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On Stationary Values of Rayleigh Quotient of an Operator*

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1. INTRODUCTION

In this paper, A will denote a bounded linear transformation, which for the sake of simplicity will be called an operator, on a complex Hilbert space H with inner product (,) and associated norm $\| \ \|$. The ratio $((Ag, g)/(g, g))$, $g \in H$ is known as the Rayleigh quotient of A and will be denoted here by $R(g)$. The numerical radius of A, that is, the supremum of $|R(g)|$, will be denoted by $w(A)$.

It may be shown that if f is a vector that maximizes the quotients

$$|R(g)| = \frac{|(Ag, g)|}{(g, g)}$$

so that $(Af, f) = \lambda(f, f)$ where $|\lambda| = w(A)$, then

$$\operatorname{Re}(\bar{\lambda}A) f = \tfrac{1}{2}(\bar{\lambda}A + \lambda A^*) f = |\lambda|^2 f.$$

A proof follows from the fact that $\operatorname{Re}(\bar{\lambda}A)$ is a self-adjoint operator and $|\lambda|^2$ is its norm. The main result of this paper is the observation that this eigenvalue equation actually characterizes all those, and only those vectors f, at which $R(g)$ has a stationary value, and in particular at the supremum of $|R(g)|$.

By our method we not only obtain the eigenvalue relation at the maximum of $|R(g)|$ as an obvious consequence, but also we get a number of useful results at the said maximum (cf. Theorem 2 and its corollaries). All the results which hold at the maximue of $|R(g)|$ only by virtue of the above-mentioned eigenvalue relation also hold at the stationary values of $R(g)$, (Corollaries 2-4 of Theorem 1).

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Interestingly, it follows that all operators resemble a normal operator in some sense at stationary values of their respective Rayleigh quotients (see Corollaries 3 and 4 of Theorem 1). This observation was made also in [6]. It also follows that if an operator A behaves like a normal operator only at a vector f for which supremum of $|R(g)|$ may be attained, that is, if $A^*Af = AA^*f$, the eigenvalue relations $Af = \lambda f$ and $A^*f = \bar{\lambda}f$, $|\lambda| = w(A)$ hold (Corollary 4 of Theorem 2). Since the requirement of normality of A, that is $A^*Ag = AA^*g$ for all $g \in H$, could be dispensed with, we have in effect established existence of eigenvalue of a much larger class of operators. A new approach to the spectral theory for compact normal operators, without using the property $w(A) = \| A \|$, may also be obtained and has been indicated at the end of Corollary 4 of Theorem 2.

The properties of an operator A at the supremum of $|R(g)|$ may be seen from Theorems 1 and 2 and their Corollaries if the supremum is attained. An interesting result is given in Theorem 3 when the supremum of $|R(g)|$ is not attained. This result is a generalization of Theorem 1 of [6] where the result was proved for normal operators.

2. Stationary Values of $R(g)$

The Rayleigh quotient $R(f)$ may be defined to have a stationary value at f if the functional

$$w_g(t) = \frac{|(Af + tAg, f + tg)|}{(f + tg, f + tg)} \tag{1}$$

of a real variable t is stationary at $t = 0$ for any arbitrary but fixed $g \in H$. Obviously the supremum and infimum of $|R(g)|$ are also stationary values of $R(g)$ if they are attained. On differentiating $w_g(t)$, we obtain after some calculations,

$$|\lambda| \, w_g'(0) = \operatorname{Re}(g, \bar{\lambda}Af + \lambda A^*f - 2|\lambda|^2 f), \tag{2}$$

where $\lambda = (Af, f)$. From Eq. (2) we may easily see that the following theorem holds.

Theorem 1. *The Rayleigh quotient $R(g)$ is stationary at a vector f, if and only if, the eigenvalue relation*

$$\operatorname{Re}(\bar{\lambda}A) f = |\lambda|^2 f \tag{3}$$

holds where $\lambda = (Af, f)$.

We note that (3) must hold at the maximum of $|R(g)|$.

COROLLARY 1. *A vector f is an eigenvector of a self-adjoint operator A, if and only if, $R(f)$ is stationary at f.*

In Corollaries 2–5, f will denote a vector for which $R(f)$ is stationary.

COROLLARY 2. *The vectors Af and A^*f can be expressed as $Af = \lambda f + h$ and $A^*f = \bar{\lambda} f - (\bar{\lambda}/\lambda)\, h$, where h is a vector such that $(f, h) = 0$.*

COROLLARY 3. *The eigenvalue relation $Af = \lambda f$ holds, if and only if, $A^*f = \bar{\lambda} f$.*

COROLLARY 4. *The norms of Af and A^*f are equal, that is, $\| Af \| = \| A^*f \|$.*

Corollaries 3 and 4 show how an operator resembles a normal operator at the stationary values of its Rayleigh quotient.

COROLLARY 5. $(A^2f, f) = (Af, f)^2$ *implies* $Af = \lambda f$ *and* $A^*f = \bar{\lambda} f$.

3. SUPREMUM OF $|R(g)|$

We shall now confine our attention to the supremum of $|R(g)|$ and assume that the same is attained for a unit vector f. Considering the second derivative of $w_g(t)$ at $t = 0$, we get

$$[\mathrm{Re}(Af, g) + \mathrm{Re}(Ag, f)]^2 + [\mathrm{Im}(Af, g) + \mathrm{Im}(Ag, f)]^2 + 2\mathrm{Re}(Af, f)\,\mathrm{Re}(Ag, g) + 2\,\mathrm{Im}(Af, f)\,\mathrm{Im}(Ag, g) - 2\,|\lambda|^2\,(g, g) - 4\,|\lambda|^2\,\mathrm{Re}^2(f, g) \leqslant 0. \tag{4}$$

Putting $Af = \lambda f + h$ and $A^*f = \bar{\lambda} f - (\bar{\lambda}/\lambda)\, h$ in (4), with $(f, h) = 0$ as in Corollary 2, we obtain after a lengthy calculation

$$\frac{2}{|\lambda|^2\,(g, g)}\,[\mathrm{Re}(\lambda h, g)]^2 + \lambda_x\,\mathrm{Re}\,\frac{(Ag, g)}{(g, g)} + \lambda_y\,\mathrm{Im}\,\frac{(Ag, g)}{(g, g)} - |\lambda|^2 \leqslant 0, \tag{5a}$$

if $h \neq 0$.

Since we may put ig in place of g, we get

$$\frac{2}{|\lambda|^2\,(g, g)}\,[\mathrm{Im}(\lambda h, g)]^2 + \lambda_x\,\mathrm{Re}\,\frac{(Ag, g)}{(g, g)} + \lambda_y\,\mathrm{Im}\,\frac{(Ag, g)}{(g, g)} - |\lambda|^2 \leqslant 0. \tag{5b}$$

Obviously since

$$|\lambda| = |(Af, f)| = \sup_{g \in H} \frac{|(Ag, g)|}{(g, g)},$$

we have

$$\lambda_x \operatorname{Re} \frac{(Ag, g)}{(g, g)} + \lambda_y \operatorname{Im} \frac{(Ag, g)}{(g, g)} - |\lambda|^2 \leqslant 0.$$

Hence the positive term

$$\frac{2}{|\lambda|^2 (g, g)} [\operatorname{Re}(\lambda h, g)]^2 \qquad \text{in (5a)}$$

or

$$\frac{2}{|\lambda|^2 (g, g)} [\operatorname{Im}(\lambda h, g)]^2 \qquad \text{in (5b)},$$

whichever is greater, puts a restriction on the values of $((Ag, g)/(g, g))$.
Combining these results we formulate Theorem 2.

THEOREM 2. *If*

$$|\lambda| = |(Af, f)| = \sup_{g \in H} \frac{(Ag, g)}{(g, g)}, \qquad \|f\| = 1,$$

then for any arbitrary but fixed g, *the following inequalities hold true.*

$$\frac{2}{|\lambda|^2 (g, g)} [\operatorname{Re}(\lambda h, g)]^2 + \lambda_x \operatorname{Re} \frac{(Ag, g)}{(g, g)} + \lambda_y \operatorname{Im} \frac{(Ag, g)}{(g, g)} - |\lambda|^2 \leqslant 0,$$

and

$$\frac{2}{|\lambda|^2 (g, g)} [\operatorname{Im}(\lambda h, g)]^2 + \lambda_x \operatorname{Re} \frac{(Ag, g)}{(g, g)} + \lambda_y \operatorname{Im} \frac{(Ag, g)}{(g, g)} - |\lambda|^2 \leqslant 0,$$

where

$$h = Af - \lambda f.$$

Putting $g = \lambda h$, we get

$$2(h, h) + \lambda_x \operatorname{Re} \frac{(Ah, h)}{(h, h)} + \lambda_y \operatorname{Im} \frac{(Ah, h)}{(h, h)} - |\lambda|^2 \leqslant 0. \tag{6}$$

The least possible value of

$$\lambda_x \operatorname{Re} \frac{(Ah, h)}{(h, h)} + \lambda_y \operatorname{Im} \frac{(Ah, h)}{(h, h)}$$

is $-|\lambda|^2$; hence we conclude $\|h\|^2 \leqslant |\lambda|^2$.

COROLLARY 1. *$\| h \|^2$ is less than or equal to the smaller of $| \lambda |^2$ and $\| A \|^2 - | \lambda |^2$.*

Proof. We have already proved $\| h \|^2 \leqslant | \lambda |^2$. For the second part we only note that

$$\| Af \|^2 = | \lambda |^2 + \| h \|^2 \leqslant \| A \|^2.$$

If A is nonnormal, then A need not have any eigenvalue at all at the supremum of $| R(g) |$, the quantity $\| h \| = \| Af - \lambda f \|$ gives a measure of departure from the eigenvalue relation. Corollary 1 gives an upper bound for $\| h \|$. Since $| \lambda |^2 = \| A \|^2 - | \lambda |^2$ implies $| \lambda |^2 = \| A \|^2/2$, we always have $\| h \|^2 \leqslant \| A \|^2/2$.

COROLLARY 2. *If f is a unit vector for which* $\sup[|(Ag, g)|/(g, g)]$ *and* $\sup[\| Ag \|/\| g \|]$ *are attained, then $| \lambda | \geqslant \| A \|/\sqrt{2}$.*

Proof. $\| Af \|^2 = | \lambda |^2 + \| h \|^2 = \| A \|^2$ or $\| A \|^2 \leqslant 2 | \lambda |^2$ by Corollary 1 of Theorem 2.

In general $| \lambda | \geqslant \| A \|/2$ (see [2, p. 114]). This inequality is improved if $(|(Ag, g)|/(g, g))$ and $(\| Ag \|/\| g \|)$ attain supremum for the same vector f.

COROLLARY 3. *In no case does $((Ah, h)/(h, h))$ equal λ.*

Proof. If $((Ah, h)/(h, h)) = \lambda$, we easily see from (4) that $(h, h) = 0$.

COROLLARY 4. *If $AA^*f = A^*Af$, then $Af = \lambda f$ and $A^*f = \bar{\lambda} f$.*

Proof. From the relations $Af = \lambda f + h$ and $A^*f = \bar{\lambda} f - (\bar{\lambda}/\lambda) h$, we get $A^*Af = | \lambda |^2 f - \bar{\lambda} h + A^*h$ and $AA^*f = | \lambda |^2 f + \bar{\lambda} h - (\bar{\lambda}/\lambda) Ah$. Hence, $\bar{\lambda} Ah + \lambda A^*h = 2 | \lambda |^2 h$ or $((Ah, h)/(h, h)) = \lambda$, since $| \lambda |$ is the supremum of $| R(g) |$. So, by Corollary 3, $h = 0$, and hence the result follows.

This shows that if the operator is normal only at the supremum of $| R(g) |$ then also A has an eigenvalue λ, $| \lambda | = w(A)$. We can now develop in the usual manner the spectral theorem for a compact normal operator (cf. [7]). The spectral theorem for compact self-adjoint operators is based on the property that $w(A) = \| A \|$, whose proof is elementary. The spectral theorem for compact normal operators also rests on the same property. Halmos observed long ago [1, p. 111] that if an elementary proof of the above property could be found, the spectral theorem for normal operators would be immensely simplified. Bernau and Smithies gave in [3] an elementary proof of it. The method just indicated for the development of spectral theory for a compact normal operator is also elementary and completely different from the other approaches in that we do not make use of the property that $w(A) = \| A \|$ for all normal operators.

We have till now dealt with cases where $\sup[|(Ag, g)|/(g, g)]$, $g \in H$ is attained. Now we shall examine what happens if the supremum is not attained. Let $\{f_n\}$ be a sequence of vectors with unit norm, i.e., $\|f_n\| = 1$, such that $|(Af_n, f_n)| \to w(A)$. Since the unit sphere in H is weakly compact, it is always possible to get a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ weakly converges to f and $\{(Af_{n_k}, f_{n_k})\}$ converges. We assume that such a choice has already been made, that is, $\{f_n\}$ weakly converges to f.

Let $(Af_n, f_n) \to \lambda$. Consider now the operator $2 | \lambda |^2 I - \bar{\lambda} A - \lambda A^*$. It is easy to see that it is a positive operator and as $(Af_n, f_n) \to \lambda$, $(2 | \lambda |^2 f_n - \bar{\lambda} Af_n - \lambda A^* f_n, f_n) \to 0$. Therefore, by a property of positive operators we have

$$2 | \lambda |^2 f_n - \bar{\lambda} Af_n - \lambda A^* f_n \to 0.$$

If $f \neq 0$, then

$$2 | \lambda |^2 (f_n, f) - \bar{\lambda}(Af_n, f) - \lambda(A^* f_n, f) \to 0,$$

or

$$2 | \lambda |^2 (f, f) = \bar{\lambda}(Af, f) + \lambda(A^* f, f),$$

or

$$| \lambda |^2 (f, f) = \text{Re}(\bar{\lambda}(Af, f)),$$

or $(Af, f)/(f, f) = \lambda$, since

$$| \lambda | = \sup_{g \in H} | R(g) | .$$

Hence we have Theorem 3.

THEOREM 3. *Let $\{f_n\}$ be a weakly convergent sequence such that $\{(Af_n, f_n)\}$ converges and $|(Af_n, f_n)| \to w(A)$. If the weak limit f is nonzero, then the supremum of $| R(g)|$ is attained for the vector f. If the supremum is not attained, then all such sequences must tend weakly to zero.*

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