Chromatic capacities of graphs and hypergraphs

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Received 2 January 2001; received in revised form 21 January 2003; accepted 24 June 2003

Abstract

Given a hypergraph $H$, the chromatic capacity $\gamma_{\text{cap}}(H)$ of $H$ is the largest $k$ for which there exists a $k$-coloring of the edges of $H$ such that, for every coloring of the vertices of $H$ with the edge colors, there exists an edge that has the same color as all its vertices. We prove that if $G$ is a graph on $n$ vertices with chromatic number $\chi$ and chromatic capacity $\gamma_{\text{cap}}$, then $\gamma_{\text{cap}} \geq (1-o(1)) \sqrt{\frac{2n}{\ln n}}$, extending a result of Brightwell and Kohayakawa. We also answer a question of Archer by constructing, for all $r$ and $\chi$, $r$-uniform hypergraphs attaining the bound $\gamma_{\text{cap}} = \chi - 1$. Finally, we show that a connected graph $G$ has $\gamma_{\text{cap}}(G) = 1$ if and only if it is almost bipartite. In proving this result, we also obtain a structural characterization of such graphs in terms of forbidden subgraphs.

Keywords: Chromatic capacity; Emulsive edge coloring; Compatible vertex coloring

1. Introduction

When the vertices and edges of a hypergraph $H$ are colored simultaneously, a monochromatic edge will refer to one colored the same as all its vertices. Given an edge coloring of $H$, a compatible vertex coloring is one under which there are no monochromatic edges. An emulsive edge coloring of $H$ is one for which there does not exist a compatible vertex coloring using the edge colors. The chromatic capacity $\gamma_{\text{cap}}(H)$ of $H$ is the largest $k$ for which there exists an emulsive $k$-coloring of the edges of $H$.

The notion of chromatic capacity has evolved from a number of separate sources. Cochand and Duchet [4] first invented the concept of emulsive edge colorings, using a graph of large chromatic capacity to construct a graph $G(D)$ all of whose acyclic
orientations contain an induced subgraph isomorphic to some specified acyclic digraph \( D \). (The existence of such a graph was proved several years earlier by Rödl \[8\].) Using probabilistic methods, they prove that, for an \( r \)-uniform hypergraph \( \mathcal{H} \) with maximum degree \( \Delta \), \( \chi_{\text{cap}}(\mathcal{H}) < \sqrt{er \Delta} \), where \( e \) denotes the base of the natural logarithm. Cocchand and Karolyi \[5\] constructively show that for the complete graph \( K_n \) on \( n \) vertices, \( \chi_{\text{cap}}(K_n) \geq (1 - o(1))\sqrt{n} \). Combined with the upper bound of \( \chi_{\text{cap}}(K_n) < \sqrt{2en} \) implied by the general estimate cited above, this establishes \( \sqrt{n} \) as the correct order of growth of \( \chi_{\text{cap}}(K_n) \). Erdős and Gyárfás \[7\] raised a related question which reduces, as a special case, to the chromatic capacity question for complete graphs. Independent of \[4,5\], they constructively establish \( (1 - o(1))\sqrt{n} \leq \chi_{\text{cap}}(K_n) \leq \sqrt{2en} \), thereby improving the upper estimate on \( \chi_{\text{cap}}(K_n) \). Archer \[2\] introduced the concept of compatible vertex colorings in connection with the upper chromatic numbers of the real line, independently proving that \( \chi_{\text{cap}}(G) < \sqrt{2e\Delta} \) for graphs \( G \), as well as \( \chi_{\text{cap}}(K_n) > (1 - o(1))\sqrt{n} \).

The preceding summary indicates little investigation into chromatic capacity beyond the determination of upper bounds and the estimation of \( \chi_{\text{cap}}(K_n) \). In this paper we prove an assortment of results intended to broaden the study of this chromatic quantity. In Section 2, we prove a general lower bound on the chromatic capacity of a graph; specifically, if \( G \) is a graph on \( n \) vertices with chromatic number \( \chi \) and chromatic capacity \( \chi_{\text{cap}} \), then we show that \( (\chi_{\text{cap}})^2 \ln(\chi_{\text{cap}}) > (1 - o(1))(\chi^2/2n) \), which yields the bound stated in the abstract. This extends a result of Brightwell and Kohayakawa \[3\], who establish this lower bound for complete graphs \( G \), although for complete graphs this bound is weaker than the one mentioned in the preceding paragraph. Since a proper vertex coloring of a hypergraph \( \mathcal{H} \) is compatible with every edge coloring, the bound \( \chi_{\text{cap}}(\mathcal{H}) \leq \chi(\mathcal{H}) - 1 \) is immediate, where \( \chi \) denotes the ordinary chromatic number. In Section 3, we provide a simple inductive construction that shows that for every \( r, k \geq 2 \), there exists an \( r \)-uniform hypergraph \( \mathcal{H} \) attaining the equality \( \chi_{\text{cap}}(\mathcal{H}) + 1 = \chi(\mathcal{H}) = k \). Moreover, in the case of graphs \( (r = 2) \), we can even require the graphs to be triangle-free. These results affirmatively answer and extend a question of Archer \[2\]. In Section 4, we prove that \( \chi_{\text{cap}}(G) = 1 \) if and only if each connected component of \( G \) is almost bipartite. This result relies on a lemma characterizing such graphs as being exactly those that contain neither a pair of edge-disjoint odd cycles nor an odd subdivision of \( K_4 \), which is interesting in its own right. Finally, in Section 5, we raise a host of possibilities for further work on chromatic capacity.

2. A lower bound

Brightwell and Kohayakawa \[3\] prove the lower bound \( n \leq 2\chi_{\text{cap}}(K_n)^2(\ln(\chi_{\text{cap}}(K_n))) + 2 \) for sufficiently large \( n \). Although this bound is weaker than the \( (1 - o(1))\sqrt{n} \) \( \chi_{\text{cap}}(K_n) \) bound cited above, we can adapt their method to prove a general lower bound on the chromatic capacity of a graph in terms of its number of vertices and chromatic number. We first require the following proposition.

**Proposition 1.** Let \( G \) be a graph with chromatic number \( m \), let \( k \) be a positive integer less than \( m \), and let \( n \) denote the minimum number of edges we can delete from \( G \)
to obtain a graph with chromatic number \( k \). Then
\[
\frac{m(m - k)}{2k} \leq n. \tag{1}
\]

**Proof.** Delete \( n \) edges of \( G \) such that the resulting graph has chromatic number \( k \). Give the vertices of this new graph a proper \( k \)-coloring, and partition the vertices into classes \( V_1, \ldots, V_k \), where \( V_i \) is the set of vertices colored \( i \). Now add in all missing edges between distinct \( V_i \) and \( V_j \) to obtain a complete \( k \)-partite graph \( G' \). If we add to \( G' \) the \( n \) edges deleted from \( G \), the resulting graph contains \( G \), so it has chromatic number at least \( m \).

Therefore, \( n \) is at least as large as the minimum number \( n' \) of edges we can add to \( G' \) to obtain a graph \( G'' \) of chromatic number \( m \). Let \( G_i \) denote the subgraph of \( G'' \) induced by \( V_i \). Also, let \( \chi_i = \chi(G_i) \) and \( E_i \) denote the number of edges in \( G_i \) for each such \( i \). Then
\[
n \geq n' = \sum_{i=1}^{k} E_i \geq \sum_{i=1}^{k} \left( \frac{\chi_i}{2} \right). \tag{2}
\]
Since \( \left( \frac{x}{2} \right) \) is a convex function of \( x \) and \( \sum_{i=1}^{k} \chi_i = m \), it follows that the last expression in (2) is minimized when all \( \chi_i = m/k \). Substituting this value into (2) yields inequality (1).

We remark that the bound in Proposition 1 can be improved to \( \left( \frac{m}{2} \right) - t(m, k + 1) \leq n \), where the **Turán number** \( t(m, k + 1) \) is the maximum number of edges a graph on \( m \) vertices containing no clique of size \( k + 1 \) can have (see [6]). We obtain this bound at once from the above proof by using the fact that the \( \chi_i \) are integral in estimating inequality (2). This improved bound is tight, as we see by considering \( G = K_m \). However, this sharper inequality does not yield any real advantage for our current purposes, so we omit the details.

The proof of the following theorem is now an adaptation of the proof technique in [3] combined with Proposition 1.

**Theorem 2.** Let \( G \) be a graph on \( n \) vertices with chromatic number \( \chi \) and chromatic capacity \( \chi_{cap} \). Then
\[
(\chi_{cap})^2 \ln(\chi_{cap}) > (1 - o(1)) \frac{\chi^2}{2n}, \tag{3}
\]
where the \( o(1) \) term approaches zero as \( \chi \to \infty \).

**Proof.** Fix a positive integer \( r \) and arbitrarily \( r \)-color the vertices of \( G \). Then, by Proposition 1, at least \( f(\chi, r) = \chi(\chi - r)/2r \) edges have both their endpoints colored the same. Now color the edges of \( G \) randomly, independently, and uniformly with the vertex colors. Then the probability the given vertex coloring is compatible with this edge coloring is at most \( (1 - 1/r)^{f(\chi, r)} \). Since there are \( r^n \) possible \( r \)-colorings of the vertices of \( G \), we find that if
\[
r^n(1 - 1/r)^{f(\chi, r)} < 1, \tag{4}
\]
then with positive probability the random edge coloring is not compatible with any vertex coloring; equivalently, this edge coloring is emulsive. So if \((4)\) holds, then \(r \leq \chi_{\text{cap}}\). Now, since \((1 - 1/r) < e^{-1/r}, (4)\) holds if \(f(\chi, r) > r n \ln r\). Therefore, if \(r > \chi_{\text{cap}}\), then \(f(\chi, r) \leq r n \ln r\). Setting \(r = \chi_{\text{cap}} + 1\) gives

\[
\frac{\chi (\chi - \chi_{\text{cap}} - 1)}{2(\chi_{\text{cap}} + 1)} = f(\chi, \chi_{\text{cap}} + 1) \leq (\chi_{\text{cap}} + 1) n \ln(\chi_{\text{cap}} + 1).
\]

Rearranging (5) then yields the bound stated in the theorem.

Estimating the \(\ln(\chi_{\text{cap}})\) term in Theorem 2 from above by \(\ln(\chi)\) gives rise to the bound stated in the abstract.

3. A pair of constructions

In [2], Archer observes the trivial inequality \(\chi_{\text{cap}}(G) \leq \chi(G) - 1\) and asks whether there exist graphs \(G\) of arbitrarily large chromatic number attaining this bound. The following theorem answers this question affirmatively with an explicit construction, and generalizes the result to \(r\)-uniform hypergraphs.

**Theorem 3.** For every \(r, k \geq 2\), there exists an \(r\)-uniform hypergraph with chromatic number \(k\) and chromatic capacity \(k - 1\).

**Proof.** Fix \(r \geq 2\). We recursively define a sequence of \(r\)-uniform hypergraphs \(\{H_k\}\) such that \(\chi(H_k) = k\), along with an emulsive \((k - 1)\)-coloring for each \(H_k\). First, \(H_2\) consists of a single edge colored with color 1. Clearly, \(\chi(H_2) = 2\) and this edge coloring is emulsive. Given \(H_k\), we construct \(H_{k+1}\) by taking \(k\) disjoint hypergraphs \(H_k^1, \ldots, H_k^k\), where for each \(i\), \(H_k^i\) is a copy of \(H_k\) with an emulsive edge-coloring using the color palette \(\{1, \ldots, i - 1, i + 1, \ldots, k\}\). We then add an additional set of vertices \(X = \{x_1, \ldots, x_n\}\), where \(n = (r - 2)k + 1\), and for each subset of \(r - 1\) vertices in \(X\) and a vertex in \(H_k^i\), form an edge colored \(i\) (see Fig. 1 for depictions of the graphs \(G_3\) and \(G_4\) corresponding to \(r = 2\) and \(k = 3, 4\), respectively).

We need to exhibit a proper \((k + 1)\)-coloring of the vertices and show that the given edge \(k\)-coloring is emulsive. Combined with the trivial bound \(\chi_{\text{cap}}(H_{k+1}) \leq \chi(H_{k+1}) - 1\), this will show that \(\chi_{\text{cap}}(H_{k+1}) = k\) and \(\chi(H_{k+1}) = k + 1\). We can properly \((k + 1)\)-color the vertices of \(H_{k+1}\) by inductively \(k\)-coloring the vertices of each \(H_k^i\) with a common set of \(k\) colors and coloring the vertices in \(X\) with an additional color. Now, let us \(k\)-color the vertices of \(H_{k+1}\) arbitrarily using \(1, \ldots, k\); by the pigeon-hole principle, some \(r - 1\) of the vertices in \(X\) receive the same color, say \(i\). If some vertex in \(H_k^i\) receives color \(i\), then it is contained in a monochromatic edge with those \(r - 1\) vertices in \(X\) colored \(i\). Otherwise, the colors of the vertices of \(H_k^i\) are restricted to \(1, \ldots, i - 1, i + 1, \ldots, k\), and by induction there exists a monochromatic edge in this subhypergraph. Hence the \(k\)-coloring of the edges of \(H_{k+1}\) described is emulsive, completing the proof of the theorem. □
Letting \( G_k \) denote the (2-uniform) graph with chromatic number \( k \) in Theorem 3, we see that each \( G_k \) also has clique number \( \omega(G_k) = k \). However, as the forthcoming theorem asserts, there exist triangle-free graphs \( (\omega = 2) \) of arbitrarily large chromatic number attaining the equality \( \chi_{\text{cap}}(G) = \chi(G) - 1 \) as well. We will rely on Zykov’s construction of a family of triangle-free graphs \( \{Z_k\} \) satisfying \( \chi(Z_k) = k \) for all \( k \) [10].

First, let \( Z_1 \) consist of a single vertex. Now given \( Z_k \) with \( k \geq 1 \), construct \( Z_{k+1} \) by taking \( k \) disjoint copies of \( Z_k \) and an independent vertex set \( A_k \) of \( |V(Z_k)|^k \) elements. Identify the vertices of \( A_k \) with \( k \)-tuples \((v_1, \ldots, v_k)\), where \( v_j \) is a vertex in the \( j \)th copy of \( Z_k \), and join each vertex in \( A_k \) to the corresponding vertices in its \( k \)-tuple. It is a straightforward exercise to verify that the family \( \{Z_k\} \), so constructed, has the properties mentioned above. Moreover, we have the following result.

**Theorem 4.** For all \( k \geq 1 \), the Zykov graph \( Z_k \) satisfies \( \chi_{\text{cap}}(Z_k) + 1 = \chi(Z_k) = k \).

**Proof.** We proceed by induction on \( k \). For \( k = 1 \) the assertion of the theorem is trivial, so for the inductive step assume that \( k \geq 1 \) and the assertion holds for \( k \). Clearly \( \chi_{\text{cap}}(Z_{k+1}) \leq k \) since \( \chi(Z_{k+1}) = k + 1 \). Denote the \( j \)th copy of \( Z_k \) in the construction of \( Z_{k+1} \) by \( Z^{(j)} \) and inductively endow it with an emulsive \( (k - 1) \)-coloring using the color palette \( \{1, \ldots, j-1, j+1, \ldots, k\} \). Also, color each edge between a vertex in \( Z^{(j)} \)
and one in $A_k$ with the color $j$. We claim that this is an emulsive $k$-coloring of $Z_{k+1}$.

For suppose we $k$-color $V(Z_{k+1})$; in order for this vertex-coloring to be compatible with the edge-coloring just described, we must use the color $j$ on some vertex $v_j$ of $Z^{(j)}$ for each $j$. But then the vertex $(v_1, \ldots, v_k)$ in $A_k$ cannot be colored. It follows that $\chi_{\text{cap}}(Z_{k+1}) = k$, and the theorem follows by induction. \qed

4. A characterization

A non-bipartite graph is said to be \textit{almost bipartite} if it becomes bipartite on the deletion of some edge. With this definition we can give a simple characterization of graphs $G$ with $\chi_{\text{cap}}(G) = 1$.

\textbf{Theorem 5.} A graph has chromatic capacity 1 if and only if each of its connected components are either bipartite or almost bipartite.

To prove this theorem, we require the following lemma, which characterizes graphs that are bipartite or almost bipartite in terms of forbidden subgraphs. An \textit{odd subdivision of $K_4$} refers to a subdivision of $K_4$ in which the induced subdivision of every triangle is an odd cycle.

\textbf{Lemma 6.} A graph is either bipartite or almost bipartite if and only if it does not contain a pair of edge-disjoint odd cycles nor an odd subdivision of $K_4$.

While Lemma 6 possesses independent interest, we postpone its proof until after deducing Theorem 5 from it.

\textbf{Proof of Theorem 5.} It suffices to assume that $G$ is a connected graph. If $G$ is bipartite, then $\chi_{\text{cap}}(G) = 1$ follows immediately from the inequality $\chi_{\text{cap}}(G) \leq \chi(G) - 1$. If $G$ is bipartite plus an edge, arbitrarily color the edges of $G$ red and blue. Suppose that $G$ is not bipartite, but becomes bipartite upon removal of the edge $(x, y)$ and, without loss of generality, that this edge is colored red. Then we may properly color the vertices of $G - (x, y)$ with red and blue in such a way that $x$ and $y$ both get colored blue, and this coloring is compatible with the given edge coloring.

Now assume that $G$ is neither bipartite nor bipartite plus an edge. By Lemma 6, it contains either a pair of nearly disjoint odd cycles or an odd subdivision of $K_4$. We now construct an emulsive 2-coloring of the edges of $G$.

Suppose first that $G$ contains two nearly disjoint odd cycles $C_1$ and $C_2$. Since $G$ is connected, there exist vertices $u$ on $C_1$ and $v$ on $C_2$ such that there exists a (possibly degenerate) path $P$ with endpoints $u$ and $v$ which is otherwise disjoint from $C_1 \cup C_2$. Starting at $u$, color the edges of $C_1$ alternately red and blue (so that both edges incident to $u$ are red), and in case $P$ is non-degenerate, color its edges alternately blue and red. If $u = v$, let $c$ denote the color blue. Otherwise, let $c$ denote the color that is not used on the edge of $P$ incident with $v$. Starting at $v$, color the edges of $C_2$ alternately, beginning with the color $c$. We claim that this edge coloring, extended arbitrarily to the
rest of $G$, is emulsive. For suppose we had a compatible two-coloring of the vertices of $G$. If $u$ is blue, then moving around $C_1$ in either direction we are forced to color the vertices alternately blue and red; but then on arriving back at $u$ we must use blue, a contradiction. If $u$ is red, then moving along the path $P$, we are forced to color the vertex $v$ with color $c$. But then, as was the case when $u$ was colored red, we obtain a contradiction (see Fig. 2).

Next, suppose that $G$ contains an odd subdivision of $K_4$. Let $a, b, c$, and $d$ denote the vertices in the subdivision corresponding to the vertices of the $K_4$, and let $P_{x,y}$ denote the path between $x$ and $y$ for each pair of distinct $x, y \in \{a, b, c, d\}$. We obtain one of four separate cases to consider, depending on the number of paths $P_{a,x}$, $x \in \{b, c, d\}$, of odd length. For each we claim there exists an emulsive 2-coloring of the edges. As the colorings for each of the cases are very similar, we only treat the case when exactly one of the $P_{a,x}$ has odd length, and the other cases follow similarly. These various cases are shown in Fig. 3.

So suppose that only $P_{a,b}$ has odd length. Note that $P_{c,d}$ also has odd length, whereas $P_{b,c}$ and $P_{b,d}$ both have even length. Along the six paths, we color the edges alternately red and blue, such that the two end edges of $P_{a,b}$ are both blue, the end edges of $P_{c,d}$ are both red, and the other four end edges adjacent to $a$ or $b$ are red.

Now suppose we had a compatible vertex coloring of $G$ and that $a$ were colored red. Then, considering the cycle spanned by $P_{a,c}$, $P_{c,d}$, and $P_{d,a}$, we obtain a situation identical to when $u$ was colored red in $C_1$ above, so there must exist a monochromatic edge. If $a$ were colored blue, then following $P_{a,b}$ we find that $b$ must be red. But considering the cycle spanned by $P_{b,c}$, $P_{c,d}$, and $P_{d,b}$, we again see that there must exist a monochromatic edge. Hence this coloring is emulsive.

**Corollary 7.** If $\chi(G) \geq 4$, then $\chi_{cap}(G) \geq 2$.

**Proof.** This follows immediately from Theorem 5.

We refer the reader to [6] for the terminology regarding separation, Menger’s theorem, and paths used in the following proof.

**Proof of Lemma 6.** The forward implication is trivial. For the reverse implication, it suffices to consider a graph $G$ that is neither bipartite nor almost bipartite, but for which there exist two distinct edges, $(v_1, v_2)$ and $(v_3, v_4)$, whose deletion leaves $G$ bipartite. Observe that a cycle in $G$ is odd if and only if it contains precisely one
Fig. 3. Emulsive edge-colorings of odd subdivisions of $K_4$ with $P_{x,y}$ odd for three, two, one, and no $x \in \{b, c, d\}$ shown in (a), (b), (c), and (d), respectively.

of the two edges $(v_1, v_2)$ and $(v_3, v_4)$. Moreover, $G$ must contain an odd cycle $C_1$ containing $(v_1, v_2)$ and an odd cycle $C_2$ containing $(v_3, v_4)$.

Now set $G' = G - (v_1, v_2) - (v_3, v_4)$, and suppose that there is a vertex $v$ whose deletion separates $\{v_1, v_2\}$ from $\{v_3, v_4\}$ in $G'$. Then $C_1$ and $C_2$ form a pair of edge-disjoint odd cycles in $G$, since they meet in at most the vertex $v$. If no such vertex exists, then by (the vertex version of) Menger’s theorem, there is a pair of vertex-disjoint $\{v_1, v_2\} - \{v_3, v_4\}$ paths $P$ and $Q$ in $G'$. Without loss of generality, we may assume that $P$ has endpoints $v_1$ and $v_3$, and that $Q$ has endpoints $v_2$ and $v_4$.

Suppose now that there exists an edge $(x, y)$ that separates $P$ from $Q$ in $G'$. Then every cycle in $G$ that contains $(v_1, v_2)$ must also contain one of $(x, y)$ and $(v_3, v_4)$, and similarly every cycle that contains $(v_3, v_4)$ must contain one of $(x, y)$ and $(v_1, v_2)$. It follows from this observation and the third sentence in the proof that $G$ is bipartite on removal of the edge $(x, y)$, a contradiction.
Therefore, by (the edge version of) Menger’s theorem, there exists a pair of edge-disjoint $P$–$Q$ paths $S$ and $T$ in $G'$. Let $a$ denote the endpoint of $S$ in $P$ and $b$ its endpoint in $Q$, and let $c$ denote the endpoint of $T$ in $P$ and $d$ its endpoint in $Q$. Without loss of generality, suppose that $a$ is at least as close to the vertex $v_1$ along $P$ as $c$ is (observe that $a = c$ is possible). If $b$ is at least as close to the vertex $v_2$ along $Q$ as $d$ is, then $v_1PaSbQv_2v_1$ and $v_3PcTdQv_4v_3$ are two edge-disjoint odd cycles in $G$ (Fig. 4(a)). Otherwise $d$ is closer to $v_2$ along $Q$ than $b$. In this case, if $S$ and $T$...
are vertex-disjoint, then the subgraph of $G$ induced by the paths $P$, $Q$, $S$, and $T$ and edges $(v_1, v_2)$ and $(v_3, v_4)$ is an odd subdivision of $K_4$ with degree 3 vertices $a$, $b$, $c$, and $d$, since each cycle in this subgraph contains exactly one of the edges $(v_1, v_2)$ and $(v_3, v_4)$ (Fig. 4(b)). If instead $S$ and $T$ intersect at a vertex $z$, then $v_1PaSzTdQv_2v_1$ and $v_3PcTzShQv_4v_3$ are two edge-disjoint odd cycles in $G$ (Fig. 4(c)). This completes the proof of the theorem. □

5. Directions for further study

The results contained herein give rise to numerous directions for continued research into the chromatic capacities of graphs and hypergraphs. Here we mention just a few possibilities.

1. The lower bound in Theorem 2 is most effective for a graph with large chromatic number $\chi$ relative to its number of vertices $n$. In fact, it yields no information on the chromatic capacity of a graph with $\chi < \sqrt{2n}$. However, Corollary 7 shows that for any graph with $\chi \geq 4$, we have $\chi_{cap} \geq 2$, regardless of $n$. We are therefore tempted to conjecture the following.

**Conjecture 8.** There exists an unbounded function $f: \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $\chi_{cap}(G) > f(\chi(G))$ for every graph $G$.

By analogy to the lower bound on $\chi_{cap}(K_n)$, perhaps even $\chi_{cap}(G) = \Omega\left(\sqrt{\chi(G)}\right)$ holds.

2. The *girth* of a graph $G$ is the length of a shortest cycle in $G$. Theorem 4 therefore shows that there exist graphs of girth 4 and arbitrarily large chromatic number for which $\chi_{cap} = \chi - 1$. In [2], Archer remarks that there exist graphs of arbitrarily large girth and chromatic capacity. His proof of this fact is probabilistic, and parallels Erdős’s well-known proof that there exist graphs of arbitrarily large girth and chromatic number (see [1, pp. 38–39]). In light of this result and Theorem 4, we are naturally led to pose:

**Problem 1.** Do there exist graphs of arbitrarily large girth and chromatic capacity for which $\chi_{cap} = \chi - 1$?

3. The precise values of $\chi_{cap}$ remain uncalculated or loosely estimated for all but a very small class of graphs. For the complete multipartite graph $K^r_n$, which consists of $n$ disjoint independent $r$-sets with all possible edges formed between the vertices of distinct $r$-sets, our best estimates so far give

$$(1 - o(1))\sqrt{n} < \chi_{cap}(K^r_n) < \min\left\{\sqrt{2er(n-1)}, n\right\}.$$  

The lower bound in (6) arises from the lower bound on $\chi_{cap}$ for the subgraph $K_n$ of $K^r_n$, while the upper bound follows from the maximum degree bound stated in the introduction and the trivial chromatic number bound. Clearly, these bounds leave quite a gap as $r$ grows relative to fixed $n$. 
Problem 2. Obtain tighter estimates on $\chi_{\text{cap}}(K_n^r)$ and, more generally, the chromatic capacity of other classes of graphs.

4. Is it possible to obtain an analogue of constructibility for chromatic capacity? For a given $k$, the program would be to find a simple set of graphs $S_k$ and graph operations such that a graph has chromatic capacity of at least $k$ if and only if it contains a subgraph which can be obtained by successive applications of the given operations to the graphs in $S_k$. We note, in passing, that the operations used in Hajós’s and Ore’s constructions of $k$-chromatic graphs [9] do not preserve chromatic capacity.

5. Let $b(G)$ denote the smallest integer $n$ for which there exists a two-coloring of the vertices of $G$ in which there are $n$ monochromatic edges in the usual sense. Then $b(G) = 0$ if and only if $G$ contains no odd cycles, and Lemma 6 provides an analogous structural characterization for those graphs $G$ with $b(G) \leq 1$. We plan to address the general problem of characterizing those graphs with $b(G) \leq n$, $n$ a non-negative integer, in a forthcoming manuscript.

Acknowledgements

I would like to thank Joseph Gallian for organizing and supervising the Research Experience for Undergraduates (REU) at the University of Minnesota, Duluth, thereby making this research possible. The program was supported by grants from the NSF (DMS 92820179) and the NSA (904-00-1-0026). I would also like to thank David Witte for numerous helpful and inspiring comments, as well as two referees whose comments significantly improved the presentation and content of this paper. In particular, one of the referees identified a major error in a previous version of this work, and another indirectly inspired the present proof of Lemma 6.

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