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Approximation Theorems for Double Orthogonal Series

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Let $\{\phi_{ik}(x): i, k = 1, 2,...\}$ be a double orthonormal system on a positive measure space $(X, \mathscr{F}, \mu)$ and $\{a_{ik}\}$ a double sequence of real numbers for which $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 < \infty$. Then the sum $f(x)$ of the double orthogonal series $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}\phi_{ik}(x)$ exists in the sense of L^2-metric. If, in addition, $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \kappa^2(i, k) < \infty$ with an appropriate double sequence $\{\kappa(i, k)\}$ of positive numbers, then a rate of approximation to $f(x)$ can be concluded by the rectangular partial sums $s_{mn}(x) = \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik}\phi_{ik}(x)$, by the first arithmetic means of the rectangular partial sums $\sigma_{mn}(x) = (1/mn) \sum_{i=1}^{m} \sum_{k=1}^{n} s_{ik}(x)$, by the first arithmetic means of the square partial sums $\sigma_r(x) = (1/r) \sum_{k=1}^{r} s_{kk}(x)$, etc. The so-called strong approximation to $f(x)$ by $s_{mn}(x)$ is also studied.

1. Introduction

Let $(X, \mathscr{F}, \mu)$ be an arbitrary positive measure space and $\{\phi_{ik}(x): i, k = 1, 2,...\}$ an orthonormal system (abbreviated ONS) on X. We will consider the double orthogonal series

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}\phi_{ik}(x), \tag{1.1}$$

where $\{a_{ik}: i, k = 1, 2,...\}$ is a double sequence of real numbers (coefficients) for which

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 < \infty. \tag{1.2}$$

By the Riesz–Fischer theorem there exists a function $f(x) \in L^2 = L^2(X, \mathscr{F}, \mu)$ such that the series (1.1) is the Fourier series of $f(x)$ with respect to the system $\{\phi_{ik}(x)\}$. In particular, the rectangular partial sums

$$s_{mn}(x) = \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik}\phi_{ik}(x) \qquad (m, n = 1, 2,...),$$

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converge to $f(x)$ in the L^2-metric:

$$\int [s_{mn}(x) - f(x)]^2 \, d\mu(x) \to 0 \qquad \text{as} \quad \min\{m, n\} \to \infty.$$

Here and in the sequel the integrals are taken over the entire space X.

By the extension of the Rademacher–Menšov theorem (see, e.g., [1, 9]), if

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log(i+1)]^2 [\log(k+1)]^2 < \infty, \tag{1.3}$$

then the rectangular partial sums $s_{mn}(x)$ regularly converge a.e., a fortiori converge in Pringsheim's sense to $f(x)$ a.e., and there exists a function $F(x) \in L^2$ such that

$$\sup_{m,n \geqslant 1} |s_{mn}(x)| \leqslant F(x), \text{ a.e.}$$

In this paper the logarithms are to the base 2. As for the notion of regular convergence, see [7 and 10], and for convergence in Pringsheim's sense see, e.g., [14, p. 303; or 10].

Denote by $\sigma_{mn}(x)$ the first arithmetic means of the rectangular partial sums:

$$\begin{aligned}\sigma_{mn}(x) &= \frac{1}{mn} \sum_{i=1}^{m} \sum_{k=1}^{n} s_{ik}(x) \\ &= \sum_{i=1}^{m} \sum_{k=1}^{n} \left(1 - \frac{i-1}{m}\right)\left(1 - \frac{k-1}{n}\right) a_{ik} \phi_{ik}(x) \qquad (m, n = 1, 2, \ldots).\end{aligned}$$

By the extension of the Menšov–Kaczmarz theorem if

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log\log(i+3)]^2 [\log\log(k+3)]^2 < \infty, \tag{1.4}$$

then the $(C, 1, 1)$-means $\sigma_{mn}(x)$ regularly converge a.e., a fortiori converge in Pringsheim's sense to $f(x)$ a.e., and there exists a function $f(x) \in L^2$ such that

$$\sup_{m,n \geqslant 1} |\sigma_{mn}(x)| \leqslant F(x), \quad \text{a.e.}$$

This extension was firstly stated by Fedulov [5]. Unfortunately, his proof contains two essential defects. Later on, Csernyák [4] restated this theorem, but he corrected only the first defect in Fedulov's proof. A complete proof was given by the present author in [12].

We will consider the arithmetic means of the rectangular partial sums with respect to only m:

$$\tau_{mn}(x) = \tau_{mn}^{(1)}(x) = \frac{1}{m} \sum_{i=1}^{m} s_{in}(x)$$

$$= \sum_{i=1}^{m} \sum_{k=1}^{n} \left(1 - \frac{i-1}{m}\right) a_{ik}\phi_{ik}(x),$$

and those with respect to only n:

$$\tau_{mn}^{(2)}(x) = \frac{1}{n} \sum_{k=1}^{n} s_{mk}(x) = \sum_{i=1}^{m} \sum_{k=1}^{n} \left(1 - \frac{k-1}{n}\right) a_{ik}\phi_{ik}(x) \qquad (m, n = 1, 2, \ldots).$$

These means are called the $(C, 1, 0)$ and $(C, 0, 1)$-means of series (1.1), respectively.

2. Main Results: Approximation by Rectangular Partial Sums and Their Means

First we make the following convention. Given a double sequence $\{f_{mn}(x)\}$ of functions in L^2 and a double sequence $\{\lambda(m, n)\}$ of positive numbers, we write

$$f_{mn}(x) = o_x\{\lambda(m, n)\}, \text{ a.e.} \qquad \text{as } \min\{m, n\} \to \infty \qquad (2.1)$$

$$(\text{or } \max\{m, n\} \to \infty),$$

if

$$\frac{f_{mn}(x)}{\lambda(m, n)} \to 0, \text{ a.e.} \qquad \text{as } \min\{m, n\} \to \infty$$

$$(\text{or } \max\{m, n\} \to \infty),$$

and, in addition, there exists a function $F(x) \in L^2$ such that

$$\sup_{m,n} \frac{|f_{mn}(x)|}{\lambda(m, n)} \leqslant F(x), \quad \text{a.e.}$$

Here m ranges over either $0, 1, \ldots$, or $1, 2, \ldots$; and so does n. Furthermore, we agree to omit the expression "as $\min\{m, n\} \to \infty$" in (2.1). Also, in o_x estimates containing both m and n as free parameters we mean that $\min\{m, n\} \to \infty$, unless it is specified otherwise. A similar meaning is assigned to the symbol

$$f_m(x) = o_x\{\lambda(m)\}, \text{ a.e.} \qquad \text{as } m \to \infty,$$

where $\{f_m(x)\}$ is a sequence of functions in L^2 and $\{\lambda(m)\}$ is a sequence of positive numbers, both defined either for $m = 0, 1, \ldots,$ or for $m = 1, 2, \ldots$. The specification "as $m \to \infty$" is also omitted if m is the only free parameter involved.

In Section 1 we have mentioned that conditions (1.3) and (1.4) are sufficient for the a.e. convergence of $s_{mn}(x)$ and $\sigma_{mn}(x)$ to $f(x)$, respectively. Now the main point is that if we require somewhat more than (1.3) and (1.4), then we can even state an approximation rate for the deviations $s_{mn}(x) - f(x)$ and $\sigma_{mn}(x) - f(x)$, respectively. A part of the theorems obtained can be considered the extensions of the two theorems of Tandori [13] from single orthogonal series to double ones.

In the sequel the double sequence $\{\lambda(m, n)\}$ will be specified as

$$\lambda(m, n) = \max\{\lambda_1(m), \lambda_2(n)\} \qquad (m, n = 1, 2, \ldots; \lambda_1(1) = \lambda_2(1) = 1), \tag{2.2}$$

where $\{\lambda_1(m)\colon m = 1, 2, \ldots\}$ and $\{\lambda_2(n)\colon n = 1, 2, \ldots\}$ are nondecreasing sequences of positive numbers tending to ∞.

THEOREM 1. *If*

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log(i+1)]^2 [\log(k+1)]^2 [\max\{\lambda_1(i), \lambda_2(k)\}]^2 < \infty, \tag{2.3}$$

then

$$s_{mn}(x) - f(x) = o_x \left\{ \frac{1}{\lambda_1(m+1)} + \frac{1}{\lambda_2(n+1)} \right\}, \quad \text{a.e.} \tag{2.4}$$

We note that the right-hand side of conclusion (2.4) can be equivalently rewritten as $o_x\{\max\{1/\lambda_1(m+1), 1/\lambda_2(n+1)\}\}$, a.e.

The next theorem provides an approximation rate when a double subsequence of the rectangular partial sums is considered, instead of the whole sequence.

THEOREM 2. *Let $\{i_p\colon p = 1, 2, \ldots\}$ and $\{k_q\colon q = 1, 2, \ldots\}$ be two strictly increasing sequences of positive integers. If*

$$\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \left(\sum_{i=i_{p-1}+1}^{i_p} \sum_{k=k_{q-1}+1}^{k_q} a_{ik}^2 \right) [\log(p+1)]^2 [\log(q+1)]^2 \times [\max\{\lambda_1(i_p), \lambda_2(k_q)\}]^2 < \infty \qquad (i_0 = k_0 = 0), \tag{2.5}$$

then

$$s_{i_p, k_q}(x) - f(x) = o_x \left\{ \frac{1}{\lambda_1(i_{p+1})} + \frac{1}{\lambda_2(k_{q+1})} \right\}, \quad \text{a.e.} \tag{2.6}$$

This theorem is of special interest in the cases where $i_p = 2^p$, $k_q = q$ and $i_p = 2^p$, $k_q = 2^q$, respectively. (See Part 1 in Sects. 6 and 7.)

THEOREM 3. *If*

$$\lambda_1(2m) \leqslant C\lambda_1(m) \qquad \text{with} \quad C < 2 \text{ for } m \geqslant m_0, \tag{2.7}$$

$$\lambda_2(2n) = O\{\lambda_2(n)\}, \tag{2.8}$$

and

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log\log(i+3)]^2 [\log(k+1)]^2 [\max\{\lambda_1(i), \lambda_2(k)\}]^2 < \infty, \tag{2.9}$$

then

$$\tau_{mn}(x) - f(x) = o_x \left\{ \frac{1}{\lambda_1(m)} + \frac{1}{\lambda_2(n)} \right\}, \quad a.e. \tag{2.10}$$

Here and in the sequel, by C we denote positive constants not necessarily the same at each occurrence. We note that, under (2.7), condition (2.5) in the special case $i_p = 2^p$ and $k_q = q$ is equivalent to (2.9).

THEOREM 4. *If condition* (2.7) *is satisfied,*

$$\lambda_2(2n) \leqslant C\lambda_2(n) \qquad \text{with } C < 2 \text{ for } n \geqslant n_0, \tag{2.11}$$

and

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log\log(i+3)]^2 [\log\log(k+3)]^2 \times [\max\{\lambda_1(i), \lambda_2(k)\}]^2 < \infty, \tag{2.12}$$

then

$$\sigma_{mn}(x) - f(x) = o_x \left\{ \frac{1}{\lambda_1(m)} + \frac{1}{\lambda_2(n)} \right\}, \quad a.e. \tag{2.13}$$

It is clear that, under (2.7) and (2.11), condition (2.5) for $i_p = 2^p$ and $k_q = 2^q$ is equivalent to (2.12). If we assume that m and n tend restrictedly to ∞, i.e., there exists a constant $\theta \geqslant 1$ such that $\theta^{-1} \leqslant n/m \leqslant \theta$, then we can achieve essentially the same rate of approximation as in (2.13) under a weaker assumption.

THEOREM 5. *If condition* (2.7) *is satisfied and*

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log \log(\max\{i, k\} + 3)]^2 \lambda_1^2(\max\{i, k\}) < \infty, \tag{2.14}$$

then for every $\theta \geqslant 1$,

$$\max_{n:\theta^{-1} \leqslant n/m \leqslant \theta} |\sigma_{mn}(x) - f(x)| = o_x \left\{ \frac{1}{\lambda_1(m)} \right\}, \ a.e. \tag{2.15}$$

It is a simple observation that

$$\sigma_{mn}(x) - f(x) = \frac{1}{mn} \sum_{i=1}^{m} \sum_{k=1}^{n} [s_{ik}(x) - f(x)].$$

The next theorem reveals that the average of the deviations $s_{ik}(x) - f(x)$ is of $o_x\{1/\lambda_1(m)\}$ in (2.15), not because of the cancellation of positive and negative terms, but because the pairs (i, k) for which $|s_{ik}(x) - f(x)|$ is not small are sparse, at least in the case where the ratio k/i is bounded both from below and from above.

THEOREM 6. *If conditions* (2.7) *and* (2.14) *are satisfied and* $\{m/\lambda_1(m)\}$ *is nondecreasing, then for every* $\theta \geqslant 1$,

$$\left\{ \frac{1}{m^2} \sum_{i=1}^{m} \sum_{k=\theta^{-1}i}^{\theta i} [s_{ik}(x) - f(x)]^2 \right\}^{1/2} = o_x \left\{ \frac{1}{\lambda_1(m)} \right\}, a.e. \tag{2.16}$$

By $\sum_{k=\theta^{-1}i}^{\theta i}$ we mean that the summation is extended over those integers k for which $\theta^{-1}i \leqslant k \leqslant \theta i$.

Remark 1. Condition (2.7) is satisfied, e.g., if $\lambda_1(m) = m^\alpha$ with $0 < \alpha < 1$ or $\lambda_1(m) = m^\alpha [\log(m+1)]^\beta$ with $0 \leqslant \alpha < 1$ and $\beta > 0$.

Remark 2. Following Alexits [3], the property expressed in (2.16) can be called a strong approximation to $f(x)$ by the rectangular partial sums. In particular, via the Cauchy inequality (2.16) implies

$$\frac{1}{m^2} \sum_{i=1}^{m} \sum_{k=\theta^{-1}i}^{\theta i} |s_{ik}(x) - f(x)| = o_x \left\{ \frac{1}{\lambda_1(m)} \right\}, \text{ a.e.}$$

Remark 3. By slightly modifying the proof of Theorem 6, one can conclude the following somewhat stronger statement: If condition (2.14) is satisfied,

$$\lambda_1(2m) \leqslant C\lambda_1(m) \quad \text{with} \quad C < \sqrt{2} \quad \text{for } m \geqslant m_0, \tag{2.17}$$

and $\{m/\lambda_1^2(m)\}$ is nondecreasing, then for every $\theta \geqslant 1$,

$$\left\{\frac{1}{m}\sum_{i=1}^{m}\frac{1}{i}\sum_{k=\theta^{-1}i}^{\theta i}[s_{ik}(x)-f(x)]^2\right\}^{1/2}=o_x\left\{\frac{1}{\lambda_1(m)}\right\}, \text{ a.e.}$$

3. Approximation by Special Partial Sums and Their Means

We fix a single sequence $Q=\{Q_r\colon r=1, 2,...\}$ of finite sets in $\mathbb{N}^2=\{(i, k) : i, k=1, 2,...\}$ such that

$$Q_1\subset Q_2\subset\cdots, \qquad \text{and} \qquad \bigcup_{r=1}^{\infty} Q_r=\mathbb{N}^2.$$

The sums

$$s_r(Q; x)=\sum_{(i,k)\in Q_r} a_{ik}\phi_{ik}(x) \qquad (r=1, 2,...),$$

can be also regarded as a certain kind of partial sums of series (1.1). The following two special cases are well known:

$$Q_r=\{(i, k)\in\mathbb{N}^2 : i, k=1, 2,..., r\}$$

provides the square partial sums, while

$$Q_r=\{(i, k)\in\mathbb{N}^2 : i^2+k^2\leqslant r^2\} \qquad (r=1, 2,...),$$

provides the spherical partial sums of series (1.1).

Denote by $\sigma_r(Q; x)$ the first arithmetic means of the $s_r(Q; x)$:

$$\sigma_r(Q; x)=\frac{1}{r}\sum_{\rho=1}^{r} s_\rho(Q; x)$$
$$=\sum_{\rho=1}^{r}\left(1-\frac{\rho-1}{r}\right)\sum_{(i,k)\in Q_\rho\backslash Q_{\rho-1}} a_{ik}\phi_{ik}(x) \qquad (r=1, 2,...; Q_0=\varnothing).$$

The one-parameter versions of Theorems 1, 2, 3, and 6 read as follow. In these theorems $\{\lambda_1(r)\colon r=1, 2,...\}$ is a nondecreasing sequence of positive numbers tending to ∞.

Theorem 1′. *If*

$$\sum_{r=1}^{\infty}\left(\sum_{(i,k)\in Q_r\backslash Q_{r-1}} a_{ik}^2\right)[\log(r+1)]^2\,\lambda_1^2(r)<\infty, \tag{3.1}$$

then

$$s_r(Q;x)-f(x)=o_x\left\{\frac{1}{\lambda_1(r+1)}\right\}, \quad a.e.$$

For the square partial sums, (3.1) is equivalent to the condition

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2[\log(\max\{i,k\}+1)]^2\,\lambda_1^2(\max\{i,k\})<\infty.$$

THEOREM 2′. *Let* $\{r_p: p=1,2,\ldots\}$ *be a strictly increasing sequence of positive integers. If*

$$\sum_{p=1}^{\infty}\left(\sum_{(i,k)\in Q_{r_p}\setminus Q_{r_{p-1}}} a_{ik}^2\right)[\log(p+1)]^2\,\lambda_1^2(r_p)<\infty \qquad (r_0=0,\ Q_0=\varnothing), \tag{3.2}$$

then

$$s_{r_p}(Q;x)-f(x)=o_x\left\{\frac{1}{\lambda_1(r_{p+1})}\right\}, \quad a.e.$$

In the special case where $r_p=2^p$ and

$$\lambda_1(2r)=O\{\lambda_1(r)\},$$

(3.2) goes over to the condition

$$\sum_{r=1}^{\infty}\left(\sum_{(i,k)\in Q_r\setminus Q_{r-1}} a_{ik}^2\right)[\log\log(r+3)]^2\,\lambda_1^2(r)<\infty. \tag{3.3}$$

Specialized further, in the case of square partial sums (3.3) is equivalent to condition (2.14).

THEOREM 4′. *If conditions* (2.7) *and* (3.3) *are satisfied, then*

$$\sigma_r(Q;x)-f(x)=o_x\left\{\frac{1}{\lambda_1(r)}\right\}, \quad a.e.$$

THEOREM 6′. *If conditions* (2.17) *and* (3.3) *are satisfied and* $\{r/\lambda_1^2(r)\}$ *is nondecreasing, then*

$$\left\{\frac{1}{r}\sum_{\rho=1}^{r}[s_\rho(Q;x)-f(x)]^2\right\}^{1/2}=o_x\left\{\frac{1}{\lambda_1(r)}\right\}, \quad a.e.$$

The last theorem expresses a strong approximation to $f(x)$ by the $s_\rho(Q;x)$, in a particular case by the square partial sums.

4. Auxiliary Results on Numerical Sequences

Given a double sequence $\{\lambda(m, n): m, n = 1, 2, \ldots\}$ of numbers, we write

$$\Delta_{10}\lambda(m, n) = \lambda(m, n) - \lambda(m + 1, n),$$
$$\Delta_{01}\lambda(m, n) = \lambda(m, n) - \lambda(m, n + 1),$$
$$\Delta_{11}\lambda(m, n) = \lambda(m, n) - \lambda(m + 1, n) - \lambda(m, n + 1) + \lambda(m + 1, n + 1).$$

We say that $\{\lambda(m, n)\}$ is nonincreasing if both $\Delta_{10}\lambda(m, n) \geqslant 0$ and $\Delta_{01}\lambda(m, n) \geqslant 0$, while $\{\lambda(m, n)\}$ is nondecreasing if both $\Delta_{10}\lambda(m, n) \leqslant 0$ and $\Delta_{01}\lambda(m, n) \leqslant 0$ for all m and n. Furthermore, $\{\lambda(m, n)\}$ is said to be convex if $\Delta_{11}\lambda(m, n) \geqslant 0$ for all m and n.

Lemma 1. *If $\{\lambda_1(m): m = 1, 2, \ldots\}$ and $\{\lambda_1(n): n = 1, 2, \ldots\}$ are nondecreasing sequences of positive numbers and $\{\lambda(m, n)\}$ is defined by* (2.2), *then $\{1/\lambda(m, n)\}$ is nonincreasing and convex.*

Proof. It is clear that $\{1/\lambda(m, n)\}$ is nonincreasing. We will prove that it is convex. To this effect, let a pair (m, n) of positive integers be given. Without loss of generality, we may assume $\lambda_1(m) \geqslant \lambda_2(n)$. Then, by definition $\lambda(m, n) = \lambda_1(m)$ and $\lambda(m + 1, n) = \lambda_1(m + 1)$.

We distinguish two cases: either

(a) $\lambda(m, n + 1) = \lambda_1(m) \geqslant \lambda_1(n + 1)$ or

(b) $\lambda(m, n + 1) = \lambda_2(n + 1) \geqslant \lambda_1(m)$.

In case (a), by definition $\lambda(m + 1, n + 1) = \lambda_1(m + 1)$, consequently

$$\Delta_{11} \frac{1}{\lambda(m, n)} = 0.$$

In case (b), there are two subcases: either

(b_1) $\lambda(m + 1, n + 1) = \lambda_1(m + 1) \geqslant \lambda_2(n + 1)$ or

(b_2) $\lambda(m + 1, n + 1) = \lambda_2(n + 1) \geqslant \lambda_1(m + 1)$.

In case (b_1), by definition and property (b),

$$\Delta_{11} \frac{1}{\lambda(m, n)} = \frac{1}{\lambda_1(m)} - \frac{1}{\lambda_2(n + 1)} \geqslant 0,$$

while in case (b_2), by the monotony of $\{\lambda_1(m)\}$,

$$\Delta_{11} \frac{1}{\lambda(m, n)} = \frac{1}{\lambda_1(m)} - \frac{1}{\lambda_1(m + 1)} \geqslant 0. \quad \blacksquare$$

LEMMA 2. *If $\{\lambda(m,n)\}$ is a nondecreasing sequence of positive numbers for which $\{1/\lambda(m,n)\}$ is convex and the condition*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2[\log(i+1)]^2[\log(k+1)]^2\,\lambda^2(i,k)<\infty \tag{4.1}$$

is satisfied, then there exists a nondecreasing sequence $\{\lambda^*(m,n)\}$ of positive numbers for which $\{1/\lambda^*(m,n)\}$ is convex,

$$\frac{\lambda(m,n)}{\lambda^*(m,n)}\to 0 \qquad \text{as} \quad \max\{m,n\}\to\infty, \tag{4.2}$$

and

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2[\log(i+1)]^2[\log(k+1)]^2[\lambda^*(i,k)]^2<\infty. \tag{4.3}$$

Proof. By (4.1), there exists a strictly increasing sequence $\{m_p\}$ of positive numbers such that

$$\sum_{i=1}^{\infty}\sum_{\substack{k=1\\ \max\{i,k\}\geqslant m_p}}^{\infty} a_{ik}^2[\log(i+1)]^2[\log(k+1)]^2\,\lambda^2(i,k)\leqslant\frac{1}{p^3} \qquad (p=1,2,\ldots).$$

Define

$$\begin{aligned}\lambda^*(i,k)&=\lambda(i,k) && \text{for}\quad i,k=1,2,\ldots,m_2-1;\\ &=p\lambda(i,k) && \text{for}\quad m_p\leqslant\max\{i,k\}<m_{p+1}\qquad (p=2,3,\ldots).\end{aligned}$$

The fulfillment of (4.2) and (4.3) are obvious. To prove that $\{1/\lambda^*(i,k)\}$ is convex, we distinguish four cases.

Case (a). $\max\{i,k\}<m_1$. Then, by assumption,

$$\Delta_{11}\frac{1}{\lambda^*(i,k)}=\Delta_{11}\frac{1}{\lambda(i,k)}\geqslant 0.$$

Case (b). $m_p\leqslant\max\{i,k\}<m_{p+1}-1$ for some $p\geqslant 1$. Then, by definition,

$$\Delta_{11}\frac{1}{\lambda^*(i,k)}=\frac{1}{p}\Delta_{11}\frac{1}{\lambda(i,k)}\geqslant 0.$$

Case (c). $\max\{i,k\}=m_{p+1}-1$, but $\min\{i,k\}<m_{p+1}-1$. If $i=m_{p+1}-1$, say, then

$$\begin{aligned}\Delta_{11}\frac{1}{\lambda^*(i,k)}&=\frac{1}{p}\Delta_{01}\frac{1}{\lambda(i,k)}-\frac{1}{p+1}\Delta_{01}\frac{1}{\lambda(i+1,k)}\\ &=\frac{1}{p}\Delta_{11}\frac{1}{\lambda(i,k)}+\frac{1}{p(p+1)}\Delta_{01}\frac{1}{\lambda(i+1,k)}\geqslant 0.\end{aligned}$$

Case (d). $i=k=m_{p+1}-1$. Then

$$\Delta_{11}\frac{1}{\lambda^*(i,k)}=\frac{1}{p\lambda(i,k)}-\frac{1}{p+1}\left(\frac{1}{\lambda(i+1,k)}+\frac{1}{\lambda(i,k+1)}-\frac{1}{\lambda(i+1,k+1)}\right)$$
$$=\frac{1}{p+1}\Delta_{11}\frac{1}{\lambda(i,k)}+\frac{1}{p(p+1)\lambda(i,k)}\geqslant 0. \quad ∎$$

LEMMA 3. *If* $\{\lambda_1(m)\}$ *is a nondecreasing sequence of positive numbers for which condition* (2.7) *is satisfied, then*

$$\text{(i)}\quad \frac{m}{\lambda_1(m)}\to\infty, \qquad \text{as} \quad m\to\infty, \tag{4.4}$$

$$\text{(ii)}\quad \sum_{m=0}^{p}\frac{2^m}{\lambda_1(2^m)}=O\left\{\frac{2^p}{\lambda_1(2^p)}\right\} \quad (p=0,1,\ldots), \tag{4.5}$$

$$\text{(iii)}\quad \frac{1}{i}\sum_{m=1}^{i}\frac{1}{\lambda_1(m)}=O\left\{\frac{1}{\lambda_1(i)}\right\} \quad (i=1,2,\ldots), \tag{4.6}$$

$$\text{(iv)}\quad \sum_{m=p}^{\infty}\frac{\lambda_1^2(2^m)}{2^{2m}}=O\left\{\frac{\lambda_1^2(2^p)}{2^{2p}}\right\} \quad (p=0,1,\ldots), \tag{4.7}$$

$$\text{(v)}\quad \sum_{m=i}^{\infty}\frac{\lambda_1^2(m)}{m^3}=O\left\{\frac{\lambda_1^2(i)}{i^2}\right\} \quad (i=1,2,\ldots). \tag{4.8}$$

Proof. Here we drop the one subscript on $\lambda_1(m)$.

(i) By (2.7),

$$\lambda(2^p m_0)\leqslant C^p\lambda(m_0) \qquad (p=0,1,\ldots;\, C<2),$$

whence

$$\frac{2^p m_0}{\lambda(2^p m_0)}\geqslant\left(\frac{2}{C}\right)^p\frac{m_0}{\lambda(m_0)}\to\infty \qquad \text{as} \quad p\to\infty.$$

In the case where $2^{p-1}m_0<m\leqslant 2^p m_0$, we suffice to take into account the inequality

$$\frac{m}{\lambda(m)}\geqslant\frac{2^{p-1}m_0}{\lambda(2^p m_0)}.$$

(ii) Let $2^{m_1}\geqslant m_0$. Then for every m and p such that $m_1\leqslant m\leqslant p$,

$$\lambda(2^p)\leqslant C^{p-m}\lambda(2^m),$$

and by (2.7),

$$\sum_{m=m_1}^{p} \frac{2^m}{\lambda(2^m)} \leqslant \frac{2}{2-C} \frac{2^p}{\lambda(2^p)}.$$

(iii) Let $2^p \leqslant i < 2^{p+1}$. Then by (ii) and (2.7),

$$\frac{1}{i} \sum_{m=1}^{i} \frac{1}{\lambda(m)} \leqslant \frac{1}{2^p} \sum_{q=0}^{p} \sum_{i=2^q}^{2^{q+1}-1} \frac{1}{\lambda(m)}$$

$$\leqslant \frac{1}{2^p} \sum_{q=0}^{p} \frac{2^q}{\lambda(2^q)} = \frac{1}{2^p} O\left\{\frac{2^p}{\lambda(2^p)}\right\} = O\left\{\frac{1}{\lambda(i)}\right\}.$$

(iv) Let $2^{m_1} \geqslant m_0$. Then for every p and m, $m_1 \leqslant p \leqslant m$,

$$\lambda(2^m) \leqslant C^{m-p} \lambda(2^p).$$

Consequently, by (2.7),

$$\sum_{m=p}^{\infty} \frac{\lambda^2(2^m)}{2^{2m}} \leqslant \frac{\lambda^2(2^p)}{2^{2p}} \sum_{m=p}^{\infty} \left(\frac{C^2}{4}\right)^{m-p} = \frac{4}{4-C^2} \frac{\lambda^2(2^p)}{2^{2p}}.$$

(v) Let $2^p \leqslant i < 2^{p+1}$. Then by (iv) and (2.7),

$$\sum_{m=i}^{\infty} \frac{\lambda^2(m)}{m^3} \leqslant \sum_{q=p}^{\infty} \sum_{m=2^q}^{2^{q+1}-1} \frac{\lambda^2(m)}{m^3}$$

$$\leqslant \sum_{q=p}^{\infty} \frac{\lambda^2(2^{q+1})}{2^{2q}} = O\left\{\frac{\lambda^2(2^{p+1})}{2^{2p+2}}\right\} = O\left\{\frac{\lambda^2(i)}{i^2}\right\}. \quad \blacksquare$$

5. Proofs of Theorems 1 and 2

Proof of Theorem 1. First we apply Lemmas 1 and 2, then the extended Rademacher–Menšov theorem to the double orthogonal series

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik} \lambda^*(i, k) \phi_{ik}(x),$$

resulting in a function $F(x) \in L^2$ such that

$$|s^*_{mn}(x)| = \left| \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik} \lambda^*(i, k) \phi_{ik}(x) \right| \leqslant F(x), \text{ a.e.} \qquad (m, n = 1, 2, \ldots). \quad (5.1)$$

We represent the difference $f(x)-s_{mn}(x)$ figuring in (2.4) as follows

$$f(x)-s_{mn}(x)=\left\{\sum_{i=1}^{m}\sum_{k=n+1}^{\infty}+\sum_{i=m+1}^{\infty}\sum_{k=1}^{n}+\sum_{i=m+1}^{\infty}\sum_{k=n+1}^{\infty}\right\}a_{ik}\phi_{ik}(x)$$
$$=A_{mn}^{(1)}(x)+A_{mn}^{(2)}(x)+A_{mn}^{(3)}(x),\ \text{say.} \tag{5.2}$$

Applying a double Abel transformation (see [6 or 11]) yields

$$A_{mn}^{(1)}(x)=\sum_{i=1}^{n}\sum_{k=n+1}^{\infty}a_{ik}\lambda^*(i,k)\,\phi_{ik}(x)\frac{1}{\lambda^*(i,k)}$$
$$=\sum_{i=1}^{m-1}\sum_{k=n+1}^{\infty}s_{ik}^*(x)\Delta_{11}\frac{1}{\lambda^*(i,k)}+\sum_{k=n+1}^{\infty}s_{mk}^*(x)\Delta_{01}\frac{1}{\lambda^*(m,k)}$$
$$-\sum_{i=1}^{m-1}s_{in}^*(x)\Delta_{10}\frac{1}{\lambda^*(i,n+1)}-\frac{s_{mn}^*(x)}{\lambda^*(m,n+1)}.$$

On account of (5.1) and the convexity of $\{1/\lambda^*(i,k)\}$,

$$|A_{mn}^{(1)}(x)|\leqslant F(x)\left\{\left(\frac{1}{\lambda^*(1,n+1)}-\frac{1}{\lambda^*(m,n+1)}\right)+\frac{1}{\lambda^*(m,n+1)}\right.$$
$$\left.+\left(\frac{1}{\lambda^*(1,n+1)}-\frac{1}{\lambda^*(m,n+1)}\right)+\frac{1}{\lambda^*(m,n+1)}\right\}$$
$$=\frac{2F(x)}{\lambda^*(1,n+1)},\quad \text{a.e.,} \tag{5.3}$$

independently of m.

Similarly, independently of n,

$$|A_{mn}^{(2)}(x)|\leqslant\frac{2F(x)}{\lambda^*(m+1,1)}. \tag{5.4}$$

Finally, applying again a double Abel transformation,

$$A_{mn}^{(3)}(x)=\sum_{i=m+1}^{\infty}\sum_{k=n+1}^{\infty}s_{ik}^*(x)\Delta_{11}\frac{1}{\lambda^*(i,k)}-\sum_{k=n+1}^{\infty}s_{mk}^*(x)\Delta_{01}\frac{1}{\lambda^*(m+1,k)}$$
$$-\sum_{i=m+1}^{\infty}s_{in}^*(x)\Delta_{10}\frac{1}{\lambda^*(i,n+1)}-\frac{s_{mn}^*(x)}{\lambda^*(m+1,n+1)},$$

whence

$$|A_{mn}^{(3)}(x)|\leqslant\frac{2F(x)}{\lambda^*(m+1,n+1)},\quad \text{a.e.} \tag{5.5}$$

Putting (5.2)–(5.5) together, we find

$$|f(x)-s_{mn}(x)| \leqslant 4F(x)\left\{\frac{1}{\lambda^*(m+1,1)}+\frac{1}{\lambda^*(1,n+1)}\right\},\quad \text{a.e.}$$

By (2.2) and (4.2), this implies the wanted inequality (2.4). ∎

Proof of Theorem 2. We set

$$a^*_{pq}=\left\{\sum_{i=i_{p-1}+1}^{i_p}\ \sum_{k=k_{q-1}+1}^{k_q} a^2_{ik}\right\}^{1/2} \qquad (p, q=1, 2,...;\ i_0=k_0=0)$$

and

$$\begin{aligned}\phi^*_{pq}(x)&=\frac{1}{a^*_{pq}}\sum_{i=i_{p-1}+1}^{i_p}\ \sum_{k=k_{q-1}+1}^{k_q} a_{ik}\phi_{ik}(x) \qquad &&\text{if}\quad a^*_{pq}\neq 0,\\ &=\phi_{i_p,k_q}(x) &&\text{if}\quad a^*_{pq}=0.\end{aligned}$$

It is obvious that $\{\phi^*_{pq}(x): p, q=1, 2,...\}$ is an ONS and by (2.5),

$$\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}[a^*_{pq}]^2[\log(p+1)]^2[\log(q+1)]^2\{\max\{\lambda_1(i_p), \lambda_2(k_q)\}\}^2<\infty.$$

Thus, the application of Theorem 1 yields

$$\begin{aligned}s_{i_p,k_q}(x)-f(x)&=\sum_{r=1}^{p}\sum_{t=1}^{q} a^*_{rt}\phi^*_{rt}(x)-f(x)\\ &=o_x\left\{\frac{1}{\lambda_1(i_{p+1})}+\frac{1}{\lambda_2(k_{q+1})}\right\},\quad \text{a.e.}\end{aligned}$$

This is (2.6) to be proved. ∎

6. Proof of Theorem 3

Let $2^p<m\leqslant 2^{p+1}$ with an integer $p\geqslant 0$. (For $m=1$ we have $\tau_{1n}(x)=s_{1n}(x)$.) Then clearly

$$\begin{aligned}\tau_{mn}(x)-f(x)=&\,[s_{2^p,n}(x)-f(x)]\\ &+[\tau_{2^p,n}(x)-s_{2^p,n}(x)]+[\tau_{mn}(x)-\tau_{2^p,n}(x)]. \qquad (6.1)\end{aligned}$$

Accordingly, the proof of (2.10) is split into three parts.

Part 1. By Theorem 2 (in the special case $i_p = 2^p$ and $k_q = q$), condition (2.9) implies

$$s_{2^p,n}(x) - f(x) = o_x \left\{ \frac{1}{\lambda_1(2^p)} + \frac{1}{\lambda_2(n)} \right\}, \text{ a.e.} \tag{6.2}$$

Part 2. We will prove that, under the condition

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log(k+1)]^2 \lambda_1^2(i) < \infty, \tag{6.3}$$

we have

$$\sup_{n \geqslant 1} |s_{2^p,n}(x) - \tau_{2^p,n}(x)| = o_x \left\{ \frac{1}{\lambda_1(2^p)} \right\}, \text{ a.e.} \quad \text{as} \quad p \to \infty. \tag{6.4}$$

The proof of (6.4) is done in two steps, while using the representation

$$s_{2^p,n}(x) - \tau_{2^p,n}(x) = \sum_{i=2}^{2^p} \sum_{k=1}^{n} \frac{i-1}{2^p} a_{ik} \phi_{ik}(x) \qquad (p, n = 1, 2, \ldots). \tag{6.5}$$

Step 1. First we treat the special case where $n = 2^q$ $(q = 0, 1, \ldots)$ and prove

$$\sup_{q \geqslant 0} |s_{2^p,2^q}(x) - \tau_{2^p,2^q}(x)| = o_x \left\{ \frac{1}{\lambda_1(2^p)} \right\}, \text{ a.e.} \quad \text{as} \quad p \to \infty. \tag{6.6}$$

To this end, by the Cauchy inequality and (6.5),

$$\begin{aligned} |s_{2^p,2^q}(x) - \tau_{2^p,2^q}(x)| &\leqslant \sum_{r=0}^{q} \left| \sum_{i=2}^{2^p} \sum_{k=2^{r-1}+1}^{2^r} \frac{i-1}{2^p} a_{ik} \phi_{ik}(x) \right| \\ &\leqslant \left\{ \sum_{r=0}^{q} (r+1)^2 \left[\sum_{i=2}^{2^p} \sum_{k=2^{r-1}+1}^{2^r} \frac{i-1}{2^p} a_{ik} \phi_{ik}(x) \right]^2 \right\}^{1/2} \left\{ \sum_{r=0}^{q} \frac{1}{(r+1)^2} \right\}^{1/2}, \end{aligned}$$

with the agreement that by 2^{-1} we mean 0 in this paper. Taking into account that the last factor on the right does not exceed $\{\pi^2/6\}^{1/2}$, we can conclude that

$$\lambda_1(2^p) \left[\sup_{q \geqslant 0} |s_{2^p,2^q}(x) - \tau_{2^p,2^q}(x)| \right] \leqslant \frac{\pi}{\sqrt{6}} \left\{ \sum_{r=0}^{\infty} (r+1)^2 \lambda_1^2(2^p) \left[\sum_{i=2}^{2^p} \sum_{k=2^{r-1}+1}^{2^r} \frac{i-1}{2^p} a_{ik} \phi_{ik}(x) \right]^2 \right\}^{1/2}. \tag{6.7}$$

Setting

$$F_1(x) = \left\{ \sum_{p=1}^{\infty} \sum_{r=0}^{\infty} (r+1)^2 \lambda_1^2(2^p) \left[\sum_{i=2}^{2^p} \sum_{k=2^{r-1}+1}^{2^r} \frac{i-1}{2^p} a_{ik} \phi_{ik}(x) \right]^2 \right\}^{1/2},$$

we have to show that $F_1(x) \in L^2$. Indeed, by (4.7) and (6.3),

$$\int F_1^2(x)\,d\mu(x) = \sum_{p=1}^{\infty} \sum_{r=0}^{\infty} (r+1)^2 \lambda_1^2(2^p) \sum_{i=2}^{2^p} \sum_{k=2^{r-1}+1}^{2^r} \frac{(i-1)^2}{2^{2p}} a_{ik}^2$$
$$\leqslant \sum_{p=1}^{\infty} \sum_{k=1}^{\infty} \sum_{i=2}^{2^p} \frac{(i-1)^2}{2^{2p}} a_{ik}^2 [\log 4k]^2 \lambda_1^2(2^p)$$
$$= \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} (i-1)^2 a_{ik}^2 [\log 4k]^2 \sum_{p:2^p \geqslant i} \frac{\lambda_1^2(2^p)}{2^{2p}}$$
$$= O\{1\} \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log 4k]^2 \lambda_1^2(i) < \infty. \tag{6.8}$$

Hence B. Levi's theorem implies (6.6) via (6.7).

Step 2. Let $2^q < n \leqslant 2^{q+1}$ with some $q \geqslant 1$. Then by (6.5),

$$s_{2^p,n}(x) - \tau_{2^p,n}(x) = [s_{2^p,2^q}(x) - \tau_{2^p,2^q}(x)] + \sum_{i=2}^{2^p} \sum_{k=2^q+1}^{n} \frac{i-1}{2^p} a_{ik} \phi_{ik}(x),$$

whence

$$\max_{2^q < n \leqslant 2^{q+1}} |s_{2^p,n}(x) - \tau_{2^p,n}(x)| \leqslant |s_{2^p,2^q}(x) - \tau_{2^p,2^q}(x)| + M_{pq}^{(1)}(x), \tag{6.9}$$

where

$$M_{pq}^{(1)}(x) = \max_{2^q < n \leqslant 2^{q+1}} \left| \sum_{i=2}^{2^p} \sum_{k=2^q+1}^{n} \frac{i-1}{2^p} a_{ik} \phi_{ik}(x) \right|.$$

We are going to prove that, under condition (6.3),

$$M_{pq}^{(1)}(x) = o_x \left\{ \frac{1}{\lambda_1(2^p)} \right\}, \text{ a.e.} \quad \text{as} \quad \max\{p, q\} \to \infty. \tag{6.10}$$

To this effect, we apply the Rademacher–Menšov inequality (see, e.g., [2, p. 79; or 8, Theorem 3]) to obtain

$$\int [M_{pq}^{(1)}(x)]^2\,d\mu(x) \leqslant [\log 2^{q+1}]^2 \sum_{i=2}^{2^p} \sum_{k=2^q+1}^{2^{q+1}} \frac{(i-1)^2}{2^{2p}} a_{ik}^2.$$

Setting

$$F_2(x) = \left\{ \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \lambda_1^2(2^p) [M_{pq}^{(1)}(x)]^2 \right\}^{1/2},$$

we can obtain, in the same manner as in (6.8),

$$\int F_2^2(x)\,d\mu(x) \leqslant \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \lambda_1^2(2^p)[\log 2^{q+1}]^2 \sum_{i=2}^{2^p} \sum_{k=2^q+1}^{2^{q+1}} \frac{(i-1)^2}{2^{2p}} a_{ik}^2$$
$$\leqslant \sum_{p=1}^{\infty} \sum_{k=2}^{\infty} \sum_{i=2}^{2^p} \frac{(i-1)^2}{2^{2p}} a_{ik}^2 [\log 2k]^2 \lambda_1^2(2^p) < \infty.$$

Now (6.10) follows from B. Levi's theorem. Combining (6.6), (6.9), and (6.10), we get (6.4).

Part 3. We will prove that, under condition (6.3),

$$\sup_{n\geqslant 1} \max_{2^p < m \leqslant 2^{p+1}} |\tau_{mn}(x) - \tau_{2^p,n}(x)| = o_x \left\{ \frac{1}{\lambda_1(2^p)} \right\}, \text{ a.e.,} \quad \text{as } p \to \infty. \quad (6.11)$$

Taking into account that

$$\max_{2^p < m \leqslant 2^{p+1}} |\tau_{mn}(x) - \tau_{2^p,n}(x)| \leqslant \sum_{m=2^p+1}^{2^{p+1}} |\tau_{mn}(x) - \tau_{m-1,n}(x)| = A_{pn}^{(4)}(x), \quad (6.12)$$

we will prove somewhat more, namely,

$$\sup_{n\geqslant 1} A_{pn}^{(4)}(x) = o_x \left\{ \frac{1}{\lambda_1(2^p)} \right\}, \text{ a.e.} \quad \text{as } p \to \infty. \quad (6.13)$$

We carry out the proof again in two steps, using the representation

$$\tau_{mn}(x) - \tau_{m-1,n}(x) = \sum_{i=2}^{m} \sum_{k=1}^{n} \frac{i-1}{m(m-1)} a_{ik}\phi_{ik}(x) \qquad (m = 2, 3, \ldots;\ n = 1, 2, \ldots). \quad (6.14)$$

Step 3. First we verify (6.13) in the special case $n = 2^q$, i.e.,

$$\sup_{q\geqslant 1} A_{p,2^q}^{(4)}(x) = o_x \left\{ \frac{1}{\lambda_1(2^p)} \right\}, \text{ a.e.} \quad \text{as } p \to \infty. \quad (6.15)$$

To achieve this goal, we use (6.14) and the Cauchy inequality:

$$A_{p,2^q}^{(4)}(x) \leqslant \sum_{m=2^p+1}^{2^{p+1}} \sum_{r=0}^{q} \left| \sum_{i=2}^{m} \sum_{k=2^{r-1}+1}^{2^r} \frac{i-1}{m(m-1)} a_{ik}\phi_{ik}(x) \right|$$
$$\leqslant \frac{\pi}{\sqrt{6}} \left\{ \sum_{m=2^p+1}^{2^{p+1}} \sum_{r=0}^{q} m(r+1)^2 \left[\sum_{i=2}^{m} \sum_{k=2^{r-1}+1}^{2^r} \frac{i-1}{m(m-1)} a_{ik}\phi_{ik}(x) \right]^2 \right\}^{1/2}. \quad (6.16)$$

This inequality suggests defining

$$F_3(x)=\left\{\sum_{m=2}^{\infty}\sum_{r=0}^{\infty} m(r+1)^2\lambda_1^2(m)\right.$$
$$\left.\times\left[\sum_{i=2}^{\infty}\sum_{k=2^{r-1}+1}^{2^r}\frac{i-1}{m(m-1)}a_{ik}\phi_{ik}(x)\right]^2\right\}^{1/2}.$$

By (4.8) and (6.3),

$$\begin{aligned}\int F_3^2(x)\,d\mu(x) &= \sum_{m=2}^{\infty}\sum_{r=0}^{\infty} m(r+1)^2\lambda_1^2(m)\sum_{i=2}^{m}\sum_{k=2^{r-1}+1}^{2^r}\frac{(i-1)^2}{m^2(m-1)^2}a_{ik}^2\\
&\leqslant \sum_{m=2}^{\infty}\sum_{r=0}^{\infty}\sum_{i=2}^{m}\sum_{k=2^{r-1}+1}^{2^r}\frac{i^2}{m^3}a_{ik}^2[\log 4k]^2\lambda_1^2(m)\\
&= \sum_{m=2}^{\infty}\sum_{k=1}^{\infty}\sum_{i=2}^{m}\frac{i^2}{m^3}a_{ik}^2[\log 4k]^2\lambda_1^2(m)\\
&= \sum_{i=2}^{\infty}\sum_{k=1}^{\infty} i^2a_{ik}^2[\log 4k]^2\sum_{m=i}^{\infty}\frac{\lambda_1^2(m)}{m^3}\\
&= O\{1\}\sum_{i=2}^{\infty}\sum_{k=1}^{\infty}a_{ik}^2[\log 4k]^2\lambda_1^2(i)<\infty. \qquad (6.17)\end{aligned}$$

Hence B. Levi's theorem implies (6.15) through (6.16).

Step 4. We proceed similarly to Step 2. By (6.14),

$$\max_{2^q<n\leqslant 2^{q+1}} A_{pn}^{(4)}(x)\leqslant A_{p,2^q}^{(4)}(x)+\sum_{m=2^p+1}^{2^{p+1}} M_{mq}^{(2)}(x), \qquad (6.18)$$

where

$$M_{mq}^{(2)}(x)=\max_{2^q<n\leqslant 2^{q+1}}\left|\sum_{i=2}^{m}\sum_{k=2^q+1}^{n}\frac{i-1}{m(m-1)}a_{ik}\phi_{ik}(x)\right|$$
$$(m=2,3,\ldots;\ q=1,2,\ldots).$$

Applying the Cauchy inequality:

$$\sum_{m=2^p+1}^{2^{p+1}} M_{mq}^{(2)}(x)\leqslant\left\{\sum_{m=2^p+1}^{2^{p+1}} m[M_{pq}^{(2)}(x)]^2\right\}^{1/2}, \qquad (6.19)$$

then the Rademacher–Menšov inequality separately for each fixed m:

$$\begin{aligned}\int [M_{mq}^{(2)}(x)]^2\,d\mu(x) &\leqslant [\log 2^{q+1}]^2\sum_{i=2}^{m}\sum_{k=2^q+1}^{2^{q+1}}\frac{(i-1)^2}{m^2(m-1)^2}a_{ik}^2\\
&\leqslant \sum_{i=2}^{m}\sum_{k=2^q+1}^{2^{q+1}}\frac{i^2}{m^4}a_{ik}^2[\log 2k]^2.\end{aligned}$$

Setting

$$F_4(x)=\left\{\sum_{m=2}^{\infty}\sum_{q=1}^{\infty} m\lambda_1^2(m)[M_{mq}^{(2)}(x)]^2\right\}^{1/2},$$

we can get, in the same way as in (6.17),

$$\begin{aligned}\int F_4^2(x)\,d\mu(x) &\leqslant \sum_{m=2}^{\infty}\sum_{q=1}^{\infty}\sum_{i=2}^{m}\sum_{k=2^q+1}^{2^{q+1}} \frac{i^2}{m^3}\,a_{ik}^2[\log 2k]^2\,\lambda_1^2(m)\\ &= \sum_{m=2}^{\infty}\sum_{k=3}^{\infty}\sum_{i=2}^{m} \frac{i^2}{m^3}\,a_{ik}^2[\log 2k]^2\,\lambda_1^2(m)<\infty.\end{aligned}$$

Hence B. Levi's theorem implies, through (6.19),

$$\sup_{q\geqslant 0}\sum_{m=2^p+1}^{2^{p+1}} M_{mq}^{(2)}(x)=o_x\left\{\frac{1}{\lambda_1(2^p)}\right\},\ \text{a.e.}\qquad \text{as } p\to\infty. \tag{6.20}$$

Putting (6.15), (6.18), and (6.20) together, we find (6.13) to be proved. Finally, (2.10) follows from (6.1), (6.2), (6.4), and (6.11). ∎

7. Proof of Theorem 4

We start with the identity

$$\begin{aligned}\sigma_{mn}(x)-f(x) &= [s_{2^p,2^q}(x)-f(x)]+[\sigma_{2^p,2^q}(x)-s_{2^p,2^q}(x)]\\ &\quad+[\sigma_{m,2^q}(x)-\sigma_{2^p,2^q}(x)]+[\sigma_{2^p,n}(x)-\sigma_{2^p,2^q}(x)]\\ &\quad+[\sigma_{mn}(x)-\sigma_{m,2^q}(x)-\sigma_{2^p,n}(x)+\sigma_{2^p,2^q}(x)],\end{aligned} \tag{7.1}$$

where $2^p\leqslant m\leqslant 2^{p+1}$ and $2^q\leqslant n\leqslant 2^{q+1}$, p and q being nonnegative integers. Accordingly, the proof is accomplished in five parts.

Part 1. In the special case $i_p=2^p$ and $k_q=2^q$ Theorem 2 states that, under condition (2.12),

$$s_{2^p,2^q}(x)-f(x)=o_x\left\{\frac{1}{\lambda_1(2^p)}+\frac{1}{\lambda_2(2^q)}\right\},\ \text{a.e.} \tag{7.2}$$

Part 2. We prove that

$$s_{2^p,2^q}(x)-\sigma_{2^p,2^q}(x)=o_x\left\{\frac{1}{\lambda_1(2^p)}+\frac{1}{\lambda_2(2^q)}\right\},\ \text{a.e.} \tag{7.3}$$

To this goal, we use the representation

$$
\begin{aligned}
& s_{2^p,2^q}(x) - \sigma_{2^p,2^q}(x) \\
&\quad = \sum_{i=1}^{2^p} \sum_{k=1}^{2^q} \left(\frac{i-1}{2^p} + \frac{k-1}{2^q} - \frac{(i-1)(k-1)}{2^p 2^q} \right) a_{ik} \phi_{ik}(x) \\
&\quad = [s_{2^p,2^q}(x) - \tau^{(1)}_{2^p,2^q}(x)] + [s_{2^p,2^q}(x) - \tau^{(2)}_{2^p,2^q}(x)] \\
&\quad\quad - \sum_{i=2}^{2^p} \sum_{k=2}^{2^q} \frac{(i-1)(k-1)}{2^p 2^q} a_{ik} \phi_{ik}(x).
\end{aligned} \tag{7.4}
$$

Thus, the proof of (7.2) is divided into three steps.

Step 1. First we are going to prove that if

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log\log(k+3)]^2 \lambda_1^2(i) < \infty, \tag{7.5}$$

then

$$\sup_{q \geqslant 0} |s_{2^p,2^q}(x) - \tau^{(1)}_{2^p,2^q}(x)| = o_x \left\{ \frac{1}{\lambda_1(2^p)} \right\}, \text{ a.e.} \qquad \text{as} \quad p \to \infty. \tag{7.6}$$

This statement is a simple consequence of (6.4). In fact, setting

$$\tilde{a}_{ir} = \left\{ \sum_{k=2^{r-1}+1}^{2^r} a_{ik}^2 \right\}^{1/2} \qquad (r = 0, 1, \ldots),$$

and

$$
\begin{aligned}
\tilde{\phi}_{ir}(x) &= \frac{1}{\tilde{a}_{ir}} \sum_{k=2^{r-1}+1}^{2^r} a_{ik} \phi_{ik}(x) \qquad && \text{if} \quad \tilde{a}_{ir} \neq 0, \\
&= \phi_{i,2^r}(x) && \text{if} \quad \tilde{a}_{ir} = 0;
\end{aligned}
$$

we obtain a new ONS $\{\tilde{\phi}_{ir}(x) : i = 1, 2, \ldots;\ r = 0, 1, \ldots\}$. By (7.5),

$$\sum_{i=1}^{\infty} \sum_{r=0}^{\infty} \tilde{a}_{ir}^2 [\log(r+2)]^2 \lambda_1^2(i) < \infty,$$

i.e., condition (6.3) is fulfilled. Thus, by (6.4),

$$\sup_{q \geqslant 0} |\tilde{s}_{2^p,q}(x) - \tau^{(1)}_{2^p,q}(x)| = o_x \left\{ \frac{1}{\lambda_1(2^p)} \right\}, \text{ a.e.} \qquad \text{as} \quad p \to \infty, \tag{7.7}$$

where

$$\tilde{s}_{2^p,q}(x) - \tau^{(1)}_{2^p,q}(x) = \sum_{i=2}^{2^p} \sum_{r=0}^{q} \frac{i-1}{2^p} \tilde{a}_{ir} \tilde{\phi}_{ir}(x)$$

$$= \sum_{i=2}^{2^p} \sum_{k=1}^{2^q} \frac{i-1}{2^p} a_{ik} \phi_{ik}(x) = s_{2^p,2^q}(x) - \tau^{(1)}_{2^p,2^q}(x).$$

That is, (7.7) is equivalent to (7.6) to be proved.

Step 2. In the same way one can deduce that if

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log\log(i+3)]^2 \lambda_2^2(k) < \infty, \tag{7.8}$$

then

$$\sup_{p \geq 0} |s_{2^p,2^q}(x) - \tau^{(2)}_{2^p,2^q}(x)| = o_x \left\{ \frac{1}{\lambda_2(2^q)} \right\}, \text{ a.e.} \quad \text{as } q \to \infty. \tag{7.9}$$

Step 3. We show that under the condition

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \max\{\lambda_1^2(i), \lambda_2^2(k)\} < \infty, \tag{7.10}$$

we have

$$A^{(5)}_{pq}(x) = \sum_{i=2}^{2^p} \sum_{k=2}^{2^q} \frac{(i-1)(k-1)}{2^p 2^q} a_{ik} \phi_{ik}(x)$$

$$= o_x \left\{ \min \left\{ \frac{1}{\lambda_1(2^p)}, \frac{1}{\lambda_2(2^q)} \right\} \right\},$$

$$\text{a.e.} \quad \text{as } \max\{p, q\} \to \infty. \tag{7.11}$$

Indeed, setting

$$F_5(x) = \left\{ \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \lambda_1^2(2^p) [A^{(5)}_{pq}(x)]^2 \right\}^{1/2},$$

we get by (4.7),

$$\int F_5^2(x)\, d\mu(x) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \lambda_1^2(2^p) \sum_{i=2}^{2^p} \sum_{k=2}^{2^q} \frac{(i-1)^2(k-1)^2}{2^{2p} 2^{2q}} a_{ik}^2$$

$$= \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} (i-1)^2 (k-1)^2 a_{ik}^2 \sum_{p: 2^p \geq i} \frac{\lambda_1^2(2^p)}{2^{2p}} \sum_{q: 2^q \geq k} \frac{1}{2^{2q}}$$

$$= O\{1\} \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} a_{ik}^2 \lambda_1^2(i) < \infty.$$

Hence B. Levi's theorem implies

$$A^{(5)}_{pq}(x) = o_x\left\{\frac{1}{\lambda_1(2^p)}\right\}, \text{ a.e.} \quad \text{as} \quad \max\{p, q\} \to \infty,$$

which is the first half of statement (7.11). The second half can be proved analogously.

Collecting (7.6), (7.9), and (7.11) we find (7.3).

Part 3. We will prove that under condition (7.5)

$$\sup_{q\geqslant 0} \max_{2^p < m \leqslant 2^{p+1}} |\sigma_{m,2^q}(x) - \sigma_{2^p,2^q}(x)| = o_x\left\{\frac{1}{\lambda_1(2^p)}\right\},$$
$$\text{a.e.} \quad \text{as} \quad p \to \infty. \tag{7.12}$$

We even prove a bit more; under (7.5),

$$\sup_{q\geqslant 0} \sum_{m=2^p+1}^{2^{p+1}} |\sigma_{m,2^q}(x) - \sigma_{m-1,2^q}(x)| = o_x\left\{\frac{1}{\lambda_1(2^p)}\right\}, \text{ a.e.} \quad \text{as} \quad p \to \infty \tag{7.13}$$

(cf. (6.12) and (6.13)). Using the representation

$$\sigma_{mn}(x) - \sigma_{m-1,n}(x) = \sum_{i=2}^{m} \sum_{k=1}^{n} \frac{i-1}{m(m-1)}\left(1 - \frac{k-1}{n}\right) a_{ik}\phi_{ik}(x)$$
$$(m = 2, 3,\ldots; n = 1, 2,\ldots), \tag{7.14}$$

and taking (6.5) into account, we can write

$$\sigma_{m,2^q}(x) - \sigma_{m-1,2^q}(x) = [\tau_{m,2^q}(x) - \tau_{m-1,2^q}(x)]$$
$$- \sum_{i=2}^{m} \sum_{k=2}^{2^q} \frac{(i-1)(k-1)}{m(m-1)2^q} a_{ik}\phi_{ik}(x).$$

Hence

$$\sum_{m=2^p+1}^{2^{p+1}} |\sigma_{m,2^q}(x) - \sigma_{m-1,2^q}(x)| \leqslant A^{(4)}_{p,2^q}(x) + A^{(6)}_{pq}(x), \tag{7.15}$$

where $A^{(4)}_{pn}(x)$ was defined in (6.12) (now $n = 2^q$) and

$$A^{(6)}_{pq}(x) = \sum_{m=2^p+1}^{2^{p+1}} \left| \sum_{i=2}^{m} \sum_{k=2}^{2^q} \frac{(i-1)(k-1)}{m(m-1)2^q} a_{ik}\phi_{ik}(x)\right|.$$

We divide the proof of (7.13) into two steps.

Step 4. Using the same "contraction" technique as in Step 1 above, from estimate (6.13) one can deduce that, under condition (7.5),

$$\sup_{q\geqslant 0} A^{(4)}_{p,2^q}(x)=o_x\left\{\frac{1}{\lambda_1(2^p)}\right\}, \text{ a.e.} \qquad \text{as} \quad p\to\infty. \tag{7.16}$$

Step 5. We will check that, under condition (7.10),

$$A^{(6)}_{pq}(x)=o_x\left\{\min\left\{\frac{1}{\lambda_1(2^p)},\frac{1}{\lambda_2(2^q)}\right\}\right\}, \text{ a.e.} \qquad \text{as } \max\{p,q\}\to\infty. \tag{7.17}$$

In fact, by the Cauchy inequality,

$$A^{(6)}_{pq}(x)\leqslant\left\{\sum_{m=2^p+1}^{2^{p+1}} m\left[\sum_{i=2}^{m}\sum_{k=2}^{2^q}\frac{(i-1)(k-1)}{m(m-1)2^q}a_{ik}\phi_{ik}(x)\right]^2\right\}^{1/2}.$$

Setting

$$F_6(x)=\left\{\sum_{m=2}^{\infty}\sum_{q=1}^{\infty} m\lambda_1^2(m)\left[\sum_{i=2}^{m}\sum_{k=2}^{2^q}\frac{(i-1)(k-1)}{m(m-1)2^q}a_{ik}\phi_{ik}(x)\right]^2\right\}^{1/2},$$

by (4.8) and (7.10),

$$\begin{aligned}\int F_6^2(x)\,d\mu(x)&=\sum_{m=2}^{\infty}\sum_{q=1}^{\infty} m\lambda_1^2(m)\sum_{i=2}^{m}\sum_{k=2}^{2^q}\frac{(i-1)^2(k-1)^2}{m^2(m-1)^2 2^{2q}}a_{ik}^2\\ &\leqslant\sum_{m=2}^{\infty}\sum_{q=1}^{\infty}\sum_{i=2}^{m}\sum_{k=2}^{2^q}\frac{i^2k^2}{m^3 2^{2q}}a_{ik}^2\lambda_1^2(m)\\ &=\sum_{i=2}^{\infty}\sum_{k=2}^{\infty} i^2k^2a_{ik}^2\sum_{m=i}^{\infty}\frac{\lambda_1^2(m)}{m^3}\sum_{q:2^q\geqslant k}\frac{1}{2^{2q}}\\ &=O\{1\}\sum_{i=2}^{\infty}\sum_{k=2}^{\infty}a_{ik}^2\lambda_1^2(i)<\infty.\end{aligned}$$

Hence B. Levi's theorem implies the one half of (7.17). The other half can be proved similarly. Combining (7.15)–(7.17) yields (7.13).

Part 4. The companion statement to (7.12) reads as follows: Under condition (7.8),

$$\sup_{p\geqslant 0}\max_{2^q<n\leqslant 2^{q+1}}|\sigma_{2^p,n}(x)-\sigma_{2^p,2^q}(x)|=o_x\left\{\frac{1}{\lambda_2(2^q)}\right\}, \text{ a.e.} \qquad \text{as} \quad q\to\infty. \tag{7.18}$$

Part 5. Finally, we prove that, under condition (7.8),

$$A_{pq}^{(7)}(x)=\max_{2^p\leqslant m\leqslant 2^{p+1}}\ \max_{2^q\leqslant n\leqslant 2^{q+1}}|\sigma_{mn}(x)-\sigma_{m,2^q}(x)-\sigma_{2^p,n}(x)+\sigma_{2^p,2^q}(x)|$$

$$=o_x\left\{\min\left\{\frac{1}{\lambda_1(2^p)},\frac{1}{\lambda_2(2^q)}\right\}\right\},\ \text{a.e.}\quad \text{as}\ \max\{p,q\}\to\infty. \tag{7.19}$$

The proof is based on the following estimation:

$$A_{pq}^{(7)}(x)\leqslant\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}|\sigma_{mn}(x)-\sigma_{m-1,n}(x)-\sigma_{m,n-1}(x)+\sigma_{m-1,n-1}(x)|$$

$$\leqslant\left\{\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}mn[\sigma_{mn}(x)-\sigma_{m-1,n}(x)-\sigma_{m,n-1}(x)+\sigma_{m-1,n-1}(x)]^2\right\}^{1/2} \tag{7.20}$$

$$=\left\{\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}mn\left[\sum_{i=2}^{m}\sum_{k=2}^{n}\frac{(i-1)(k-1)}{m(m-1)\,n(n-1)}a_{ik}\phi_{ik}(x)\right]^2\right\}^{1/2}.$$

Now we define

$$F_7(x)=\left\{\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}mn\lambda_1^2(m)\left[\sum_{i=2}^{\infty}\sum_{k=2}^{n}\frac{(i-1)(k-1)}{m(m-1)\,n(n-1)}a_{ik}\phi_{ik}(x)\right]^2\right\}^{1/2}.$$

A simple computation gives, by (4.8) and (7.10),

$$\int F_7^2(x)\,d\mu(x)=\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}mn\lambda_1^2(m)\sum_{i=2}^{m}\sum_{k=2}^{n}\frac{(i-1)^2(k-1)^2}{m^2(m-1)^2\,n^2(n-1)^2}a_{ik}^2$$

$$\leqslant\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\sum_{i=2}^{m}\sum_{k=2}^{n}\frac{i^2k^2}{m^3n^3}a_{ik}^2\lambda_1^2(m)$$

$$=\sum_{i=2}^{\infty}\sum_{k=2}^{\infty}i^2k^2a_{ik}^2\sum_{m=i}^{\infty}\frac{\lambda_1^2(m)}{m^3}\sum_{n=k}^{\infty}\frac{1}{n^3}$$

$$=O\{1\}\sum_{i=2}^{\infty}\sum_{k=2}^{\infty}a_{ik}^2\lambda_1^2(i)<\infty.$$

It remains to apply B. Levi's theorem in order to obtain the part $o_x\{1/\lambda_1(2^p)\}$ in (7.19). The proof of the part $o_x\{1/\lambda_2(2^q)\}$ is quite similar.

Collecting (7.1)–(7.3), (7.12), (7.18), and (7.19) we obtain (2.13) to be proved. ∎

8. Proofs of the Theorems in Section 3

We set

$$\bar{a}_r = \left\{ \sum_{(i,k)\in Q_r\setminus Q_{r-1}} a_{ik}^2 \right\}^{1/2} \qquad (r=1, 2,...;\ Q_0=\varnothing),$$

and

$$\begin{aligned}\bar{\phi}_r(x) &= \frac{1}{\bar{a}_r} \sum_{(i,k)\in Q_r\setminus Q_{r-1}} a_{ik}\phi_{ik}(x) && \text{if} \quad \bar{a}_r \neq 0,\\ &= \phi_{ik}(x) \quad \text{with some } (i,k)\in Q_r\setminus Q_{r-1} && \text{if} \quad \bar{a}_r = 0.\end{aligned}$$

It is clear that $\{\bar{\phi}_r(x): r=1,2,...\}$ is an ONS and conditions (3.1)–(3.3) turn into the following ones:

$$\sum_{r=1}^{\infty} \bar{a}_r^2[\log(r+1)]^2 \lambda_1^2(r) < \infty,$$

$$\sum_{p=1}^{\infty} \left(\sum_{r=r_{p-1}+1}^{r_p} \bar{a}_r^2 \right) [\log(p+1)]^2 \lambda_1^2(r_p) < \infty,$$

and

$$\sum_{r=1}^{\infty} \bar{a}_r^2[\log\log(r+3)]^2 \lambda_1^2(r) < \infty.$$

Thus, we can apply the two theorems of Tandori [13] in order to conclude Theorems 1′ and 4′. Theorem 2′ can be deduced from Theorem 1′ in the same way as Theorem 2 is deduced from Theorem 1 in Section 5. It remains to prove Theorem 6′.

To this effect, let $\{\psi_i(x): i=1,2,...\}$ be an (ordinary) ONS and consider the single orthogonal series

$$\sum_{i=1}^{\infty} b_i \psi_i(x), \tag{8.1}$$

where $\{b_i: i=1,2,...\}$ is a sequence of real numbers with $\sum b_i^2 < \infty$. By the Riesz–Fischer theorem there exists a function $g(x) \in L^2$ such that the partial sums

$$s_m(x) = \sum_{i=1}^{m} b_i \psi_i(x) \qquad (m=1,2,...),$$

of series (8.1) converge to $g(x)$ in L^2-metric:

$$\int [s_m(x) - g(x)]^2 \, d\mu(x) \to 0 \qquad \text{as} \quad m \to \infty.$$

Denote by $\sigma_m(x)$ the first arithmetic means of the partial sums:

$$\sigma_m(x) = \frac{1}{m} \sum_{i=1}^{m} s_i(x) = \sum_{i=1}^{m} \left(1 - \frac{i-1}{m}\right) b_i \psi_i(x) \qquad (m = 1, 2, \ldots).$$

The following theorem seems to be new.

THEOREM 7. *If conditions* (2.17) *and*

$$\sum_{i=1}^{\infty} b_i^2 [\log \log(i+3)]^2 \lambda_1^2(i) < \infty \tag{8.2}$$

are satisfied and $\{m/\lambda_1^2(m)\}$ *is nondecreasing, then*

$$\left\{\frac{1}{m} \sum_{i=1}^{m} [s_i(x) - g(x)]^2\right\}^{1/2} = o_x \left\{\frac{1}{\lambda_1(m)}\right\}, \text{ a.e.} \tag{8.3}$$

After these preliminaries, Theorem 6′ can be deduced from Theorem 7 in the same manner as Theorems 1′ and 4′ are deduced from the corresponding theorems of [13].

Proof of Theorem 7. We begin with the obvious inequality

$$\left\{\frac{1}{m} \sum_{i=1}^{m} [s_i(x) - g(x)]^2\right\}^{1/2} \leqslant \left\{\frac{1}{m} \sum_{i=1}^{m} [s_i(x) - \sigma_i(x)]^2\right\}^{1/2} + \left\{\frac{1}{m} \sum_{i=1}^{m} [\sigma_i(x) - g(x)]^2\right\}^{1/2}. \tag{8.4}$$

On the one hand, by (8.2) we can apply [13, Theorem 2] resulting in

$$\sigma_m(x) - g(x) = o_x \left\{\frac{1}{\lambda_1(m)}\right\}, \text{ a.e.} \tag{8.5}$$

We note that in the Tandori theorem in question a stronger requirement is imposed on the sequence $\{\lambda_1(m)\}$ than (2.17), namely

$$\lambda_1(m^2) = O\{\lambda_1(m)\} \qquad (m = 1, 2, \ldots).$$

But an analysis of his proof reveals that even condition (2.7) is actually enough.

Due to (2.17), $\{\lambda_1^2(m)\}$ satisfies condition (2.7). By (4.4), (4.6), and (8.5), one can conclude that

$$\left\{\frac{1}{m}\sum_{i=1}^{m}[\sigma_i(x)-g(x)]^2\right\}^{1/2}=o_x\left\{\frac{1}{\lambda_1(m)}\right\},\ \text{a.e.} \tag{8.6}$$

On the other hand, letting

$$F_8(x)=\left\{\sum_{m=1}^{\infty}\frac{\lambda_1^2(m)}{m}[s_m(x)-\sigma_m(x)]^2\right\}^{1/2},$$

the termwise integration gives

$$\begin{aligned}\int F_8^2(x)\,d\mu(x)&=\sum_{m=1}^{\infty}\frac{\lambda_1^2(m)}{m}\sum_{i=2}^{m}\frac{(i-1)^2}{m^2}b_i^2\\&=\sum_{i=2}^{\infty}(i-1)^2b_i^2\sum_{m=i}^{\infty}\frac{\lambda_1^2(m)}{m^3}=O\{1\}\sum_{i=2}^{\infty}b_i^2\lambda_1^2(i)<\infty,\end{aligned}$$

where we used (4.8) and (8.2). By B. Levi's theorem $F_8(x)\in L^2$. We can apply the well known Kronecker lemma (see, e.g., [2, p. 72]) since $\{m/\lambda_1^2(m)\}$ is nondecreasing by assumption and tends to ∞ by (4.4). As a result we get

$$\left\{\frac{1}{m}\sum_{i=1}^{m}[s_i(x)-\sigma_i(x)]^2\right\}^{1/2}=o_x\left\{\frac{1}{\lambda_1(m)}\right\},\ \text{a.e.} \tag{8.7}$$

To sum up, (8.4), (8.6), and (8.7) result in (8.3) to be proved. ∎

9. Proofs of Theorems 5 and 6

Proof of Theorem 5. It resembles the proof of Theorem 4. Therefore we only sketch the proof. We again use identity (7.1), this time with $p=q$.

Part 1. Theorem 2′ in the special case $Q_r=\{(i.\ k)\in\mathbb{N}^2: i,k=1,2,\ldots,r\}$ (square partial sums) and $r_p=2^p$ states that, under condition (2.14),

$$s_{2^p,2^p}(x)-f(x)=o_x\left\{\frac{1}{\lambda_1(2^p)}\right\},\ \text{a.e.} \tag{9.1}$$

Part 2. If

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}a_{ik}^2\lambda_1^2(\max\{i,k\})<\infty, \tag{9.2}$$

then

$$s_{2^p,2^p}(x)-\sigma_{2^p,2^p}(x)=o_x\left\{\frac{1}{\lambda_1(2^p)}\right\}, \text{ a.e.} \tag{9.3}$$

Indeed, setting

$$F_9(x)=\left\{\sum_{p=0}^{\infty}\lambda_1^2(2^p)[s_{2^p,2^p}(x)-\sigma_{2^p,2^p}(x)]^2\right\}^{1/2},$$

by (7.4), (4.7), and (9.2) one can show that $F_9(x)\in L^2$. Applying B. Levi's theorem yields (9.3).

Part 3. If (9.2) is satisfied, then for every $\theta\geqslant 1$

$$\begin{aligned}M_{p,\theta}^{(3)}(x)&=\max_{\theta^{-1}2^p\leqslant m\leqslant\theta 2^{p+1}}|\sigma_{m,2^p}(x)-\sigma_{2^p,2^p}(x)|\\&=o_x\left\{\frac{1}{\lambda_1(2^p)}\right\}, \text{ a.e.}\end{aligned} \tag{9.4}$$

It is clear that

$$\begin{aligned}M_{p,\theta}^{(3)}(x)&\leqslant\max_{\theta^{-1}2^p\leqslant m\leqslant 2^p}|\sigma_{m,2^p}(x)-\sigma_{2^p,2^p}(x)|\\&\quad+\max_{2^p\leqslant m\leqslant\theta 2^{p+1}}|\sigma_{m,2^p}(x)-\sigma_{2^p,2^p}(x)|\\&=M_{p,\theta}^{(4)}(x)+M_{p,\theta}^{(5)}(x),\end{aligned} \tag{9.5}$$

say. For instance, we treat $M_{p,\theta}^{(5)}(x)$ in detail. By the Cauchy inequality

$$\begin{aligned}M_{p,\theta}^{(5)}(x)&\leqslant\sum_{m=2^p+1}^{\theta 2^{p+1}}|\sigma_{m,2^p}(x)-\sigma_{m-1,2^p}(x)|\\&\leqslant\left\{(2\theta-1)\sum_{m=2^p+1}^{\theta 2^{p+1}}[\sigma_{m,2^p}(x)-\sigma_{m-1,2^p}(x)]^2\right\}^{1/2}.\end{aligned}$$

Using (7.14), (4.7), and (9.2) one can check that

$$F_{10}(x)=\left\{\sum_{p=0}^{\infty}\lambda_1^2(2^p)[M_{p,\theta}^{(5)}(x)]^2\right\}^{1/2}\in L^2,$$

whence B. Levi's theorem implies

$$M_{p,\theta}^{(5)}(x)=o_x\left\{\frac{1}{\lambda_1(2^p)}\right\}, \text{ a.e.}$$

The same estimate can be deduced for $M^{(4)}_{p,\theta}(x)$. This completes the proof of (9.4).

Part 4. The symmetric counterpart of (9.4) reads as follows: If (9.2) is satisfied, then for every $\theta \geqslant 1$

$$\max_{\theta^{-1}2^p \leqslant n \leqslant \theta 2^{p+1}} |\sigma_{2^p,n}(x) - \sigma_{2^p,2^p}(x)| = o_x \left\{ \frac{1}{\lambda_1(2^p)} \right\}, \text{ a.e.} \tag{9.6}$$

Part 5. Under (9.2), for every $\theta \geqslant 1$,

$$\max_{2^p \leqslant m \leqslant 2^{p+1}} \max_{\theta^{-1}2^p \leqslant n \leqslant \theta 2^{p+1}} |\sigma_{mn}(x) - \sigma_{m,2^p}(x) - \sigma_{2^p,n}(x) + \sigma_{2^p,2^p}(x)|$$
$$= o_x \left\{ \frac{1}{\lambda_1(2^p)} \right\}, \text{ a.e.} \tag{9.7}$$

In fact, it is enough to estimate

$$M^{(6)}_{pq}(x) = \max_{2^p \leqslant m \leqslant 2^{p+1}} \max_{2^p \leqslant n \leqslant \theta 2^{p+1}} |\sigma_{mn}(x) - \sigma_{m,2^p}(x) - \sigma_{2^p,n}(x) + \sigma_{2^p,2^p}(x)|$$

(cf. (9.5)). Introducing

$$F_{11}(x) = \left\{ \sum_{p=0}^{\infty} \lambda_1^2(2^p)[M^{(6)}_{pq}(x)]^2 \right\}^{1/2}$$

and using an estimate similar to (7.20) (this time $p = q$), one can conclude $F_{11}(x) \in L^2$ and (9.7). Putting (9.1), (9.3), (9.4), (9.6), and (9.7) together, we find (2.15). ∎

Proof of Theorem 6. It will be done in two parts.

Part 1. Due to the monotony of $\{m/\lambda_1(m)\}$ and (4.6),

$$\frac{1}{m^2} \sum_{i=1}^{m} \frac{i}{\lambda_1^2(i)} = O\left\{ \frac{1}{\lambda_1^2(m)} \right\} \qquad (m = 1, 2, \ldots).$$

Consequently, by Theorem 5,

$$\left\{ \frac{1}{m^2} \sum_{i=1}^{m} \sum_{k=\theta^{-1}i}^{\theta i} [\sigma_{ik}(x) - f(x)]^2 \right\}^{1/2}$$
$$= \left\{ \frac{1}{m^2} \sum_{i=1}^{m} o_x \left\{ \frac{i}{\lambda_1^2(i)} \right\} \right\}^{1/2} = o_x \left\{ \frac{1}{\lambda_1(m)} \right\}, \text{ a.e.} \tag{9.8}$$

Part 2. We will prove that if (9.2) is satisfied and $\{m/\lambda_1(m)\}$ is nondecreasing, then for every $\theta \geqslant 1$,

$$\left\{\frac{1}{m^2} \sum_{i=1}^{m} \sum_{k=\theta^{-1}i}^{\theta i} [s_{ik}(x) - \sigma_{ik}(x)]^2\right\}^{1/2} = o_x\left\{\frac{1}{\lambda_1(m)}\right\}, \text{ a.e.} \tag{9.9}$$

This can be verified by showing

$$F_{12}(x) = \left\{\sum_{m=1}^{\infty} \frac{\lambda_1^2(m)}{m^2} \sum_{n=\theta^{-1}m}^{\theta m} [s_{mn}(x) - \sigma_{mn}(x)]^2\right\}^{1/2} \in L^2.$$

To this end, one has to use a representation analogous to (7.4), then (4.8) and (9.2).

So, the series

$$\sum_{m=1}^{\infty} \frac{\lambda_1^2(m)}{m^2} \sum_{n=\theta^{-1}m}^{\theta m} [s_{mn}(x) - \sigma_{mn}(x)]^2$$

converges a.e. One can apply the Kronecker lemma, since $m/\lambda_1(m) \to \infty$ as $m \to \infty$ in a nondecreasing way, and obtain (9.9). Combining (9.8) and (9.9), we get (2.16) to be proved. ∎

REFERENCES

1. P. R. AGNEW, On double orthogonal series, *Proc. London Math. Soc.* (2) **33** (1932), 420–434.
2. G. ALEXITS, "Convergence Problems of Orthogonal Series," Hungar. Acad. Sci., Budapest, 1961.
3. G. ALEXITS, Über die Approximation im starken Sinne, Approximationstheorie, *in* "Proceedings, Conf. Oberwolfach, 1963," pp. 89–95, Birkhäuser, Basel, 1964.
4. L. CSERNYÁK, Bemerkung zur Arbeit von V. S. Fedulov "Über die Summierbarkeit der doppelten Orthogonalreihen," *Publ. Math. Debrecen* **15** (1968), 95–98.
5. V. S. FEDULOV, On $(C, 1, 1)$-summability of a double orthogonal series, *Ukrain. Mat. Zh.* **7** (1955), 433–442. [Russian]
6. G. H. HARDY, On the convergence of certain multiple series, *Proc. London Math. Soc.* (2) **1** (1903–1904), 124–128.
7. G. H. HARDY, On the convergence of certain multiple series, *Proc. Cambridge Philos. Soc.* **19** (1916–1919), 86–95.
8. F. MÓRICZ, Moment inequalities and the strong laws of large numbers, *Z. Wahrsch. verw. Gebiete* **35** (1976), 299–314.
9. F. MÓRICZ, Multiparameter strong laws of large numbers. I. Second order moment restrictions, *Acta Sci. Math. (Szeged)* **40** (1978), 143–156.
10. F. MÓRICZ, On the convergence in a restricted sense of multiple series, *Anal. Math.* **5** (1979), 135–147.

11. F. Móricz, The Kronecker lemmas for multiple series and some applications, *Acta Math. Acad. Sci. Hungar.* **37** (1981), 39–50.
12. F. Móricz, On the a.e. convergence of the arithmetic means of double orthogonal series, *Trans. Amer. Math. Soc.*, in press.
13. K. Tandori, Über die orthogonalen Funktionen. VII. Approximationssätze, *Acta Sci. Math. (Szeged)* **20** (1959), 19–24.
14. A. Zygmund, "Trigonometric Series, II," Cambridge Univ. Press, London, 1959.