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Möbius functions and semigroup representation theory

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Abstract

This paper explores several applications of Möbius functions to the representation theory of finite semigroups. We extend Solomon's approach to the semigroup algebra of a finite semilattice via Möbius functions to arbitrary finite inverse semigroups. This allows us to explicitly calculate the orthogonal central idempotents decomposing an inverse semigroup algebra into a direct product of matrix algebras over group rings. We also extend work of Bidigare, Hanlon, Rockmore and Brown on calculating eigenvalues of random walks associated to certain classes of finite semigroups; again Möbius functions play an important role.

Keywords: Inverse semigroups; Representation theory; Semigroup algebras; Möbius functions; Random walks on semigroups

1. Introduction

The characters of commutative semigroups were studied independently by Schwarz [29] and by Hewitt and Zuckerman [9,10]. In particular the characters of a finite semilattice (that is idempotent, commutative semigroup) were shown to correspond to prime ideals and the semigroup algebra was shown to be a direct product of fields (see [7, Chapter 5] for an account of this work). Solomon [30] later gave a very explicit isomorphism between the semigroup algebra of a finite semilattice and a direct product of fields, using the Möbius function of the semilattice, from which the character results described above are easily deduced.

A semilattice is just an idempotent inverse semigroup. Inverse semigroups are semigroups that admit a faithful representation as a semigroup of partial permutations closed under inversion [7,11]. By a *partial permutation* of a set X, we mean a bijection $\varphi : Y \to Z$ between subsets Y

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and Z of X; in other words an injective partial function from X to X. For example

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 3 & 2 & 1 \end{pmatrix}$$

is a partial permutation of the set $\{1, 2, 3, 4\}$. Inverse semigroups capture partial symmetry in much the same way that groups capture symmetry [11]; they have recently become quite fashionable in Analysis in the guise of semigroups of partial isometries of a Hilbert space [19]. Inverse semigroups also play an important role in the theory of algebraic monoids, where the so-called Renner inverse monoid takes the place of the Weyl group in the analogue of the Bruhat decomposition [20,21,31,22]. The semilattice of idempotents in this case is isomorphic to the face lattice of a rational polytope [20] and so the Möbius function can be computed by the classical formula [33].

Munn [14,15] studied the semigroup algebra of a finite inverse semigroup and showed that in good characteristic it is always semisimple. His approach was to construct an ideal series in which the quotients are matrix algebras over the group rings of the maximal subgroups (see [7, Chapter 5]); in fact, the associated exact sequences are easily seen to split [18] and so the semigroup algebra is isomorphic to a direct product of such matrix algebras.

In this paper, we generalize Solomon's approach and we explicitly calculate the orthogonal central idempotents giving the direct sum decomposition of the algebra of an inverse semigroup into matrix algebras over group rings using the Möbius function of the natural partial order on the inverse semigroup. This involves calculating an isomorphism between the algebra of an inverse semigroup and the algebra [19] of its associated groupoid [11]. The relationship with the Schützenberger representation by monomial matrices is studied, as well (cf. [24]). Our results then allow for an explicit formula, in terms of characters, for the central primitive idempotents corresponding to the irreducible representations of the semigroup algebra of a finite inverse semigroup, again using Möbius functions.

More generally, Munn's results show that whenever a semigroup algebra is semisimple, there is a natural ideal series such that the quotients are matrix algebras over the group rings of maximal subgroups. The Möbius function of the $\mathcal{J}$-order can be used to show how to relate the identities of these ideals to the identities of these matrix algebras.

Bidigare et al. [4] discovered a pleasant formula for the eigenvalues, and their multiplicities, of the transition matrix of the random walk on the minimal ideal of the face semigroup associated to a hyperplane arrangement [4]; such random walks include well-known Markov chains like the Tsetlin library [4]. Brown extended these results, first to left regular bands [5] and then to bands in general [6]. These results all make use of Möbius functions. We extend these results to the largest class of finite semigroups for which they hold: the so-called semilattices of combinatorial archimedean semigroups, also called semilattices of locally trivial semigroups and also called members of the pseudovariety **DA** [1]. These are precisely the finite semigroups that can be faithfully represented by upper triangular matrices, with diagonal entries zero or one, over a field of characteristic 0. They can be characterized algebraically as semigroups whose von Neumann regular elements are precisely the idempotents.

There is also a well-known description of eigenvalues for random walks on Cayley graphs of finite Abelian groups in terms of irreducible characters. By combining the ideas of [4] with the proof of [17], we simultaneously generalize these results to random walks on minimal left ideals of semigroups in the pseudovariety **DO**(**Ab**) of finite semigroups whose regular $\mathcal{J}$-classes are

orthodox semigroups with Abelian maximal subgroups. A more natural way to describe **DO**(**Ab**) is as the pseudovariety of semigroups with a faithful representation over the complex field $\mathbb{C}$ by upper triangular matrices.

Hopefully these various results make a convincing case for the importance of Möbius functions in semigroup representation theory.

2. Möbius functions

Let $(P, \leqslant)$ be a locally finite partially ordered set (that is, intervals are finite) and A a unital ring. We identify the order $\leqslant$ with its graph

$$\{(p, q) \in P \times P \mid p \leqslant q\}.$$

The incidence algebra over A of P, which we denote $A[\![\leqslant]\!]$, is the algebra of functions $f : \leqslant \to A$ with the convolution product

$$f * g(x, y) = \sum_{x \leqslant z \leqslant y} f(x, z)g(z, y).$$

$A[\![\leqslant]\!]$ is a ring with unit the Kronecker delta function δ [8,33]. A function $f \in A[\![\leqslant]\!]$ is invertible if and only if $f(x, x)$ is a unit of A for all $x \in P$. In fact

- $f^{-1}(x, x) = f(x, x)^{-1}$,
- $f^{-1}(x, y) = -f(x, x)^{-1} \sum_{x<z\leqslant y} f(x, z) f^{-1}(z, y)$,

serves as an inductive definition of f^{-1}.

The *zeta function* of P is the element of $A[\![\leqslant]\!]$ that takes on the constant value of 1. It is invertible over any ring A and the inverse is called the *Möbius function* of P, denoted $\mu(x, y)$. Recall that an *order ideal* in a partially ordered set P is a subset I such that $x \leqslant y \in I$ implies $x \in I$. If $p \in P$, then $p^{\downarrow}$ denotes the principal order ideal generated by p. We observe that the Möbius function for $p^{\downarrow}$ is the restriction of the Möbius function for P.

The following is the Möbius Inversion Theorem [8,33].

Theorem 2.1 (*Möbius Inversion Theorem*). *Let $(P, \leqslant)$ be a locally finite partially ordered set in which each principal ideal has a minimum and G be an Abelian group. Suppose that $f : P \to G$ is a function and define $g : P \to G$ by*

$$g(x) = \sum_{y \leqslant x} f(y).$$

Then

$$f(x) = \sum_{y \leqslant x} g(y)\mu(y, x).$$

An *inverse semigroup* is a semigroup S such that, for each $x \in S$, there is a unique $y \in S$ such that $xyx = x,\ yxy = y$; one denotes y by x^{-1} [7,11]. The idempotents of S commute and hence form a semilattice subsemigroup [7,11]. If S is an inverse semigroup, then $(S, \leqslant)$ is a partially

ordered set where $x \leqslant y$ if and only if $xx^{-1}y = x$ [7,11]. This is equivalent to $yx^{-1}x = x$ and to $x \in E(S)y = yE(S)$ where $E(S)$ is the semilattice of idempotents of S. Notice that the order restricted to $E(S)$ is the usual order on idempotents given by $e \leqslant f$ if and only if $e = ef = fe$. Assume that $E(S)$ is finite. Then $(S, \leqslant)$ is locally finite. In fact, $s^{\downarrow} = sE(S)$ is finite; moreover, it is isomorphic as a poset to the lattice of idempotents $f \leqslant ss^{-1}$ and hence has a minimum. It follows that the Möbius function μ_S of S is completely determined by the Möbius function $\mu_{E(S)}$ of $E(S)$ via

$$\mu_S(t, s) = \mu_{E(S)}(tt^{-1}, ss^{-1}) = \mu_{E(S)}(t^{-1}t, s^{-1}s). \tag{1}$$

The symmetric inverse monoid I_X on a finite set X [7,11] (also known as the rook monoid [32]) is the inverse monoid of all partial permutations of the set X. It can also be described as the monoid of all $n \times n$ monomial matrices with entries consisting of only zeroes and ones (also called rook matrices since they correspond to placements of non-attacking rooks on an $n \times n$ chessboard) [7,32]. In this case, $E(I_X) \cong 2^X$ ordered by inclusion. The Möbius function of 2^X is well known [8,33]:

$$\mu_{2^X}(Y, Z) = (-1)^{|Z|-|Y|}.$$

Hence by (1), for $t \leqslant s$,

$$\mu_{I_X}(t, s) = (-1)^{\mathrm{rank}(s)-\mathrm{rank}(t)}, \tag{2}$$

where $\mathrm{rank}(s) = |\mathsf{dom}(s)| = |\mathsf{ran}(s)|$. In this note we shall view elements of I_X as acting on the right of X.

3. Groupoids and groupoid algebras

By a *groupoid*, we mean a small category in which all arrows are isomorphisms. A small category is a category whose objects form a set rather than a class [12]. We use the notation dom, ran for, respectively, the domain and range of an arrow. We follow the convention above and define composition fg when $\mathsf{ran}(f) = \mathsf{dom}(g)$. One says G is connected if $G(e, f) \neq \emptyset$ for all objects e and f. Connected components are defined in the natural way. If e is an object, the local group is $G_e = G(e, e)$. If e is isomorphic to f, then G_e and G_f are conjugate in G and hence isomorphic. We identify G with its arrow set, rather than having two distinct notations.

A typical example is the groupoid whose objects are the subsets of a finite set X and whose arrows are isomorphisms between these subsets. Clearly this groupoid is entirely encoded by I_X, but I_X has extra structure since all elements can be multiplied. We shall see that at the level of algebras, this difference disappears. This holds more generally.

Let A be a unital ring and G a groupoid. Then the *groupoid algebra* [19] of G is the ring AG of all functions $f : G \to A$ of finite support with multiplication given by convolution:

$$fg(z) = \sum_{w,\ \mathsf{dom}(w)=\mathsf{dom}(z)} f(w)g(w^{-1}z).$$

We identify an arrow $g \in G$ with the characteristic function of $\{g\}$. One can make $G^0 = G \cup 0$ into an inverse semigroup by defining all undefined products to be 0; AG is then the contracted semigroup algebra of G^0 [7].

If S is an inverse semigroup, then there is an associated groupoid [11,26] $G(S)$ with objects $E(S)$ and arrows

$$G(s) = ss^{-1} \xrightarrow{s} s^{-1}s.$$

The composition is the restriction of the multiplication of S. The identity at e is $G(e)$. The groupoid $G(I_X)$ is the groupoid discussed above.

The following proposition is obvious:

Proposition 3.1. *Let G be a groupoid with connected components G_i, $i \in I$. Then $AG \cong \bigoplus_{i\in I} AG_i$.*

Now we study the structure of the algebra of a connected groupoid with finitely many objects. This is essentially due to Munn [14,7], using inverse semigroup language. If A is a ring, $M_n(A)$ will denote the ring of $n \times n$ matrices over A.

Theorem 3.2. *Let G be a connected groupoid with object set $\{1, \ldots, n\}$. Then $AG \cong M_n(AG_{[1]})$ where $G_{[1]}$ is the connected component of the object 1. The identity matrix is given by $\sum_{i=1}^{n} 1_i$.*

Proof. Fix, for each i, an element $p_i \in G(1, i)$. We take $p_1 = 1_1$. Define a map $\varphi : AG \to M_n(AG_{[1]})$ on a basis element $g \in G(i, j)$ by

$$\varphi(g) = p_i g p_j^{-1} E_{ij},$$

where E_{ij} is the matrix unit. Then if $g \in G(i, j)$, $g' \in G(j, k)$, one calculates

$$\varphi(g)\varphi(h) = p_i g p_j^{-1} p_j h p_k^{-1} = p_i g h p_k^{-1} = \varphi(gh).$$

Thus φ induces a homomorphism.

If we define $\psi : M_n(AG_{[1]}) \to AG$ on basis elements by $\psi(gE_{ij}) = p_i^{-1} g p_j$, then clearly $\varphi\psi$ and $\psi\varphi$ are the identity on basis elements and so φ is an isomorphism.

The last statement is clear since

$$\varphi\left(\sum_{i=1}^{n} 1_i\right) = \sum_{i=1}^{n} p_i p_i^{-1} E_{ii} = \sum_{i=1}^{n} E_{ii}. \qquad \square$$

In the case of $G(I_X)$, two elements are in the same connected component if and only if they have the same rank. If J_r is the component of elements of rank r, the natural object to choose is the identity of the set $[r] = \{1, \ldots, r\}$. If K is any subset of size r, the natural choice of p_{1_K} is the unique order-preserving map from $[r]$ to K.

4. The Schützenberger representation and the isomorphism of algebras

Let S be an inverse semigroup with $E(S)$ finite and let μ be its Möbius function. Let A be a unital ring. Two elements s, t of an inverse semigroup are said to be $\mathcal{D}$-equivalent if $G(s)$ and $G(t)$ are in the same connected component [11,7]. If $e \in E(S)$, the maximal subgroup at e is denoted

H_e; it is the same as the local group at e of $G(S)$. If D is a $\mathcal{D}$-class of S, then the *Schützenberger representation* with respect to D is the map $\varphi_D : S \to AG(D)$ given by

$$\varphi_D(s) = \sum_{t \leqslant s,\ t \in D} G(t). \tag{3}$$

This representation is normally expressed in terms of monomial matrices (or so-called rook matrices) over H_e [7,27] and was defined by Schützenberger [27] in a more general context. Indeed, suppose $E(D) = \{e_1, \ldots, e_n\}$ and consider the isomorphism $AG(D) \cong M_n(H_e)$. Then (using the above notation)

$$\varphi_D(s) = \sum_{e_i \leqslant ss^{-1},\ e_j = s^{-1}e_i s} p_i e_i s p_j^{-1} E_{ij}, \tag{4}$$

so $\varphi_D(s)$ is a monomial matrix over H_e. See [24,23] for more on the connection between the Schützenberger representation and semigroup representation theory in the general case.

Proposition 4.1. *The Schützenberger representation $\varphi_D : S \to AG(D)$ is a homomorphism.*

Proof. First we calculate:

$$\varphi_D(s)\varphi_D(t) = \left(\sum_{s' \leqslant s,\ s' \in D} G(s')\right)\left(\sum_{t' \leqslant t,\ t' \in D} G(t')\right) \tag{5}$$

$$= \sum_{s' \leqslant s,\ t' \leqslant t,\ s', t', s't' \in D} G(s't'). \tag{6}$$

Suppose now that $u \in D$ and $u \leqslant st$. Then $uu^{-1}stu^{-1}u = u$. Hence $uu^{-1}s, tu^{-1}u \in D$ and $uu^{-1}s \leqslant s, tu^{-1}u \leqslant t$. Thus $G(u)$ appears (6). It follows that $\varphi_D(st) = \varphi_D(s)\varphi_D(t)$. $\square$

Since each $\mathcal{D}$-class contains an idempotent, our assumption that $E(S)$ is finite implies we have finitely many $\mathcal{D}$-classes. Thus we can define the *Schützenberger representation* $\varphi : S \to AG(S)$ by

$$\varphi(s) = \sum_D \varphi_D(s) = \sum_{t \leqslant s} G(t), \tag{7}$$

where the middle sum runs over all the $\mathcal{D}$-classes of S. Since $AG(S) \cong \bigoplus_D AG(D)$, this is a well-defined homomorphism.

Theorem 4.2. *Let S be an inverse semigroup with $E(S)$ finite. Let μ be the Möbius function for $(S, \leqslant)$. Then the Schützenberger representation extends to an isomorphism $\varphi : AS \to AG(S)$. The inverse $\psi : AG(S) \to AS$ is given on the basis by*

$$\psi(G(s)) = \sum_{t \leqslant s} t\mu(t, s). \tag{8}$$

Proof. All that remains to be proved is that φ and ψ are inverses. Indeed

$$\begin{aligned}\varphi\psi(G(s)) &= \varphi\left(\sum_{t\leqslant s} t\mu(t,s)\right)\\ &= \sum_{t\leqslant s}\varphi(t)\mu(t,s)\\ &= G(s),\end{aligned}$$

where the last equality follows from the Möbius Inversion Theorem and (7).

Conversely,

$$\begin{aligned}\psi(\varphi(s)) &= \psi\left(\sum_{t\leqslant s} G(t)\right)\\ &= \sum_{t\leqslant s}\psi(G(t))\\ &= \sum_{t\leqslant s}\sum_{u\leqslant t} u\mu(u,t)\\ &= \sum_{u\leqslant s} u\sum_{u\leqslant t\leqslant s}\mu(u,t)\zeta(t,s)\\ &= \sum_{u\leqslant s} u\delta(u,s) = s.\end{aligned}$$

Thus ψ is inverse to φ and hence they are isomorphisms. □

We remark that if S is a finite semilattice, then the groupoid $G(S)$ consists of just the identities at the idempotents and so $AG(S)$ is then just the direct product of $|E(S)|$ copies of the ring A. The isomorphism then reduces to the case studied in [30,33].

The following consequence of Theorems 3.2 and 4.2 is essentially due to Munn [14,15,18,7], the new twist being the explicit description of the central idempotents (which had been observed by Solomon [32] and Dlab, amongst others for I_X). If $s \in S$, we denote by D_s the $\mathcal{D}$-class of s.

Theorem 4.3. *Let S be an inverse semigroup with $|E(S)| < \infty$ and A a unital ring. Then*

$$AS \cong AG(S) \cong \bigoplus_{e\in E(S)/\mathcal{D}} M_{|E(D_e)|}(H_e).$$

The central idempotent corresponding to a $\mathcal{D}$-class D in this decomposition is given by the formula:

$$e_D = \sum_{f\in D}\sum_{e\leqslant f} e\mu(e,f).$$

The special case of $S = I_X$ (X finite) and $D = J_r$, gives the famous formula

$$e_{J_r} = \sum_{Z \subseteq X,\ |Z|=r} \sum_{Y \subseteq Z} (-1)^{|Z|-|Y|} 1_Y.$$

In fact, one has the following theorem:

Theorem 4.4. *Let I_n be the symmetric inverse monoid on n letters, S_r the symmetric group on r letters and A a unital ring. If $K \subseteq \{1, \ldots, n\}$ with $|K| = r$, let $p_K : \{1, \ldots, r\} \to K$ be the unique order-preserving bijection. Then there is an isomorphism*

$$\varphi : AI_n \to \bigoplus_{r=0}^{n} M_{\binom{n}{r}}(AS_r)$$

given by

$$\varphi(s) = \sum_{K \subseteq \mathsf{dom}(s)} p_K s|_K p_{Ks}^{-1} E_{K,Ks}$$

with inverse given on basis elements $\sigma \in S_r$ and $E_{K,J}$ with $|K| = r = |J|$ by

$$\sigma E_{K,J} \longmapsto \sum_{X \subseteq K} (-1)^{|K|-|X|} (p_K^{-1} \sigma p_J)|_X.$$

Theorem 4.3 recovers Munn's result on the semisimplicity of finite inverse semigroup algebras.

Corollary 4.5. *Let S be a finite inverse semigroup and K a field. Then KS is semisimple if and only $\operatorname{char} K \nmid |H_e|$ for each idempotent e.*

5. Characters of inverse semigroups

Let K be a field and S a finite inverse semigroup. Theorem 4.3 implies that the irreducible representations of S are in bijection with the irreducible representations of maximal subgroups. Suppose $\psi : H_e \to M_n(K)$ is an irreducible representation. Set $D = D_e$ and $|E(D)| = k$. Then ψ extends to an irreducible representation $\psi^* : M_k(H_e) \to M_{nk}(K)$ by

$$\psi^* \left(\sum_{ij} g_{ij} E_{ij} \right) = \psi(g_{ij}) \otimes E_{ij}$$

(see [24,23] for a more general result relating irreducible representations to the Schützenberger representation). Hence the corresponding irreducible representation $\psi^S_{H_e} : S \to M_{nk}(K)$ is given by $\psi^* \varphi_D$ where φ_D is the Schützenberger representation (4) by monomial matrices over H_e. The representation $\psi^S_{H_e}$ is hence by block monomial matrices, the blocks coming from ψ. To calculate the character $\chi_{\psi^S_{H_e}}$, it suffices to calculate the diagonal elements of $\varphi_D(s)$ and take their character

with respect to ψ. From (4) we obtain the formula

$$\chi_{\psi^s_{H_e}}(s) = \sum_{f\in D,\ f\leqslant ss^{-1},\ f=s^{-1}fs} \chi_\psi(p_f s p_f^{-1}), \tag{9}$$

where for each $f \in E(D)$, we have chosen $p_f \in D$ with $p_f p_f^{-1} = e$, $p_f^{-1} p_f = f$.

Using this, one can compute in principle the characters of a finite inverse semigroup, knowing the characters of the maximal subgroups. We mention the observation, made independently by McAlister [13], Munn [16] and Rhodes and Zalcstein [24], that if $s \in S$ and s^ω is the unique idempotent power of s, then it follows easily from (9) that, for any character χ of s, $\chi(s) = \chi(ss^\omega)$. Thus one need only calculate characters on group elements. This allows one to define a character table, which is invertible as a matrix [16,13,24].

We end this section by computing the central primitive idempotent associated to an irreducible representation of S.

Theorem 5.1. *Suppose S is a finite inverse semigroup, K is a field such that $char(K) \nmid |H_e|$ for all $e \in E(S)$. Let μ be the Möbius function of S. Let χ be an irreducible character of S coming from a $\mathcal{D}$-class D. Then the central primitive idempotent corresponding to χ is given by*

$$e_\chi = \sum_{e\in E(D)} \left(\frac{\chi(e)}{|H_e|} \sum_{s\in H_e} \chi(s) \sum_{t\leqslant s} t^{-1}\mu(t,s) \right). \tag{10}$$

Proof. If H is a finite group, then it is well known that the central primitive idempotent associated to a character χ_0 is

$$e_{\chi_0} = \frac{\chi_0(1)}{|H|} \sum_{h\in H} \chi_0(h)h^{-1}.$$

Hence if χ_0^* is the induced character on $M_n(KH)$, clearly the associated central primitive idempotent is

$$e_{\chi_0^*} = \sum_{i=1}^{n} e_{\chi_0} E_{ii}.$$

The theorem now follows from our previous results and the fact that inversion is an order isomorphism between $s^\downarrow$ and $(s^{-1})^\downarrow$. $\square$

Of course (9) shows how to write (10) entirely in terms of the characters of the maximal subgroups.

5.1. Semisimple semigroup algebras and the Möbius function

We briefly remark on another application of Möbius functions, although this is really more of a remark than anything else.

Let S be a finite semigroup. Then Munn [7] gave necessary and sufficient conditions for the semigroup algebra KS over a field K to be semisimple. An element $s \in S$ is called (von Neumann)

regular if there exists $t \in S$ such that $sts = s$. A necessary condition for KS to be semisimple is that all elements of S are regular [7]. Let us recall Green's quasi-order $\leqslant_{\mathcal{J}}$ [7]. Define $s \leqslant_{\mathcal{J}} t$ if s is in the principal two-sided ideal generated by t. We write $s \mathcal{J} t$ if $s \leqslant_{\mathcal{J}} t$ and $t \leqslant_{\mathcal{J}} s$. Then the set of $\mathcal{J}$-classes is a partially ordered set denoted $S/\mathcal{J}$. For finite semigroups, $\mathcal{J}$ and $\mathcal{D}$ coincide [7]. Let μ be the Möbius function of $\leqslant_{\mathcal{J}}$ on $S/\mathcal{J}$.

There is a sequence of principal two-sided ideals $I_0 \subset I_1 \cdots \subset I_n = S$ of S such that $I_j/I_{j-1} = J_j^0$ for some $\mathcal{J}$-class J_j; moreover all $\mathcal{J}$-classes arise in this way [7]. In the case that KS is semisimple, then the algebras KI_j and $KI_j/KI_{j-1} = K_0 J_j^0$ are semisimple (here $K_0 J^0$ is the contracted semigroup algebra of J^0 [7]). Hence KI_j has an identity η_{J_j}. By semisimplicity, $K_0 J_j^0$ is an internal direct summand of KI_j. Let ε_{J_j} be the corresponding central orthogonal idempotent. Then it is easy to see that

$$\eta_{J_j} = \sum_{J \leqslant_{\mathcal{J}} J_j} \varepsilon_J.$$

We thus obtain the following formula via Möbius inversion:

$$\varepsilon_{J_j} = \sum_{J \leqslant_{\mathcal{J}} J_j} \eta_J \mu(J, J_j).$$

M. Putcha (unpublished) has a related formula relating these idempotents in terms of coset representatives.

6. Random walks on finite semigroups

A semigroup is called a *band* if every element is idempotent. A *left regular band* is a band satisfying the identity $xyx = xy$; in other words left regular bands are $\mathcal{R}$-trivial bands, where $\mathcal{R}$ is Green's relation associated to principal right ideals [7]. Examples include the face semigroups associated to hyperplane arrangements [4,5].

Recall that if S is a finite semigroup with finite generating set X and $\{w_x\}_{x \in X}$ are weights describing a probability measure supported by X, then the left random walk on S is the Markov chain on the set S whose probability of transition from s to t is $\sum_{x \in X,\ xs = t} w_x$. The interpretation is that a walker starts at the vertex s of the left Cayley graph of S and moves along the edge labelled by x with probability w_x. Results of Rosenblatt [25] show that such a walk eventually enters a minimal left ideal L and so it is customary to restrict the Markov chain to the subset L. The transition matrix is easily seen to be independent of the choice of L since all minimal left ideals are isomorphic via right multiplication by a semigroup element [7].

An algebraic way to describe the transition matrix is to consider the semigroup algebra $\mathbb{C}S$. Then $\mathbb{C}L$ is a left ideal and one easily verifies (cf. [5,6]) that if we take L as a basis for $\mathbb{C}L$ and take the matrix for the element

$$M = \sum_{x \in X} w_x x,$$

then the transition matrix for our Markov chain is the transpose of the matrix for M.

Bidigare et al. used representation theory to compute the eigenvalues and their multiplicities for face algebras of hyperplane arrangements [4]. Brown extended this to left regular bands [5]

and then to bands [6]. In this section we generalize these results to the largest class of semigroups for which they hold.

On the other hand, it is well known [17] that the eigenvalues of the random walk on an Abelian group G correspond to the irreducible characters. More precisely, if G is a finite Abelian group generated by a subset X and $\{w_x\}_{x\in X}$ is a probability distribution on X, then the eigenvalues of the transition matrix are of the form λ_χ where χ is an irreducible character of G and

$$\lambda_\chi = \sum_{x\in X} w_x \cdot \chi(x). \tag{11}$$

In this section we obtain a simultaneous generalization of the results of [4–6] and (11). Unfortunately, this section will also assume quite a bit more of background in semigroup theory than the previous sections.

6.1. *Representations for semigroups in* **DO(Ab)**

The class **DO(Ab)** consists of all finite semigroups whose regular $\mathcal{J}$-classes are orthodox semigroups and whose maximal subgroups are Abelian [1]. The *maximal subgroup* of a finite semigroup S at an idempotent e is the group of units of the submonoid eSe; it depends only on the $\mathcal{J}$-class of e up to isomorphism [1,7]. A regular semigroup is called *orthodox* if its idempotents form a subsemigroup [7].

Alternatively, $S \in$ **DO(Ab)** if and only if it admits a surjective homomorphism $\varphi : S \twoheadrightarrow I$, with I a commutative inverse semigroup, such that the inverse image under φ of each idempotent is a locally trivial semigroup (see [2, Proposition 3.2, 28,1]). A semigroup T is *locally trivial* if $eTe = e$ for all idempotents $e \in T$, or equivalently T is a nilpotent extension of a simple semigroup. A nicer description of **DO(Ab)** is as the class of finite semigroups admitting a faithful complex representation by upper triangular matrices (see below).

A semigroup is called *aperiodic* if its maximal subgroups are trivial. The class of aperiodic members of **DO(Ab)** is denoted **DA**. It consists of those finite semigroups whose regular elements are precisely the idempotents [1]. In particular, every band belongs to **DA** and the regular semigroups in **DA** are precisely bands. Alternatively, **DA** consists of all finite semigroups admitting a surjective homomorphism $\varphi : S \twoheadrightarrow E$, with E a semilattice, such that the inverse image under φ of each element is a locally trivial semigroup [28,1,2].

The following result was proved in [3], although one could deduce it with a little work from the results of [23,24].

Theorem 6.1. *Let S be a finite semigroup. Then the following are equivalent*:

(1) $S \in$ **DO(Ab)** (*respectively*, $S \in$ **DA**);
(2) *every irreducible complex representation of S is a homomorphism $\varphi : S \to \mathbb{C}$* (*respectively*, *every irreducible representation of S over a field of characteristic* 0 *is a homomorphism $\varphi : S \to \{0, 1\}$*);
(3) *every complex representation of S is equivalent to one by upper triangular matrices* (*respectively*, *every representation of S over a field of characteristic* 0 *is equivalent to one by upper triangular matrices with zeroes and ones on the diagonal*);

(4) *S admits a faithful complex representation by upper triangular matrices* (*respectively*, *S admits a faithful representation over the rationals by upper triangular matrices with zeroes and ones on the diagonal*).

The key ingredient to prove this theorem is the following result from [3]:

Theorem 6.2. *Let K be a field of characteristic* 0 *and* $\varphi : S \to T$ *a homomorphism of finite semigroups. Then the induced map* $KS \to KT$ *has nilpotent kernel if and only if, for each idempotent* $e \in T$, $\varphi^{-1}(e)$ *is a locally trivial semigroup.*

It follows that if $S \in \mathbf{DO}(\mathbf{Ab})$ (respectively, $S \in \mathbf{DA}$) and $\varphi : S \to I$ (respectively, $\varphi : S \to E$) is the map alluded to above with I a commutative inverse semigroup (respectively, E a semilattice) such that the inverse image of each idempotent is locally trivial [1], then the induced map $\varphi : KS \to KI$ (respectively, $\varphi : S \to E$) is the semisimple quotient for any field K of characteristic 0. Here we are using that the semigroup algebra of a finite inverse semigroup is semisimple in characteristic 0. From here we can easily compute the irreducible representations and the eigenvalues (with multiplicities for semigroups in **DA**) for the random walk on a minimal left ideal of a semigroup in $\mathbf{DO}(\mathbf{Ab})$.

To do this we first describe I and φ. Details can be found, for instance, in [2, Proposition 3.2] or [1]. Since we care primarily about the representations of I, we just describe $G(I)$ and $E(I)$. When $S \in \mathbf{DO}(\mathbf{Ab})$, it turns out that the set E of regular $\mathcal{J}$-classes ordered by $\leqslant_{\mathcal{J}}$ is a meet semilattice. Fix a maximal subgroup H_J for each regular $\mathcal{J}$-class $J \in E$ (the choice of idempotent from J does not change the group up to isomorphism [7,1]); let e_J be the identity of H_J. The inverse semigroup I has semilattice of idempotents E. The groupoid $G(I)$ has object set E. If J, J' are regular $\mathcal{J}$-classes of S, then

$$G(I)(J, J') = \begin{cases} H_J, & J = J', \\ \emptyset, & J \neq J'. \end{cases}$$

The map $\varphi : S \to I$ is given by

$$s \longmapsto e_{J_{s^\omega}} s e_{J_{s^\omega}} \in H_{J_{s^\omega}} = G(I)(J_{s^\omega}, J_{s^\omega}),$$

where J_{s^ω} is the $\mathcal{J}$-class of s^ω. The multiplication in I is the unique multiplication making φ a homomorphism. In the case $S \in \mathbf{DA}$, the semilattice E is the set of regular $\mathcal{J}$-classes of S and the map φ takes s to the $\mathcal{J}$-class of s^ω.

The results of Section 4, applied to the commutative inverse semigroup I, show that the irreducible representations of I are parameterized by pairs of the form (J, χ), with $J \in E$ and χ an irreducible character of the Abelian group H_J. The associated irreducible representation $\psi_{(J,\chi)}$ of I has the following formula, where $s \in I$ and we recall that we are viewing regular $\mathcal{J}$-classes of S as idempotents of I,

$$\psi_{(J,\chi)}(s) = \begin{cases} \chi(s \cdot J), & s \geqslant_{\mathcal{J}} J, \\ 0, & \text{else}. \end{cases}$$

In the case that $S \in \mathbf{DA}$ we have, using [30], that the irreducible representations of E are of the form ψ_J, $J \in E$, where

$$\psi_J(J') = \begin{cases} 1, & J' \geqslant_{\mathcal{J}} J, \\ 0, & \text{else.} \end{cases}$$

So if $S \in \mathbf{DO}(\mathbf{Ab})$, then its irreducible complex representations are of the form $\varphi_{(J,\chi)}$ where J is a regular $\mathcal{J}$-class of S, χ is an irreducible character of H_J and

$$\varphi_{(J,\chi)}(s) = \begin{cases} \chi(e_J s e_J), & s \geqslant_{\mathcal{J}} J, \\ 0, & \text{else.} \end{cases} \tag{12}$$

In particular, each complex irreducible representation of S is one dimensional. If the maximal subgroups of S have exponent 2, then each irreducible representation of S is rational-valued.

Specializing to the case $S \in \mathbf{DA}$, we see that the irreducible complex representations are parameterized by regular $\mathcal{J}$-classes. The representation φ_J associated to a regular $\mathcal{J}$-class J is given by

$$\varphi_J(s) = \begin{cases} 1, & s \geqslant_{\mathcal{J}} J, \\ 0, & \text{else.} \end{cases}$$

6.2. Eigenvalues

We proceed to our main theorem of this section, which extends the results of [4–6] as well as the results for Abelian groups [17]:

Theorem 6.3. *Let $S \in \mathbf{DO}(\mathbf{Ab})$ with generating set X and let L be a minimal left ideal. Assume that S has a left identity. Choose a maximal subgroup H_J, with identity e_J, for each regular $\mathcal{J}$-class J. Let $\{w_x\}_{x\in X}$ be a probability distribution on X. Then the transition matrix for the left random walk on L can be placed in upper triangular form over $\mathbb{C}$. Moreover, there is an eigenvalue $\lambda_{(J,\chi)}$ for each regular J-class J and irreducible character χ of H_J given by the formula*:

$$\lambda_{(J,\chi)} = \sum_{x\in X,\ x \geqslant_{\mathcal{J}} J} w_x \cdot \chi(e_J x e_J). \tag{13}$$

Proof. Let $\rho : S \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}L)$ be the representation induced via left multiplication. Let

$$M = \sum_{x\in X} w_x x \in \mathbb{C}S.$$

Then the matrix in which we are interested is the transpose of the matrix for the operator $\rho(M)$, so we may restrict attention to M.

By choosing a composition series for the left $\mathbb{C}S$-module $\mathbb{C}L$ and recalling that the complex irreducible representations of S are one dimensional, we may choose a basis for $\mathbb{C}L$ so that ρ is in upper triangular form with the irreducible constituents of ρ on the diagonal (cf. Theorem 6.1). Our assumption that S has a left identity guarantees that the zero representation does not appear in our composition series. Hence $\rho(M)$ is in upper triangular form and the diagonal elements

are just the values of $\varphi_{(J,\chi)}(M)$, J a regular $\mathcal{J}$-class and χ an irreducible character of H_J, with multiplicity the same as the multiplicity of $\varphi_{(J,\chi)}$ as a constituent of ρ. But (12) gives us that

$$\varphi_{(J,\chi)}(M) = \sum_{x\in X} w_x \cdot \varphi_{(J,\chi)}(x) = \sum_{x\in X,\ x \geqslant_{\mathcal{J}} J} w_x \cdot \chi(e_J x e_J) = \lambda_J,$$

establishing (13). □

We remark that the case where S is an Abelian group gives (11). It is possible that the eigenvalue $\lambda_{(J,\chi)}$ can appear with multiplicity zero. We leave it as an open question to determine the multiplicities in this generality. However, if we restrict ourselves to **DA**, we can calculate the multiplicities of the eigenvalues as well, generalizing the results of [4–6].

To state our result, we need one last notation. Let J be a regular $\mathcal{J}$-class of S and $e \in J$ an idempotent (recall that each regular $\mathcal{J}$-class has an idempotent [7]). Define

$$c_J = |eS \cap L|. \tag{14}$$

First we observe that c_J is independent of the choice of e. Indeed, any two principal right ideals generated elements of J are in bijection via left multiplication by an element of S by Green's Lemma [7]. Since left multiplication preserves L (L being a left ideal), it follows that if $e, f \in J$ are idempotents, then $eS \cap L$ and $fS \cap L$ are in bijection via left multiplication by some element of S. Thus c_J is well defined.

Theorem 6.4. *Let $S \in$ **DA** with generating set X and let L be a minimal left ideal. Suppose that S has a left identity. Let $\{w_x\}_{x\in X}$ be a probability distribution on X. Then the transition matrix for the left random walk on L can be placed in upper triangular form over $\mathbb{R}$. Moreover, there is an eigenvalue λ_J for each regular $\mathcal{J}$-class J given by the formula*

$$\lambda_J = \sum_{x\in X,\ x \geqslant_{\mathcal{J}} J} w_x \tag{15}$$

with multiplicity m_J given by

$$m_J = \sum_{J' \leqslant_{\mathcal{J}} J} c_{J'} \cdot \mu(J', J), \tag{16}$$

where μ is the Möbius function of the semilattice of regular $\mathcal{J}$-classes of S (and the sum in (16) *only includes regular $\mathcal{J}$-classes J').*

Proof. This time we let $\rho : S \to \mathrm{End}_{\mathbb{R}}(\mathbb{R}L)$ be the representation induced via left multiplication and let

$$M = \sum_{x\in X} w_x x \in \mathbb{R}S.$$

As before we are interested in the eigenvalues of M. Since the irreducible representations over $\mathbb{R}$ of S take on only the values 0 and 1 (Theorem 6.1), we can place, as in the previous theorem, ρ into upper triangular form, but this time over $\mathbb{R}$, with the irreducible constituents of ρ appearing

on the diagonal with their multiplicities as constituents of ρ. Again the zero representation does not appear because S has a left identity. Theorem 6.3 implies (15) (recalling that H_J is trivial for each J), so we just need to calculate the multiplicities. We follow the approach of [5,6].

To calculate the multiplicity of φ_J, choose an idempotent $e \in J$. Since $e = e^2$, $\rho(e)$ is a projection and hence its eigenvalues are zero and one. The multiplicity of one is just the rank of $\rho(e)$. But it is clear that $\rho(e)(\mathbb{R}L)$ is the linear span of $eL = eS \cap L$. Hence ρ has rank c_J. But counting this multiplicity via the $\varphi_{J'}$, we obtain

$$c_J = \sum_{J' \leqslant_{\mathcal{J}} J} m_{J'}. \tag{17}$$

An application of Möbius inversion to (17) then gives (16), completing the proof. □

We remark that it is possible that $m_J = 0$, meaning that λ_J does not really occur.

Many well-known Markov chains [4–6] have been obtained by considering left regular bands, including the Tsetlin library [5,6], which can be obtained by taking the free left regular band. It would be interesting to find more examples using semigroups from **DA**.

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