Rigidity of Truncated Quiver Algebras

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1. Introduction

A quiver $Q$ is a finite oriented graph which can contain more than one arrow between two vertices, as well as loops and oriented cycles. For $n$ a positive integer, $Q_n$ is the set of oriented paths of length $n$ of $Q$, where the length is the number of arrows of the oriented path. Notice that $Q_0$ is the set of vertices and $Q_1$ the set of arrows. Let $k$ be a field and $kQ_0 = \times_{s \in Q_0} k_s$ be the commutative semi-simple algebra with $Q_0$ as a $k$-basis of idempotents. For each arrow $a \in Q_1$ with source vertex $s(a) = s$ and end vertex $t(a) = t$, the one dimensional vector space $k_a$ has an evident $kQ_0$-bimodule structure: for $u \in Q_0$, we have $au = \delta_{u,s} a$ and $ua = \delta_{u,t} a$, where $\delta$ is the Kronecker symbol. In this way $kQ_1 = \oplus_{a \in Q_1} ka$ is a $kQ_0$-bimodule and the quiver algebra $kQ$ is the tensor algebra over $kQ_0$ of the $kQ_0$-bimodule $kQ_1$. Of course, $kQ$ can also be described as the vector space $kQ_0 \oplus kQ_1 \oplus kQ_2 \oplus \cdots$ where the multiplication of $\beta \in Q_i$ and $\alpha \in Q_j$ is $\beta \alpha \in Q_{i+j}$ if $t(\alpha) = s(\beta)$ and 0 otherwise.

Let now $A$ be a finite dimensional $k$-algebra. We suppose that $A$ is Morita reduced and that the endomorphism ring of each simple $A$-module is $k$. This is equivalent to $A/r = k \times \cdots \times k$, were $r$ is the Jacobson radical of $A$. By definition, the set of vertices of the Gabriel's quiver $Q$ of $A$ is the set of isomorphism classes of simple $A$-modules. If $S$ and $T$ are simple $A$-modules, the number of arrows from $S$ to $T$ is $\dim_k \text{Ext}_A^1(S, T)$.

By an observation of Gabriel [6, 7, 4.3] every $k$-algebra $A$ such that $A/r = k \times \cdots \times k$ admits a presentation, that is, an algebra surjection $\varphi: kQ_A \rightarrow A$ whose kernel $I$ verifies $F^m \subset I \subset F^2$, where $F$ is the two-sided ideal of $kQ_A$ generated by $Q_1$ and $m$ is some positive integer. Such ideals $I$ of a quiver algebra are called admissible. In general an algebra $A$ has not a unique presentation, that is, two different admissible two-sided ideals $I$ and $J$ of a quiver algebra $kQ$ can give isomorphic $k$-algebras $kQ/I$ and $kQ/J$.

By definition, we say that a $k$-algebra $A$ is a truncated quiver algebra if

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it admits a presentation \( \varphi: kQ_A \to A \) such that \( \ker \varphi = F^m \) for \( m \geq 2 \). In fact this is an intrinsic notion. More precisely, let us suppose that \( kQ/I \) is isomorphic to \( kQ/F^m \) for \( I \) an admissible two-sided ideal of \( kQ \). The radical of \( kQ/I \) is \( F/I \) and we must have \( (F/I)^m = 0 \) because \( \text{rad}^m(kQ/F^m) = 0 \). That means \( F^m \subseteq I \) and we have \( (kQ/F^m)/(I/F^m) = kQ/I \). We deduce \( \dim_k(I/F^m) = 0 \) so \( I = F^m \).

From this point of view, the truncated quiver algebras are a natural family of algebras to study. The main purpose of this paper is to determine which algebras in this family are rigid, giving a solution in this case to the problem of Gerstenhaber (see [8, p. 66]).

Notice that if \( m = 2 \) the truncated quiver algebras coincide with the algebras \( A \) such that \( A/r = k \times \cdots \times k \) and \( r^2 = 0 \). We have studied their rigidity in [4]. However, we use new methods in this paper. The basic tool is always Hochschild cohomology (see [8]). To study it we need a projective resolution of an algebra \( A \) (considered as bimodule over itself) smaller than the usual one and first considered by Gerstenhaber and Schack [9] using relative Hochschild cohomology. We construct it in Section 2 for \( A \) a finite dimensional \( k \)-algebra such that \( A/r \) is a separable algebra. We infer an Hochschild cocomplex which enables us to make explicit considerations on cocycles.

In Section 3 we consider the spectral sequence associated to the filtration of an algebra \( A \) by the powers of its radical. It converges to the Hochschild cohomology of \( A \) and the differential at the first level is obtained using the cup-product with a canonical element of \( H^1(A, r/r^2) \). This description enables us to give in Section 4 conditions (called \( m \)-vanishing conditions) on \( Q \) equivalent to the fact that the terms of total degree 2 at level 2 of the spectral sequence vanish, for \( A \) a truncated quiver algebra.

In the last section we give a non-trivial one-parameter deformation of the algebra \( kQ/F^m \) when the \( m \)-vanishing conditions on \( Q \) are not satisfied. Collecting the results of Sections 4 and 5 we obtain the main result of this paper: a truncated quiver algebra \( kQ/F^m \) is rigid if and only if \( Q \) satisfies the \( m \)-vanishing conditions.

2. HOCHSCHILD COHOMOLOGY

Let \( k \) be a field and \( A \) a \( k \)-algebra. The Hochschild cohomology groups \( H^i(A, M) \) of \( A \) with coefficients in some \( A \)-bimodule \( M \) are, by definition, the groups \( \text{Ext}^i_{A \otimes_k A^{op}}(A, M) \) where \( A \otimes_k A^{op} \) is the enveloping algebra of \( A \). Clearly there is a natural identification between \( A \)-bimodules and left \( A \otimes_k A^{op} \)-modules which gives the definition of a free or projective \( A \)-bimodule as well as the definition of a \( A \)-bimodule homomorphism.

There is a standard resolution of the \( A \)-bimodule \( A \) using \( A \)-free
bimodules which gives, applying the functor $\text{Hom}_{A-A}(-, M)$, the cocomplex defined by Hochschild in [10], see [12, 2].

From now on we suppose $A$ to be finite dimensional over $k$ and such that the semisimple $k$-algebra $A/r$ is separable, where $r$ is the Jacobson radical of $A$. In this situation, we are going to give a resolution of $A$ smaller than the standard one using $A$-projective bimodules. Recall that a $k$-algebra $E$ is separable if $K \otimes_k E$ is a semisimple $K$-algebra for each field $K$ containing $k$. This is equivalent for $E$ to be a projective $E$-bimodule, and also to the fact that the center of the endomorphism skew field of each simple left $A$-module is a separable (in the Galois sense) extension of $k$, see [5, 12]. In particular, if $k$ is a perfect field our hypothesis on the finite dimensional algebra $A$ is always satisfied.

In case $A/r$ is separable, the theorem of Wedderburn-Malcev (see [10, 5, 13]) asserts that there exist a subalgebra $E$ of $A$ such that $A = E \oplus r$. In that case we prove directly the following result of [9]:

**Lemma 2.1.** Let $A$ be a finite dimensional $k$-algebra such that $A/r$ is separable, and let $A = E \oplus r$ be a decomposition of $A$ as above. The following is a projective resolution of the $A$-bimodule $A$:

$$
\cdots \xrightarrow{\delta} A \otimes E \otimes^{E,i} E A \xrightarrow{\delta} \cdots \xrightarrow{\delta} A \otimes E \otimes^E A \xrightarrow{\delta} A \otimes E A \xrightarrow{e} A \xrightarrow{\delta} 0,
$$

where

$$e(\lambda \otimes \mu) = \lambda \mu$$

and

$$
\delta(\lambda \otimes x_1 \otimes \cdots \otimes x_i \otimes \mu) = \lambda x_1 \otimes \cdots \otimes x_i \otimes \mu + \sum_{j=1}^{i} (-1)^j \lambda \otimes x_1 \otimes \cdots \otimes x_j x_{j+1} \otimes \cdots \otimes x_i \otimes \mu
$$

$$+ (-1)^i \lambda \otimes x_1 \otimes \cdots \otimes x_i \mu.
$$

**Proof.** First we verify that the modules are all $A$-projective bimodules. Notice that $E$ is separable, so the $k$-algebra $E \otimes_k E^{op}$ is semisimple (see [5]) and all the $E$-bimodules are projective. In particular $r \otimes^{E,i} i$ is projective and it follows immediately that $A \otimes E \otimes^{E,i} E A$ is a projective $A$-bimodule for $i \geq 1$. In order to include $A \otimes E A$, just notice that $A \otimes E A = A \otimes E E \otimes E A$.

It is trivial to verify that $\delta$ is a well-defined morphism of $A$-bimodules and that $\delta^2 = 0$. We write $\lambda = \lambda_E + \lambda_r$ the decomposition of each $\lambda \in A$ in the direct sum $E \oplus r$. We have an homotopy contraction

$$t: A \otimes E \otimes^{E,i} E A \rightarrow A \otimes E \otimes^{E,i+1} E A$$
given by
\[ t(\lambda \otimes x_1 \otimes \cdots \otimes x_i \otimes \mu) = 1 \otimes \lambda \otimes x_1 \otimes \cdots \otimes x_i \otimes \mu. \]

This \( t \) is well defined as a morphism of \( k \)-vector spaces, because \( A = E \oplus r \) is a decomposition of \( E \)-bimodules. When we verify that \( \delta t + t \delta = 1 \), we obtain
\[
(\delta t + t \delta)(\lambda \otimes x_1 \otimes \cdots \otimes x_i \otimes \mu) = 1 \otimes (\lambda x_1) \otimes \cdots \otimes x_i \otimes \mu + \lambda_r \otimes x_1 \otimes \cdots \otimes x_i \otimes \mu - 1 \otimes \lambda_r x_1 \otimes \cdots \otimes x_i \otimes \mu.
\]

But \( (\lambda x_1)_r = \lambda x_1 \) because \( x_1 \in r \). Using that \( \lambda x_1 = \lambda_E x_1 + \lambda_r x_1 \) we have
\[
(\delta t + t \delta)(\lambda \otimes x_1 \otimes \cdots \otimes x_i \otimes \mu) = 1 \otimes \lambda_E x_1 \otimes \cdots \otimes x_i \otimes \mu + \lambda_r \otimes x_1 \otimes \cdots \otimes x_i \otimes \mu = \lambda_E \otimes x_1 \otimes \cdots \otimes x_i \otimes \mu + \lambda_r \otimes x_1 \otimes \cdots \otimes x_i \otimes \mu = \lambda \otimes x_1 \otimes \cdots \otimes x_i \otimes \mu.
\]

We consider now a \( \Lambda \)-bimodule \( M \). In order to obtain the Hochschild cohomology \( H^i(\Lambda, M) \) we have to apply the functor \( \text{Hom}_{\Lambda-\Lambda}(\cdot, M) \) to the resolution of \( \Lambda \) in the lemma above. Using the easy identification
\[
\text{Hom}_{\Lambda-\Lambda}(\Lambda \otimes_E X \otimes_E \Lambda, M) = \text{Hom}_{E-E}(X, M),
\]
where \( X \) is an \( E \)-bimodule, we obtain the following:

**Proposition 2.2.** Let \( \Lambda \) be a finite dimensional algebra such that \( \Lambda/r \) is separable, where \( r \) is the Jacobson radical of \( \Lambda \). Let \( E \) be a subalgebra of \( \Lambda \) such that \( \Lambda = E \oplus r \) and let \( M \) be a \( \Lambda \)-bimodule.

The Hochschild cohomology vector spaces \( H^i(\Lambda, M) \) are the cohomology groups of the cocomplex:
\[
0 \longrightarrow M^E \overset{d}{\longrightarrow} \text{Hom}_{E-E}(r, M) \overset{d}{\longrightarrow} \text{Hom}_{E-E}(r \otimes_E r, M) \overset{d}{\longrightarrow} \cdots
\]

\[
\cdots \overset{d}{\longrightarrow} \text{Hom}_{E-E}(r \otimes_E \cdots \otimes_E r, M) \overset{d}{\longrightarrow} \cdots,
\]
where \( M^E = \{ m \in M \mid sm = ms \text{ for all } s \in E \} \), \( (dm)(x) = mx - xm \) for \( m \in M^E \) and \( x \in r \), and
\[
(df)(x_1 \otimes \cdots \otimes x_{i+1}) = x_1 f(x_2 \otimes \cdots \otimes x_{i+1}) + \sum (-1)^j f(x_1 \otimes \cdots \otimes x_j x_{j+1} \otimes \cdots \otimes x_{i+1}) + (-1)^{i+1} f(x_1 \otimes \cdots \otimes x_i) x_{i+1}.
\]
Remark. This cocomplex has been obtained by Gerstenhaber and Schack (see [9, Sect. 1]). The only difference is that we use here the bimodule \( r \) instead of \( A/E \); this will be useful in Section 3.

Example. Let \( A = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix} \) and \( \alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \tau = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \). Clearly \( A = ks \oplus k\tau \oplus k\alpha \) and \( E = ks \oplus kt \), \( r = k\alpha \) gives a Wedderburn–Malcev decomposition of \( A \). The tensor product \( r \otimes_E r \) is zero because \( \alpha = \alpha \beta \) and \( s\alpha = 0 \). For \( M \) a \( A \)-bimodule, the cocomplex of the above proposition is

\[
0 \longrightarrow M^E \overset{d}{\longrightarrow} \text{Hom}_{E-E}(r, M) \longrightarrow 0.
\]

In contrast, the Hochschild cocomplex is infinite:

\[
0 \longrightarrow M \longrightarrow \text{Hom}_k(A, M) \longrightarrow \text{Hom}_k(A \otimes_k A, M) \longrightarrow \cdots.
\]

Remark. By definition, the Hochschild homology groups of a \( A \)-bimodule \( M \) are defined by

\[
H_i(A, M) = \text{Tor}^A_{i} A^{ve}(M, A).
\]

With the same hypothesis as in Proposition 1.2, we obtain that they are the homology groups of the complex.

\[
\cdots \overset{d}{\longrightarrow} r \otimes s \otimes E \otimes E M \overset{d}{\longrightarrow} \cdots \overset{d}{\longrightarrow} (r \otimes_E r) \otimes E \otimes E M \overset{d}{\longrightarrow} r \otimes E \otimes E M \overset{d}{\longrightarrow} M/\langle sm - ms \mid m \in M, s \in E \rangle \longrightarrow 0,
\]

where

\[
d(x_1 \otimes \cdots \otimes x_j \otimes m) = x_2 \otimes \cdots \otimes x_i \otimes mx_1
\]

\[
\times \sum (-1)^j x_1 \otimes \cdots \otimes x_jx_{j+1} \otimes \cdots \otimes x_1 \otimes m
\]

\[
+ (-1)^j x_1 \otimes \cdots \otimes x_{i-1} \otimes x_im.
\]

Using this complex the following result about homology follows immediately (compare with [3]). We are not going to use it in this paper, we include it here as an application of the resolution of Lemma 2.1.

Proposition 2.3. Let \( A \) be a finite dimensional \( k \)-algebra such that \( A/r = k \times \cdots \times k \). Let \( Q_A \) be its Gabriel quiver. Assume \( Q_A \) has no oriented cycle.

Then

\[
H_i(A, A) = 0 \quad \text{for} \quad i > 0
\]
and

\[ H_0(A, A) = kQ_0. \]

**Proof.** Consider a presentation \( \varphi : kQ_A \rightarrow A \) (see the Introduction) and the complex of the remark above for \( M = A \) and \( E = kQ_0 \subset kQ_A/\ker \varphi \). The \( E \)-bimodules \( r^\otimes E \otimes_{E - E} A \) are zero for \( i > 0 \) if \( Q_A \) has no oriented cycles. Indeed

\[ r^\otimes E \otimes_{E - E} A = (r^\otimes E \otimes_{E - E} r) \oplus (r^\otimes E)^E. \]

The first term is generated by the \((i + 1)\)-path partitions of oriented cycles of length greater than \( i + 1 \), and the second one is generated by the \( i \)-path partitions of oriented cycles of length greater than \( i \).

### 3. The Spectral Sequence

As in the previous section, we consider \( A \) a finite dimensional algebra over a field \( k \) such that \( A/r \) is a separable \( k \)-algebra, where \( r \) is the Jacobson radical of \( A \). The Wedderburn–Malcev theorem ensures that we have a decomposition \( A = E \oplus r \), where \( E \) is a subalgebra of \( A \). For each \( A \)-bimodule \( M \) we denote \( C(M) \) the cocomplex we have obtained in Proposition 2.2.

The powers of the radical gives a filtration by \( A \)-bimodules of the \( A \)-bimodule \( A \). We deduce a filtration by sub-cocomplexes of the cocomplex \( C(A) \)

\[ 0 = C(r^m) \subset C(r^{m - 1}) \subset \cdots \subset C(r^2) \subset C(r) \subset C(A), \]

where \( m \) is the positive integer such that \( r^m = 0 \) and \( r^{m - 1} \neq 0 \).

From this filtration we construct a spectral sequence (see for instance [11]). The vector spaces at the zero level are

\[ E_0^{p, q} = C^{p + q}(r^p/r^{p + 1}) \]

with differentials \( d : E_0^{p, q} \rightarrow E_0^{p + 1, q} \) equal to the cobords of \( C(r^p/r^{p + 1}) \). At the first level we have \( E_1^{p, q} = H^{p + q}(A, r^p/r^{p + 1}) \).

The filtration of \( C(A) \) is finite, so the spectral sequence converges trivially to the Hochschild cohomology. In particular we have the following trivial lemma, which we shall use in Section 4.

**Lemma 3.1.** If \( E_2^{p, q} = 0 \) for all \( p \) and \( q \) such that \( p + q = 2 \), then \( H^2(A, A) = 0 \).
Recall that the term $E^{p,q}_2$ is the cohomology at $E^{p,q}_1$ of

$$E^{p-1,q}_1 \xrightarrow{d} E^{p,q}_1 \xrightarrow{d} E^{p+1,q}_1,$$

where $d$ is the differential at the first level. The next proposition gives a computation of $d$ using a cup-product which we define first.

There is a morphism of $E$-bimodules

$$r^p/r^{p+1} \otimes_E r^q/r^{q+1} \to r^p+q/r^{p+q+1}$$

$$\alpha \otimes \beta \mapsto \alpha \cdot \beta.$$

Here $\alpha \cdot \beta$ is the projection modulo $r^{p+q+1}$ of the product $ab$, where $a \in r^p$ and $b \in r^q$ are arbitrary representatives of $\alpha$ and $\beta$.

This gives a cup-product

$$H^i(A, r^p/r^{p+1}) \otimes_E H^j(A, r^q/r^{q+1}) \to H^{i+j}(A, r^{p+q}/r^{p+q+1})$$

defined as follows: let $f: r^{\otimes i} \to r^p/r^{p+1}$ and $g: r^{\otimes j} \to r^q/r^{q+1}$ be $E$-bimodules morphisms which are cocycles of $C^i(r^p/r^{p+1})$ and $C^j(r^q/r^{q+1})$.

The cup-product of their cohomology classes is the cohomology class of

$$f \square g: r^{\otimes i+j} \to r^{p+q}/r^{p+q+1},$$

where

$$f \square g(x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_{i+j}) = f(x_1 \otimes \cdots \otimes x_i) \cdot g(x_{i+1} \otimes \cdots \otimes x_{i+j}).$$

**Proposition 3.2.** There exist a canonical element $\pi \in H^1(A, r/r^2)$ such that

$$\Delta: H^i(A, r^p/r^{p+1}) \to H^{i+1}(A, r^{p+1}/r^{p+2})$$

is given by

$$\Delta f = \pi \square f + (-1)^{i+1} f \square \pi.$$

We need the following lemma in order to obtain the element $\pi \in H^1(A, r/r^2)$ of the Proposition:

**Lemma 3.3.** For $M$ a semisimple $A$-bimodule, we have

$$H^1(A, M) = \text{Hom}_{E-E}(r/r^2, M).$$
Proof. Consider the cocomplex $C(M)$

$$0 \longrightarrow M^E \xrightarrow{d_0} \text{Hom}_{E-E}(r, M) \xrightarrow{d_1} \text{Hom}_{E-E}(r \otimes_E r, M) \longrightarrow \cdots.$$

The radical of $A \otimes_k A^{op}$ is $r \otimes E^{op} \oplus E \oplus r^{op} \oplus r \otimes r^{op}$, so $M$ is a semisimple $A$-bimodule if and only if $rM = Mr = 0$. Using this, we obtain $d_0 = 0$. So

$$H^1(A, r/r^2) = \ker d_1 = \{ f | xf(y) - f(xy) + f(x)y = 0 \ \forall x, y \in r \}.$$

Again using that $M$ is semisimple, we have

$$\ker d_1 = \{ f | f(xy) = 0 \ \forall x, y \in r \} = \{ f | f|_{r^2} = 0 \}.$$

In case $M = r/r^2$ we have that

$$H^1(A, r/r^2) = \{ f \in \text{Hom}_{E-E}(r, r/r^2) | f|_{r^2} = 0 \}.$$

So the canonical projection $\pi: r \rightarrow r/r^2$ is an element of $H^1(A, r/r^2)$ and this is the canonical element of Proposition 3.2.

Proof of Proposition 3.2. For each $p \geq 0$ let $\pi_p: r^p \rightarrow r^p/r^{p+1}$ be the canonical projection. The algebra $E \otimes_k E^{op}$ is semisimple because $E$ is separable, so all the $E$-bimodules are projective. In particular we can choose an $E \rightarrow E$ section $\sigma_p: r^p/r^{p+1} \rightarrow r^p$ of each $\pi_p$.

From the spectral sequence we know that $A = \pi_{p+1} \delta$ where

$$\delta: H^i(A, r^p/r^{p+1}) \rightarrow H^{i+1}(A, r^{p+1})$$

is the connecting homomorphism of the long cohomology exact sequence associated to the exact sequence $0 \rightarrow r^{p+1} \rightarrow r^p \rightarrow r^p/r^{p+1} \rightarrow 0$. For $f$ an $i$-cocycle of $C(r^p/r^{p+1})$ we have $\delta f = d \sigma_p f$. For $x_1 \otimes \cdots \otimes x_{i+1} \in r^p \otimes_{E^{op}} \otimes E$ we obtain

$$(\delta f)(x_1 \otimes \cdots \otimes x_{i+1}) = x_1[\sigma_p f(x_2 \otimes \cdots \otimes x_{i+1})] + \sum (-1)^j \sigma_p f(x_1 \otimes \cdots \otimes x_j x_{j+1} \otimes \cdots \otimes x_{i+1}) + (-1)^{i+1} [\sigma_p f(x_2 \otimes \cdots \otimes x_i)] x_{i+1}.$$

But $f$ is an $i$-cocycle, so we have

$$x_1(f(x_2 \otimes \cdots \otimes x_{i+1})) + \sum (-1)^j f(x_1 \otimes \cdots \otimes x_j x_{j+1} \otimes \cdots \otimes x_{i+1}) + (-1)^{i+1} f(x_1 \otimes \cdots \otimes x_i) x_{i+1} = 0.$$

The first and the last terms of this sum are zero, because $f$ takes its values in the semisimple $A$-bimodule $r^p/r^{p+1}$. 

Applying $\sigma_p$ to the middle term of preceding sum, we obtain

$$\sum (-1)^j \sigma_p f(x_1 \otimes \cdots \otimes x_j x_{j+1} \otimes \cdots \otimes x_{i+1})$$

and

$$(f)(x_1 \otimes \cdots \otimes x_{i+1}) = \pi_{p+1} \{ x_1 [\sigma_p f(x_2 \otimes \cdots \otimes x_{i+1})] \}
+ (-1)^{i+1} \pi_{p+1} \{ [\sigma_p f(x_1 \otimes \cdots \otimes x_i)] x_{i+1} \}
= \pi(x_1) \cdot f(x_2 \otimes \cdots \otimes x_{i+1})
+ (-1)^{i+1} f(x_1 \otimes \cdots \otimes x_i) \cdot \pi(x_{i+1}).$$

We are going to use the above proposition in case $A = kQ/F^m$ is a truncated quiver algebra (see the Introduction). For this sort of algebras, it is possible to give an explicit interpretation of the vector spaces at the first level of the spectral sequence as well as a modelisation for $A$. Once this work will be done, we shall compute $\text{ker } A: H^2(A, r^p/r^{p+1}) \rightarrow H^3(A, r^p/r^{p+2})$ in Section 4.

**Definition.** Let $Q$ be a quiver and $Q_i$ be the set of oriented paths of length $i$. Recall that we denote $s(\gamma)$ and $t(\gamma)$ the source and end vertices of an oriented path $\gamma$. Two oriented paths $\gamma \in Q_i$ and $\rho \in Q_j$ are parallel (notation: $\gamma//\rho$) if $s(\gamma) = s(\rho)$ and $t(\gamma) = t(\rho)$.

Let $Q_i//Q_j$ be the set $\{(\gamma, \rho) \in Q_i \times Q_j | \gamma//\rho \}$ and $k(Q_i//Q_j)$ be the $k$-vector space with $Q_i//Q_j$ as a basis.

**Proposition 3.4.** Let $A = kQ/F^m$ be a truncated quiver algebra and $p$ an integer such that $0 < p < m - 1$. There are canonical isomorphisms

$$H^1(A, r^p/r^{p+1}) = k(Q_i//Q_p),$$

$$H^2(A, r^p/r^{p+1}) = k(Q_m//Q_p),$$

$$H^3(A, r^p/r^{p+1}) = k(Q_{m+1}//Q_p).$$

**Remark.** More generally, one can prove

$$H^i(A, r^p/r^{p+1}) = \begin{cases} k(Q_{nm}//Q_p) & \text{if } i = 2n, n \geq 1 \\ k(Q_{nm+1}//Q_p) & \text{if } i = 2n+1, n \geq 0 \end{cases}$$

but the proof is easier in low degrees and we only need the result for $i = 1, 2,$ and 3.

**Proof.** The semisimple $A$-bimodule $r^p/r^{p+1}$ has a canonical decomposition $\bigoplus_{\rho \in Q_p} k\rho$, where $k\rho$ is the one dimensional vector space equipped
with the evident simple \(A\)-bimodule structure determined by the source and end vertices of \(p\). It follows that \(H^i(A, r^p/r^{p+1}) = \bigoplus_{p \in Q} H^i(A, k\rho)\).

In order to study \(H^i(A, k\rho)\) we define cohomology groups attached to oriented paths of \(Q\).

**Definition.** Let \(\gamma \in Q_i\) for \(l > 0\). For each \(i > 0\), we consider the set \(P^i_\gamma\) of partitions of length \(i\) of \(\gamma\). More precisely \((x_1, \ldots, x_i)\) is an element of \(P^i_\gamma\) if each \(x_j\) is an oriented path of positive length strictly less than \(m\) such that the composition \(x_1 \cdots x_i\) equals \(\gamma\) in the quiver algebra \(kQ\). Of course \(P^i_\gamma \neq \emptyset\) if and only if \(i \leq l \leq i(m-1)\).

We consider the cocomplex

\[
0 \to \text{Hom}_k(kP^i_\gamma, k) \xrightarrow{d} \cdots \xrightarrow{d} \text{Hom}_k(kP^i_\gamma, k) \xrightarrow{d} \cdots
\]

where \(kP^i_\gamma\) is the vector space with basis \(P^i_\gamma\) and

\[
df(x_1, \ldots, x_{i+1}) = \sum (-1)^{i+1} f(x_1, \ldots, x_jx_{j+1}, \ldots, x_{i+1}) \quad \text{for} \quad f \in \text{Hom}_k(kP^i_\gamma, k).
\]

We agree that if the length of \(x_jx_{j+1}\) is more than \(m\) for some \(j\) then the element \((x_1, \ldots, x_jx_{j+1}, \ldots, x_{i+1})\) is zero in \(kP^i_\gamma\).

We denote by \(C_\gamma\) this cocomplex and by \(H_\gamma\) its cohomology.

**Lemma 3.5.** Let \(\rho \in tQ_\rho s = \{ \rho \in Q_\rho | s(\rho) = s \text{ and } t(\rho) = t \}\) for \(s\) and \(t\) two fixed vertices of \(Q_0\).

For \(i > 0\), \(H^i(A, k\rho) = \bigoplus_{i > 0} \bigoplus_{\gamma \in tQ_\rho s} H^i_\gamma\).

**Proof of Lemma 3.5.** Consider the cocomplex \(C(k\rho)\) of the preceding section:

\[
0 \to (k\rho)^E \xrightarrow{d_0} \text{Hom}_{E-E}(r, k\rho) \xrightarrow{d_1} \text{Hom}_{E-E}(r \otimes_E r, k\rho) \to \cdots
\]

where \(E = kQ_0 \subset A = kQ/F^m\).

The differential \(d_0\) is zero because \(k\rho\) is a semisimple bimodule. Moreover \(\text{Hom}_{E-E}(r \otimes_{k^i}s, k\rho) = \text{Hom}_k(tr \otimes_{k^i}s, k)\) and \(tr \otimes_{k^i}s = \bigoplus_{i > 0} \bigoplus_{\gamma \in tQ_\rho s} kP^i_\gamma\).

Now it is easy to see that

\[
C(k\rho) = \left[ \bigoplus_{i > 0} \bigoplus_{\gamma \in tQ_\rho s} C_\gamma \right] \oplus [(k\rho)^E]_0,
\]

where \([(k\rho)^E]_0\) is the cocomplex with \((k\rho)^E\) in degree 0 and 0 in other degrees.
Lemma 3.6. For $l > 0$, consider $\gamma \in \mathcal{Q}_l$. There are canonical isomorphisms

$$
H^1_\gamma = \begin{cases} 0 & \text{if } l \neq 1 \\ k(\mathcal{Q}_1) & \text{if } l = 1 \end{cases}
$$

$$
H^2_\gamma = \begin{cases} 0 & \text{if } l \neq m \\ k(\mathcal{Q}_m) & \text{if } l = m \end{cases}
$$

$$
H^3_\gamma = \begin{cases} 0 & \text{if } l \neq m + 1 \\ k(\mathcal{Q}_{m+1}) & \text{if } l = m + 1 \end{cases}
$$

Proof of Lemma 3.6. Let $\rho \in \mathcal{Q}_l$. The $A$-bimodule $k\rho$ is isomorphic to $\text{Hom}_A(ks, kt)$ where $ks$ and $kt$ are the simple left $A$-modules corresponding to $s$ and $t \in \mathcal{Q}_0$ (see the Introduction).

So we have

$$H^i(A, k\rho) = H^i(A, \text{Hom}_A(ks, kt)) = \text{Ext}^i_A(ks, kt).$$

For the last isomorphism between Hochschild cohomology and the functor $\text{Ext}$ over the algebra $A$, see for instance [2, p. 179]. The dimension of $\text{Ext}^i_A(ks, kt)$ has been computed by K. Bongartz in [1, p. 463]. For $A$ a truncated quiver algebra we have

$$\dim_k H^1(A, k\rho) = \dim_k \text{Ext}^1_A(ks, kt) = |\mathcal{Q}_1|$$

$$\dim_k H^2(A, k\rho) = \dim_k \text{Ext}^2_A(ks, kt) = |\mathcal{Q}_m|$$

$$\dim_k H^3(A, k\rho) = \dim_k \text{Ext}^3_A(ks, kt) = |\mathcal{Q}_{m+1}|.$$ 

From the preceding lemma we have

$$H^i(A, k\rho) = \bigoplus_{l > 0} \bigoplus_{\gamma \in \mathcal{Q}_l} H^i_\gamma \quad \text{for } i > 0.$$ 

For each element $\gamma$ of $\mathcal{Q}_l$ (resp. $\mathcal{Q}_m$, $\mathcal{Q}_{m+1}$) we are going to associate a canonical non-zero element of $H^1_\gamma$ (resp. $H^2_\gamma$, $H^3_\gamma$). By evident dimensions reasons, this will prove the lemma.

Let $a \in \mathcal{Q}_l$ and consider $f_a : kP^1_a = ka \to k$, $f_a(a) = 1$. Clearly $f_a$ is a non-zero element of $H^1_a$ (observe that $P^0_a = P^2_a = \emptyset$).

Let $\gamma \in \mathcal{Q}_m$ and consider $f_\gamma : kP^2_\gamma = k$ where $f_\gamma(x, y) = 1$ for all $(x, y) \in P^2_\gamma$. We have $df_\gamma(x, y, z) = -f_\gamma(xy, z) + f_\gamma(x, yz) = 0$, for all $(x, y, z) \in P^3_\gamma$ (observe that the lengths of $xy$ and $yz$ are less than $m$, because the lengths of $x$ and $z$ are at least 1 and $\gamma$ is of length $m$). This 2-cocycle $f_\gamma$ is not a cobord: indeed for $g : P^1_\gamma = k\gamma \to k$ we have $dg(x, y) = g(xy) = 0$ for $(x, y) \in P^2_\gamma$ because the length of $xy$ is the length of $\gamma$ which is $m$. 

Let $\gamma \in tQ_{m+1}s$ and denote $c$ the last arrow of $\gamma$. We consider $f_\gamma: kP^3_\gamma \to k$,

$$f_\gamma(x, y, z) = \begin{cases} 1 & \text{if } x = c \\ 0 & \text{if } x \neq c. \end{cases}$$

We have $df_\gamma(x, y, z, u) = -f_\gamma(xy, z, u) + f_\gamma(x, yz, u) - f_\gamma(x, y, zu) = 0$ for all $(x, y, z, u) \in P^4_\gamma$. Indeed, if $x \neq c$ all the terms are 0; if $x = c$ we have $df_\gamma(c, y, z, u) = f_\gamma(c, yz, u) - f_\gamma(c, y, zu) = 1 - 1 = 0$ (observe that the lengths of $yz$ and $zu$ are less than $m$ because $c$ is of length one, $u$ of length at least one, and the total length of $\gamma$ is $m + 1$).

This 3-cocycle $f_\gamma$ is not a cobord: let $g: P^2_\gamma \to k$ be a $k$-linear map. For the particular element $(c, \beta, a) \in P^3_\gamma$ where $c$ is the last arrow of $\gamma$ and $a$ the first one (so $\beta$ is the oriented path of length $m - 1$ such that $\beta a = \gamma$ in the quiver algebra), we have $dg(c, \beta, a) = -g(\beta a) + g(c, \beta a) = 0$ because $\beta a$ and $\beta a$ are of length $m$. We infer $dg \neq f_\gamma$ because $f_\gamma(c, \beta, a) = 1$.

The canonical isomorphisms of Proposition 3.3 are now obtained collecting all the previous informations. To get them explicitly we define $P^i = \bigcup_{i > 0} \bigcup_{\gamma \in Q} P^i_\gamma$ which is a $k$-basis of $r^{\otimes e^i}$ for $i > 0$.

We obtain

$$k(Q_m//Q_p) \to H^1(A, r^p/r^{p+1})$$

$$(a, \rho) \mapsto f_{(a, \rho)}: r \to r^p/r^{p+1},$$

where $f_{(a, \rho)}(a) = \rho$ and $f_{(a, \rho)}(\gamma) = 0$ for all $\gamma \in P^1 \setminus \{a\}$ (notice that $P^1$ is just the set $Q_1 \cup Q_2 \cup \cdots \cup Q_{m-1}$).

$$k(Q_m//Q_p) \to H^2(A, r^p/r^{p+1})$$

$$(\gamma, \rho) \mapsto f_{(\gamma, \rho)}: r \otimes r \to r^p/r^{p+1},$$

where $f_{(\gamma, \rho)}(x, y) = \rho$ for all $(x, y) \in P^2_\gamma$ and $f_{(\gamma, \rho)}(u, v) = 0$ for all other elements of $P^2$.

$$k(Q_{m+1}//Q_p) \to H^3(A, r^p/r^{p+1})$$

$$(\gamma, \rho) \mapsto f_{(\gamma, \rho)}: r \otimes r \otimes r \to r^p/r^{p+1},$$

where $f_{(\gamma, \rho)}(x, y, z) = \rho$ if $(x, y, z) \in P^3_\gamma$ and $x$ is the last arrow of $\gamma$, and $f_{(\gamma, \rho)}(u, v, w) = 0$ for all other elements of $P^3$.

**Theorem 3.7.** Let $A = kQ/F^m$ be a truncated quiver algebra. The following diagram is commutative for $0 \leq p \leq m - 2$:

$$\begin{array}{c}
H^2(A, r^p/r^{p+1}) \\ \downarrow \\
H^3(A, r^{p+1}/r^{p+2}) \\
\downarrow \\
k(Q_m//Q_p) \to k(Q_{m+1}//Q_{p+1}).
\end{array}$$
where $A$ is the differential at level 1 of the spectral sequence, the vertical isomorphisms are the canonical ones of the previous proposition, and

$$D(\gamma, \rho) = \sum_{a \in Q_1} (a\gamma, a\rho) - \sum_{a \in Q_1} (\gamma a, \rho a).$$

**Remark.** The element $(a\gamma, a\rho)$ of $Q_{m+1} / Q_{p+1}$ is obtained by adding the arrow $a$ at the end of $\gamma$ and $\rho$. If $s(a) \neq t(\gamma) = t(\rho)$, then $(a\gamma, a\rho) = 0 \in k(Q_{m+1} / Q_{p+1})$.

**Proof.** Let $(\gamma, \rho) \in Q_m / Q_p$ and $f_{(\gamma, \rho)}: r \otimes_E r \to r^p / r^{p+1}$ be the corresponding 2-cocycle. From Proposition 3.2 we have $\Delta f_\gamma = \pi \sqcup f_\gamma - f_\gamma \sqcup \pi$, where $\pi: r \to r^2$ is the canonical projection. Notice that $\pi = \sum_{a \in Q_1} f((a, a))$, using the identification of $k(Q_1 / Q_1)$ with $H^1(A, r/r^2)$. Then a simple verification gives

$$\pi \sqcup f_{(\gamma, \rho)} - f_{(\gamma, \rho)} \sqcup \pi = \sum_{a \in Q_1} f_{(a\gamma, a\rho)} - \sum_{a \in Q_1} f_{(\gamma a, \rho a)}.$$

**Lemma 3.8.** Let $A = kQ/E^m$ be a truncated quiver algebra. The differential at level 1 of the spectral sequence $\Delta: H^1(A, r^p / r^{p+1}) \rightarrow H^2(A, r^p / r^{p+2})$ is zero for $0 \leq p \leq m-1$ and $m > 2$.

**Proof.** Let $(a, \rho) \in Q_1 / Q_p$ and $f_{(a, \rho)}$ be the corresponding 1-cocycle. From Proposition 3.2 we have

$$\Delta f_{(a, \rho)} = \pi \sqcup f_{(a, \rho)} + f_{(a, \rho)} \sqcup \pi.$$

Recall that $P^2$ is a $k$-basis of $r \otimes_E r$.

A simple inspection shows that $\pi \sqcup f_{(a, \rho)}$ and $f_{(a, \rho)} \sqcup \pi$ decomposes in a sum of 2-cocycles (using $\pi = \sum_{c \in Q_1} f_{(c, c)}$). Each one of these 2-cocycles takes values in $kcp$ or $kpc$ and has non-zero values only on the elements of $P^2$ for $\gamma$ an oriented path of length 2. When $m \neq 2$, we know from the proof of Lemma 3.6 that these cocycles are zero in $H^2(A, r^p / r^{p+2})$.

**Remark.** It is possible to prove a generalisation of Theorem 3.7 and Lemma 3.8. Namely

$$\Delta: H^{2n}(A, r^p / r^{p+1}) \rightarrow H^{2n+1}(A, r^p / r^{p+2})$$

is given (through the canonical identifications) by

$$D: k(Q_m / Q_p) \rightarrow k(Q_{m+1} / Q_{p+1})$$

$$(\gamma, \rho) \mapsto \sum_{a \in Q_1} (a\gamma, a\rho) - \sum_{a \in Q_1} (\gamma a, \rho a)$$

for $n > 0$ and $0 \leq p \leq m-2$. 

We have also that

\[ \Delta: H^{2n+1}(A, r^p/r^{p+1}) \to H^{2n+2}(A, r^p/r^{p+2}) \]

is zero for \( n \geq 0 \) if \( m > 2 \) and for \( n \geq 1 \) if \( m = 2 \).

4. Vanishing Conditions at Level 2

As before \( Q \) is a quiver, \( m \) an integer \( (m \geq 2) \), and \( A = kQ/F^m \) a truncated quiver algebra over an arbitrary field \( k \).

The objective of this section is to give necessary and sufficient conditions on \( Q \) called “vanishing conditions” such that the terms of total degree 2 at level 2 of the spectral sequence of \( A \) vanishes. This conditions will imply that \( H^2(A, A) = 0 \) by evident considerations on the spectral sequence. However, we cannot directly assert the converse, namely that if \( H^2(A, A) = 0 \) then the vanishing conditions are satisfied. To do so we should know that the spectral sequence collapses at level 2, but we don’t know if this is true.

Nevertheless we shall prove in Section 5 that if the vanishing conditions on \( Q \) are not verified, then the algebra \( A \) is not rigid (for a definition, see Section 5). Using the result of Gerstenhaber (if \( H^2(A, A) = 0 \) then \( A \) is rigid) we obtain the main result of this paper: a truncated quiver algebra is rigid if and only if the vanishing conditions are satisfied.

As a consequence we obtain that a truncated quiver algebra is rigid if and only if its \( H^2 \) is zero. For other sorts of algebras this is false: in [9] Gerstenhaber and Schack gave examples of incidence algebras of finite posets which are rigid but with non-zero \( H^2 \).

Let us first prove that we can assume that \( Q \) has no loop.

**Lemma 4.1.** Let \( Q \) a quiver with a loop \( b \) at a vertex \( s \) and \( A = kQ/F^m \) for some \( m \geq 2 \). Then \( H^2(A, A) \neq 0 \).

**Proof.** Let \( P^2 \) be the canonical basis of \( r \otimes_k r \). Define \( f: r \otimes_k r \to A \) by \( f(b^i b^j) = b^{m-1} \) for \( i > 0, j > 0, i + j \geq m \), and \( f(x, y) = 0 \) for all other couples \( (x, y) \in P^2 \). This \( E \)-bimodule homomorphism is a 2-cocycle which is not a coboundary by an easy computation.

In order to obtain the terms \( E^{p,q}_2 \) with \( p + q = 2 \), notice that as an immediate consequence of Lemma 3.8, we have for \( m > 2 \)

\[ E^{p,q}_2 = \ker \Delta: H^2(A, r^p/r^{p+1}) \to H^3(A, r^p/r^{p+2}) \hspace{1cm} (\ast) \]

for \( p + q = 2 \). If \( m = 2 \) the differential \( \Delta: H^1(A, A/r) \to H^2(A, r) \) is not zero in general. But if we assume that \( Q \) has no loop, then \( H^1(A, A/r) \) is zero by Lemma 3.3, so we obtain \((\ast)\) for all \( m \geq 2 \).
The pattern of $\Lambda$ we gave in Theorem 3.7 enables us to study its kernel. To do so, we need some definitions which arise naturally when trying to compute $\dim_k \ker \Lambda$.

From now on $Q$ is a quiver without loop. Recall that if $s \in Q_0$, we have

$$Q_1s = \{ a \in Q_1 \mid s(a) = s \}$$

$$sQ_1 = \{ a \in Q_1 \mid t(a) = s \}.$$ 

We want to define an equivalence relation on the set $Q_m/Q_p$ for $0 < p < m - 1$.

**Definitions.** Let $(\gamma, \rho) \in Q_m/Q_p$ be a couple of parallel paths from $s \in Q_0$ to $t \in Q_0$.

- We say that $(\gamma, \rho)$ *ends at a sink* (resp. *starts at a source*) if $Q_1t = \emptyset$ (resp. $sQ_1 = \emptyset$).
- We say that $(\gamma, \rho)$ *ends together* (resp. *starts together*) if the last (resp. the first) arrows of $\gamma$ and $\rho$ coincide.

Suppose $(\gamma, \rho) \in Q_m/Q_p$ starts together, so $\gamma = \gamma a$ and $\rho = \rho a$ for $a \in Q_1$ and $(\hat{\gamma}, \hat{\rho}) \in Q_{m-1}/Q_{p-1}$ (recall that $p$ is strictly positive). Moreover, suppose $(\gamma, \rho)$ does not end at a sink.

A *+movement* of $(\gamma, \rho)$ is the couple $(c\hat{\gamma}, c\hat{\rho}) \in Q_m/Q_p$, where $c$ is some arrow in $Q_1t$.

If $(\gamma, \rho) \in Q_m/Q_p$ does not start together or does not end at a sink, no +movement is defined on $(\gamma, \rho)$. We say that $(\gamma, \rho)$ is a *+extreme*.

Similarly, a *−movement* of $(\gamma, \rho) \in Q_m/Q_p$ is defined if $(\gamma, \rho)$ ends together and did not start at a source. If no −movement can be defined, we say that $(\gamma, \rho)$ is a *−extreme*.

The equivalence relation on $Q_m/Q_p$ is now the obvious one: $(\gamma, \rho)$ is equivalent to $(\gamma', \rho')$ if there exists an arbitrary sequence of + and −movements transforming $(\gamma, \rho)$ in $(\gamma', \rho')$.

There are some special equivalence classes in $Q_m/Q_p$: we call an equivalence class a *$p$−medal* if all its +extreme elements end at a sink and if all its −extreme elements start at a source.

**Proposition 4.2.** Let $Q$ be a quiver without loop and $\Lambda = kQ/F'' m$ a truncated quiver algebra for $m \geq 2$.

Let $\Lambda: H^2(A, r^p/r^{p+1}) \to H^3(A, r^{p+1}/r^{p+2})$ be the differential at the first level of the spectral sequence.

Then, for $0 < p < m - 1$, we have

$$\dim_k \ker \Lambda = \text{number of } p \text{−medals}.$$
We need two easy lemmas which follow the next definition.

**Definition.** Let \( x = \sum_{(\gamma, \rho) \in Q_m/Q_p} x_{(\gamma, \rho)} \) be an element of the vector space \( k(Q_{m+1}/Q_{p+1}) \). Its \((\gamma, \rho)\)-coordinate which we denote \( (\gamma, \rho)^* (x) = x_{(\gamma, \rho)} \). The support of \( x \) is the set of \((\gamma, \rho) \in Q_{m+1}/Q_{p+1} \) such that \((\gamma, \rho)^* (x) = x_{(\gamma, \rho)} \neq 0 \).

**Lemma 4.3.** Let \((\gamma, \rho) \in Q_m/Q_p\), with common source vertex \( s \) and common end vertex \( t \). The support of \( D(\gamma, \rho) \) is the set
\[
\{(a\gamma, a\rho) | a \in Q_1 t \} \cup \{(a, \rho a) | a \in sQ_1 \}.
\]
Moreover, \((a\gamma, a\rho)^* (D(\gamma, \rho)) = 1 \) for \( a \in Q_1 t \) and \((a, \rho a)^* (D(\gamma, \rho)) = -1 \) for \( a \in sQ_1 \).

**Proof.** By definition,
\[
D(\gamma, \rho) = \sum_{a \in Q_1} (a\gamma, a\rho) - \sum_{a \in Q_1} (a, \rho a)
\]
with \((a\gamma, a\rho) = 0 \) if \( a \notin Q_1 t \) and \((a, \rho a) = 0 \) if \( a \notin sQ_1 \). We only need the following facts:
\[
(a\gamma, a\rho) \neq (ca, c\rho) \quad \text{if} \quad a, c \in Q_1 t \quad \text{and} \quad a \neq c \\
(a, \rho a) \neq (c\rho, c\rho) \quad \text{if} \quad a, c \in sQ_1 \quad \text{and} \quad a \neq c \\
(a\gamma, a\rho) \neq (c\rho, c\rho) \quad \text{for} \quad a \in Q_1 t \quad \text{and} \quad c \in sQ_1.
\]

The two first assertions are trivial, the last one is true because \( Q \) has no loop.

**Lemma 4.4.** Let \((\gamma, \rho)\) be a couple of parallel paths in \( Q_m/Q_p \). Let \((a\gamma, a\rho)\) (resp. \((a, \rho a)\)) be an element of the support of \( D(\gamma, \rho) \).

Let \((\gamma', \rho') \in Q_m/Q_p\) with \((\gamma', \rho') \neq (\gamma, \rho)\).

The elements \((a\gamma, a\rho)\) (resp. \((a, \rho a)\)) is in the support of \( D(\gamma', \rho') \) if and only if \((\gamma, \rho)\) starts (resp. ends) together and \((\gamma', \rho')\) is the \(+\) (resp. \(-\)) movement of \((\gamma, \rho)\) corresponding to \( a \). Moreover \((a\gamma, a\rho)^* D(\gamma', \rho') = -1 \) (resp. \(+1\)).

**Proof.** Let \((\gamma', \rho') \in Q_m/Q_p\) such that \((a\gamma, a\rho)\) is in the support of \( D(\gamma', \rho') \). Then either \((a\gamma, a\rho) = (ca', c\rho')\) or \((a\gamma, a\rho) = (c\gamma', c\rho')\) for some arrow \( c \) (use the lemma above). In the first case we obtain \( a = c, \gamma = \gamma' \), and \( \rho = \rho' \). In the second one we infer that \( \gamma \) and \( \rho \) start together with the common arrow \( c \) and we can write \((\gamma, \rho) = (c\gamma', c\rho')\) for \((\gamma', \rho') \in Q_{m-1}/Q_{p-1}\). We have \((\gamma', \rho') = (a\gamma', a\rho')\) so \((\gamma', \rho')\) is the \(+\) movement of \((\gamma, \rho)\) associated to \( a \).
Proof of Proposition 4.2. For each $p$-medal $M$, let us consider the medal sum $\mathcal{M}$, an element of $k(Q_m/Q_p)$:

$$\mathcal{M} = \sum_{(\gamma, \rho) \in M} (\gamma, \rho).$$

We assert that the set $\{\mathcal{M}\}$ where $M$ runs over the set of $p$-medals is a basis of $\ker D$.

First we prove that $\mathcal{M} \in \ker D$. Let $(\gamma, \rho) \in M$ and suppose $(a\gamma, a\rho)$ is in the support of $D(\gamma, \rho)$. Clearly $(\gamma, \rho)$ does not end at a sink, so it is not a $+$-extreme, because $M$ is a $p$-medal. This means that $(\gamma, \rho)$ starts together and Lemma 4.4 gives us a unique element $(\gamma', \rho')$ in the medal such that $(a\gamma, a\rho)$ is in the support of $D(\gamma', \rho')$. Moreover the $(a\gamma, a\rho)$ coordinate of $D(\gamma', \rho')$ is $-1$.

Second we have that if $x = \sum_{(\gamma, \rho) \in Q_m/Q_p} x_{(\gamma, \rho)}(\gamma, \rho)$ is in the kernel of $D$, then $x$ is constant on the equivalence classes. More precisely if $x_{(\gamma, \rho)} \neq 0$ for some $(\gamma, \rho)$ and if $(\gamma', \rho')$ is a $+$ or $-$-movement of $(\gamma, \rho)$, then $x_{(\gamma', \rho')} = x_{(\gamma, \rho)}$. This is clear from Lemma 4.4. From this we infer formally that if $x_{(\gamma, \rho)} = 0$, then $x_{(\gamma', \rho')} = 0$.

Third we record that if $x \in \ker D$ and $(\gamma, \rho)$ is a $+$ (resp. $-$) extreme which does not end at a sink (resp. starts at a source), then $x_{(\gamma, \rho)} = 0$. Indeed, $(a\gamma, a\rho)$ is in the support of $D(\gamma, \rho)$ and there is not any other $(\gamma', \rho')$ with $(a\gamma, a\rho)$ in the support of $D(\gamma', \rho')$ (such an $(\gamma', \rho')$ should be a $+$ movement of $(\gamma, \rho)$ by Lemma 4.4).

Finally we note that the set of medal-sums is linearly independent because the supports of any two different medal-sums are disjoint.

We need now to study the cases $p = 0$ and $p = m - 1$. For $p = m - 1$ we have $A: H^2(A, r^{m-1}) \rightarrow 0$ because $r^m = 0$. Using Proposition 3.4 we obtain that $E_2^{m-1,1-m} = 0$ if and only if $Q_m/Q_{m-1} = \emptyset$. For $p = 0$, we have the following:

**Proposition 4.5.** In the situation of Proposition 4.2, consider


The dimension of $\ker \Delta$ is the number of connected components of $Q$ which are oriented crowns of length $m$ like the following one.
Proof. We study the kernel of
\[ D : k(Q_m//Q_0) \to k(Q_{m+1}//Q_1). \]

Let \( x \in \ker D \) and consider \( \gamma = \gamma_1 \cdots \gamma_1 \) an oriented cycle at the vertex \( s \), i.e., \( \gamma_i \in Q_1 \) for \( 1 \leq i \leq m \) and \( s(\gamma_1) = t(\gamma_m) = s \). So \( (\gamma, s) \) is an element of \( Q_m//Q_0 \) and we consider the \( (\gamma, s) \) coordinate \( x_{(\gamma, s)} \) of \( x \).

We assert that if there exist some arrow \( a \in sQ_1 \cup Q_1 s \), with \( a \neq \gamma_m \) and \( a \neq \gamma_1 \), then \( x_{(\gamma, s)} = 0 \). Indeed, \( (ay, a) \) (or \( (ya, a) \)) is in the support of \( D(\gamma, s) \) and is not in the support of \( D(\gamma', s') \) for any \( (\gamma', s') \in Q_m//Q_0 \) with \( (\gamma', s') \neq (\gamma, s) \).

Moreover if \( sQ_1 \cup Q_1 s = \emptyset \), we consider \( (\gamma^1, s^1) \in Q_m//Q_0 \) where \( \gamma^1 = \gamma_1 \gamma_m \cdots \gamma_2 \) and \( s^1 = s(\gamma_2) = t(\gamma_1) \). Clearly \( x_{(\gamma, s)} = x_{(\gamma^1, s^1)} \) using the fact that \( (\gamma_1 \gamma_m \cdots \gamma_2 \gamma_1, \gamma_1) \) is in the support of both \( D(\gamma, s) \) and \( D(\gamma^1, s^1) \) but is not in the support of \( D(\gamma', s') \) for any other \( (\gamma', s') \in Q_m//Q_0 \).

These two facts together easily imply the proposition.

Definition. The \( m \)-vanishing conditions on a quiver \( Q \) are the following ones:

(a) \( Q \) has no loop.

(b) There is no connected component of \( Q \) which is an oriented crown of length \( m \).

(c) \( Q_m//Q_{m-1} = \emptyset \).

(d) \( Q \) has no \( p \)-medals for \( 0 < p < m - 1 \).

Collecting the results of this section, we obtain the

Theorem 4.6. Let \( Q \) be a quiver, \( k \) a field, and \( A = kQ/F^m \) where \( m \geq 2 \). If \( Q \) satisfies the \( m \)-vanishing conditions, then \( H^2(A, A) = 0 \).

5. Deformations

Let \( A \) be a finite dimensional algebra over a field \( k \), and let \( K = k((x)) \) be the power series field in one variable. A one-parameter family of deformations of \( A \) is a structure of \( K \)-algebra over the \( K \)-vector space \( L = K \otimes_k A \) given by

\[ f : A \otimes_k A \to L, \]

where

\[ f(a, b) = ab + xf_1(a, b) + x^2f_2(a, b) + \cdots. \]
Remarks. For each positive i, the map \( f_i : A \otimes_k A \to A \) is \( k \)-linear. The term \( ab \) denotes the original product of \( A \). The \( k \)-map \( f \) can be extended in a unique way to a \( K \)-linear map \( f : L \otimes_k L \to L \) which gives the product of \( L \).

A result of Gerstenhaber (see [8]) states that if \( H^2(A, A) = 0 \), then \( A \) is rigid (which means that any one-parameter family of deformations of \( A \) is isomorphic over \( k \) to the trivial one obtained when \( f_i = 0 \) for \( i \geq 1 \)).

**Theorem 5.1.** Let \( k \) be a field, \( Q \) be a quiver, and \( m \) an integer \( (m \geq 2) \). The truncated quiver algebra \( A = kQ/F^m \) is rigid if and only if the \( m \)-vanishing conditions hold for \( Q \).

**Proof.** We know that the \( m \)-vanishing conditions imply that \( H^2(A, A) = 0 \), so \( A \) is rigid.

To prove the converse, we construct a non-trivial one-parameter family of deformations in each of the following cases:

(a) \( Q \) contains a loop \( b \).
(b) \( Q \) is an oriented crown of length \( m \).
(c) \( Q \) has no loop and \( Q_{m-1}/Q_m \neq \emptyset \).
(d) \( Q \) has no loop and contains a \( p \)-medal \( M \) for some \( p \) such that \( 0 < p < m - 1 \).

The cases (a), (b), and (c) are easy:

(a) Let \( b \) be a loop of \( Q \). We consider the algebra \( KQ/I \), where

\[ I = \langle b^m - xb^{m-1}, y | y \in Q_m \setminus \{ b^m \} \rangle. \]

We assert that the set \( \mathcal{C} = Q_0 \cup Q_1 \cup \cdots \cup Q_{m-1} \) is a \( K \)-basis of \( KQ/I \). This fact gives an identification between \( K \otimes_k A \) and \( KQ/I \) which gives immediately that \( KQ/I \) is a one-parameter family of deformations of \( kQ/F^m \). Moreover, this deformation is not trivial: the ideal \( I \) is different from \( F^m \subset KQ \) where \( F \) is the two-sided ideal of \( KQ \) generated by \( Q_1 \) (see the Introduction). Alternatively, notice that \( KQ/I \) has a non-trivial central idempotent: \( (xb)^{m-1} \).

If \( \delta \) is an oriented path of \( Q \), we denote \( l(\delta) \) its length. In order to prove the assertion, consider the set

\[ \{ \delta | l(\delta) \geq m \text{ and } \delta \neq b^i \} \cup \{ b^{m+i} - x^{i+1}b^{m-1} \text{ for } i > 0 \}. \]

This set is a \( K \)-basis of \( I \) and from this it is easy to prove that \( KQ = K\mathcal{C} \oplus I \).

(b) Suppose \( Q \) is an oriented crown:
We consider the $K$-algebra $KQ/I$ where

$$I = \langle \gamma_{i-1} \cdots \gamma_1 \gamma_m \cdots \gamma_i - xs_i \mid 1 \leq i \leq m \rangle.$$ 

It can be proven that the set $\mathcal{E}$ is again a $K$-basis. In fact this algebra is isomorphic to $M_m(K)$, the algebra of $m \times m$ matrices over $K$.

(c) Suppose $Q$ has no loop and $Q_{m-1}/Q_m \neq \emptyset$. Let $(\gamma, \rho)$ be a couple of parallel oriented paths in $Q_{m-1}/Q_m$, and consider $KQ/I$, where

$$I = \langle \beta, \gamma - xp \mid \beta \in Q_m \setminus \{\gamma\} \rangle.$$

There is no difficulty to prove that the set $\mathcal{E}$ of oriented paths of length less than $m$ is a $K$-basis of $KQ/I$. Moreover $KQ/I$ is not isomorphic to $K \otimes_k (kQ/F^m) = KQ/F^m$ because its radical $r$ verifies $r^m = k \gamma \neq 0$.

The case where $Q$ contains a $p$-medal is more difficult. The next proposition states the result.

**Definition.** Let $Q$ be a quiver with a $p$-medal $M \subset Q_m/Q_\rho$, where $m \geq 2$ and $0 < p < m - 1$.

We set

$$Q_m^M = \{ \gamma \in Q_m \mid \exists \rho \in Q_\rho \text{ s.t. } (\gamma, \rho) \in M \}.$$

**Proposition 5.2.** Let $Q$ be a quiver without loop and $M$ a $p$-medal for $0 < p < m - 1$ ($m \geq 2$).

Let $I$ be the two-sided ideal of $KQ$ generated by $Q_m \setminus Q_m^M$ and $\gamma - xp$ for all $(\gamma, \rho) \in M$.

The $K$-algebra $KQ/I$ is a non-trivial one-parameter family of deformations of $kQ/F^m$.

In section 4 we have defined $+$ and $-$ movements of elements of $Q_m/Q_\rho$. We can also define $+$ and $-$ movements of elements of $Q_m$. More precisely we have

**Definition.** Let $\gamma = \gamma_m \cdots \gamma_1$ be an oriented path from $s \in Q_0$ to $t \in Q_0$.

A $+$ movement $\gamma'$ of $\gamma$ is $\gamma' = a\gamma_m \cdots \gamma_2$, where $a \in Q_1 t$. If $Q_1 t = \emptyset$, no
+ movement of $\gamma$ exists. We define similarly $-$ movements of $\gamma$. Collecting + and $-$ movements we obtain an equivalence relation on $Q_m$.

**Lemma 5.3.** Let $Q$ be a quiver and $M$ a $p$-medal. Then $Q^M_m$ is an entire equivalence class in $Q_m$.

**Proof.** Let $\gamma \in Q^M_m$ and $\rho \in Q_p$ such that $(\gamma, \rho) \in M$. If a + movement $\gamma'$ of $\gamma$ exists then $t$ is not a sink, so $(\gamma, \rho)$ is not a + extreme because $M$ is a $p$-medal. The corresponding + movement of $(\gamma, \rho)$ gives then $\gamma' \in Q^M_m$. Similarly, if a $-$ movement $\gamma'$ of $\gamma$ exists, we obtain $\gamma' \in Q^M_m$.

**Definition.** Let $X$ be a set of oriented paths of a quiver $Q$. We denote $\tilde{X}$ the set of oriented paths "generated by $X"$, namely

$$\tilde{X} = \{ \beta \gamma \alpha \mid \gamma \in X, \beta \text{ and } \alpha \text{ are oriented paths such that } t(\alpha) = s(\gamma) \text{ and } t(\gamma) = s(\beta) \}.$$ 

**Lemma 5.4.** Let $Q$ be a quiver and $M$ a $p$-medal. Then

$$Q^M_m \cap Q_m \setminus Q^M_m = \emptyset.$$ 

**Proof.** Let $\beta \gamma \alpha \in Q^M_m$, where $\gamma \in Q^M_m$. We assert that all subpaths of length $m$ of $\beta \gamma \alpha$ are in $Q^M_m$. Indeed such a path $\gamma'$ is obtained from $\gamma$ by a sequence of movements. Using the lemma above, we obtain that $\gamma' \in Q^M_m$.

**Definition.** For $M$ a $p$-medal, let $\tilde{M}$ be the following set of couples of oriented paths:

$$\tilde{M} = \{ (\beta \gamma \alpha, \beta \rho \alpha) \mid (\gamma, \rho) \in M, \beta \text{ and } \alpha \text{ are oriented paths such that } t(\alpha) = s(\gamma) = s(\rho) \text{ and } s(\beta) = t(\gamma) = t(\rho) \}.$$ 

**Lemma 5.5.** Let $Q$ be a quiver and $M$ a $p$-medal. Let $(A, B)$ and $(A', B')$ be elements of $\tilde{M}$. Then $A = A'$ if and only if $B = B'$.

We postpone the proof of this lemma (which is technical) after the proof of the proposition.

**Proof of Proposition 5.2.** It is enough to prove that the set $\mathcal{C}$ of oriented paths of length less than $m$ is a $K$-basis of $KQ/I$. If it is so, $KQ/I$ is a one-parameter family of deformations of $kQ/F^m$ which is not trivial because $I \neq F^m \subset KQ$ (see the Introduction).

First we record that $\mathcal{C}$ generates $KQ/I$ as a vector space. Certainly the set of all oriented paths generates $KQ/I$. 


Let $\delta \in \overline{Q}_m$. If $\delta \in Q_m \setminus Q_m^M$, then $\delta \in I$ and $\delta = 0$ in $KQ/I$. If $\delta \in Q_m^M$, then there exist $(A, B) \in \bar{M}$ with $\delta = A$. By definition $A = xB$ in $KQ/I$, and we notice that $l(B) < l(A)$ (recall that $p < m$). If $l(B) \geq m$, we reapply the procedure to $B$ instead of $\delta$. If $l(B) < m$, then $B \in \mathcal{C}$ and we have expressed $\delta$ as a multiple of a path of $\mathcal{C}$.

In order to prove that the set $\mathcal{C}$ is linearly independent in $KQ/I$, suppose there is a non-zero $K$-linear combination $\sum_{c \in \mathcal{C}} \lambda_c c$ which lies in $I$. Clearly the ideal $I$ is $K$-generated by $Q_m \setminus Q_m^M \cup \{ A - xB \mid (A, B) \in \bar{M} \}$. We have

$$\sum_{c \in \mathcal{C}} \lambda_c c = \sum_{(A, B) \in \bar{M}} \lambda_{(A, B)} (A - xB) + \sum_{\delta \in Q_m \setminus Q_m^M} \lambda_\delta \delta. \tag{\ast}$$

Let $c \in \mathcal{C}$ such that $\lambda_c \neq 0$. Recall that $c \notin Q_m \setminus Q_m^M$ and $c \neq A$ for all $(A, B) \in \bar{M}$, by length reasons. So there exist $(A, B) \in \bar{M}$ with $c = B$. Moreover $(A, B)$ is unique by Lemma 5.5 and $\lambda_c = -x\lambda_{(A, B)}$. Notice that $l(A) > m$, so the final coefficient of $A$ at the right hand side of (\ast) must be zero.

By Lemma 5.4 we have $Q_m^M \cap Q_m \setminus Q_m^M = \emptyset$ and we know that $A \in Q_m^M$. We infer there exist a unique (Lemma 5.5) $(A_1, B_1) \in \bar{M}$ with $B_1 = A$ and $\lambda_{(A_1, B_1)} = x\lambda_{(A_1, B_1)}$.

But $l(A_1) = l(B_1) = l(A)$ and the final coefficient of $A_1$ at the right hand side of (\ast) must be zero. We reapply the procedure to $(A_1, B_1)$ instead of $(A, B)$, and continuing this way we obtain a contradiction using that the sum at the right hand side of (\ast) is finite.

**Definition.** A movement is a sequence of $+$ and $-$ movements, either in $Q_m/Q_p$ or in $Q_m$.

**Proof of Lemma 5.5.** We just need to prove the following: for $(\gamma, \rho)$ and $(\gamma', \rho')$ elements of $M$, we have $\gamma = \gamma'$ iff $\rho = \rho'$. Indeed for $(A, B)$ and $(A, B')$ elements of $\bar{M}$, we have $A = B \gamma x$, $B = B \rho x$, $A = B' \gamma' x'$, and $B' = B' \rho' x'$ where $(\gamma, \rho)$ and $(\gamma', \rho')$ are elements of $M$. There is a movement along $A$ taking $\gamma$ to $\gamma'$. Because $M$ is a medal, we know that the same movement can be applied to $(\gamma, \rho)$, transforming it into $(\gamma', \rho' \rho' x')$. We have $B = B' \rho' x'$. If we know that if $(\gamma', \rho')$ and $(\gamma', \rho'' \rho' x')$ are in the same medal then $\rho' = \rho''$, we infer $B = B'$. The same thing can be done for $(A, B)$ and $(A', B)$ in $\bar{M}$; we obtain $A = A'$.

Let us prove now that if $(\gamma, \rho)$ and $(\gamma, \rho')$ are in $M$ then $\rho = \rho'$. The proof of $(\gamma, \rho)$ and $(\gamma', \rho)$ in $M$ implies $\gamma = \gamma'$ is completely analogous.

As a first case, suppose there exist $(\beta, x) \in M$ such that there is an oriented cycle at the common end vertex. Because $M$ is a medal (\(+\) extreme elements have to end at a sink) all the $+$ movements for $\beta$ along this oriented cycle can be performed for $(\beta, x)$. This implies that $(\beta, x)$ starts
together, their + movement also, and so on. We infer that \( \beta = \beta \alpha \), and if 
\((\beta, \alpha')\) is in \( M \) we obtain also \( \beta = \beta \alpha' \), so \( \alpha = \alpha' \).

Now choose a movement taking \((\gamma, \rho)\) to \((\beta, \alpha)\). The same movement can
be applied to \((\gamma, \rho')\) (because it can be applied to \( \gamma \) and \( M \) is a medal). This
movement takes \((\gamma, \rho')\) to \((\beta, \alpha')\). But we have proved that \( \alpha = \alpha' \). So \( \rho \) and
\( \rho' \) gives the same path by the same movement. This implies that they are equal.

The similar thing can be done if there is an element \((\beta, \alpha)\) in \( M \) with an
oriented cycle at the source vertex.

The second case is the following: suppose that for each \((\gamma, \rho)\) in \( M \) there
is no oriented cycle at the source or end vertex.

For \( \rho \) an oriented path, we define the character \( \chi_\rho : Q_1 \to \mathbb{N} \) by
\( \chi_\rho(a) = \text{number of arrows of } \rho \text{ which are equal to } a \). Notice that if \( \rho \) does
not contain an oriented cycle and if \( \rho' \) is some path such that \( \chi_\rho - \chi_{\rho'} \) then
\( \rho = \rho' \).

If \((\gamma, \rho)\) and \((\gamma', \rho')\) are elements of \( M \), we have \( \chi_{\gamma'} - \chi_{\gamma} = \chi_{\rho'} - \chi_{\rho} \). So if
\( \gamma = \gamma' \), we have \( \chi_{\rho} = \chi_{\rho'} \). Moreover if \( \rho \) has no oriented cycle, we obtain
\( \rho = \rho' \).

We assume now that \( \rho \) has an oriented cycle. Let \( \rho = \rho_p \cdots \rho_1 \) where
\( \rho_i \in Q_1 \) and let
\[
(s_0 = s(\rho_1), s_1 = s(\rho_2), \ldots, s_{p-1} = s(\rho_p), s_p = t(\rho_p))
\]
be the sequence of vertices of \( \rho \). Let \( s_a \) be the first vertex such that there
exist \( i > a \) with \( s_i = s_a \). Let \( s_b \) be the last vertex such that there exist \( i < b \)
with \( s_i = s_b \). Notice that \( a \) and \( b \) exists because \( \rho \) has an oriented cycle. We
have \( a \leq b \).

This defines a canonical decomposition of \( \rho \), which we write \( \rho = \rho_+ \rho_0 \rho_- \) where

\[
\rho_- = \rho_a \cdots \rho_1 \\
\rho_+ = \rho_p \cdots \rho_{b+1}
\]

and

\( \rho_0 = \rho_b \cdots \rho_{a+1} = \text{the core path of } \rho. \)

The assumption that there is no oriented cycles at the source or at the
cend of \((\gamma, \rho)\) implies \( a > 0 \) and \( b < p \), so \( \rho_a \) and \( \rho_b \) are oriented paths of
positive length.

**Assertion.** Let \( M \) be a medal such that for each \((\gamma, \rho)\) in \( M \) there is no
oriented cycle at their source and end vertices. Let \((\gamma, \rho)\) be an element of \( M \)
such that \( \rho \) has an oriented cycle. Then for any \((\gamma', \rho')\) in \( M \), the path \( \rho' \) has
an oriented cycle. Moreover, the canonical core paths of \( \rho \) and \( \rho' \) are equal.
Proof of the assertion. Let \((\gamma, \rho) \in M\) where \(\rho = \rho_p \cdots \rho_1\) \((\rho_i \in Q)\). Let \(\rho = \rho_+ + \rho_0 \rho_-\) be the canonical decomposition of \(\rho\). It is enough to prove the assertion for \((\gamma', \rho')\) a \(+\) or \(-\) movement of \((\gamma, \rho)\).

Suppose \((\gamma', \rho')\) is a \(+\) movement of \((\gamma, \rho)\): there is an arrow \(a \in s_{p+1}Q_1s_p\) such that \(\rho' = a \rho_p \cdots \rho_2\). The sequence of vertices of \(\rho'\) is \((s_1, \ldots, s_p, s_{p+1})\). We assert that the vertex \(s_a\) of \(\rho\) (first vertex of \(\rho\) such that there exist \(i > a\) with \(s_i = s_a\)) is still in the sequence of \(\rho'\) and that \(s_a\) is the first vertex of \(\rho'\) such that there exist \(j > a\) with \(s_j = s_a\). Indeed, if \(s_a = s_0\) we would have an oriented cycle at the source vertex of \((\gamma, \rho)\). Moreover, if there exist a vertex \(s_d\) \((d < a)\) such that \(s_d = s_i\) for \(i > d\), we obtain \(s_i = s_{p+1}\) and \((\gamma', \rho')\) have an oriented cycle at the end vertex which contradicts our hypothesis.

We prove similarly that the vertex \(s_b\) of \(\rho\) is the good one for \(\rho'\): first we notice that \(s_b = s_0\) is impossible. Second we record that the vertex \(s_{p+1}\) cannot have the property that there exist \(i \leq p\) such that \(s_i = s_{p+1}\) because \((\gamma', \rho')\) has no oriented vertex at \(s_{p+1}\) by hypothesis.

The case where \((\gamma', \rho')\) is a \(-\) movement of \((\gamma, \rho)\) is completely analogous.

We return now to our problem: if \((\gamma, \rho)\) and \((\gamma', \rho')\) are in this special medal \(M\), we want to prove \(\rho = \rho'\). Until now we have the following: let \(\rho = \rho_+ + \rho_0 \rho_-\) and \(\rho' = \rho'_+ + \rho'_0 \rho'_-\) be the canonical decompositions. Then \(\rho_0 = \rho'_0\). Recall that we know also that \(\chi_\rho = \chi_{\rho'}\), so we obtain \(\chi_{\rho_-} + \chi_{\rho_+} = \chi_{\rho'_-} + \chi_{\rho'_+}\). We would like to infer that \(\rho_- = \rho'_-\), \(\rho_+ = \rho'_+\).

To do so we notice that \(\rho_-\) and \(\rho'_-\) are parallel paths: they have \(s_0\) as common source vertex and their end vertex is the source of the core path. Similarly, \(\rho_+\) and \(\rho'_+\) are also parallel.

Moreover, the sequence of sources of the arrows of \(\rho_a\) and \(\rho_b\) has no repetitions (this follows immediately from the definition of \(s_a\) and \(s_b\) for \(\rho\)).

With these two facts, it is clear that \(\chi_{\rho_-} + \chi_{\rho_+} = \chi_{\rho'_-} + \chi_{\rho'_+}\) implies \(\rho_- = \rho'_-\) and \(\rho_+ = \rho'_+\).

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