i}\}$, all the groups $H_{p_i}$ are abelian, and hence by Corollary 4.10, the Zassenhaus conjecture holds for $\Gamma_G(E) = \Gamma_H(E)$. Consequently, also the $p$-version of the Zassenhaus conjecture is true.

Assume now again that $\Gamma_G(E) = \Gamma_H(E)$. The main result now describes $G$ in terms of $H$ and the cocycle $\sigma$ from (8).

**Theorem 4.17.** Assume that $\Gamma_0 = \Gamma_G(E) = \Gamma_H(E)$ as augmented algebras $21$, and let the cocycle $\sigma \in \mathbb{Z}(H, \text{Aut}_r(H))$ be defined as in (8). Then $G \cong H(\sigma)$, where $H(\sigma) = \{(h_i)_{1 \leq i \leq n} \mid h_i \in H_i, \sigma_j(h_i) = h_i\}$ (cf. Lemma 3.8).

**Proof.** Since $\Gamma_G(E) = \Gamma_H(E)$, we get induced equations $\mathbb{Z}G_i = \mathbb{Z}H_i$. However, $O_{p_i}(G_i) = 1$ and so by Theorem 4.6 there are automorphisms $\sigma_i$ of $\mathbb{Z}G_i = \mathbb{Z}H_i$, which leave the centre elementwise fixed and map $G_i$ to $H_i$. Thus the cocycle $\sigma = (\sigma_i) - (\bar{\sigma_i})\bar{\sigma_i}^{-1}$ lies in $\mathbb{Z}(H, \text{Aut}_r(H))$. Let us recall the construction of $H(\sigma)$ (cf. Lemma 3.8):

$$H(\sigma) = \{(h_i)_{1 \leq i \leq n} \mid h_i \in H_i, \sigma_j(h_i) = h_i\},$$

where $\phi_{p_i} : \mathbb{Z}G_i \rightarrow \mathbb{Z}G_{p_i}$ are the induced maps. We know that

$$G = \{(g_i)_{1 \leq i \leq n} \mid \phi_{p_i}(g_i) = \phi_{p_i}(g_i)\}.$$

The map $\rho : G \rightarrow H(\sigma)$ defined by $\rho((g_i)_{1 \leq i \leq n} = (\sigma_i(g_i))_{1 \leq i \leq n}$ is then an isomorphism. In fact, the condition $\phi_{p_i}(g_i) = \phi_{p_i}(g_i)$ translates to $\sigma_i(g_i) = \bar{\sigma_i}\bar{\sigma_i}^{-1}\phi_{p_i}(g_i)$, and we have constructed the desired isomorphism. □

**Remark 4.18.** (1) The formulation of the analogous theorem according to the three cases in Note 4.7 with $\mathcal{N}_0$ instead of $\mathcal{O}$ is left to the reader.

(2) Assume that $\mathbb{Z}G = \mathbb{Z}H$ as augmented algebras for $G$ a finite soluble group, then also $H$ is soluble, and as explained in Note 4.7 we have $\Gamma_G(E) = \Gamma_H(E)$ and so the conclusion of Theorem 4.17 says $G \cong H(\sigma)$ for the associated cocycle $\sigma$.

(3) Given a central cocycle $\sigma = (\sigma_i) \in \mathbb{Z}(H, \text{Aut}_r(H))$, we can interpret $\sigma$ also as an element $\tau_x = \sigma \in \mathbb{Z}(\Gamma_H(E), \text{Aut}_r(\Gamma_H(E)))$. We can then form the group $H(\sigma)$ and the ring $\Gamma_H(E)(\tau_x) = \Gamma_0(H(\sigma))$. Then $H(\sigma) \cong H$ if, and only if,

21 This implies that $\Gamma_0(E) = \Gamma_H(E)$ as projective limits.
22 Here $\text{Aut}_r(R)$ are the ring automorphisms of the ring $R$, which leave the centre of the ring $R$ elementwise fixed.
\[ \sigma \in B(H, \text{Aut}(H)); \text{i.e. } \sigma \text{ is a coboundary with respect to all automorphisms of } H. \] Similarly,

\[ \Gamma_H(\mathcal{C})(\sigma_2) = \Gamma_H(\mathcal{C}) \text{ if, and only if, } \sigma \in B(\Gamma_H(\mathcal{C}), \text{Aut}(\Gamma_H(\mathcal{C}))); \]

i.e. \( \sigma_2 \) is a coboundary with respect to all automorphisms of \( \Gamma_H(\mathcal{C}) \).

(4) In order to find two non-isomorphic soluble groups \( G \) and \( H \) with \( \Gamma_G(\mathcal{C}) = \Gamma_H(\mathcal{C}) \) it is thus necessary and sufficient, to a group \( H \) and \( \sigma \in Z(H, \text{Aut}_H(H)) \), such that

\[ 1 \neq [\sigma] \in \tilde{H}(H, \text{Aut}_c(H)), \text{ but } 1 = [\sigma] \in \tilde{H}(\Gamma_H(\mathcal{C}), \text{Aut}(\Gamma_H(\mathcal{C}))). \]

We shall construct such an example later. Here we just point out that it is necessary to have such an example if one wants to construct a counterexample to the Isomorphism problem.

5. Special classes of groups

**Theorem 5.1.** Let \( F(G) \) be the Fitting subgroup of \( G \) and assume that \( F(G) \) is non-trivial.

1. Assume that for each non-trivial Sylow subgroup \( S \) of \( F(G) \) the centralizer \( C_G(S) \) is contained in \( F(G) \).
2. Assume that \( \text{Aut}_c(G/F(G)) \) consists only of inner automorphisms.

Then the \( p \)-version of the Zassenhaus conjecture holds for every prime \( p \) simultaneously.

**Proof.** By Theorem 4.6 we may assume that \( F(G) \) is not a \( p \)-group. Let \( S_p \) be the Sylow \( p \)-subgroup of \( F(G) \). Then \( O_p(G) \) centralizes \( S_p \), which is normal in \( G \). By the first assumption \( O_p(G) \) lies in \( F(G) \). Consequently, since \( f(G) \) is nilpotent, we get for all different primes \( p, q \in \pi(F(G)) \) that

\[ O_p(G) \cdot O_q(G) = F(G). \]

We now put \( N_i = O_{p_i}(F(G)) \) for all elements \( p_i \) in \( \pi(F(G)) \). We point out that

1. for each \( p \in \pi(G) \) there is an index \( i \) with \( (p, |N_i|) = 1 \), and so Lemma 2.3 and Proposition 4.3 can be applied;
2. the groups \( G_i = G/N_i \cdot N_i \) are the same for each \( i \neq j \), say \( G_0 = G_{i_0}, i \neq j \), and so we can apply Claim 2.5 and Lemma 3.7.

The discussion in Section 4 and Lemma 3.7 now yields the desired result, since inner automorphisms lift to inner automorphisms. \( \square \)

**Corollary 5.2.** Let \( G \) be a Frobenius group with metacyclic Frobenius complements or let \( G \) be a 2-Frobenius group. Then the \( p \)-version of the Zassenhaus conjecture is valid simultaneously for all primes \( p \).
Proof. If $G$ is a Frobenius group, then the Frobenius kernel $K$ is nilpotent and coincides with $F(G)$. Since $G$ is a Frobenius group, the centralizer of each Sylow subgroup of $K$ is contained in $K$. By assumption $G/F(G)$ is a metacyclic group. Obviously metacyclic groups have the property that $\text{Aut}_r(G)$ consists of inner automorphisms. Thus both hypotheses of Theorem 5.1 are satisfied, and we get the first part of the corollary.

If $G$ is a 2-Frobenius group, then the structure theorems for Frobenius groups, (cf. [4, Chapter V]) show that the upper Frobenius group $U$ has a cyclic Frobenius kernel. Consequently $U$ is metacyclic. $F(G)$ coincides with the Frobenius kernel of the lower Frobenius group. Let $S$ be a Sylow $p$-subgroup of $F(G)$. Because $S$ is normal in $G$ its centralizer $C = C_G(S)$ is a normal subgroup of $G$. If $C$ is not contained in $F(G)$, then $C \cdot F(G)/F(G)$ is a non-trivial normal subgroup of $U$. Since $U$ is a Frobenius group, it follows that $C$ contains non-trivial elements of complements of the lower Frobenius group, a contradiction. Now again both hypotheses of Theorem 5.1 have been established and the second part of the corollary follows. □

Corollary 5.3. Let $G$ be a soluble group such that its integral augmentation ideal decomposes. Then for each prime $p$ the $p$-version of the Zassenhaus conjecture is valid for $\mathbb{Z}G$.

Proof. If $G$ is soluble, then its integral augmentation ideal decomposes if, and only if, $G$ is a Frobenius or a 2-Frobenius group [3]. By Corollary 5.2 the result follows for 2-Frobenius groups and Frobenius groups with metacyclic complements. For the remaining Frobenius groups it follows that [4, Chapter V] two divides the order of a complement and that the Frobenius kernel $K$ is abelian. By Theorem 4.6 even the Zassenhaus conjecture is valid, if the Frobenius kernel is a $p$-group. So we may assume that at least two different primes divide the order of $K$. Let $N_i/O_{p_i}$, where $p_i$ ranges over $\pi(K)$. With the family $N_i$ we can apply Theorem 4.14 and the result follows. □

Note that the second part of the previous proof also shows that the $p$-version of the Zassenhaus conjecture is valid for non soluble Frobenius groups. In particular, Theorem 1.5 from the Introduction is established. We shall now prove Theorem 1.3 from the Introduction.

Proof of Theorem 1.3. If $F(G) = P$ is a $p$-group, then $O_p(G) = 1$ and the statement follows from Theorem 4.6.

Let $p$ be a prime dividing the order of $N$ and let $P$ be a Sylow $p$-subgroup of $N$. Put $Q = O_p(G)$ and $G_1 = G/P$ and $G_2 = G/Q$, furthermore $\tilde{G} = G/Q \cdot P$. Then $G$ is the pullback

$$
\begin{array}{ccc}
G & \rightarrow & G_1 \\
\downarrow & & \downarrow \\
G_2 & \rightarrow & \tilde{G}
\end{array}
$$

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Clearly the hypotheses on $G$ hold as well for images of $G$. By induction we may assume that for both $G_1$ and $G_2$ the $p$-version of the Zassenhaus conjecture is true.

But $\tilde{G}$ is nilpotent, and hence $\operatorname{Aut}_p(\tilde{G}) = \operatorname{Aut}_*(\tilde{G})$. Moreover, $\tilde{G}$ is an image of $G/N$. By assumption $\operatorname{Aut}_*(\tilde{G})$ consists of inner automorphisms only. Now the statement follows from Corollary 4.15. \qed

A direct consequence is the following result, which was proved in a different way by the first author in [6].

**Corollary 5.4.** Assume that $G/F(G)$ is abelian. Then the $p$-version of the Zassenhaus conjecture holds simultaneously for all primes $p$. In particular it holds for supersoluble groups.

**Proof.** If $X$ is an abelian group, then $\operatorname{Aut}_c(X)$ is trivial. Consequently, also the second hypothesis of Theorem 1.3 is satisfied and the result follows immediately. Finally the class of supersoluble groups is contained in the class of finite groups with nilpotent commutator subgroup. \qed

6. **A counterexample to the Isomorphism problem for $I_G^*(\mathcal{O})$**

The following result, which is also of interest for itself, is the basis for the construction in this section.

**Proposition 6.1.** Let $H$ be a soluble group and write it as the projective limit with respect to $\{O_{p_i}(H)\}_{1 \leq i \leq n}$. Assume that $H_0 = H_{ij}$ is the same for all pairs $\{i, j\}$, $i \neq j$. For the kernels $K_i = \operatorname{Ker}(H_i \to H_0)$ we require that $K_i$ is a Sylow $p_i$-subgroup of $H_i$. If

$$\sigma \in \mathbb{Z} \circ (H, Aut_c(H))$$

such that $1 \neq [\sigma] \in \hat{H} \circ (H, Aut(H))$, then there is a group $G$ not isomorphic to $H$ with

$$\mathbb{Z}_n \otimes \mathbb{Z} \Gamma_0(\mathcal{O}) \simeq \mathbb{Z}_n \otimes \mathbb{Z} \Gamma_H(\mathcal{O}),$$

where $\mathbb{Z}_n$ is the semilocalisation of $\mathbb{Z}$ at all the prime divisors of $|G|$.

**Note 6.2.** The above conditions just mean that we have central automorphisms $\sigma_i$ of $H_0$, $2 \leq i \leq n$, such that there cannot be found automorphisms $\sigma_i : H_i \to H_i$ with $\sigma_i = \sigma_i \cdot \sigma_j$.\[\sigma_i = \sigma_i \cdot \sigma_j\]

**Proof.** We shall show that $\sigma_i$ can be lifted to—even—a central automorphism of $\mathbb{Z}_n H$ for $2 \leq i \leq n$. Then we can choose $\sigma_i = \operatorname{id}_{H_i}$. Since $\pi$ involves only finitely many primes, we are done if we can show (cf. [11]) the following:
Claim 6.3. Given an exact sequence of groups with $K$ a $p$-group

$$1 \to K \to G \to H \to 1.$$ If $(|K|, |H|) = 1$, then a central automorphism $\sigma_0 \in \text{Aut}_c(\mathbb{Z}[H])$ can be lifted to a central automorphism $\sigma \in \text{Aut}_c(\mathbb{Z}[G]).$

Proof. We have the induced exact sequence of group rings

$$0 \to I(G, K) \to \mathbb{Z}[G] \to \mathbb{Z}[H] \to 0.$$

(1) Assume that the prime $q$ divides $|K|$; then $|H|$ is invertible in $\mathbb{Z}_q$, and so $\mathbb{Z}_q[H]$ is a maximal order, for which every central automorphisms is inner and so $\sigma$ is given by conjugation with a unit $u_0$. However, since $K$ is a normal $q$-subgroup of $G$, the ideal $I_q(G, K)$ lies in the radical of $\mathbb{Z}_q[G]$, and so $u_0$ can be lifted to a unit $u \in \mathbb{Z}_q[G]$, and conjugation with $u$ extends $\sigma_0$.

(2) Assume that $q$ divides $|H|$, then the exact sequence $\mathbb{Z}_q \otimes_{\mathbb{Z}_q} \mathcal{E}$ is two-sided split, and then $\sigma_0$ surely can be extended to an automorphism of $\mathbb{Z}_q[G]$. 

Theorem 6.4. There are two non-isomorphic soluble groups $G$ and $H$ such that

$$\mathbb{Z}_q \otimes_{\mathbb{Z}_q} \Gamma_G(\mathcal{E}) = \mathbb{Z}_q \otimes_{\mathbb{Z}_q} \Gamma_H(\mathcal{E}).$$

Remark 6.5. This is a slight modification of the construction which was used to find a counterexample to the Zassenhaus conjecture for the integral group rings by L.L. Scott and the second author [10]. We define the group

$$H_0 = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

as a subgroup of $\text{GL}(3, 4)$. Three elements will play an important rôle in our construction:

$$s = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

these matrices all lie in $\text{GL}(3, 2)$.

If $\mathbb{F}_4$ is the field with four elements with a $\mathbb{Z}/2 \cdot \mathbb{Z}$-basis $\{1, r\}$, then the group

$$K = \begin{pmatrix} 1 & \delta \cdot r & * \\ 0 & 1 & \delta \cdot r \\ 0 & 0 & 1 \end{pmatrix},$$

where $\delta$ is either 1 or 0, is a normal subgroup of $H_0$ with quotient generated by the images of $s$ and $t$. 

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The group $H_0$ has an automorphism $\sigma_0$ defined via the map
\[
\phi : H_0 \to H_0/\langle K \rangle \to H_0, \quad s \mapsto c, \quad t \mapsto c
\]
as $\sigma_0(h) = h \cdot \phi(h)$.

Then $\sigma_0 \in \text{Aut}_c(G)$; i.e. for every $h \in H$ the elements $h$ and $\sigma_0(h)$ are conjugate in $H$.

We now define three modules for $H_0$:

1. On $\langle m \mid m^3 \rangle$ the normal subgroup $K$ centralizes $m$, but $^m m = m^{-1}$ and $^\mu m = m^\mu$. We denote by $H$ the semi-direct product $H = \langle m \rangle \rtimes H_0$.

2. On $D = \langle d \mid d^3 \rangle$ the normal subgroup $K$ centralizes $f$, but $^f d = d$ and $^\mu d = d^\mu$. We denote by $H$ the semi-direct product $D \rtimes H$.

3. On $F = \langle f \mid f^5 \rangle$ the normal subgroup $K$ centralizes $f$, but $^f f = f$ and $^\mu f = f$. We denote by $H_5$ the semi-direct product $F \rtimes H$.

The following result was shown in [10]; cf. also [7].

**Lemma 6.6.** (1) $\sigma_0$ can be extended to $\sigma \in \text{Aut}_c(H)$, by letting $\sigma$ centralize $m$.

(2) $\sigma$ can be extended to $\sigma_i \in \text{Aut}(H_i)$, by letting $\sigma_i$ centralize $d$ and $f$ resp. for $i = 3, 5$.

(3) $\sigma$ does not extend to an automorphism in $\text{Aut}_c(H_i)$. Even more is true:

(4) There do not exist $\gamma_i \in \text{Aut}_c(H_i)$, $i = 3, 5$ such that $\gamma_i = \gamma_3 \cdot \gamma_5$ in the common quotient $H$.

We now look at all possibilities of pairs
\[(\rho_3^k, \rho_5^k) \in \text{Aut}(H_3) \times \text{Aut}(H_5)\]such that $\sigma = \rho_3^k \cdot \rho_5^k$, $k = 1, \ldots, n$.

By the above lemma, not both $\rho_3$ and $\rho_5$ can lie in $\text{Aut}_c(H_3)$ and $\text{Aut}_c(H_5)$ resp. After renumbering we may assume that
\[
\{ \rho_3^k \mid 1 \leq k \leq n_1 \} \subset \text{Aut}(H_3) \setminus \text{Aut}_c(H_3) \quad \text{and}
\{ \rho_5^k \mid n_1 + 1 \leq k \leq n \} \subset \text{Aut}(H_5) \setminus \text{Aut}_c(H_5).
\]

A result of R. Brauer states that for an automorphism $\lambda$ of a group $G$ the number of fixed points of $\lambda$ on the conjugacy classes is the same as the number of fixed points of $\lambda$ on the irreducible characters. Hence a non-central automorphism moves an irreducible character. We now let $p$ and $q$ be primes such that $\mathbb{Q}_p$ is a splitting field for $H_3$ and $\mathbb{Q}_q$ is a splitting field for $H_5$. For each $\rho_3^k$, $1 \leq k \leq n_1$ we pick an irreducible $\mathbb{F}_p H_3$-lattice $M^k_p$ such that $M^k_p$ and $\rho_3^k M^k_p$ are non-isomorphic. This can be done, since these $\rho_3^k$ move conjugacy classes of $H_3$.

We point out that among the modules $\{ M^k_p \}$ there may be repetitions, and it is not clear that $\bigoplus_{1 \leq k \leq n_1} M^k_p$ is not invariant under each of the homomorphisms $\rho_3^k$. However, we can modify these modules accordingly:
Claim 6.7. There exists a faithful finite \( \mathbb{Z}_p H_3 \)-module \( \tilde{M} \) with \( \rho^* \tilde{M} \not\cong \tilde{M} \) for any \( \rho \in \{ \rho_3^k \}_{1 \leq k \leq n_1} = T \).

**Proof.** We choose a strictly decreasing chain of natural numbers \( \nu_1 > \nu_2 > \cdots > \nu_{n_1} \) and pick a \( \mathbb{Z}_p H_3 \)-lattice \( M_1 \) such that—if necessary after renumbering—

\[
M_1 \not\cong^\rho M_1 \quad \text{for} \quad \rho \in \{ \rho_3^1, \ldots, \rho_3^{n_1} \} = S_1,
\]

\[
M_1 = \rho^* M_1 \quad \text{for} \quad \rho \in T \setminus S_1 = T_1.
\]

We repeat this process until we have found \( \mathbb{Z}_p H_3 \)-lattices \( M_1, \ldots, M_{m_1} \) and subsets \( S_i \subset T \) with \( T = \bigcup S_i \), a disjoint union, and if \( T_i = T \setminus ( \bigcup_{i=1}^{i-1} S_i ) \), then

\[
\rho^* M_i \not\cong M_i \quad \text{for} \quad \rho \in S_i \quad \text{but} \quad \rho^* M_i = M_i \quad \text{for} \quad \rho \in T_i.
\]

We now put \( M = \bigoplus_{i=1}^{m_1} M_i(\rho) \oplus M_q \), where \( M_q \) is chosen in such a way as to make \( M \) a faithful \( H_3 \)-module. We now reduce modulo \( p \) to get the desired module \( \tilde{M} \); note that \( p \) does not divide the order of \( H_3 \). \( \square \)

We relabel \( \tilde{M} \) and call it \( M_p \). Then by Claim 4.11 none of the maps \( \rho_3^k \) extends to an automorphism of the semidirect product \( M_p \rtimes H_3 \).

In a similar fashion we construct the \( \mathbb{Z}_3 H_5 \)-module \( M_q \).

Let us recall, where we stand:

**Definition 6.8.** We put \( N_p = M_p \rtimes D \) and \( N_q = M_q \rtimes F \), and then \( G_p = N_p \rtimes H \) and \( G_q = N_q \rtimes H \). We now define \( G \) as the pullback

\[
\begin{array}{ccc}
G_q & \to & H \\
\downarrow & & \downarrow \\
G & \to & G_p
\end{array}
\]

Then it is clear that \( O_p(G) = N_q \) and \( O_p(G) = N_p \), \( O_q(G) = N_q \) and \( O_q(G) = N_p \). Moreover, \( O_2(G) = N_p \times N_q \). Since \( G/(N_p \cdot N_p) \cong H \) the above diagram is the projective limit of the various \( G/O_\rho(G) \).

Claim 6.9. The groups \( G \) and \( W = G(\sigma) \) are not isomorphic. But for \( N = \{ O_p(G), O_q(G) \} \) and \( M = \{ O_p(W), O_q(W) \} \) the pullbacks

\[
\mathbb{Z}_\sigma \otimes \Gamma_c(N) \quad \text{and} \quad \mathbb{Z}_\sigma \otimes \Gamma_w(M),
\]

are isomorphic.

**Proof.** Assume there are automorphisms \( \tau_p \in \text{Aut}(G_p) \) and \( \tau_q \in \text{Aut}(G_q) \) such that \( \tau_q \cdot \tau_p = \sigma \) on the common quotient \( H \). Then \( \rho_3 = \tau_p \mod (M_p) \) and \( \rho_5 = \tau_q \mod (M_q) \).
\( \tau_q \mod (M_q) \) lie in \( \rho_3 \in \text{Aut}(H_3) \) and \( \rho_5 \in \text{Aut}(H_5) \) and they form a pair with \( \rho_3 \cdot \rho_5 = \sigma \) on \( H \).

However, our construction of \( M_p \) and \( M_q \) shows that \( \rho_3 \) or \( \rho_5 \) does not extend to \( G_p \) or \( G_q \), a contradiction. As for \( I_G(\mathcal{C}) \), our construction is such that we may apply now Proposition 6.1, and the claim follows with \( \pi = \{2, 3, 5, p, q\} \). \( \square \)

References