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ASYMPTOTIC BEHAVIOUR OF THE PERTURBATION OF A GIVEN MEAN FLOW

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Abstract—We consider an oceanic domain included in $\mathbb{R}^3$, in which there exist, at initial time, a current field $\mathbf{V}_0$ and a temperature field θ_0. Perturbations $\mathbf{V}$ and θ of the velocity and the temperature are induced by a perturbation of the mean wind-stress. $\mathbf{V}$ and θ have to satisfy a non-linear problem of Navier–Stokes type. We prove the existence of the solution, for the variational problem, and give some results about uniqueness and regularity. In order to study the asymptotic behaviour of the perturbation, we introduce some operators, deduced from the Stokes operator. Their properties allow us to make *a priori* estimations, and to prove that, under some assumptions, the perturbations $\mathbf{V}(t)$ and $\theta(t)$ remain bounded as $t \to \infty$. With stronger assumptions about the initial data, we can prove that the perturbation tends to 0 as $t \to \infty$, and that the solution of the variational problem is a strong solution for every $t \in [0, \infty[$.

NOTATION

x, y, z = The cartesian coordinates, forming a right-handed set, in which x, y are measured in the horizontal plane of the undisturbed sea surface (0x towards the East, 0y towards the North and 0z vertically ascendant)
$\mathbf{V}$ = The current velocity
p = Pressure
θ = Temperature
ρ_m = A mean value of the density over all the domain
$\mathbf{V}_0$ = The current velocity at initial time
P_0, θ_0 = Pressure and temperature at initial time
$\mathbf{F} = (0, 0, 2\omega \sin \phi)$ = The Coriolis stress
ω = Rate of rotation of the earth
ϕ = Latitude
$\mathbf{G} = (0, 0, -g)$ = The gravity stress
$\mathbf{T}$ = The wind-stress
ν = The eddy viscosity coefficient
ν' = The eddy diffusivity coefficient
$\mathbf{n}$ = The unit outward vector normal to the boundary
$\mathbf{V} = \mathbf{V}_t + \mathbf{V}_n$; $\mathbf{V}_t$ = tangential component of the velocity, $\mathbf{V}_n$ = normal component of the velocity
$\mathbf{V} = (v_1, v_2, v_3)$
$(x, y, z) \equiv (x_1, x_2, x_3)$

$$D_i = \frac{\partial}{\partial x_i}, \quad i = 1, 3.$$

∇ = The gradient operator $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \equiv (D_1, D_2, D_3)$

$$\Delta = \nabla \cdot \nabla = \sum_{i=1}^{3} D_i^2$$

$[\mathbf{V} \cdot \nabla]$ = The operator $\sum_{i=1}^{3} v_i D_i$

1. INTRODUCTION

We have undertaken this study in order to make precise some results obtained in previous works about equatorial waves [1, 2]. Our purpose was then to investigate the effects of a mean circulation, with vertical and latitudinal shear, on the equatorial oceanic waves. To this end, we have developed a numerical model to calculate the perturbation of a given mean flow. The excited waves were obtained by Fourier analysis of the perturbation of the velocity. We have dealt with two initial situations, characteristic of the circulation in the equatorial Atlantic during the summer and the winter. One important feature displayed by this study is the existence of unstable waves, in the meaning that their amplitude increases as a function of time. This instability is dependent upon the characteristics of the mean circulation. For example, with an initial circulation corresponding to the summer in the equatorial Atlantic, we get an unstable westward propagating wave of 24-day period. This wave is no longer present with an initial mean flow corresponding to the winter, and characterized by weaker westward currents. These results have been corroborated by recent observations *in situ*.

The fact that oceanic waves could be stable or unstable, depending on the characteristics of the mean circulation, induced us to consider the problem of the stability of a given initial flow. To this end, we went back to the problem of the perturbation of a mean circulation, introduced to modelize waves, in order to study its asymptotic behaviour.

Let Ω be a bounded oceanic domain, included in $\mathbb{R}^3$. We assume that there exist in Ω, at initial time, a current field $\mathbf{V}_0$ and a temperature field θ_0. A perturbation of the wind-stress induces a perturbation of the mean circulation. We are going to study its behaviour as time $t \to \infty$. Velocity of the current has to satisfy Navier–Stokes equations, to which is added a linear term resulting from Coriolis stress. The equation satisfied by temperature is of transport-diffusion type. We prove, for this non-linear problem, the existence of a solution in proper functional spaces. The uniqueness can be proved only for a more regular solution, but then we cannot assure the existence.

To study the stability, we introduce some operators deduced from the Stokes operator. These operators generate strongly continuous semi-groups, which allows us to write a weak solution of the problem. We have located the eigenvalues of these operators; they are situated inside a parabolic curve drawn in the complex plane.

We show that the perturbations of velocity and temperature remain bounded for every time t, provided that the eigenvalues have positive real parts and under some assumptions about the inital circulation and the perturbation of the wind-stress. With stronger assumptions, we can prove that the perturbation tends to 0 as time $t \to \infty$.

2. GENERAL ASSUMPTIONS AND EQUATIONS

The equations governing the circulation in an oceanic domain included in $\mathbb{R}^3$ are the Navier–Stokes equations, to which is added a term resulting from the Coriolis stress. Temperature is governed by a transport-diffusion equation. So, velocity $\mathbf{V}(x, y, z; t)$, pressure $p(x, y, z; t)$ and temperature $\theta(x, y, z; t)$ have to satisfy the following equations:

$$\begin{cases} \dfrac{\partial \mathbf{V}}{\partial t} + [\mathbf{V}\cdot\nabla]\mathbf{V} + \mathbf{F} \wedge \mathbf{V} - \nu\Delta\mathbf{V} + \dfrac{1}{\rho_m}\nabla p = \mathbf{T} + \mathbf{G} & (1) \\ \operatorname{div} \mathbf{V} = 0 & (2) \\ \dfrac{\partial \theta}{\partial t} + [\mathbf{V}\cdot\nabla)\theta - \nu'\Delta\theta = 0, & (3) \end{cases}$$

in $\Omega \times [0, T]$.

In equation (1), $\mathbf{T}$ represents the wind-stress which will be considered as a body-force.

The physical problem we are dealing with is the problem of equatorial waves developed in Ref. [2]. We assume that at the initial time, induced by a mean wind-stress $\mathbf{T}_0$, there exists in Ω a stationary flow characterized by a velocity $\mathbf{V}_0(x, y, z)$, a pressure $p_0(x, y, z)$ and a temperature $\theta_0(x, y, z)$. In order to study the stability of this stationary solution, we are going to calculate the perturbations $\mathbf{V}'(x, y, z; t)$, $p'(x, y, z; t)$, $\theta'(x, y, z; t)$ of velocity, pressure and temperature induced by a perturbation $\mathbf{T}'(t)$ of the wind-stress.

The mean situation $(\mathbf{V}_0, \theta_0)$ is given. The values of current and temperature must be characteristic of an equatorial domain. Two cases can occur, these values, resulting from physical observations, either verify the linearized equations, or are solutions of the complete, non-linear, stationary problem.

In these two cases, equations satisfied by the perturbations $\mathbf{V}'$, p' and θ' can be written:

$$\begin{cases} \dfrac{\partial \mathbf{V}'}{\partial t} + [\mathbf{V}_0 \cdot \nabla]\mathbf{V}' + [\mathbf{V}' \cdot \nabla]\mathbf{V}_0 + [\mathbf{V}' \cdot \nabla]\mathbf{V}' + \mathbf{F} \wedge \mathbf{V}' - \nu \Delta \mathbf{V}' + \dfrac{1}{\rho_m} \nabla p' = \mathbf{f}(t) & (4) \\ \operatorname{div} \mathbf{V}' = 0 & (5) \\ \dfrac{\partial \theta'}{\partial t} + [\mathbf{V}_0 \cdot \nabla]\theta' + [\mathbf{V}' \cdot \nabla]\theta_0 + [\mathbf{V}' \cdot \nabla]\theta' - \nu' \Delta \theta' = \phi, & (6) \end{cases}$$

in $\Omega \times [0, T]$.

If $(\mathbf{V}_0, \theta_0)$ satisfies the linearized stationary problem,

$$\mathbf{f}(t) = \mathbf{T}'(t) - [\mathbf{V}_0 \cdot \nabla]\mathbf{V}_0; \quad \phi = -[\mathbf{V}_0 \cdot \nabla]\theta_0 + \nu' \Delta \theta_0.$$

If $(\mathbf{V}_0, \theta_0)$ is the solution of the non-linear stationary problem,

$$\mathbf{f}(t) = \mathbf{T}'(t); \quad \phi = 0.$$

At initial time $t = 0$, the mean flow is not modified:

$$\begin{cases} \mathbf{V}'(x, y, z; 0) = \mathbf{0} & (7) \\ \theta'(x, y, z; 0) = 0. & (8) \end{cases}$$

The oceanic domain we want to study is an equatorial band, lying from 10°S to 10°N, limited by eastern and western boundaries, and of constant depth H.

On this equatorial band, we neglect curvature of the earth. So, the domain is parallelepipedic. Let Ω be the open set obtained from this oceanic domain by regularizing vertical edges, in order to suppress possible singularities in the corners. $\Gamma = \Gamma' \cup \Gamma''$ is the boundary of the open set Ω, Γ' designating the bottom of the studied layer, Γ'' the union of the sea surface and the lateral boundaries. We suppose that velocity vanishes on Γ'. On Γ'', velocity is supposed to be tangential to the boundary, and its derivative is supposed to be zero.

We also assume that perturbation of the temperature vanishes on Γ. Then, the boundary conditions on Γ can be written

$$\Gamma = \Gamma' \cup \Gamma''$$

$$\theta' = 0 \quad \text{on} \quad \Gamma \tag{9}$$

$$\mathbf{V}' = \mathbf{0} \quad \text{on} \quad \Gamma' \tag{10}$$

$$\mathbf{V}' \cdot \mathbf{n} = 0 \quad \text{and} \quad \frac{\partial (\mathbf{V}'_t)}{\partial n} = 0 \quad \text{on} \quad \Gamma''. \tag{11}$$

N.B. Hereafter, the notation ",′," for the perturbation of the mean situation will be omitted.

3. EXISTENCE AND REGULARITY OF THE PERTURBATION

We will denote $\mathbb{H}^1(\Omega) = [H^1(\Omega)]^3$, $\mathbb{L}^2(\Omega) = [L^2(\Omega)]^3$; $(\mathbf{U}, \mathbf{V}) = \int_\Omega \mathbf{U} \cdot \mathbf{V} \, d\Omega$ is the scalar product in $\mathbb{L}^2(\Omega)$, $|\mathbf{U}| = (\mathbf{U}, \mathbf{U})^{1/2}$ is the norm in $\mathbb{L}^2(\Omega)$ [$|q|$ the norm in $L^2(\Omega)$], $\|\mathbf{U}\|$ is the norm in $\mathbb{H}^1(\Omega)$ [$\|q\|$ the norm in $H^1(\Omega)$] and $[\mathbf{U}] = (\operatorname{grad} \mathbf{U}, \operatorname{grad} \mathbf{U})^{1/2}$ the semi-norm in $\mathbb{H}^1(\Omega)$.

We introduce the following functional spaces:

$$\mathscr{U} = \{\mathbf{U} \in \mathscr{D}(\Omega) / \operatorname{div} \mathbf{U} = 0\}$$

$$\mathbb{H} = \{\mathbf{U} \in \mathbb{L}^2(\Omega) / \operatorname{div} \mathbf{U} = 0, \quad \mathbf{U} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma\}$$

$$\mathbb{V} = \{\mathbf{U} \in \mathbb{H}^1(\Omega) / \operatorname{div} \mathbf{U} = 0, \quad \mathbf{U} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma'', \quad \mathbf{U} = \mathbf{0} \quad \text{on} \quad \Gamma'\},$$

where

$\mathbb{V}'$ = the dual space of $\mathbb{V}$,

$\mathbb{H}$ = identical to its dual space $\mathbb{H}'$

and

$\mathbb{V}$ = a closed subset of $\mathbb{H}^1(\Omega)$, endowed with the norm of $\mathbb{H}^1$.

Ω is an open set of $\mathbb{R}^3$ of class $\mathscr{C}^{0,1}$. The outward vector normal to the boundary $\mathbf{n}$ is therefore defined almost everywhere on Γ. On $\mathbb{V}$, the semi-norm $[\ ]$ is a norm equivalent to the norm of $\mathbb{H}^1: \|\ \|$.

Thus, there is one constant C such that

$$C\,\|\mathbf{V}\| \leqslant [\mathbf{V}] \leqslant \|\mathbf{V}\|, \quad \forall\, \mathbf{V} \in \mathbb{V}. \tag{12}$$

We know that $\mathbb{L}^2(\Omega) = \mathbb{H} \oplus \mathbb{H}^{\perp}$. Let P be the orthogonal projector in $\mathbb{L}^2(\Omega)$ onto the space $\mathbb{H}$. For $\mathbf{U}, \mathbf{V}, \mathbf{W}$ in $\mathbb{H}^1(\Omega)$, we define

$$a(\mathbf{V}, \mathbf{W}) = (\operatorname{grad} \mathbf{V}, \operatorname{grad} \mathbf{W}) = \sum_{i=1}^{3} (D_i \mathbf{V}, D_i \mathbf{W}),$$

$$d(\mathbf{V}, \mathbf{W}) = (\mathbf{F} \wedge \mathbf{V}, \mathbf{W})$$

and

$$b(\mathbf{U}, \mathbf{V}, \mathbf{W}) = ([\mathbf{U} \cdot \nabla]\mathbf{V}, \mathbf{W}) = \sum_{i,j=1}^{3} \int_\Omega u_i (D_i v_j) w_j \, \mathrm{d}\Omega.$$

Likewise, for p and q in $H^1(\Omega)$, we set

$$a_1(p, q) = (\operatorname{grad} p, \operatorname{grad} q) = \sum_{i=1}^{3} (D_i p, D_i q)$$

and

$$b_1(\mathbf{V}, p, q) = ([\mathbf{V} \cdot \nabla] p, q) = \sum_{i=1}^{3} \int_\Omega v_i (D_i p) q \, \mathrm{d}\Omega.$$

To get the perturbations $\mathbf{V}$, θ of the velocity and the temperature, we have to solve equations (4)–(6) in $\Omega \times [0, T]$, with the initial conditions (7) and (8) and the boundary conditions (9)–(11).

The variational formulation of this problem is the following (Problem I):

$$\mathbf{V}_0 \in \mathbb{V}, \quad \theta_0 \in H^1(\Omega), \quad \mathbf{f} \in \mathbb{L}^2(0, T; \mathbb{L}^2(\Omega)) \quad \text{and} \quad \phi \in L^2(\Omega) \text{ being given,}$$

we seek $\mathbf{V} \in \mathbb{L}^2(0, T; \mathbb{V})$ and $\theta \in L^2(0, T; H_0^1(\Omega))$, satisfying

$$\text{Problem I} \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{V}, \mathbf{W}) + \nu a(\mathbf{V}, \mathbf{W}) + b(\mathbf{V}_0, \mathbf{V}, \mathbf{W}) + b(\mathbf{V}, \mathbf{V}_0, \mathbf{W}) + b(\mathbf{V}, \mathbf{V}, \mathbf{W}) \\ \qquad + d(\mathbf{V}, \mathbf{W}) = (\mathbf{f}, \mathbf{W}), \quad \forall\, \mathbf{W} \in \mathbb{V}, & (13) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(\theta, q) + \nu' a_1(\theta, q) + b_1(\mathbf{V}_0, \theta, q) + b_1(\mathbf{V}, \theta_0, q) \\ \qquad + b_1(\mathbf{V}, \theta, q) = (\phi, q), \quad \forall\, q \in H_0^1(\Omega), & (14) \\ \mathbf{V}(x, y, z; 0) = \mathbf{0}, & (15) \\ \theta(x, y, z; 0) = 0. & (16) \end{cases}$$

Remark 3.1

Using a classical result from TEMAM, we can prove that, after modification on a set of measure zero, $\mathbf{V}$ is a continuous function from $[0, T]$ into $\mathbb{V}'$ and θ a continuous function from $[0, T]$ into $H^{-1}(\Omega)$. This result gives sense to initial conditions (15) and (16).

Proposition 3.1

- a and d are two bilinear continuous forms on $\mathbb{V} \times \mathbb{V}$.
- a_1 is a bilinear continuous form on $H_0^1(\Omega) \times H_0^1(\Omega)$.
- a is coercive on $\mathbb{V} \times \mathbb{V}$, a_1 is coercive on $H_0^1(\Omega) \times H_0^1(\Omega)$.

Indeed:

$$|a(\mathbf{V},\mathbf{W})| = |(\text{grad}\,\mathbf{V}, \text{grad}\,\mathbf{W})| \leqslant \|\mathbf{V}\|\,\|\mathbf{W}\|, \quad \forall\,\mathbf{V} \quad \text{and} \quad \mathbf{W} \in \mathbb{V},$$

$$|a_1(p,q)| = |(\text{grad}\,p, \text{grad}\,q)| \leqslant \|p\|\,\|q\|, \quad \forall\,p \quad \text{and} \quad q \in H_0^1(\Omega),$$

$$|d(\mathbf{V},\mathbf{W})| = |(\mathbf{F}\wedge\mathbf{V},\mathbf{W})| \leqslant |F|\,\|\mathbf{V}\|\,\|\mathbf{W}\|, \quad \forall\,\mathbf{V} \quad \text{and} \quad \mathbf{W}\in\mathbb{V},$$

and

$$a(\mathbf{V},\mathbf{V}) = [\mathbf{V}]^2 \geqslant C^2\|\mathbf{V}\|^2, \quad \forall\,\mathbf{V}\in\mathbb{V} \quad (\textit{idem.}\ \text{for}\ a_1).$$

Proposition 3.2

- b is a trilinear continuous form on $\mathbb{V}\times\mathbb{V}\times\mathbb{V}$, b_1 is a trilinear continuous form on $\mathbb{V}\times H_0^1(\Omega)\times H_0^1(\Omega)$.
- For every $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}\in\mathbb{V}$, p and $q\in H_0^1(\Omega)$, there is one constant C' such that:

  $|b(\mathbf{U},\mathbf{V},\mathbf{W})| \leqslant C'\|\mathbf{U}\|_{\mathbb{L}^4}\|\mathbf{V}\|_{\mathbb{H}^1}\|\mathbf{W}\|_{\mathbb{L}^4}$;

  $|b_1(\mathbf{U},p,q)| \leqslant C'\|\mathbf{U}\|_{\mathbb{L}^4}\|p\|_{H^1}\|q\|_{L^4}$.
- $b(\mathbf{U},\mathbf{V},\mathbf{V}) = 0, \quad \forall\,\mathbf{U}$ and $\mathbf{V}\in\mathbb{V}$,

  $b_1(\mathbf{U},q,q) = 0, \quad \forall\mathbf{U}\in\mathbb{V}, \quad q\in H_0^1(\Omega)$.
- $b(\mathbf{U},\mathbf{V},\mathbf{W}) = -b(\mathbf{U},\mathbf{W},\mathbf{V}), \quad \forall\,\mathbf{U},\mathbf{V},\mathbf{W}\in\mathbb{V}$,

  $b_1(\mathbf{U},p,q) = -b_1(\mathbf{U},q,p), \quad \forall\,\mathbf{U}\in\mathbb{V}, \quad \forall\,p,q\in \mathrm{H}_0^1(\Omega)$.

Lemma 3.1

For every function $v\in H^1(\Omega)$ (Ω being a sufficiently regular open set in $\mathbb{R}^3$), there is one constant C'' such that

$$\|v\|_{L^4(\Omega)} \leqslant C''\|v\|_{H^1(\Omega)}^{3/4}|v|_{L^2(\Omega)}^{1/4}.$$

Proofs for Propositions 3.1 and 3.2, as well as for Lemma 3.1, are given in Ref. [2].

Proposition 3.3

The variational Problem I has at least one solution:

$$\begin{cases} \mathbf{V}\in\mathbb{L}^2(0,T;\mathbb{V}) \\ \theta\in L^2(0,T;H_0^1(\Omega)). \end{cases}$$

Moreover, $\mathbf{V}\in\mathbb{L}^\infty(0,T;\mathbb{H})$ and $\theta\in L^\infty(0,T;L^2(\Omega))$. $\mathbf{V}$ is weakly continuous from $[0,T]$ into $\mathbb{H}$, θ weakly continuous from $[0,T]$ into $L^2(\Omega)$.

Proof. We set

$$\tilde a(\mathbf{V},\mathbf{W}) = \nu a(\mathbf{V},\mathbf{W}) + b(\mathbf{V}_0,\mathbf{V},\mathbf{W}) + \mathrm{b}(\mathbf{V},\mathbf{V}_0,\mathbf{W}) + d(\mathbf{V},\mathbf{W}),$$

$$\tilde a_1(\theta,q) = \nu' a_1(\theta,q) + b_1(\mathbf{V}_0,\theta,q)$$

and

$$n(\mathbf{V},q) = b_1(\mathbf{V},\theta_0,q).$$

Equations (13) and (14) can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{V},\mathbf{W}) + \tilde a(\mathbf{V},\mathbf{W}) + b(\mathbf{V},\mathbf{V},\mathbf{W}) = (\mathbf{f},\mathbf{W}), \quad \forall\,\mathbf{W}\in\mathbb{V}, \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\theta,q) + \tilde a_1(\theta,q) + n(\mathbf{V},q) + b_1(\mathbf{V},\theta,q) = (\phi,q), \quad \forall\,q\in H_0^1(\Omega). \tag{18}$$

We are going to prove the existence of the solution, using the Galerkin method. Let

$\mathbf{w}_1,\ldots,\mathbf{w}_m,\ldots$ be a free and total sequence in $\mathbb{V}$,

and

$$q_1, \ldots, q_m, \ldots \text{ be a free and total sequence in } H_0^1(\Omega).$$

For each fixed integer m, let V_m be the subspace of $\mathbb{V}$ generated by $\mathbf{w}_1, \ldots, \mathbf{w}_m$ and H_m the subspace of $H_0^1(\Omega)$ generated by $q_1, \ldots, q_m$.

We define an approximate solution of Problem I by

$$\mathbf{V}_m = \sum_{i=1}^{m} g_{im}(t)\mathbf{w}_i, \quad \theta_m = \sum_{j=1}^{m} \delta_{jm}(t) q_j.$$

$\mathbf{V}_m$ and θ_m satisfying the equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{V}_m, \mathbf{w}_k) + \tilde{a}(\mathbf{V}_m, \mathbf{w}_k) + b(\mathbf{V}_m, \mathbf{V}_m, \mathbf{w}_k) = (\mathbf{f}, \mathbf{w}_k), \quad k = 1, \ldots, m, & (19) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(\theta_m, q_k) + \tilde{a}_1(\theta_m, q_k) + n(\mathbf{V}_m, q_k) & \\ \qquad + b_1(\mathbf{V}_m, \theta_m, q_k) = (\phi, q_k), \quad k = 1, \ldots, m, & (20) \\ \mathbf{V}_m(x, y, z; 0) = \mathbf{0}, & (21) \\ \theta_m(x, y, z; 0) = 0. & (22) \end{cases}$$

Functions $g_{im}(t)$, $\delta_{jm}(t)$ are scalar functions defined on $[0, T]$. They have to check a non-linear differential system, which has a maximal solution on $[0, T]$.

To get the convergence of the approximate solution $(\mathbf{V}_m, \theta_m)$, we have to use the following lemmas.

Lemma 3.2

- The sequence $\mathbf{V}_m(t)$ is bounded in $\mathbb{L}^2(0, T; \mathbb{V})$ and in $\mathbb{L}^\infty(0, T; \mathbb{H})$.
- The sequence $\theta_m(t)$ is bounded in $L^2(0, T; H_0^1(\Omega))$ and in $L^\infty(0, T; L^2(\Omega))$.

Lemma 3.3

- The sequence $\mathbf{V}'_m(t)$ is bounded in $\mathbb{L}^1(0, T; \mathbb{V}')$.
- The sequence $\theta'_m(t)$ is bounded in $L^1(0, T; H^{-1}(\Omega))$.

Notation: $\mathbf{V}'_m(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}_m(t)$; $\theta'_m(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}\theta_m(t)$.

We now define the following functional spaces:

$$\mathcal{Y} = \{\mathbf{V} \in \mathbb{L}^2(0, T; \mathbb{V}); \quad \mathbf{V}' \in \mathbb{L}^1(0, T; \mathbb{V}')\};$$
$$\mathcal{Y}_1 = \{\theta \in L^2(0, T; H_0^1(\Omega)); \quad \theta' \in L^1(0, T; H^{-1}(\Omega))\}.$$

The injection of $\mathcal{Y}$ (respectively $\mathcal{Y}_1$) into $\mathbb{L}^2(0, T; \mathbb{H})$ [resp. into $L^2(0, T; L^2(\Omega))$] is compact [3].

We have proved (Lemmas 3.2 and 3.3) that the sequence $\mathbf{V}_m$ is bounded in $\mathcal{Y}$, the sequence θ_m bounded in $\mathcal{Y}_1$.

Therefore, we have got the following convergence results

—It is possible to extract from $(\mathbf{V}_m, \theta_m)$ a subsequence $(\mathbf{V}_{m'}, \theta_{m'})$ which converges towards $(\mathbf{V}^*, \theta^*)$ in $\mathbb{L}^2(0, T; \mathbb{H}) \times L^2(0, T; L^2(\Omega))$ strongly.

—On the other hand (cf. Lemma 3.2) $(\mathbf{V}_{m'}, \theta_{m'})$ converges towards $(\mathbf{V}^*, \theta^*)$ in $\mathbb{L}^2(0, T; \mathbb{V}) \times L^2(0, T; H_0^1(\Omega))$ weakly and in $\mathbb{L}^\infty(0, T; \mathbb{H}) \times L^\infty(0, T; L^2(\Omega))$ weak-star.

We have now to verify that $(\mathbf{V}^*, \theta^*)$ satisfy equations (13)–(16). Let $\psi(t)$ be a scalar function on $[0, T]$, of class $\mathscr{C}^1$, and such that $\psi(T) = 0$. We multiply equations (19) and (20) by $\psi(t)$ and integrate between 0 and T.

There is no problem for passing to the limit in the linear terms. In the non-linear terms, we apply the following result [3]:

Lemma 3.4

If $(\mathbf{V}_m, \theta_m)$ converges towards $(\mathbf{V}, \theta)$ in $\mathbb{L}^2(0, T; \mathbb{V}) \times L^2(0, T; H_0^1(\Omega))$ weakly and in $\mathbb{L}^2(0, T; \mathbb{H}) \times L^2(0, T; L^2(\Omega))$ strongly, then

$$\int_0^T b(\mathbf{V}_m(t), \mathbf{V}_m(t), \mathbf{W}(t))\, \mathrm{d}t \quad \text{converges towards} \quad \int_0^T b(\mathbf{V}(t), \mathbf{V}(t), \mathbf{W}(t))\, \mathrm{d}t,$$

$$\forall\, \mathbf{W} \in (\mathscr{C}^1(\Omega \times [0, T]))^3,$$

and

$$\int_0^T b_1(\mathbf{V}_m(t), \theta_m(t), q(t))\, \mathrm{d}t \quad \text{converges towards} \quad \int_0^T b_1(\mathbf{V}(t), \theta(t), q(t))\, \mathrm{d}t,$$

$$\forall\, q \in \mathscr{C}^1(\Omega \times [0, T]).$$

Thus, we get that $(\mathbf{V}^*, \theta^*)$ satisfy equations (13) and (14) in the distribution sense on $[0, T]$. It is now easy to prove that initial conditions $\mathbf{V}^*(0) = \mathbf{0}$, $\theta^*(0) = 0$ are verified.

We have proved the existence of at least one solution $(\mathbf{V}, \theta)$ for Problem I, satisfying

$$\begin{cases} \mathbf{V} \in \mathbb{L}^2(0, T; \mathbb{V}) \cap \mathbb{L}^\infty(0, T; \mathbb{H}) \\ \theta \in \mathrm{L}^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)). \end{cases}$$

We have proved moreover, that $\mathbf{V}$ is almost everywhere equal to a continuous function from $[0, T]$ into $\mathbb{V}'$, and θ almost everywhere equal to a continuous function from $[0, T]$ into $H^{-1}(\Omega)$.

This implies, using a result proved by TEMAM, that $\mathbf{V}$ is weakly continuous from $[0, T]$ into $\mathbb{H}$ and θ weakly continuous from $[0, T]$ into $L^2(\Omega)$, that is to say that

$$\forall\, \mathbf{W} \in \mathbb{H}, \quad \text{the application: } t \to (\mathbf{V}(t), \mathbf{W}) \quad \text{is continuous;}$$

$$\forall\, q \in L^2(\Omega), \quad \text{the application: } t \to (\theta(t), q) \quad \text{is continuous.}$$

The uniqueness of the solution cannot be obtained in these functional spaces. In the following proposition, we prove that a more regular solution is unique if it exists, but this time we cannot assure the existence.

Proposition 3.4

There is at most one solution of Problem I, such that

$$\begin{cases} \mathbf{V} \in \mathbb{L}^2(0, T; \mathbb{V}) \cap \mathbb{L}^\infty(0, T; \mathbb{H}) \\ \mathbf{V} \in \mathbb{L}^8(0, T; \mathbb{L}^4(\Omega)) \\ \theta \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) \\ \theta \in \mathrm{L}^8(0, T; L^4(\Omega)). \end{cases}$$

$\mathbf{V}$ is then continuous from $[0, T]$ into $\mathbb{H}$, θ continuous from $[0, T]$ into $L^2(\Omega)$.

Proof. If $\mathbf{V} \in \mathbb{L}^8(0, T; \mathbb{L}^4(\Omega))$, $\theta \in L^8(0, T; L^4(\Omega))$, we have the following result:

$$\mathbf{V} \in \mathbb{L}^2(0, T; \mathbb{V}), \quad \mathbf{V}' \in \mathbb{L}^2(0, T; \mathbb{V}'),$$

$$\theta \in \mathbb{L}^2(0, T; H_0^1(\Omega)), \quad \theta' \in L^2(0, T; H^{-1}(\Omega)).$$

This implies that $\mathbf{V}$ (resp. θ) is almost everywhere equal to a continuous function from $[0, T]$ into $\mathbb{H}$ [resp. into $L^2(\Omega)$].

Let us suppose now that there are two solution $(\mathbf{V}_1, \theta_1)$ and $(\mathbf{V}_2, \theta_2)$ satisfying the assumptions of Proposition 3.4. Set

$$\mathbf{D} = \mathbf{V}_2 - \mathbf{V}_1, \quad \gamma = \theta_2 - \theta_1.$$

$(\mathbf{D}, \gamma)$ have to verify the equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} |\mathbf{D}|^2 + 2\tilde{a}(\mathbf{D}, \mathbf{D}) = 2b(\mathbf{D}, \mathbf{D}, \mathbf{V}_2) & (23) \\ \dfrac{\mathrm{d}}{\mathrm{d}t} |\gamma|^2 + 2\tilde{a}_1(\gamma, \gamma) + 2n(\mathbf{D}, \gamma) = 2b(\mathbf{D}, \gamma, \theta_2). & (24) \end{cases}$$

Equation (23) can also be written

$$\frac{\mathrm{d}}{\mathrm{d}t} |\mathbf{D}|^2 + 2\nu[\mathbf{D}]^2 = 2b(\mathbf{D}, \mathbf{D}, \mathbf{V}_2 + \mathbf{V}_0).$$

Thus, we get the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} |\mathbf{D}(t)|^2 + 2\nu C^2 \| \mathbf{D}(t) \|^2 \leqslant 2 | b(\mathbf{D}(t), \mathbf{D}(t), \mathbf{V}_2^*(t)) |,$$

where

$$\mathbf{V}_2^*(t) = \mathbf{V}_2(t) + \mathbf{V}_0;$$

$$\begin{aligned} 2 | b(\mathbf{D}, \mathbf{D}, \mathbf{V}_2^*) | &\leqslant 2C' \| \mathbf{D}(t) \|_{\mathbb{L}^4(\Omega)} \| \mathbf{D}(t) \| \; \| \mathbf{V}_2^*(t) \|_{\mathbb{L}^4(\Omega)} \\ &\leqslant 2C'C'' \| \mathbf{D}(t) \|^{7/4} |\mathbf{D}(t)|^{1/4} \| \mathbf{V}_2^*(t) \|_{\mathbb{L}^4(\Omega)}. \end{aligned}$$

Let us apply now Young's inequality. We can prove that there is one constant C_1, such that

$$\frac{\mathrm{d}}{\mathrm{d}t} |\mathbf{D}(t)|^2 \leqslant C_1 |\mathbf{D}(t)|^2 \; \| \mathbf{V}_2^*(t) \|^8_{\mathbb{L}^4(\Omega)}. \tag{25}$$

$\mathbf{V}_0$ is given in $\mathbb{V}$, independent of time; $\mathbf{V}_2(t) \in \mathbb{L}^8(0, T; \mathbb{L}^4(\Omega))$.

Thus, the function $t \to \| \mathbf{V}_2^*(t) \|^8_{\mathbb{L}^4(\Omega)}$ can be integrated, and from inequality (25) we can deduce

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ |\mathbf{D}(t)|^2 \exp\left(-\frac{1}{C_1} \int_0^t \| \mathbf{V}_2^*(\tau) \|^8_{\mathbb{L}_4(\Omega)} \, \mathrm{d}\tau \right) \right\} \leqslant 0. \tag{26}$$

Integrating relation (26), we obtain, since $\mathbf{D}(0) = \mathbf{0}$, that

$$|\mathbf{D}(t)|^2 \leqslant 0, \quad \forall\, t \in [0, T].$$

So, we have $\mathbf{V}_1 = \mathbf{V}_2$.

Let us prove now that $\theta_1 = \theta_2$. $\gamma = \theta_2 - \theta_1$ has to satisfy equation (24) but, since $\mathbf{D} = \mathbf{0}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} |\gamma(t)|^2 + 2\tilde{a}_1(\gamma, \gamma) = 0 = \frac{\mathrm{d}}{\mathrm{d}t} |\gamma(t)|^2 + 2\nu'[\gamma(t)]^2.$$

This implies

$$\frac{\mathrm{d}}{\mathrm{d}t} |\gamma(t)|^2 + 2\nu' C^2 \| \gamma(t) \|^2 \leqslant 0.$$

Integrating this inequality, we get

$$|\gamma(t)|^2 \leqslant 0, \quad \forall\, t \in [0, T].$$

So, we have $\theta_1 = \theta_2$.

4. THE OPERATORS $\tilde{A}$ AND $\tilde{A}_1$

We have defined the two bilinear forms

$$\tilde{a}(\mathbf{V}, \mathbf{W}) = \nu a(\mathbf{V}, \mathbf{W}) + b(\mathbf{V}_0, \mathbf{V}, \mathbf{W}) + b(\mathbf{V}, \mathbf{V}_0, \mathbf{W}) + d(\mathbf{V}, \mathbf{W})$$

and

$$\tilde{a}_1(\theta, q) = \nu' a_1(\theta, q) + b_1(\mathbf{V}_0, \theta, q).$$

Let $\tilde{A}$ and $\tilde{A}_1$ be the operators associated with these forms. P being the orthogonal projector in $\mathbb{L}^2(\Omega)$ onto $\mathbb{H}$, we have

$$\tilde{A}\mathbf{V} = \nu A\mathbf{V} + L\mathbf{V}; \quad \tilde{A}_1\theta = \nu' A_1\theta + M\theta,$$

where

$$A\mathbf{V} = -P\Delta\mathbf{V}; \quad L\mathbf{V} = P([\mathbf{V}_0\cdot\nabla]\mathbf{V} + [\mathbf{V}\cdot\nabla]\mathbf{V}_0 + \mathbf{F}\wedge\mathbf{V})$$

$$A_1\theta = -\Delta\theta; \quad M\theta = [\mathbf{V}_0\cdot\nabla]\theta.$$

L being a linear continuous application from $\mathbb{V}$ onto $\mathbb{H}$, there is one constant C_1, such that

$$|L\mathbf{U}| \leqslant C_1\|\mathbf{U}\|. \tag{27}$$

Using the previous notations, $\mathbf{V}$ and θ, solutions of Problem I have to satisfy the following differential equations:

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} + \tilde{A}\mathbf{V} = \tilde{\mathbf{f}}(t) \tag{28}$$

$$\text{in } \mathbb{H}$$

$$\mathbf{V}(0) = \mathbf{0} \tag{29}$$

and

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} + \tilde{A}_1\theta = \tilde{g}(t) \tag{30}$$

$$\text{in } L^2(\Omega),$$

$$\theta(0) = 0 \tag{31}$$

where

$$\tilde{\mathbf{f}}(t) = P(\mathbf{f}(t) - [\mathbf{V}\cdot\nabla]\mathbf{V})$$

and

$$\tilde{g}(t) = \phi - [\mathbf{V}\cdot\nabla]\theta_0 - [\mathbf{V}\cdot\nabla]\theta.$$

N.B. Hereafter, $\mathbf{f}(t)$ will denote the orthogonal projection of $\mathbf{f}(t)$ onto $\mathbb{H}(P(\mathbf{f}(t)))$.
The domain of the operator $\tilde{A}$ is defined by

$$D(\tilde{A}) = \left\{\mathbf{V} \in \mathbb{H}^2(\Omega)\cap\mathbb{V};\ \frac{\partial}{\partial n}(\mathbf{V}_t) = 0 \quad \text{on } \Gamma''\right\}.$$

The domain of the operator $\tilde{A}_1$ is defined by

$$D(\tilde{A}_1) = H_0^1(\Omega)\cap H^2(\Omega) = D(A_1).$$

$\mathbf{V}_0$ and θ_0 are given. They are real values.
Hereafter, functional spaces $\mathbb{H}$, $\mathbb{V}$, $H_0^1(\Omega)$ and $L^2(\Omega)$, will be considered as complex spaces.

Proposition 4.1

- The operator $\tilde{A}$ is V-coercive.
- The operator $\tilde{A}_1$ is V-elliptic.

Indeed, applying properties of the trilinear forms b and b_1, we get

$$\mathrm{Re}(\tilde{a}(\mathbf{U},\mathbf{U})) \geqslant K_1\|\mathbf{U}\|^2 - K_2|\mathbf{U}|^2, \quad \forall\,\mathbf{U}\in\mathbb{V},$$

and

$$\mathrm{Re}(\tilde{a}_1(q,q)) \geqslant K_3\|q\|^2, \quad \forall\, q \in H_0^1(\Omega),$$

where K_1, K_2 and K_3 are positive constants.

Corollary 4.1

- For $\gamma\in\mathbb{R}$, $\gamma \leqslant -K_2$, the operator $\tilde{A} - \gamma I$ is V-elliptic.

- $(\tilde{A} - \gamma I)$ is an isomorphism from $D(\tilde{A})$ onto $\mathbb{H}$.
- $D(\tilde{A})$ is dense in $\mathbb{V}$ and in $\mathbb{H}$.
- The operator $(\tilde{A} - \gamma I)$, and hence the operator $\tilde{A}$ are closed operators.

Corollary 4.2

- The operator $\tilde{A}_1$ is an isomorphism from $D(\tilde{A}_1)$ onto $L^2(\Omega)$
- $D(\tilde{A}_1)$ is dense in $H_0^1(\Omega)$ and in $L^2(\Omega)$
- The operator $\tilde{A}_1$ is closed.

Proposition 4.2

The operator $-\tilde{A}$ (resp. $-\tilde{A}_1$) generates in $\mathbb{H}$ [resp. in $L^2(\Omega)$] a strongly continuous semi-group, denoted by $\exp(-t\tilde{A})$ [resp. $\exp(-t\tilde{A}_1)$].

Proof. Proposition 4.2 is a corollary from the Hille–Yoshida theorem. The operator $\tilde{A}$ is unbounded, closed in $\mathbb{H}$; its domain $D(\tilde{A})$ is dense in $\mathbb{H}$.

For $\gamma \leqslant -K_2$, the operator $\tilde{A} - \gamma I$ is an isomorphism from $D(\tilde{A})$ onto $\mathbb{H}$, and we can prove that, for $\gamma \in \mathbb{R}$, $\gamma < -C_2^2/4$, we have

$$\|(\tilde{A} - \gamma I)^{-1}\|_{\mathbb{H}\to\mathbb{H}} \leqslant \frac{1}{-\gamma - \dfrac{C_2^2}{4}}. \tag{32}$$

[We set $C_2 = C_1/C\sqrt{\nu}$, C and C_1 being the positive constants introduced in expressions (12) and (27).]

All the assumptions of the Hille–Yoshida theorem are satisfied by the operator $\tilde{A}$. Then $-\tilde{A}$ generates a strongly continuous semi-group in $\mathbb{H}$, denoted by $\exp(-t\tilde{A})$, and satisfying

$$\|\exp(-t\tilde{A})\| \leqslant \exp\left(\frac{C_2^2}{4}t\right). \tag{33}$$

A similar proof is valid for the operator $\tilde{A}_1$, and we get the estimation

$$\|\exp(-t\tilde{A}_1)\| \leqslant \exp(-\nu' C^2 t). \tag{34}$$

Proposition 4.3

The eigenvalues λ of the operator $\tilde{A}$ $(\lambda = \gamma + i\mu)$, are located inside the parabolic curve of the equation

$$\gamma = \mu^2 \frac{1}{4C_2^2} - C_2^2. \tag{35}$$

Proposition 4.4

- Let $\lambda \in \mathbb{C}$, $\lambda = \gamma + i\mu$.
- For every fixed $\gamma (\gamma \in \mathbb{R})$, we have

$$\|(\tilde{A} - \lambda I)^{-1}\|_{\mathbb{H}\to\mathbb{H}} = 0\left(\frac{2}{\mu}\right) \quad \text{when} \quad \mu \to \pm\infty.$$

Proposition 4.5

The eigenvalues λ_1 of $\tilde{A}_1$ are real values, and such that

$$0 < \nu' C^2 \leqslant \lambda_1.$$

Proofs for Propositions 4.3–4.5 may be found in Ref. [2].

5. THE LINEARIZED PROBLEM

If we neglect the non-linear terms, $\mathbf{V}$ and θ have to satisfy the following differential equations:

$$\left\{\begin{array}{ll} \dfrac{d\mathbf{V}}{dt} + \tilde{A}\mathbf{V} = \mathbf{f}(t) & (36) \\ & \text{in } \mathbb{H} \\ \mathbf{V}(0) = \mathbf{0} & (37) \end{array}\right.$$

and

$$\begin{cases} \dfrac{d\theta}{dt} + \tilde{A}_1\theta = g(t) & \\ & \text{in } L^2(\Omega) \\ \theta(0) = 0 & \end{cases} \tag{38, 39}$$

where

$$\mathbf{f}(t) = P(\mathbf{f}(t))$$

and

$$g(t) = \phi - [\mathbf{V}\cdot\nabla]\theta_0.$$

According to Proposition 4.2, weak solutions of the differential equations (36) and (37) can be written

$$\mathbf{V}(t) = \int_0^t \exp(-\tau\tilde{A})\mathbf{f}(t-\tau)\,d\tau. \tag{40}$$

Weak solutions of the differential equations (38) and (39) are given by

$$\theta(t) = \int_0^t \exp(-\tau\tilde{A}_1)g(t-\tau)\,d\tau. \tag{41}$$

Proposition 5.1

If all the eigenvalues of the operator $\tilde{A}$ have positive real parts, then

$$\|\mathbf{V}(t)\|^2 \leqslant C_6 \int_0^t \exp(-2\gamma'\tau)|\mathbf{f}(t-\tau)|^2\,d\tau, \tag{42}$$

where C_6 is a positive constant and γ' a real number satisfying $0 < \gamma' < \mathrm{Re}(\lambda)$ for every λ eigenvalue of $\tilde{A}$.

In order to prove inequality (42), we need the following two lemmas.

Lemma 5.1

For $0 < t \leqslant 1$,

$$\|\exp(-t\tilde{A})\|^2_{\mathbf{H}\to\mathbf{V}} \leqslant \frac{C_3}{t} \tag{43}$$

(C_3 = positive constant).

Proof:

$$\|\exp(-t\tilde{A})\|_{\mathbf{H}\to\mathbf{V}} = \sup_{\mathbf{V}^*\in\mathbf{H}} \frac{\|\exp(-t\tilde{A})\mathbf{V}^*\|}{|\mathbf{V}^*|};$$

$\mathbf{V} = \exp(-t\tilde{A})\mathbf{V}^*$ is a weak solution of the differential problem

$$\begin{cases} \dfrac{d\mathbf{V}}{dt} + \tilde{A}\mathbf{V} = \mathbf{0} & (44) \\ \mathbf{V}(0) = \mathbf{V}^*. & (45) \end{cases}$$

Then, applying properties (12) and (27), we get the inequality

$$\frac{d}{dt}[\exp(-C_2^2 t)|\mathbf{V}|^2] + \nu C^2 \exp(-C_2^2 t)\|\mathbf{V}\|^2 \leqslant 0. \tag{46}$$

Integrating inequality (46) between 0 and t, with respect to the initial condition (45), we can prove that

$$\int_0^t \|\mathbf{V}(\tau)\|^2\,d\tau \leqslant \frac{\exp(C_2^2)}{\nu C^2}|\mathbf{V}^*|^2, \quad \text{for} \quad 0 \leqslant t \leqslant 1. \tag{47}$$

From equation (44), we also obtain the inequality

$$\| \mathbf{V}(t) \|^2 \leqslant \frac{1}{C^2} \exp(\tfrac{1}{2} C_2^2) \| \mathbf{V}(t') \|^2, \quad \text{for} \quad 0 \leqslant t' < t \leqslant 1. \tag{48}$$

Integrate now inequality (48) between 0 and t. According to inequality (47), there is one constant C_3, such that

$$t \| \mathbf{V}(t) \|^2 \leqslant C_3 |\mathbf{V}^*|^2, \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

Since $\mathbf{V}(t) = \exp(-t\tilde{A})\ \mathbf{V}^*$, this implies the property (43).

Lemma 5.2

If all the eigenvalues of $\tilde{A}$ have positive real parts, and if γ is a real number satisfying $0 < \gamma < \mathrm{Re}(\lambda)$ for every λ eigenvalue of $\tilde{A}$, then

$$\| \exp(-t\tilde{A}) \|_{\mathbf{H} \to \mathbf{H}} \leqslant C_4 \exp(-\gamma t) \tag{49}$$

(C_4 = positive constant).

Proof: for $t > 0$, $\exp(-t\tilde{A})$ can be represented by

$$\exp(-t\tilde{A}) = \lim_{\omega \to \infty} \frac{-1}{2i\pi} \int_{\gamma - i\omega}^{\gamma + i\omega} \exp(-t\lambda)(\tilde{A} - \lambda I)^{-1}\, \mathrm{d}\lambda,$$

where $\lambda \in \mathbb{C}$, $\lambda = \gamma + i\mu$; $\gamma \in \mathbb{R}$ is fixed, $\mu \in \mathbb{R}$ and $-\omega \leqslant \mu \leqslant +\omega$.

Integrating by parts, we get

$$\exp(-t\tilde{A}) = \lim_{\omega \to \infty} \left[\!\left[\frac{1}{2i\pi} \frac{\exp(-\gamma t)}{t} \{[\tilde{A} - (\gamma + i\omega)I]^{-1} \exp(-i\omega t) - [\tilde{A} - (\gamma - i\omega)I]^{-1} \exp(i\omega t)\} \right]\!\right]$$
$$+ \frac{\exp(-\gamma t)}{t} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-i\mu t)[\tilde{A} - (\gamma + i\mu)I]^{-2}\, \mathrm{d}\mu.$$

We apply the result of Proposition 4.4, and we get that

$$\| \exp(-t\tilde{A}) \|_{\mathbf{H} \to \mathbf{H}} \leqslant \frac{1}{2\pi} \frac{\exp(-\gamma t)}{t} \int_{-\infty}^{+\infty} \| [\tilde{A} - (\gamma + i\mu)I]^{-1} \|^2\, \mathrm{d}\mu.$$

The value of the integral is finite. Therefore there is one constant C_4, such that

$$\| \exp(-t\tilde{A}) \|_{\mathbf{H} \to \mathbf{H}} \leqslant C_4 \frac{\exp(-\gamma t)}{t} \leqslant k \exp(-\gamma t), \quad \text{for} \quad t \geqslant 1.$$

The inequality is still true for $0 \leqslant t \leqslant 1$. to show it, we apply estimation (33).

Proof of Proposition 5.1

The weak solutions of equations (36) and (37) can be written

$$\mathbf{V}(t) = \int_0^t \exp(-\tau\tilde{A}) \mathbf{f}(t - \tau)\, \mathrm{d}\tau.$$

For $t > 1$, we set

$$\mathbf{V}_1(t) = \int_0^1 \exp(-\tau\tilde{A}) \mathbf{f}(t - \tau)\, \mathrm{d}\tau \quad \text{and} \quad \mathbf{V}_2(t) = \int_1^t \exp(-\tau\tilde{A}) \mathbf{f}(t - \tau)\, \mathrm{d}\tau.$$

Majoration of $\| \mathbf{V}_1(t) \|^2$. From equation (36), we get the estimation

$$\frac{\mathrm{d}}{\mathrm{d}t} ([\mathbf{V}]^2 \exp(-C_2^2 t)) \leqslant \frac{1}{\nu} |\mathbf{f}|^2 \exp(-C_2^2 t).$$

After integration, this implies that, for $0 \leqslant t \leqslant 1$,

$$\| \mathbf{V}(t) \|^2 = \left\| \int_0^t \exp[-(t - \tau)\tilde{A}] \mathbf{f}(\tau)\, \mathrm{d}\tau \right\|^2 \leqslant \frac{\exp(C_2^2)}{\nu C^2} \int_0^t |\mathbf{f}(\tau)|^2\, \mathrm{d}\tau. \tag{50}$$

This majoration is still true for $t = 1$. Therefore,

$$\|\mathbf{V}_1(t)\|^2 \leqslant \frac{\exp(C_2^2)}{\nu C^2} \int_0^1 |\mathbf{f}(t-\tau)|^2 \,\mathrm{d}\tau. \tag{51}$$

Majoration of $\|\mathbf{V}_2(t)\|^2$. For $t > 1$,

$$\exp(-t\tilde{A}) = \exp(-\tilde{A})\cdot\exp[-(t-1)\tilde{A}]$$

and

$$\|\exp(-t\tilde{A})\|_{\mathbf{H}\to\mathbf{V}} \leqslant \|\exp(-\tilde{A})\|_{\mathbf{H}\to\mathbf{V}}\|\exp[-(t-1)\tilde{A}]\|_{\mathbf{H}\to\mathbf{H}}.$$

We apply now the results of Lemma 5.1 and 5.2. So, we have for $t \geqslant 1$,

$$\|\exp(-t\tilde{A})\|_{\mathbf{H}\to\mathbf{V}} \leqslant C_5\exp(-\gamma t) \quad [C_5 = \sqrt{C_3}C_4\exp(\gamma)]$$

and

$$\|\mathbf{V}_2(t)\| = \left\| \int_1^t \exp(-\tau\tilde{A})\mathbf{f}(t-\tau)\,\mathrm{d}\tau \right\| \leqslant C_5 \int_1^t \exp(-\gamma\tau)|\mathbf{f}(t-\tau)|\,\mathrm{d}\tau.$$

Let γ' be a real number satisfying $0 < \gamma' < \gamma$, and set $\delta = \gamma - \gamma'$. We get

$$\|\mathbf{V}_2(t)\|^2 \leqslant C_3C_4^2 \frac{\exp(2\gamma')}{2\delta} \int_1^t \exp(-2\gamma'\tau)|\mathbf{f}(t-\tau)|^2\,\mathrm{d}\tau. \tag{52}$$

Majoration (51) for $\mathbf{V}_1(t)$ can also be written

$$\|\mathbf{V}_1(t)\|^2 \leqslant \frac{\exp(C_2^2)}{\nu C^2}\exp(2\gamma') \int_0^1 \exp(-2\gamma'\tau)|\mathbf{f}(t-\tau)|^2\,\mathrm{d}\tau \quad (0<\gamma'<\gamma).$$

We have set $\mathbf{V}(t) = \mathbf{V}_1(t) + \mathbf{V}_2(t)$, for $t > 1$. Therefore, there is one constant C_6, such that

$$\|\mathbf{V}(t)\|^2 \leqslant C_6 \int_0^t \exp(-2\gamma'\tau)|\mathbf{f}(t-\tau)|^2\,\mathrm{d}\tau.$$

It is easy to see that this majoration is still true for $t \leqslant 1$. We can then apply estimation (50).

Proposition 5.2

The solution $\theta(t)$ of the differential equations (38) and (39) satisfies the following estimation:

$$\|\theta(t)\|^2 \leqslant C_6' \int_0^t \exp(-2\gamma\tau)|g(t-\tau)|^2\,\mathrm{d}t, \tag{53}$$

where C_6' is a positive constant and γ a real number, such that $0 < \gamma < \nu' C^2$. This proposition can be proved just as Proposition 5.1.

6. STABILITY OF THE NON-LINEAR PROBLEM

We come back now to the complete non-linear problem. $\mathbf{V}$ and θ have to satisfy the differential equations (28)–(31).

Proposition 6.1

If all the eigenvalues of the operator $\tilde{A}$ have positive real parts, then

$$\|\mathbf{V}(t)\|^2 + \nu\int_0^t |A\mathbf{V}(\tau)|^2\,\mathrm{d}\tau \leqslant C_{10}\int_0^t |\tilde{\mathbf{f}}(\tau)|^2\,\mathrm{d}\tau \tag{54}$$

and moreover,

$$\|\mathbf{V}(t)\|^2 + \nu\int_0^t \exp[-2\gamma(t-\tau)]|A\mathbf{V}(\tau)|^2\,\mathrm{d}\tau \leqslant C_{10}\int_0^t \exp[-2\gamma(t-\tau)]|\tilde{\mathbf{f}}(\tau)|^2\,\mathrm{d}\tau, \tag{55}$$

where C_{10} is a positive constant, and γ a real number such that $0 < \gamma < \mathrm{Re}(\lambda)$ for every eigenvalue λ of the operator $\tilde{A}$.

Proof. From equation (28), we get the estimation

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{V}]^2 + \nu |A\mathbf{V}|^2 \leqslant \frac{2C_1^2}{\nu}\|\mathbf{V}\|^2 + \frac{2}{\nu}|\tilde{\mathbf{f}}|^2$$

and, after integration,

$$[\mathbf{V}(t)]^2 + \nu \int_0^t |A\mathbf{V}(\tau)|^2\,\mathrm{d}\tau \leqslant 2\frac{C_1^2}{\nu}\int_0^t \|\mathbf{V}(\tau)\|^2\,\mathrm{d}\tau + \frac{2}{\nu}\int_0^t |\tilde{\mathbf{f}}(\tau)|^2\,\mathrm{d}\tau. \tag{56}$$

We apply now the result proved in Proposition 5.1.

$$\|\mathbf{V}(t)\|^2 \leqslant C_6 \int_0^t \exp(-2\gamma\tau)|\tilde{\mathbf{f}}(t-\tau)|^2\,\mathrm{d}\tau,$$

which implies the two estimations

$$\int_0^t \|\mathbf{V}(\tau)\|^2\,\mathrm{d}\tau \leqslant \frac{C_6}{2\gamma}\int_0^t |\tilde{\mathbf{f}}(\tau)|^2\,\mathrm{d}\tau. \tag{57}$$

and

$$\|\mathbf{V}(t)\|^2 \leqslant C_6 \int_0^t |\tilde{\mathbf{f}}(\tau)|^2\,\mathrm{d}\tau. \tag{58}$$

From expressions (56)–(58), we can conclude that there is one constant C_{10}, such that

$$\|\mathbf{V}(t)\|^2 + \nu\int_0^t |A\mathbf{V}(\tau)|^2\,\mathrm{d}\tau \leqslant C_{10}\int_0^t |\tilde{\mathbf{f}}(\tau)|^2\,\mathrm{d}\tau.$$

This estimation can be improved in order to get expression (55). $\mathbf{V}$ satisfying the differential equation (28), we set $\mathbf{W} = \exp(\gamma t)\mathbf{V}$, where γ is a real number such that $0 < \gamma < \mathrm{Re}(\lambda)$ for every eigenvalue λ of $\tilde{A}$. $\mathbf{W}$ is solution of the differential equation

$$\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} + (\tilde{A} - \gamma I)\mathbf{W} = \exp(\gamma t)\tilde{\mathbf{f}}(t).$$

Set

$$A^* = \tilde{A} - \gamma I, \quad \mathbf{f}^*(t) = \exp(\gamma t)\tilde{\mathbf{f}}(t).$$

All the eigenvalues of the operator A^* have positive real parts. Hence, we can apply relation (54)

$$\|\mathbf{W}(t)\|^2 + \nu\int_0^t |A\mathbf{W}(\tau)|^2\,\mathrm{d}\tau \leqslant C_{10}\int_0^t |\mathbf{f}^*(\tau)|^2\,\mathrm{d}\tau,$$

which can also be written

$$\|\mathbf{V}(t)\|^2 + \nu\int_0^t \exp[-2\gamma(t-\gamma)]|A\mathbf{V}(\tau)|^2\,\mathrm{d}\tau \leqslant C_{10}\int_0^t \exp[-2\gamma(t-\gamma)]|\tilde{\mathbf{f}}(\tau)|^2\,\mathrm{d}\tau.$$

Proposition 6.2

$\theta(t)$ Satisfies the following estimations:

$$\|\theta(t)\|^2 + \nu'\int_0^t |A_1\theta(\tau)|^2\,\mathrm{d}\tau \leqslant C'_{10}\int_0^t |\tilde{g}(\tau)|^2\,\mathrm{d}\tau \tag{59}$$

and moreover,

$$\|\theta(t)\|^2 + \nu'\int_0^t \exp[-2\gamma(t-\tau)]|A_1\theta(\tau)|^2\,\mathrm{d}\tau \leqslant C'_{10}\int_0^t \exp[-2\gamma(t-\tau)]|\tilde{g}(\tau)|^2\,\mathrm{d}\tau, \tag{60}$$

where C'_{10} is a positive constant and γ a real number such that $0 < \gamma < \nu' C^2$.

The proof is the same as that of the previous proposition, taking into account estimation (53) obtained in Proposition 5.2.

As we said in Section 2, the mean situation $(\mathbf{V}_0, \theta_0)$ is given. These values either verify the linearized equations, or are solutions of the complete non-linear stationary problem. We are now going to study these two cases separately in order to get stability results.

6.1. Linearized Stationary Problem

The perturbation $(\mathbf{V}, \theta)$ has to satisfy the differential equations (28)–(31) where:

$$\tilde{\mathbf{f}}(t) = \mathbf{f}(t) - P([\mathbf{V}\cdot\nabla]\mathbf{V}); \quad \tilde{g}(t) = \phi - [\mathbf{V}\cdot\nabla]\theta_0 - [\mathbf{V}\cdot\nabla]\theta;$$

$$\mathbf{f}(t) = P(\mathbf{T}(t) - [\mathbf{V}_0\cdot\nabla]\mathbf{V}_0); \quad \phi = -[\mathbf{V}_0\cdot\nabla]\theta_0 + \nu'\Delta\theta_0;$$

$\mathbf{T}(t)$ is the perturbation of the wind-stress.

Proposition 6.3

If all the eigenvalues of the operator $\tilde{A}$ have positive real parts, and if $\mathbf{f}(t)$ remains small enough,

$$|\mathbf{f}(t)|^2 \leqslant K < \frac{\sqrt{\nu}}{kC_{10}^2}\left(\sqrt{\frac{2\gamma}{3}}\right)^3$$

then, $\mathbf{V}(t)$ and $\theta(t)$ remain bounded as $t \to \infty$, and we get the following estimations:

$$\|\mathbf{V}(t)\|^2 \leqslant K_1 < \frac{\sqrt{\nu}}{kC_{10}}\sqrt{\frac{2\gamma}{3}} \tag{61}$$

and

$$\|\theta(t)\|^2 \leqslant C_2'|\phi|^2 + C_3'K_1. \tag{62}$$

[C_2' and C_3' are positive constants, γ is a real number such that $0 < \gamma < \inf(\mathrm{Re}(\lambda), \nu' C^2)$ for every eigenvalue λ of $\tilde{\mathrm{A}}$.]

To prove this proposition, we have to use the following three lemmas.

Lemma 6.1[4]

If $\alpha(t)$ is a measurable function satisfying

$$\alpha(t) \leqslant A + \int_0^t F(\alpha, \tau)\,\mathrm{d}\tau,$$

where F is a function continuous in τ, Lipschitz continuous in α, monotone increasing in α, and A is a constant, then

$$\alpha(t) \leqslant \beta(t),$$

$\beta(t)$ satisfies the differential equation

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = F(\beta, t)$$

$$\beta(0) = A.$$

Lemma 6.2

For every $\mathbf{V} \in D(A)$,

$$|B\mathbf{V}|^2 \leqslant k\,\|\mathbf{V}\|^3\,|A\mathbf{V}| \tag{63}$$

where k is a positive constant, and $B\mathbf{V} = P([\mathbf{V}\cdot\nabla]\mathbf{V})$.

Proof. Applying Hölder's inequality, we can prove that

$$|B\mathbf{V}|^2 = |P([\mathbf{V}\cdot\nabla]\mathbf{V})|^2 \leqslant \|\mathbf{V}\|^2_{L^6(\Omega)} \times [\mathbf{V}] \times \left\{\int_\Omega \left[\sum_{i,j}(D_i v_j)^2\right]^3 \mathrm{d}x\right\}^{1/6}.$$

According to the Sobolev inclusion theorem, there is one constant k_1, such that

$$\|\mathbf{V}\|_{L^6(\Omega)} \leqslant k_1\|\mathbf{V}\|_{H^1(\Omega)} \tag{64}$$

and

$$\left\{\int_\Omega \left[\sum_{i,j}(D_i v_j)^2\right]^3 \mathrm{d}x\right\}^{1/6} \leqslant k_2 \sum_{i,j}\|D_i v_j\| \leqslant k_2\|\mathbf{V}\|_{H^2(\Omega)}$$

[we still apply inequality (64)].

We have the following estimation:

$$|B\mathbf{V}|^2 \leqslant k_1^2 k_2 \|\mathbf{V}\|_{\mathbb{H}^1}^3 \|\mathbf{V}\|_{\mathbb{H}^2}. \tag{65}$$

The operator $A = -P\Delta$ is a self-adjoint, V-elliptic operator, of domain $D(A) \subset \mathbb{H}^2(\Omega)$. A is an isomorphism from $D(A)$ onto $\mathbb{H}$. There is one constant k_3, such that

$$\|\mathbf{V}\|_{\mathbb{H}^2} \leqslant k_3 |A\mathbf{V}|, \quad \forall\, \mathbf{V} \in D(A). \tag{66}$$

Let us apply estimation (66) in inequality (65). There is one constant k such that

$$|B\mathbf{V}|^2 \leqslant k\, \|\mathbf{V}\|^3 |A\mathbf{V}|. \tag{63}$$

Lemma 6.3

For every $\theta \in \mathrm{D(A_1)}$,

$$|B_1(\mathbf{V}, \theta)|^2 \leqslant k' \|\mathbf{V}\|^2 \|\theta\| \, |A_1\theta|, \tag{67}$$

where k' is a positive constant and $B_1(\mathbf{V}, \theta) = [\mathbf{V} \cdot \nabla]\theta$.

Proof. Applying Hölder's inequality, we can prove that

$$|B_1(\mathbf{V}, \theta)|^2 \leqslant \|\mathbf{V}\|_{\mathbb{L}^6(\Omega)}^2 \|\theta\| \, \|\nabla\theta\|_{L^6}.$$

According to estimation (64), we get

$$\begin{aligned} |B_1(\mathbf{V}, \theta)|^2 &\leqslant k_1^3 \|\mathbf{V}\|^2 \|\theta\| \, \|\nabla\theta\| \\ &\leqslant k_1^3 \|\mathbf{V}\|^2 \|\theta\| \, \|\theta\|_{H^2}. \end{aligned}$$

The operator A_1 is such that, $\forall\, \theta \in D(A_1) = H_0^1(\Omega) \cap H^2(\Omega)$, there is one constant k_3', such that

$$\|\theta\|_{\mathrm{H}^2} \leqslant k_3' |A_1\theta|.$$

Thus, there is one constant k', such that

$$|B_1(\mathbf{V}, \theta)|^2 \leqslant k' \|\mathbf{V}\|^2 \|\theta\| \, |A_1\theta|, \quad \forall \theta \in D(A_1). \tag{67}$$

Proof of Proposition 6.3

$\mathbf{V}(t)$ Is solution of the differential problem (28, 29) and we can apply estimation (55) proved in Proposition 6.1.

Since $\tilde{\mathbf{f}}(t) = \mathbf{f}(t) - B\mathbf{V}$, we get that

$$|\tilde{\mathbf{f}}(t)|^2 \leqslant 2|\mathbf{f}(t)|^2 + 2k\,|A\mathbf{V}|\, \|\mathbf{V}\|^3,$$

by applying Lemma 6.2. Then, estimation (55) implies the following inequality:

$$\|\mathbf{V}(t)\|^2 \leqslant \frac{k^2 C_{10}^2}{\nu} \int_0^t \exp[-2\gamma(t-\tau)] \, \|\mathbf{V}(\tau)\|^6 \, \mathrm{d}\tau + 2C_{10} \int_0^t \exp[-2\gamma(t-\tau)] |\mathbf{f}(\tau)|^2 \, \mathrm{d}\tau.$$

We assume that $\mathbf{f}(t)$ remains bounded for every t, let

$$|\mathbf{f}(t)|^2 \leqslant K.$$

We have then the relation

$$\|\mathbf{V}(t)\|^2 \exp(2\gamma t) \leqslant \frac{k^2 C_{10}^2}{\nu} \int_0^t \|\mathbf{V}(\tau)\|^6 \exp(2\gamma\tau) \, \mathrm{d}\tau + 2KC_{10} \int_0^t \exp(2\gamma\tau) \, \mathrm{d}\tau.$$

Set

$$\alpha(t) = \|\mathbf{V}(t)\|^2 \exp(2\gamma t),$$

$$\alpha(t) \leqslant \int_0^t \left(\frac{k^2 C_{10}^2}{\nu} \alpha^3(\tau) \exp(-4\gamma\tau) + 2KC_{10} \exp(2\gamma\tau) \right) \mathrm{d}\tau.$$

We can apply Lemma 6.1, and get the following estimation: $\alpha(t) \leqslant \beta(t)$, $\beta(t)$ satisfying the differential equation

$$\begin{cases} \dfrac{d\beta}{dt} = \dfrac{k^2C_{10}^2}{\nu}\beta^3\exp(-4\gamma t) + 2KC_{10}\exp(2\gamma t) \\ \beta(0) = 0. \end{cases}$$

Set $\beta(t) = \exp(2\gamma t)\,\psi(t)\cdot\psi(t)$ is a solution of the following differential problem:

$$\begin{cases} \dfrac{d\psi}{dt} = \dfrac{k^2C_{10}^2}{\nu}\psi^3 - 2\gamma\psi + 2KC_{10} & (68) \\ \psi(0) = 0 \quad \text{and} \quad \psi(t) > 0 \quad \text{for} \quad t > 0. & (69) \end{cases}$$

The function

$$G(\psi) = \frac{k^2C_{10}^2}{\nu}\psi^3 - 2\gamma\psi + 2KC_{10}$$

is a local Lipschitz function. The differential equation (68) has thus, one and only one solution.

The differential problem (68, 69) makes sense only if the function $G(\psi)$ has positive roots, that is to say for

$$K < \frac{\sqrt{\nu}}{kC_{10}^2}\left(\sqrt{\frac{2\gamma}{3}}\right)^3$$

$\psi(t)$, the solution of the differential problem (68, 69), then remains bounded by K_1, satisfying

$$K_1 < \frac{\sqrt{\nu}}{kC_{10}}\sqrt{\frac{2\gamma}{3}}$$

and $\psi(t) \to K_1$ when $t \to \infty$.

For

$$|\mathbf{f}(t)|^2 \leqslant K < \frac{\sqrt{\nu}}{kC_{10}^2}\left(\sqrt{\frac{2\gamma}{3}}\right)^3,$$

we have proved that

$$\|\mathbf{V}(t)\|^2 \leqslant K_1 < \frac{\sqrt{\nu}}{kC_{10}}\sqrt{\frac{2\gamma}{3}}. \tag{61}$$

We are going to prove now estimation (62) for $\|\theta(t)\|^2$. $\theta(t)$ is solution of the differential problem (30, 31) and we can apply estimation (60), proved in Proposition 6.2,

$$\|\theta(t)\|^2 + \nu'\int_0^t \exp[-2\gamma(t-\tau)]|A_1\theta(\tau)|^2\,d\tau \leqslant C_{10}'\int_0^t \exp[-2\gamma(t-\tau)]|\tilde{g}(\tau)|^2\,d\tau.$$

In order to get also estimation (55), we assume that $\gamma < \inf(\nu'C^2, \mathrm{Re}(\lambda))$, for every eigenvalue λ of $\tilde{A}$. We can choose the constants C_{10} and C_{10}' in order to have

$$\frac{C_{10}}{\sqrt{\nu}} = \frac{C_{10}'}{\sqrt{\nu'}}.$$

We set $N\mathbf{V} = [\mathbf{V}\cdot\nabla]\theta_0$. N is a continuous linear operator from $H_0^1(\Omega)$ onto $L^2(\Omega)$. Hence, there is one constant C_1', such that

$$|N\mathbf{V}| \leqslant C_1'\|\mathbf{V}\|, \tag{70}$$

$$\tilde{g}(t) = \phi - [\mathbf{V}\cdot\nabla]\theta_0 - [\mathbf{V}\cdot\nabla]\theta = \phi - N\mathbf{V} - B_1(\mathbf{V},\theta).$$

Applying estimations (67) and (70), we get

$$\begin{aligned} |\tilde{g}(t)|^2 &\leqslant 3|\phi|^2 + 3C_1'^2\|\mathbf{V}\|^2 + 3k\|\mathbf{V}\|^2\|\theta\|\,|A_1\theta| \\ &\leqslant 3|\phi|^2 + 3C_1'^2K_1 + 3kK_1\|\theta\|\,|A_1\theta|, \end{aligned}$$

since estimation (61) has been proved.

$$3kC'_{10}K_1\,\|\theta\|\,|A_1\theta| \leqslant \nu'|A_1\theta|^2 + \frac{9k^2C'^2_{10}K_1^2}{4\nu'}\|\theta\|^2.$$

Relation (60) can then be written

$$\|\theta(t)\|^2 \leqslant 3C'_{10}(|\phi|^2 + C'^2_1K_1)\int_0^t \exp[-2\gamma(t-\tau)]\,\mathrm{d}\tau + \frac{9k^2C'^2_{10}K_1^2}{4\nu'}\int_0^t \|\theta(\tau)\|^2\exp[-2\gamma(t-\tau)]\,\mathrm{d}\tau$$

and since

$$K_1 < \frac{\sqrt{\nu}}{kC_{10}}\sqrt{\frac{2\gamma}{3}},$$

$$\|\theta(t)\|^2\exp(2\gamma t) \leqslant \frac{3}{2}\gamma\int_0^t \|\theta(\tau)\|^2\exp(2\gamma\tau)\,\mathrm{d}\tau + 3C'_{10}(|\phi|^2 + C'^2_1K_1)\int_0^t \exp(2\gamma\tau)\,\mathrm{d}\tau.$$

Set $\mu(t) = \|\theta(t)\|^2\exp(2\gamma t)$. We can apply Lemma 6.1, and get the following estimation: $\mu(t) \leqslant \lambda(t)$, $\lambda(t)$ satisfying the differential equation

$$\begin{cases} \dfrac{\mathrm{d}\lambda}{\mathrm{d}t} = \dfrac{3}{2}\gamma\lambda + 3C'_{10}(|\phi|^2 + C'^2_1K_1)\exp(2\gamma t) \\ \lambda(0) = 0. \end{cases}$$

Set $\lambda(t) = \exp(2\gamma t)\ \delta(t)$, $\delta(t)$ satisfying the differential problem

$$\begin{cases} \dfrac{\mathrm{d}\delta}{\mathrm{d}t} = -\dfrac{\gamma}{2}\delta + 3C'_{10}(|\phi|^2 + C'^2_1K_1) \\ \delta(0) = 0. \end{cases}$$

We have,

$$\delta(t) = \frac{6C'_{10}}{\gamma}(|\phi|^2 + C'^2_1K_1)\left[1 - \exp\left(-\frac{\gamma}{2}t\right)\right].$$

This implies the following estimation, valid for every t:

$$\|\theta(t)\|^2 \leqslant \frac{6C'_{10}}{\gamma}(|\phi|^2 + C'^2_1K_1).$$

Hence, there are two constants C'_2 and C'_3, such that

$$\|\theta(t)\|^2 \leqslant C'_2|\phi|^2 + C'_3K_1. \tag{62}$$

Proposition 6.4

Under the assumptions of Proposition 6.3, the solution $(\mathbf{V}, \theta)$ of the differential problem (28)–(31) is a strong solution on $[0, T]$, that is to say

$$\forall\, t \in [0, T],\quad \mathbf{V}(t) \in D(\tilde{A}) \quad \text{and} \quad \theta(t) \in D(\tilde{A}_1).$$

On the other hand,

$$\mathbf{V} \in \mathscr{C}(0, T; \mathbb{V}) \quad \text{and} \quad \theta \in \mathscr{C}(0, T; H^1_0(\Omega)).$$

Proof. We denote by (λ_i) the eigenvalues of the operator A: $0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_i \cdots$.

There is one orthonormal basis in $\mathbb{H}$: $\mathbf{w}_1, \ldots, \mathbf{w}_i, \ldots$, where $\mathbf{w}_i$ is an eigenvector associated to the eigenvalue λ_i. We define an approximate solution for equation (28) by setting

$$\mathbf{V}_m = \sum_{i=1}^m g_{im}(t)\mathbf{w}_i,$$

$\mathbf{V}_m \in D(A)$ and $\mathbf{V}_m$ satisfies the differential equation

$$\begin{cases} \dfrac{d\mathbf{V}_m}{dt} + \tilde{A}\mathbf{V}_m = \tilde{\mathbf{f}}_m(t) \\[2ex] \mathbf{V}_m(0) = \mathbf{0}, \end{cases}$$

where

$$\tilde{\mathbf{f}}_m(t) = \mathbf{f}(t) - B\mathbf{V}_m.$$

The assumptions of Proposition 6.3 being satisfied, we get the estimation

$$\|\mathbf{V}_m(t)\|^2 \leqslant K_1, \quad \forall\, t \in [0, \infty[.$$

We can also apply estimation (54) proved in Proposition 6.1

$$\|\mathbf{V}_m(t)\|^2 + \nu \int_0^t |A\mathbf{V}_m(\tau)|^2 \,d\tau \leqslant C_{10} \int_0^t |\tilde{\mathbf{f}}_m(\tau)|^2 \,d\tau.$$

Taking into account estimation (63) proved in Lemma 6.2, this implies

$$\int_0^t |A\mathbf{V}_m(\tau)|^2 \,d\tau \leqslant 4\frac{C_{10}}{\nu}\left(\int_0^t |\mathbf{f}(\tau)|^2 \,d\tau + k^2 \frac{C_{10}}{\nu} \int_0^t \|\mathbf{V}_m(\tau)\|^6 \,d\tau\right), \quad \forall\, t \in [0, T].$$

Since

$$|\mathbf{f}(t)|^2 \leqslant K \quad \text{and} \quad \|\mathbf{V}_m(t)\|^2 \leqslant K_1, \quad \forall\, t \in [0, T],$$

we get

$$\int_0^T |A\mathbf{V}_m(\tau)|^2 \,d\tau \leqslant 4\frac{C_{10}}{\nu} T\left(K + k^2 \frac{C_{10}}{\nu} K_1^3\right).$$

The sequence $\mathbf{V}_m(t)$ is bounded in $\mathbb{L}^2(0, T; D(A))$. We can extract a subsequence which converges towards $\mathbf{V}(t)$ in $\mathbb{L}^2(0, T; D(A))$ weakly. $\mathbf{V}(t)$ satisfies the differential problem (28, 29), $\mathbf{V}(t) \in D(A)$ and satisfies the estimations

$$\int_0^T |A\mathbf{V}(\tau)|^2 \,d\tau \leqslant 4\frac{C_{10}}{\nu} T\left(K + k^2 \frac{C_{10}}{\nu} K_1^3\right)$$

and

$$\|\mathbf{V}(t)\|^2 \leqslant K_1, \quad \forall\, t \in [0, T].$$

We can easily prove that $\mathbf{V}' = d\mathbf{V}/dt \in \mathbb{L}^2(0, T; \mathbb{H})$. Since $\mathbf{V} \in \mathbb{L}^2(0, T; D(A))$, we can conclude, according to a result from Lions–Magenes, that

$$\mathbf{V} \in \mathscr{C}(0, T; \mathbb{V}).$$

The same proof is valid for $\theta(t)$.

6.2 Non-linear Stationary Problem

The perturbations $\mathbf{V}, \theta$ of velocity and temperature always have to satisfy the differential equations (28–31) but this time the right terms are given by

$$\tilde{\mathbf{f}}(t) = P(\mathbf{f}(t)) - P([\mathbf{V}\cdot\nabla]\mathbf{V}), \quad \tilde{g}(t) = -[\mathbf{V}\cdot\nabla]\theta_0 - [\mathbf{V}\cdot\nabla]\theta, \quad \mathbf{f}(t) = \mathbf{T}(t).$$

We assume that the perturbation of the wind-stress $\mathbf{T}(t)$ acts only during some time interval $[0, T]$, and remains bounded:

$$\mathbf{f}(t) = \begin{cases} \mathbf{T}(t), & \text{for} \quad 0 \leqslant t \leqslant T \quad (T \text{ finite}) \\ \mathbf{0}, & \text{for} \quad t > T. \end{cases}$$

$$|\mathbf{f}(t)|^2 \leqslant K, \quad \forall\, t \in [0, T].$$

Proposition 6.5

If all the eigenvalues of the operator $\tilde{A}$ have positive real parts, and if the following estimation is satisfied:

$$|\mathbf{f}(t)|^2 \exp(2\gamma T) \leqslant \frac{\sqrt{2\nu\gamma^3}}{kC_{10}^2}, \quad \forall\, t \in [0, T] \tag{71}$$

[γ is a real number such that $0 < \gamma < \inf(\operatorname{Re}\lambda, \nu' C^2)$ for every eigenvalue λ of $\tilde{A}$], then, $\|\mathbf{V}(t)\|^2$ and $\|\theta(t)\|^2$ tend to 0 as $t \to \infty$.

Proof. We apply, as for the proof of Proposition 6.3, estimation (55). So we have

$$\begin{aligned}\|\mathbf{V}(t)\|^2 &\leqslant \frac{C_{10}^2 k^2}{\nu} \int_0^t \exp[-2\gamma(t-\tau)]\, \|\mathbf{V}(\tau)\|^6 \,\mathrm{d}\tau + 2C_{10} \int_0^t \exp[-2\gamma(t-\tau)]\, |\mathbf{f}(\tau)|^2 \,\mathrm{d}\tau \\ &\leqslant \frac{C_{10}^2 k^2}{\nu} \int_0^t \exp[-2\gamma(t-\tau)]\, \|\mathbf{V}(\tau)\|^6 \,\mathrm{d}\tau + 2KC_{10} \frac{\exp(2\gamma T) - 1}{2\gamma} \exp(-2\gamma t).\end{aligned}$$

We can apply Lemma 6.1. So we get the following estimation: $\|\mathbf{V}(t)\|^2 \exp(2\gamma t) \leqslant \beta(t)$, $\beta(t)$ being solution of the differential equation

$$\begin{cases} \dfrac{\mathrm{d}\beta}{\mathrm{d}t} = \dfrac{k^2 C_{10}^2}{\nu} \beta^3 \exp(-4\gamma t) \\ \beta(0) = K \exp(2\gamma T) \dfrac{C_{10}}{\gamma} = A. \end{cases}$$

Set $\beta(t) = \exp(2\gamma t)\, \psi(t)$, $\psi(t)$ is solution of the differential problem

$$\begin{cases} \dfrac{\mathrm{d}\psi}{\mathrm{d}t} = \dfrac{k^2 C_{10}^2}{\nu} \psi^3 - 2\gamma\psi \\ \psi(0) = A. \end{cases}$$

The solution of this differential equation is given by

$$\psi(t) = \frac{A \exp(-2\gamma t)}{\left\{1 - A^2 \dfrac{k^2 C_{10}^2}{2\nu\gamma} [1 - \exp(-4\gamma t)]\right\}^{1/2}},$$

provided that

$$K^2 \exp(4\gamma T) < 2\nu \frac{\gamma^3}{k^2 C_{10}^4}.$$

We get for $\|\mathbf{V}(t)\|^2$ the estimation

$$\|\mathbf{V}(t)\|^2 \leqslant K \exp(2\gamma T) \frac{C_{10}}{\gamma} \frac{\exp(-2\gamma t)}{\left\{1 - \dfrac{1}{2} \dfrac{k^2 C_{10}^4}{\nu\gamma^3} K^2 \exp(4\gamma T)[1 - \exp(-4\gamma t)]\right\}^{1/2}}. \tag{72}$$

Hence, there is one constant D, such that

$$\|\mathbf{V}(t)\|^2 \leqslant D \exp(-2\gamma t). \tag{73}$$

The estimation of $\|\theta(t)\|^2$ is obtained using relation (60). $\phi = 0$, therefore

$$|\tilde{g}(t)|^2 \leqslant 2C_1'^2 \|\mathbf{V}(t)\|^2 + 2k \|\mathbf{V}(t)\|^2 \|\theta\| \, |A_1 \theta|.$$

We get from relation (60) the estimation

$$\begin{aligned}\|\theta(t)\|^2 \leqslant{}& 2C_{10}' C_1'^2 \int_0^t \exp[-2\gamma(t-\tau)]\, \|\mathbf{V}(\tau)\|^2 \,\mathrm{d}\tau \\ &+ \frac{k^2 C_{10}'^2}{\nu'} \int_0^t \exp[-2\gamma(t-\tau)]\, \|\mathbf{V}(\tau)\|^4 \|\theta(\tau)\|^2 \,\mathrm{d}\tau,\end{aligned}$$

i.e. using expression (73),

$$\|\theta(t)\|^2 \leqslant 2C'_{10}C_1'^2 D\int_0^t \exp(-2\gamma t)\,\mathrm{d}\tau + \frac{k^2C_{10}'^2D^2}{\nu'}\int_0^t \exp(-2\gamma t)\|\theta(\tau)\|^2\exp(-2\gamma\tau)\,\mathrm{d}\tau.$$

Set $C'_4 = 2C'_{10}C_1'^2D$, $C'_5 = k^2C_{10}'^2D^2/\nu'$. We have

$$\|\theta(t)\|^2\exp(2\gamma t) \leqslant \int_0^t [C'_4 + C'_5\|\theta(\tau)\|^2\exp(-2\gamma\tau)]\,\mathrm{d}\tau.$$

Applying the result of Lemma 6.1, we get the following estimation: $\|\theta(t)\|^2\exp(2\gamma t) \leqslant \lambda(t)$, $\lambda(t)$ being the solution of the differential equation

$$\begin{cases} \dfrac{\mathrm{d}\lambda}{\mathrm{d}t} = C'_4 + C'_5\lambda\exp(-4\gamma t) \\ \lambda(0) = 0. \end{cases}$$

The solution of this differential equation is given by

$$\lambda(t) = C'_4\int_0^t \exp\left\{-\frac{C'_5}{4\gamma}[\exp(-4\gamma t) - \exp(-4\gamma\tau)]\right\}\mathrm{d}\tau$$
$$\leqslant C'_4\exp\left(\frac{C'_5}{4\gamma}\right)t.$$

Hence, there is one constant D', such that

$$\|\theta(t)\|^2 \leqslant D't\exp(-2\gamma t). \tag{74}$$

Proposition 6.6

Under the assumptions of Proposition 6.5, the solution $(\mathbf{V}, \theta)$ of the differential problem (28)–(31) is a strong solution on $[0, +\infty[$.

Proof. The design of the proof is similar to that of Proposition 6.4. We define an approximate solution $\mathbf{V}_m \in D(A)$, $\mathbf{V}_m$ satisfying the differential equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}_m + \tilde{A}\mathbf{V}_m + B\mathbf{V}_m = \mathbf{f}(t) \\ \mathbf{V}_m(0) = \mathbf{0}. \end{cases}$$

Estimations (54) and (63) imply that

$$\int_0^t |A\mathbf{V}_m(\tau)|^2\,\mathrm{d}\tau \leqslant 4\frac{C_{10}}{\nu}\left[\int_0^t |\mathbf{f}(\tau)|^2\,\mathrm{d}\tau + k^2\frac{C_{10}}{\nu}\int_0^t \|\mathbf{V}_m(\tau)\|^6\,\mathrm{d}\tau\right].$$

We have assumed that

$$|\mathbf{f}(t)|^2 \leqslant K, \quad \forall\, t \in [0, T]$$
$$\mathbf{f}(t) = \mathbf{0}, \quad \text{for} \quad t > T.$$

On the other hand, according to Proposition 6.5, we have the estimation

$$\|\mathbf{V}_m(t)\|^2 \leqslant D\exp(-2\gamma t), \quad \forall\, t \in [0, +\infty[.$$

This implies

$$\int_0^t |A\mathbf{V}_m(\tau)|^2\,\mathrm{d}\tau \leqslant 4\frac{C_{10}}{\nu}KT + \frac{4k^2C_{10}^2D^3}{\nu^2 6\gamma}[1 - \exp(-6\gamma t)], \quad \forall\, t \in [0, +\infty[.$$

The sequence $\mathbf{V}_m(t)$ is therefore bounded in $\mathbb{L}^2(0, t; D(A))$. We can extract a subsequence converging towards $\mathbf{V}(t)$ in $\mathbb{L}^2(0, t; D(A))$ weakly. $\mathbf{V}(t)$ satisfies the differential problem (28, 29), $\mathbf{V}(t) \in D(A)$ and $\mathbf{V}(t)$ is such that

$$\int_0^t |A\mathbf{V}(\tau)|^2\,\mathrm{d}\tau \leqslant 4\frac{C_{10}}{\nu}KT + 2\frac{k^2C_{10}^2}{\nu^2}\frac{D^3}{3\gamma}[1 - \exp(-6\gamma t)], \quad \forall\, t \in [0, +\infty[,$$

and

$$\| \mathbf{V}(t) \|^2 \leqslant D \exp(-2\gamma t), \quad \forall\, t \in [0, +\infty[.$$

We can easily prove that, $\forall\, t \in [0, \infty[$,

$$\mathbf{V}' = \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} \in \mathbb{L}^2(0, t; \mathbb{H}).$$

Since $\mathbf{V} \in \mathbb{L}^2(0, t; D(A))$, we can conclude that $\mathbf{V} \in \mathscr{C}(0, t; \mathbb{V})$.

The same proof is valid for $\theta(t)$.

7. CONCLUSION

We have obtained, in this paper, some theoretical results for the perturbation of a given mean flow. This problem was introduced in order to modelize equatorial waves [2]. The oceanic domain Ω is an open set included in $\mathbb{R}^3$. At initial time, induced by a mean wind-stress, there exist in Ω a velocity field $\mathbf{V}_0$ and a temperature field θ_0. We are going to calculate the perturbations $\mathbf{V}$ and θ of the velocity and the temperature induced by a perturbation of the mean wind-stress. The values of the initial situation $\mathbf{V}_0$, θ_0 are given. They must be characteristic of the circulation in an equatorial oceanic domain and these values, resulting from physical observations, either verify the linearized equations, or are solution of the complete non-linear stationary problem. In these two cases, the perturbations $\mathbf{V}$ and θ have to satisfy the same equations: an equation of Navier–Stokes type for $\mathbf{V}$ and of transport-diffusion type for θ. Only the right members are different. We prove that this non-linear problem has always one solution, in proper functional spaces. The method used is the Galerkin method. We get some *a priori* estimations which imply weak convergence for the approximate solution. Strong convergence is necessary for passing to the limit in the non-linear terms. To get this result, we apply a theorem of compactness [3]. Then, we give some results about regularity and uniqueness of the solution: we prove that a more regular solution is unique, but then, the existence of the solution cannot be assured.

These results about existence, uniqueness and regularity of the perturbation being proved, our purpose is to study the stability of the given initial situation $\mathbf{V}_0, \theta_0$, and, therefore, to determine under what condition the perturbations $\mathbf{V}(t)$ and $\theta(t)$ remain bounded as time $t \to \infty$. To this end, we generalize a method introduced by Prodi [5] based on the properties of operators deduced from the Stokes operator. We have located the eigenvalues of these operators, they are situated inside a parabolic curve drawn in the complex plane. The results about stability of the perturbation are obtained provided that these eigenvalues have positive real parts. The physical significance of this assumption cannot be clearly explained. If the initial values $\mathbf{V}_0$ and θ_0 are solutions of the linear stationary problem, the right members of the equations are dependent on the perturbation of the wind-stress, and on the initial values $\mathbf{V}_0, \theta_0$. We prove then that the perturbations $\mathbf{V}(t), \theta(t)$ of the velocity and the temperature remain bounded provided that the perturbation of the wind-stress, and the initial current $\mathbf{V}_0$, are small enough. Under the same assumptions, we get also more regularity for the perturbation. If the initial data $\mathbf{V}_0, \theta_0$, are solutions of the complete non-linear stationary problem, the right members of the equations depend only on the perturbation of the wind-stress. In this case, we get stronger results for stability. Assuming that the perturbation of the wind-stress acts only during a finite time and is small enough, we prove that the perturbations $\mathbf{V}(t), \theta(t)$ tend to zero as time $t \to \infty$. Moreover, the perturbation is then a strong solution for every $t \in [0, \infty[$.

It is the fact that oceanic waves could be stable or unstable depending on the characteristics of the mean situation $\mathbf{V}_0$, θ_0 which induced us to undertake this study. The results given here show that a stable initial situation is not to be expected. For example, in the case of a wind-stress acting only for a short time, the fact that the perturbation is a decreasing function of time is quite intuitive. Nevertheless, we need stronger assumptions to prove this result: real parts of the eigenvalues have to be positive, and the wind-stress must be small enough. We cannot prove that the perturbation induced by a strong wind-stress, acting for a finite time, tends to zero as time $t \to \infty$. Then, it is not surprising to get unstable oceanic waves, whose amplitude is an increasing function of time.

REFERENCES

1. F. Brossier, Analyse numérique de problèmes de type Stokes ou Navier–Stokes intervenant en modélisation de phénomènes océanographiques. Thèse d'Etat, Univ. Rennes (1985).
2. F. Brossier, Mathematical modelisation of equatorial waves. *Acta appl. Math.* **5**, 37–85 (1986).
3. R. Temam, *Navier–Stokes Equations. Theory and Numerical Analysis*, 2nd edn. North-Holland, Amsterdam (1979).
4. D. H. Sattinger, The mathematical problem of hydrodynamic stability. *J. Math. Mech.* **19**(9), 797–817 (1970).
5. G. Prodi, Theoremi di tipo locale per il sistema di Navier–Stokes a stabilita delle soluzioni stazionare. *Rc. Semin. Mat. Univ. Padova* **32**, 374–397 (1962).