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# Existence and Nonexistence of Nontrivial Solutions of Some Nonlinear Degenerate Elliptic Equations

MITSUHARU ÔTANI

*Department of Mathematics, Faculty of Science, Tokai University, Hiratsuka, Kanagawa, Japan 259-12*

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In connection with the maximizing problem for the functional $R(u) = \|u\|_{L^q}/\|\nabla u\|_{L^p}$ in $W_0^{1,p}(\Omega)\setminus\{0\}$, we consider the equation

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2}\nabla u(x)) &= |u|^{q-2}u(x), && x\in\Omega,\ 1<p,\ q<\infty,\ p\neq q, \\ u(x) &= 0, && x\in\partial\Omega. \end{aligned} \tag{E}$$

It is shown that for the case $q<p^*$ ($p^*=\infty$ if $p\geqq N$, and $p^*=Np/(N-p)$ if $p<N$), (E) has always a nonnegative nontrivial solution belonging to $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, and for the case $p<N$ and $q>p^*$ (resp. $q=p^*$), (E) has no nontrivial (resp. nonnegative nontrivial) solution belonging to the class $P=\{u\in W_0^{1,p}(\Omega)\cap L^q(\Omega);\ x_i|u|^{q-2}u\in L^{p/(p-1)}(\Omega),\ i=1,2,\dots,N\}\subset W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, provided that $\Omega$ is star shaped. The crucial point of the proof of our result is to obtain an $L^\infty$-estimate of weak solutions and to verify a certain "Pohozaev-type *inequality*" for weak solutions belonging to $P$. 

## 0. INTRODUCTION

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. For $p, q\in(1,+\infty)$, consider the Sobolev–Poincaré-type inequality

$$\|u\|_{L^q}\leqq C\|\nabla u\|_{L^p} \qquad \text{for all}\quad u\in W_0^{1,p}(\Omega), \tag{SP}$$

where $\|u\|_{L^q}$ and $\|\nabla u\|_{L^p}$ denote the $L^q(\Omega)$-norm of $u$ and the $L^p(\Omega)$-norm of $|\nabla u| = (\sum_{i=1}^N |\partial u/\partial x_i|^2)^{1/2}$, respectively. Suppose that $p$ and $q$ satisfy

$$q<p^*, \tag{C}$$

where $p^*=\infty$ if $p\geqq N$ and $p^*=Np/(N-p)$ if $p<N$. Then, by Rellich's compactness theorem, there exists an element $w\in W_0^{1,p}(\Omega)$ which gives the best possible constant for (SP), i.e., $w$ maximizes the functional $R(v) = \|v\|_{L^q}/\|\nabla v\|_{L^p}$ in $W_0^{1,p}(\Omega)\setminus\{0\}$. Furthermore, if $p\neq q$, then there exists a con-



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stant $\lambda \neq 0$ such that $u = \lambda w$ becomes a nontrivial solution of the nonlinear elliptic equation of the form (see Theorem I)

$$\left.\begin{aligned} -\Delta_p u(x) &= |u|^{q-2} u(x), & x \in \Omega, \quad (0.1)\\ u(x) &= 0, & x \in \partial\Omega, \quad (0.2) \end{aligned}\right\} \quad (\mathrm{E})$$

where $\Delta_p u(x) = \operatorname{div}(|\nabla u|^{p-2} \nabla u(x))$.

When $N = 1$ and $p \neq q$, it is shown in [14] that the set of all nontrivial solutions $S$ of (E) consists of a countable number of functions $u_n$ with $(n-1)$ isolated zeros ($n \in \mathbb{N}$), which are characterized as critical points of $R(\cdot)$ on certain families of subsets in $W_0^{1,p}(\Omega)$. As for the critical case $p = q$ ($N = 1$), we are led to a nonlinear eigenvalue problem for (E), and it is shown in [15] that there exist a positive number $a_p$ and functions $e_n$ with $(n-1)$ isolated zeros ($n \in \mathbb{N}$) such that $S$ is not empty and $S = \{\lambda e_n\}_{\lambda \in R^1 \setminus \{0\}}$ if and only if the best possible constant $C(\Omega) = \sup\{R(v);\, v \in W_0^{1,p}(\Omega) \setminus \{0\}\}$ for (SP) is equal to $na_p$ (cf. Remark 2.4).

When $p = 2$ (i.e., $\Delta_p = \Delta$), this type of problem is studied by many authors. It is well known that if condition (C) is satisfied (resp. (C) is not satisfied and $\Omega$ is star shaped), then (E) has a (resp. no) nontrivial solution belonging to $C^2(\Omega) \cap C^1(\bar{\Omega})$; see, e.g., Berger [1], Browder [4], Coffman [5], and Pohozaev [16]. For the case where the nonlinear term $|u|^{q-2} u$ is replaced by more complicated ones, some interesting results on existence and nonexistence of nontrivial solutions can be found in Brézis and Nirenberg [3], Coron [6], and Ni [12].

The main purpose of this paper is to discuss the regularity of weak solutions of (E) and to give a nonexistence result for (E) in a class of weak solutions. In studying the nonexistence of solutions of (E), we must first note that nontrivial solutions of (E) with $p \neq 2$ do not always belong to $C^2(\Omega) \cap C^1(\bar{\Omega})$ (see [14]), and that even for the case $p = 2$, if (C) is not satisfied, then the so-called "boot strap method" does not work any more to show that every weak solution belongs to $C^2(\Omega) \cap C^1(\bar{\Omega})$. Second, the proof for the nonexistence results usually relies much on "Pohozaev-type *identity*," which is valid only for solutions in $C^2(\Omega) \cap C^1(\bar{\Omega})$ in general. In the special case where $\Omega$ is a ball $B$, DeThélin [18] showed the usual Pohozaev-type identity for the radially symmetric solutions of (E) belonging to $C^2(B \setminus \{0\}) \cap C^1(\bar{B})$, and quite recently Ni and Serrin [13] introduced a generalized Pohozaev-type identity for the radially symmetric solutions of (E) in $B \setminus \{0\}$ belonging to $C^2(B \setminus \{0\})$. As a matter of course, one can show by the standard arguments that (E) has a nontrivial solution in $W_0^{1,p}(\Omega) \cap L^q(\Omega)$ if (C) is satisfied, and that (E) has no nontrivial solution in $C^2(\Omega) \cap C^1(\bar{\Omega})$ if (C) is not satisfied and $\Omega$ is star shaped. However, this type of result is incomplete. In other words, one would not be convinced that condition (C) leaves no room for improvement as a sufficient condition for the existence of nontrivial solutions, unless the existence and nonexistence should be discussed in the same function space. In order to unravel these difficulties, we provide two types of results. First, we give (in Theorem II) a sufficient condition, weaker than (C), under which all weak solutions of (E) belong to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. The proof of this regularity result relies on Moser's iteration sheme. Second, we give a nonexistence result (in Theorem III) in the class $P = \{u \in W_0^{1,p}(\Omega) \cap L^q(\Omega);\ x_i|u|^{q-2}u \in L^{p/(p-1)}(\Omega),\ i = 1, 2, ..., N\} \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. To this end, we shall introduce a certain "Pohozaev-type *inequality*" which is valid for all weak solutions in $P$. Then, in particular, it is shown that (E) with $p \neq q$ has a nontrivial nonnegative solution in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ or $P$ if and only if $q < p^*$, under some starshapedness assumptions on $\Omega$.

## 1. Main Results

Before stating our results, we fix some terminology and notations which will be frequently used in this paper. We say that a function $u$ is a *weak solution* of (E) if $u$ belongs to $W_0^{1,p}(\Omega) \cap L^q(\Omega)$ and satisfies (0.1) in the sense of distribution. It is obvious that (E) has always the *trivial solution* $u(x) \equiv 0$, so our concern here is the existence and nonexistence of *nontrivial solutions* $u(x) \not\equiv 0$. In the nonexistence result, we shall be concerned with star shapedness of domain in the following sense. The domain $\Omega$ is said to be *star shaped* (resp. *strictly star shaped*) if $(x \cdot n(x)) \geqq 0$ (resp. $(x \cdot n(x)) \geqq \rho > 0$) holds for all $x \in \partial\Omega$ with a suitable choice of the origin, where $n(x) = (n_1(x), n_2(x), ..., n_N(x))$ denotes the outward normal unit vector at $x \in \partial\Omega$.

Then our main results are stated as follows.

Theorem I (Existence). *Let $q < p^*$ and $q \neq p$, where $p^* = \infty$ if $p \geqq N$ and $p^* = Np/(N-p)$ if $p < N$. Then* (E) *has at least one nonnegative nontrivial weak solution.*

Theorem II (Regularity). *Let $q_0 = p^*(q-p)/(p^*-p)$ if $p < N$, and $q_0 = (q-p)$ if $p \geqq N$. If a weak solution $u$ of* (E) *belongs to $L^{q_1}(\Omega)$ with $q_1 \geqq q$ and $q_1 > q_0$, then $u$ belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Theorem III (Nonexistence). *Let $P = \{u \in W_0^{1,p}(\Omega) \cap L^q(\Omega);\ x_i|u|^{q-2}u \in L^{p'}(\Omega),\ p' = p/(p-1),$ for $i = 1, 2, ..., N\}$. Then we have*:

(i) *Let $\Omega$ be star shaped. If $p < N$ and $q > p^*$, then* (E) *has no nontrivial weak solution belonging to $P$.*

(ii) *Let $\Omega$ be strictly star shaped. If $p > N$ and $q = p^*$, then* (E) *has no nontrivial weak solution of definite sign belonging to $P$.*

*Remark* 1.1. (1) Since condition (C), $q < p^*$, implies $q > q_0$, one can take $q_1 = q$ in Theorem II. Therefore, Theorems I and II say that if (C) is satisfied, then (E) has a nonnegative nontrivial weak solution in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. On the other hand, Theorem III assures that if $\Omega$ is (strictly) star shaped, then (E) has no nonnegative nontrivial solution in $P \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. In this sense, condition (C) is best possible as a sufficient condition for the existence of nontrivial solutions of (E).

(2) The $L^\infty$-estimate of weak solutions as in Theorem II is very important information. In fact, the result of DiBenedetto [7] or Lewis [10] assures that every bounded weak solution of (0.1) enjoys $C^{1+\alpha}_{\text{loc}}$-regularity. Furthermore, Theorem 1.1 of Ladyzhenskaya and Ural'tseva [9, p. 251] says that every weak solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of (0.1)–(0.2) belongs to $C^\beta(\bar{\Omega})$ for some $\beta \in (0, 1)$, and a more minute estimate such that $u \in C^{1+\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$ can be derived from the result of Tolksdorf [19].

(3) The class of weak solutions treated in our nonexistence result is larger than that in [18] or [13] if $q > p^*$. In [13], Ni and Serrin dealt with an apparently different class of solutions, "singular ground states." However, as is shown below, this class turns out to fall within our class of weak solutions $P$, provided that $q > p^*$.

PROPOSITION 1.2. *Let $p < N$ and $q > p^*$. Suppose that $u \in C^2(B_R \setminus \{0\})$ is a radially symmetric solution of the problem*

$$\left.\begin{aligned} -\Delta_p u(x) &= |u|^{q-2} u(x) \qquad \text{in } B_R \setminus \{0\}, && (1.1) \\ u(x) &= 0 \text{ on } \partial B_R, \qquad u(x) > 0 \text{ in } B_R, && (1.2) \end{aligned}\right\} \quad (\mathrm{E})_0$$

*where $B_R$ is a ball of radius $R$ in $\mathbb{R}^N$. Then $u$ becomes a weak solution of* (E) *with $\Omega = B_R$ belonging to $P$.*

*Proof.* By virtue of Theorem 5.1 and Lemma 5.1 of [13] (with obvious modifications), there exist a constant $C$ and a sequence $\{r_k\}$ which tends to 0 as $k \to +\infty$ such that

$$v(r) \leqq C r^{-p/(q-p)} \qquad \text{for all} \quad r \in (0, R], \tag{1.3}$$

$$|v'(r_k)| \leqq C r_k^{-q/(q-p)} \qquad \text{for all} \quad k \in \mathbb{N}, \tag{1.4}$$

where $v(r) = u(x)$, $v'(r) = dv(r)/dr$, and $r = |x|$. Then it is easy to see that if $q > p^*$, then (1.3) assures that $u \in L^q(\Omega)$ and $x_i |u|^{q-2} u \in L^{p/(p-1)}(\Omega)$, $i = 1, 2, ..., N$.

On the other hand, multiplying (1.1) by $w \in C_0^2(B_R \setminus \{0\}) =$

$\{w \in C^2(B_R \setminus \{0\});\ w(x)=0$ on $\partial B_R\}$ and integrating on $\Omega_k = \{x \in \mathbb{R}^N;\ r_k \leqq |x| \leqq R\}$, we have

$$\int_{\Omega_k} |\nabla u|^{p-2} \nabla u(x) \nabla w(x)\, dx = \int_{\Omega_k} |u|^{q-2} u(x)\, w(x)\, dx + I_k(u, w),$$
$$I_k(u, w) = \int_{|x| = r_k} |\nabla u|^{p-2} \nabla u(x) \cdot n(x)\, w(x)\, dS. \tag{1.5}$$

Here (1.3) and (1.4) give

$$|I_k(u, u)| \leqq C r_k^{\alpha}, \qquad \alpha = N - 1 - (pq + p - q)/(q - p).$$

Therefore, if $q > p^*$, then $I_k(u, u) \to 0$ and $I_k(u, w) \to 0$ for all $w \in C_0^\infty(\Omega)$ as $k \to +\infty$. Hence. from (1.5), we deduce that $\|\nabla u\|_{L^p}^p = \|u\|_{L^q}^q$ and

$$\int_\Omega |\nabla u|^{p-2} \nabla u(x) \nabla w(x)\, dx = \int_\Omega |u|^{q-2} u(x)\, w(x)\, dx \qquad \text{for all} \quad w \in C_0^\infty(\Omega).$$

Thus $u$ is shown to be a weak solution of (E) belonging to $P$. Q.E.D.

## 2. Existence of Nontrivial Solutions

### 2.1. *Proof of Theorem I*

As a matter of course, Theorem I can be proved by the standard argument such as in Berger [1] and DeThélin [18]. However, we here introduce another type of abstract treatment, which seems to be of independent interest. To this end, we first fix some notations.

Given a real Banach space $X$ and its dual space $X^*$, we denote by $(\cdot,\cdot)_X: X^* \times X \to R^1$ the natural duality pairing between $X^*$ and $X$. In particular, if $X$ is a real Hilbert space and $X$ is identified with $X^*$, then $(\cdot,\cdot)_X$ designates the inner product in $X$. Let $\Phi(X)$ be the set of all lower semicontinuous convex functions from $X$ into $(-\infty, +\infty]$ which are not identically $+\infty$. For each $\phi \in \Phi(X)$, the *effective domain* $D(\phi)$ is the set

$$D(\phi) = \{z \in X;\ \phi(z) < +\infty\},$$

and the *subdifferential* $\partial\phi$ of $\phi$ is defined by

$$\partial\phi(z) = \{z^* \in X^*;\ \phi(w) - \phi(z) \geqq (z^*, w - z)_X \qquad \text{for all} \quad w \in D(\phi)\}$$

with domain

$$D(\partial\phi) = \{z \in X;\ \partial\phi(z) \neq \varnothing\}.$$

Then it is well known that $\partial\phi$ becomes a (possibly multivalued) maximal monotone operator from $X$ into $X^*$ (see Moreau [11], Rockafellar [17], and Brézis [2]). Now we are ready to state our abstract result.

PROPOSITION 2.1. *Let $X$ be a real Banach space and $\phi^i$ $(i=1,2)$ be nonnegative functions in $\Phi(X)$ satisfying*:

(i) *There exist two exponents $\alpha_i > 1$ with $\alpha_1 \neq \alpha_2$ such that $\phi^i(\lambda v) = \lambda^{\alpha_i}\phi^i(v)$ for all $v \in D(\phi^i)$ and $\lambda > 0$.*

(ii) $(\phi^2(v))^{1/\alpha_2} \leqq C(\phi^1(v))^{1/\alpha_1}$ *for all $v \in D(\phi^1)$.*

*Suppose that there exists an element $u \in D(\partial\phi^2)$ such that*

(iii) *$u$ gives the best possible constant for* (ii), *i.e.*,

$$R(u) = \max\{R(v);\ v \in D(\phi^1),\ \phi^1(v) \neq 0\}, \qquad \textit{where } R(v) = \frac{(\phi^2(v))^{1/\alpha_2}}{(\phi^1(v))^{1/\alpha_1}}.$$

(iv) $\alpha_1\phi^1(u) = \alpha_2\phi^2(u)$.

*Then $u$ belongs to $D(\partial\phi^1)$ and $\partial\phi^2(u) \subset \partial\phi^1(u)$. In particular, if $\partial\phi^1$ is single valued, then $u$ becomes a nontrivial solution of $\partial\phi^1(u) = \partial\phi^2(u)$.*

*Proof.* First of all, note that (i) implies

$$(z^*, z)_X = \alpha_i\phi^i(z) \qquad \text{for all} \quad z \in D(\partial\phi^i) \qquad \text{and} \qquad z^* \in \partial\phi^i(z). \tag{2.1}$$

In fact, since

$$(\lambda^{\alpha_i} - 1)\,\phi^i(z) = \phi^i(\lambda z) - \phi^i(z) \geqq (z^*, \lambda z - z)_X = (\lambda - 1)(z^*, z)_X,$$

dividing both sides by $(\lambda - 1)$ and letting $\lambda \downarrow 1$ and $\lambda \uparrow 1$, we obtain (2.1). In order to show $\partial\phi^2(u) \subset \partial\phi^1(u)$, it suffices to verify

$$(u^*, w - u)_X \leqq \phi^1(w) - \phi^1(u) \qquad \text{for all} \quad u^* \in \partial\phi^2(u) \qquad \text{and} \qquad w \in D(\phi^1). \tag{2.2}$$

We are going to prove this in two cases: $\phi^1(w) = 0$ and $\phi^1(w) > 0$.

(1) *The case $\phi^1(w) = 0$.* By (ii), we get $\phi^2(w) = 0$. Then

$$(u^*, \lambda w - u)_X \leqq \lambda^{\alpha_2}\phi^2(w) - \phi^2(u) = -\phi^2(u). \tag{2.3}$$

Hence, by virtue of (2.1), we have

$$(u^*, w)_X \leqq (\alpha_2 - 1)\,\phi^2(u)/\lambda \qquad \text{for all} \quad \lambda > 0.$$

Here, letting $\lambda \to +\infty$, we obtain $(u^*, w)_X \leqq 0$. Then it follows from (2.1), (2.3), and assumption (iv) that

$$(u^*, w-u)_X \leqq (u^*, -u)_X = -\alpha_2 \phi^2(u) = -\alpha_1 \phi^1(u)$$
$$\leqq \phi^1(w) - \phi^1(u).$$

(2) *The case* $\phi^1(w) > 0$. Put $\lambda = \{R_1/\phi^1(w)\}^{1/\alpha_1} > 0$ and $R_1 = \{(\alpha_2/\alpha_1)^{1/\alpha_2} R(u)\}^{\alpha_1\alpha_2/(\alpha_1-\alpha_2)}$. Then, by (i), (ii), (iii), and (iv), we easily see that $\phi^1(\lambda w) = \phi^1(u) = R_1$ and $\phi^2(u) = \max\{\phi^2(v); \phi^1(v) = R_1\}$. Then $\phi^2(\lambda w) \leqq \phi^2(u)$, so we have

$$(u^*, \lambda w - u)_X \leqq \phi^2(\lambda w) - \phi^2(u) \leqq 0.$$

Hence we obtain, by (iv) and (2.1),

$$\begin{aligned}
&\phi^1(w) - \phi^1(u) - (u^*, w-u)_X \\
&\quad = R_1 \lambda^{-\alpha_1} - R_1 - \lambda^{-1}(u^*, \lambda w - u)_X - \lambda^{-1}(1-\lambda)(u^*, u)_X \\
&\quad \geqq R_1 \lambda^{-\alpha_1} - R_1 - \lambda^{-1}(1-\lambda) R_1 \alpha_1 = R_1(\lambda^{-\alpha_1} - 1 - \lambda^{-1}(1-\lambda)\alpha_1).
\end{aligned}$$

Here the elementary calculation shows that $\lambda^{-\alpha_1} - 1 - \lambda^{-1}(1-\lambda)\alpha_1 \geqq 0$ for all $\lambda > 0$. Thus (2.2) is verified. Q.E.D.

*Remark* 2.2. If $\phi^i$ are not homogeneous functions, then the equation $\partial\phi^1(u) = \partial\phi^2(u)$ does not always have nontrivial solutions. For example, let $X = X^* = R^1$, $\phi^1(u) = u^6/6 + u^2/2$, and $\phi^2(u) = u^4/4$. Then $\partial\phi^1(u) = u^5 + u$, $\partial\phi^2(u) = u^3$, and $D(\partial\phi^1) = D(\partial\phi^2) = R^1$. Hence the equation $\partial\phi^1(u) = \partial\phi^2(u)$ has only the trivial solution $u = 0$.

We are now ready to prove Theorem I.

*Proof of Theorem* I. Let $X = W_0^{1,p}(\Omega)$ and $X^* = W^{-1,p'}(\Omega)$, $1/p + 1/p' = 1$. Put $\phi^1(u) = \|\nabla u\|_{L^p}^p/p$ and $\phi^2(u) = \|u\|_{L^q}^q/q$. Since $W_0^{1,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ by assumption, we find that $\phi^i \in \Phi(X)$ and $D(\phi^i) = D(\partial\phi^i) = X$, $i = 1, 2$. Moreover, $\partial\phi^1(u)(x) = -\Delta_p u(x)$ and $\partial\phi^2(u)(x) = |u|^{q-2} u(x)$. Clearly $\phi^1$ and $\phi^2$ are homogeneous functions of degree $p$ and $q$, respectively. In addition, Sobolev's embedding theorem and Rellich's compactness theorem assure (ii) and (iii) of Proposition 2.1. Let $w$ be an element satisfying (iii) and suppose that $w$ does not satisfy (iv), then we can choose an appropriate $\lambda > 0$ so that $u = \lambda w$ satisfies both (iii) and (iv), since $\phi^i$ are homogeneous functions of degree $\alpha_i$. Then all assumptions of Proposition 2.1 are fulfilled. Thus it is proved that (E) has a nontrivial solution $u$. Furthermore, since $|u(\cdot)|$ also belongs to $W_0^{1,p}(\Omega)$ and satisfies (iii) and (iv) of Proposition 2.1, $|u(x)|$ becomes a nonnegative solution of (E). Q.E.D.

2.2. *Remark for the Critical Case* $p = q$

Recalling the well-known case $p = q = 2$, one can easily imagine that the critical case $p = q$ gives rise to the eigenvalue problem for (E). In fact, when $N = 1$, it is shown in [15] that the existence of nontrivial solutions is determined only by the value of the best possible constant $C(\Omega)$ for (SP) (i.e., by the length of the domain $\Omega$), but not by the relations between $q$ and $N$ such as in Theorems I and III. Taking account of these observation, let us here consider the following *eigenvalue problem*:

$$\left.\begin{aligned} -\Delta_p u(x) &= \lambda |u|^{p-2} u(x), \qquad x \in \Omega,\ \lambda \in \mathbb{R}^1,\ 1 < p < \infty, \qquad &(2.4)\\ u(x) &= 0, \qquad x \in \partial\Omega. \qquad &(2.5) \end{aligned}\right\} \quad (\mathrm{E})_\lambda$$

Then we can say at least that $(\mathrm{E})_\lambda$ has the first eigenvalue $\lambda_1 > 0$ in the following sense.

THEOREM IV. *There exists a positive number* $\lambda_1$ *such that*

(i) $(\mathrm{E})_\lambda$ *has no nontrivial solution for* $\lambda \in (-\infty, \lambda_1)$, *and that*

(ii) $(\mathrm{E})_\lambda$ *with* $\lambda = \lambda_1$ *has a nonnegative nontrivial solution.*

*Furthermore, we have* $\lambda_1 = C(p, \Omega)^{-p}$, *where* $C(p, \Omega) = \sup(\|v\|_{L^p}/\|\nabla v\|_{L^p};\ v \in W_0^{1,p}(\Omega)\backslash\{0\}\}$, *i.e.,* $C(p, \Omega)$ *is the best possible constant for* (SP) *with* $p = q$.

To prove this theorem, we rely on another abstract result (cf. Lemma 5 of [15]):

PROPOSITION 2.3. *Let* $X$ *be a real Banach space and* $\phi^i \in \Phi(X)$ $(i = 1, 2)$ *such that* $D(\phi^1) \subset D(\phi^2)$. *Suppose that there exists an element* $u \in D(\partial\phi^2)$ *satisfying*

$$J(u) = \min\{J(v);\ v \in D(\phi^1)\}, \tag{2.6}$$

*where* $J(u) = \phi^1(u) - \phi^2(u)$. *Then* $u$ *belongs to* $D(\partial\phi^1)$ *and* $\partial\phi^2(u) \subset \partial\phi^1(u)$.

*Proof.* By (2.6), we get

$$\phi^1(v) - \phi^1(u) \geqq \phi^2(v) - \phi^2(u) \qquad \text{for all} \quad v \in D(\phi^1).$$

Hence, for all $v \in D(\phi^1)$ and $g \in \partial\phi^2(u)$, we obtain

$$\phi^1(v) - \phi^1(u) \geqq \phi^2(v) - \phi^2(u) \geqq (g, v - u)_X,$$

which implies that $u \in D(\partial\phi^1)$ and $g \in \partial\phi^1(u)$. Q.E.D.

We now proceed to the proof of Theorem IV.

*Proof of Theorem* IV. Let $X = W_0^{1,p}(\Omega)$, $\phi^1(u) = \|\nabla u\|_{L^p}^p/p$, and $\phi^2(u) = \|u\|_{L^p}^p/p$. Then we find that $\phi^1, \phi^2 \in \Phi(X)$ and $\partial\phi^1(u) = -\Delta_p u$, $\partial\phi^2(u) = |u|^{p-2} u$ with domain $D(\partial\phi^1) = D(\partial\phi^2) = X$. We further have

$$\phi^2(v) \leqq C(p, \Omega)^p \, \phi^1(v) \qquad \text{for all} \quad v \in X. \tag{2.7}$$

Let $\lambda \in (-\infty, \lambda_1)$, $\lambda_1 = C(p, \Omega)^{-p}$, and $u$ be a nontrivial solution of $(\mathrm{E})_\lambda$. Then, multiplying (2.4) by $u$, we obtain $\phi^1(u) = \lambda\phi^2(u)$. Hence, we get, by (2.7),

$$\phi^1(u) = \lambda\phi^2(u) < \lambda_1 \phi^2(u) \leqq \phi^1(u),$$

which is a contradiction. Thus the first assertion (i) is proved.

Let $\lambda = \lambda_1$, then (2.7) implies that $J_\lambda(v) = \phi^1(v) - \lambda\phi^2(v) \geqq 0$ for all $v \in X$. On the other hand, since $W_0^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$, there exists an element $u \in W_0^{1,p}(\Omega) \backslash \{0\}$ satisfying

$$0 = J_\lambda(u) = \min\{J_\lambda(v);\ v \in W_0^{1,p}(\Omega) \backslash \{0\}\}. \tag{2.8}$$

Thus, by Proposition 2.3, $u$ turns out to be a nontrivial solution of $(\mathrm{E})_\lambda$. Furthermore, since $|u|$ also satisfies (2.8), $|u|$ becomes a nonnegative nontrivial solution of $(\mathrm{E})_\lambda$. Q.E.D.

*Remark* 2.4. When $\Omega = (0, 1)$, all the other eigenvalues $\lambda_k$ of $(\mathrm{E})_\lambda$ are given explicitly in [15], i.e., $(\mathrm{E})_\lambda$ admits a nontrivial solution if and only if $\lambda \in \{\lambda_k = (k a_p)^p;\ k \in \mathbb{N}\}$, where $a_p = 2(p-1)^{1/p} B(1/p, 1/p + 1)$ and $B(\cdot, \cdot)$ is the beta function.

## 3. Regularity of Nontrivial Solutions

In this section, we shall prove Theorem II. When $p > N$, the assertion of Theorem II is obvious, since $W_0^{1,p}(\Omega)$ is continuously embedded in $L^\infty(\Omega)$. So, we have only to give a proof for the case $p \leqq N$.

*Proof of Theorem* II. Put $p_* = p^* = Np/(N-p)$ if $p < N$ and $p_* = \max(2p, 2q_1)$ if $p = N$. Then there exists a constant $K$ such that

$$\|u\|_{L^{p_*}} \leqq K \|\nabla u\|_{L^p} \qquad \text{for all} \quad u \in W_0^{1,p}(\Omega). \tag{3.1}$$

Here we claim the following result.

Lemma 3.1. *Let $u$ be a weak solution as in Theorem* II. *Fix two sequences of numbers $\{q_k\}$ and $\{C_k\}$ by*

$$q_{k+1} = q_k^* p_*/p, \qquad k \in \mathbb{N},\ q_k^* = q_k - q + p, \tag{3.2}$$

$$C_{k+1} = K^{p/q_k^*}(q_k - q + 1)^{-1/q_k^*}(q_k^*/p)^{p/q_k^*} C_k^{q_k/q_k^*}, \qquad k \in \mathbb{N}, \tag{3.3}$$

and $C_1 = \|u\|_{L^{q_1}}$. Then $u$ belongs to $L^{q_k}(\Omega)$ for all $k \in \mathbb{N}$, and satisfies

$$\|u\|_{L^{q_k}} \leqq C_k \qquad \text{for all} \quad k \in \mathbb{N}. \tag{3.4}$$

*Proof of Lemma* 3.1. We give a proof by induction. Relation (3.4) with $k=1$ is obvious. Suppose that (3.4) holds for $k=k$. Let $g_n$, $n \in \mathbb{N}$, be $C^1$-functions such that

$$\begin{aligned} &g_n(s) = s \text{ if } |s| \leqq n, \qquad g_n(s) = n+1 \text{ if } |s| \geqq n+2, \text{ and} \\ &0 \leqq dg_n(s)/ds \leqq 1 \qquad \text{for all } s \in \mathbb{R}^1. \end{aligned} \tag{3.5}$$

Put $u_n = g_n(u)$, then $|u_n|^{r-2} u_n$ belong to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ for all $r \in [2, \infty)$. Then we can multiply (0.1) by $|u_n|^{q_k - q} u_n(x)$ and integrate over $\Omega$ to get

$$\begin{aligned} \int_\Omega -\Delta_p u(x) |u_n|^{q_k - q} u_n(x)\, dx &= \int_\Omega |u_n(x)|^{q_k - q + 1} |u(x)|^{q-1}\, dx \\ &\leqq \int_\Omega |u(x)|^{q_k}\, dx \leqq C_k^{q_k}. \end{aligned} \tag{3.6}$$

Here we have, by (3.1),

$$\begin{aligned} &\int_\Omega -\Delta_p u(x) |\, u_n(x)|^{q_k - q} u_n(x)\, dx \\ &\quad = (q_k - q + 1) \int_\Omega |\nabla u(x)|^p\, g_n'(u(x)) |u_n(x)|^{q_k - q}\, dx \\ &\quad \geqq (q_k - q + 1) \int_\Omega |\nabla u_n(x)|^p\, |u_n(x)|^{q_k - q}\, dx \\ &\quad \geqq (q_k - q + 1)(p/q_k^*)^p \int_\Omega |\nabla(|u_n(x)|^{q_k^*/p})|^p\, dx \\ &\quad \geqq K^{-p}(q_k - q + 1)(p/q_k^*)^p\, \||u_n|^{q_k^*/p}\|_{L^{p_*}}^p. \end{aligned} \tag{3.7}$$

Then, combining (3.6) with (3.7), we deduce

$$\||u_n|^{q_k^*/p}\|_{L^{p_*}}^p = \|u_n\|_{L^{q_{k+1}}}^{q_k^*} \leqq K^p (q_k - q + 1)^{-1} (q_k^*/p)^p\, C_k^{q_k}.$$

Hence, by letting $n$ tend to $+\infty$, we obtain (3.4) with $k = k+1$. Q.E.D.

Now we go back to the proof of Theorem II. Put $E_k = q_k \log C_k$, then in view of (3.2) and (3.3), we find

$$E_{k+1} = p_*(\log K - p^{-1} \log(q_k - q + 1) + \log q_k^* - \log p) + aE_k \leqq r_k + aE_k,$$

where $r_k = p_* \log Kq_k^*$ and $a = p_*/p > 1$. Then

$$E_k \leqq a^{k-1}E_1 + r_{k-1} + ar_{k-2} + \cdots + a^{k-2}r_1. \tag{3.8}$$

Since $q_k = a^{k-1}(q_1 - \alpha) + \alpha$ with $\alpha = p_*(q-p)/(p_* - p)$ by (3.2), we easily get

$$r_k \leqq p_* \log Ka^{k-1}(q_1 - \alpha^-) \leqq (k-1)\, p_* \log a + b,$$

where $b = p_* \log K(q_1 - \alpha^-)$, $\alpha^- = \min(\alpha, 0)$. Hence, (3.7) with elementary calculations yields

$$E_k \leqq a^{k-1}E_1 + \{b(a-1) + p_* \log a\}(a^{k-1} - 1)/(a-1)^2.$$

Consequently, we deduce

$$\|u\|_{L^\infty} \leqq \limsup_{k \to +\infty} \|u\|_{L^{q_k}} \leqq \limsup_{k \to +\infty} e^{E_k/q_k} \leqq e^d,$$

with $d = [E_1 + \{b(a-1) + p_* \log a\}/(a-1)^2]/(q_1 - \alpha)$. Q.E.D.

## 4. Nonexistence of Nontrivial Solutions

### 4.1. *Pohozaev-type Inequality*

In this section, we establish a Pohozaev-type inequality for weak solutions $u$ belonging to $P$. To this end, we construct some approximate solutions for $u$. Let $u$ be a weak solution of (E) belonging to $P$, and let $u_n = g_n(u)$, where $g_n$ are the $C^1$-functions as in (3.5). Then, for each $n \in \mathbb{N}$, there exist functions $v_n^\varepsilon \in C_0^\infty(\Omega)$, $0 < \varepsilon < 1$, such that

$$|v_n^\varepsilon|_{L^\infty} \leqq C_0 \qquad \text{for all } \varepsilon, \tag{4.1}$$

$$v_n^\varepsilon \to 2|u_n|^{q-2}\, u_n \qquad \text{in } L^r(\Omega) \text{ as } \varepsilon \to 0 \text{ for all } r \in [1, \infty). \tag{4.2}$$

Let us here consider the approximate equations

$$\left.\begin{aligned} |w_n^\varepsilon|^{q-2}\, w_n^\varepsilon(x) + A_\varepsilon w_n^\varepsilon(x) &= v_n^\varepsilon(x), \qquad x \in \Omega, \qquad (4.3)\\ w_n^\varepsilon(x) &= 0, \qquad x \in \partial\Omega, \qquad (4.4) \end{aligned}\right\} \text{(E)}_n^\varepsilon$$

where $A_\varepsilon w(x) = -\operatorname{div}((|\nabla w(x)|^2 + \varepsilon)^{(p-2)/2}\, \nabla w(x))$. Then, by virtue of the result of Gilbarg and Trudinger [8, Theorem 15.10], $\text{(E)}_n^\varepsilon$ has a unique solution $w_n^\varepsilon \in C^2(\bar{\Omega})$ for each $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$. Moreover, as is shown below, $w_n^\varepsilon$ give good approximations for $w_n$, which are (unique) solutions of the problems

$$\left.\begin{aligned} |w_n|^{q-2} w_n(x) - \Delta_p w_n(x) &= 2|u_n|^{q-2} u_n(x), && x \in \Omega, && (4.5)\\ w_n(x) &= 0, && x \in \partial\Omega. && (4.6) \end{aligned}\right\} (\mathrm{E})_n$$

LEMMA 4.1. *Let $w_n^\varepsilon$ and $w_n$ be solutions of $(\mathrm{E})_n^\varepsilon$ and $(\mathrm{E})_n$, respectively. Then $w_n^\varepsilon$ converges to $w_n$ strongly in $W_0^{1,p}(\Omega)$ and $L^r(\Omega)$ for all $r \in [1, \infty)$ as $\varepsilon$ tends to zero.*

*Proof.* We denote $w_n^\varepsilon$ by $w^\varepsilon$ for the sake of simplicity, if no confusion arises. First of all, we are going to establish the following a priori estimates for $w^\varepsilon$ independent of $\varepsilon$:

$$\|w^\varepsilon\|_{L^r} \leqq C_r, \qquad \text{for all} \quad r \in [1, \infty) \quad \text{and} \quad \varepsilon \in (0, 1), \tag{4.7}$$

$$\|\nabla w^\varepsilon\|_{L^p} \leqq C, \qquad \text{for all} \quad \varepsilon \in (0, 1), \tag{4.8}$$

where $C_r$ and $C$ are constants independent of $\varepsilon$.

Multiply (4.3) by $|w^\varepsilon|^{r-1} w^\varepsilon$, $r \geqq 1$, and integrate over $\Omega$. Then, by (4.1), we get

$$r \int_\Omega (|\nabla w^\varepsilon|^2 + \varepsilon)^{(p-2)/2} |\nabla w^\varepsilon|^2 |w^\varepsilon|^{r-1} \leqq C_0 \int_\Omega |w^\varepsilon|^r \, dx. \tag{4.9}$$

Here we have

$$\begin{aligned} \text{The left-hand-side member of (4.9)} &\geqq 2^{-1} r \int_{\Omega_\varepsilon} |\nabla w^\varepsilon|^p |w^\varepsilon|^{r-1} \, dx, \\ &\geqq C_p \int_{\Omega_\varepsilon} |\nabla v_r|^p \, dx, \end{aligned} \tag{4.10}$$

where $\Omega_\varepsilon = \{x \in \Omega; |\nabla w^\varepsilon|^2 \geqq \varepsilon\}$, $C_p = r\{p/(p+r-1)\}^p/2$ and we put $v_r = |w^\varepsilon|^{(r-1)/p} w^\varepsilon$. Furthermore, since $r/(p+r-1) < 1$ and $v_r \in W_0^{1,p}(\Omega)$, there exist constants $C_1, C_2, C_3$ such that

$$\begin{aligned} C_0 \int_\Omega |w^\varepsilon|^r \, dx &= C_0 \int_\Omega |v_r|^{pr/(p+r-1)} \, dx \\ &\leqq C_1 \left( \int_\Omega |\nabla v_r|^p \, dx \right)^{r/(p+r-1)} \\ &\leqq 4^{-1} C_p \int_\Omega |\nabla v_r|^p \, dx + C_2 \\ &\leqq 4^{-1} C_p \int_{\Omega_\varepsilon} |\nabla v_r|^p \, dx + 8^{-1} r \varepsilon^{p/2} \int_{\Omega \setminus \Omega_\varepsilon} |w^\varepsilon|^{r-1} \, dx + C_2 \\ &\leqq 4^{-1} C_p \int_{\Omega_\varepsilon} |\nabla v_r|^p \, dx + 2^{-1} C_0 \int_\Omega |w^\varepsilon|^r \, dx + C_3. \end{aligned}$$

Consequently we have

$$C_0 \int_{\Omega} |w^\varepsilon|^r \, dx \leqq 2^{-1} C_p \int_{\Omega_\varepsilon} |\nabla v_r|^p \, dx + 2C_3. \tag{4.11}$$

Then, from (4.9), (4.10), and (4.11), we deduce (4.7) and that $|\nabla v_r|_{L^p(\Omega_\varepsilon)}$ is bounded. In particular, since $v_1 = w^\varepsilon$ and $|\nabla w^\varepsilon|^p_{L^p(\Omega)} \leqq |\nabla w^\varepsilon|^p_{L^p(\Omega_\varepsilon)} + |\Omega| \, \varepsilon^{p/2}$, (4.8) is also derived. Put

$$\phi_\varepsilon(z) = \int_{\Omega} (|\nabla z(x)|^2 + \varepsilon)^{p/2} \, dx \qquad \text{if} \quad z \in W_0^{1,p}(\Omega), \text{ and } = +\infty \text{ otherwise.}$$

Then, for each $\varepsilon \in [0, 1]$, $\phi_\varepsilon$ belongs to $\Phi(L^2(\Omega))$, and its sudifferential $\partial\phi_\varepsilon$ coincides with $A_\varepsilon$. Therefore, $w^\varepsilon$ satisfies

$$\phi_\varepsilon(v) - \phi_\varepsilon(w^\varepsilon) \geqq (-|w^\varepsilon|^{q-2} w^\varepsilon + v_n^\varepsilon, v - w^\varepsilon)_{L^2} \qquad \text{for all} \quad v \in W_0^{1,p}(\Omega), \tag{4.12}$$

Here, by virtue of (4.7), (4.8), and Rellich's compactness theorem, we can extract a sequence $\{\varepsilon_k\}$ which tends to zero as $k \to +\infty$ such that

$$w_n^{\varepsilon_k} \to w_n \text{ strongly in } L^r(\Omega) \text{ for all } r \in [1, \infty), \tag{4.13}$$

$$w_n^{\varepsilon_k} \to w_n \text{ weakly in } W_0^{1,p}(\Omega), \tag{4.14}$$

$$|w_n^{\varepsilon_k}|^{q-2} w_n^{\varepsilon_k} \to |w_n|^{q-2} w_n \text{ weakly in } L^2(\Omega), \tag{4.15}$$

where we used the demiclosedness of the operator $w \mapsto |w|^{q-2} w$ in $L^2(\Omega)$. Here, since $\phi_\varepsilon(z) \geqq \phi_0(z)$ for all $z \in W_0^{1,p}(\Omega)$, we have

$$\limsup_{k \to +\infty} \phi_{\varepsilon_k}(w_n^{\varepsilon_k}) \geqq \phi_0(w_n).$$

Furthermore, we note that $\phi_\varepsilon(v) \to \phi_0(v)$ as $\varepsilon \to 0$ for all $v \in W_0^{1,p}(\Omega)$. Thus, letting $k \to +\infty$ in (4.12) with $\varepsilon = \varepsilon_k$ and recalling (4.2), we obtain

$$\phi_0(v) - \phi_0(w_n) \geqq (-|w_n|^{q-2} w_n + 2|u_n|^{q-2} u_n, v - w_n)_{L^2}$$
$$\text{for all} \quad v \in W_0^{1,p}(\Omega),$$

which says that $-\Delta_p w_n = -|w_n|^{q-2} w_n + 2|u_n|^{q-2} u_n$, i.e., $w_n$ is a solution of $(\mathrm{E})_n$. Since the above argument does not depend on the choice of $\{\varepsilon_k\}$, relations (4.13)–(4.15) hold good with $\{\varepsilon_k\} = \{\varepsilon\}$.

Multiplying (4.3) by $w_n^\varepsilon$ and (4.5) by $w_n$, we note

$$(\partial\phi_\varepsilon(w_n^\varepsilon), w_n^\varepsilon)_{L^2} = \int_\Omega (|\nabla w_n^\varepsilon|^2+\varepsilon)^{(p-2)/2} |\nabla w_n^\varepsilon|^2\, dx$$
$$= -\int_\Omega |w_n^\varepsilon|^q\, dx + \int_\Omega v_n^\varepsilon w_n^\varepsilon\, dx,$$
$$(\partial\phi_0(w_n), w_n)_{L^2} = \int_\Omega |\nabla w_n|^p\, dx = -\int_\Omega |w_n|^q\, dx + 2\int_\Omega |u_n|^{q-2} u_n w_n\, dx.$$

Then these identities together with (4.2) and (4.13) give

$$\int_\Omega (|\nabla w_n^\varepsilon|^2+\varepsilon)^{(p-2)/2} |\nabla w_n^\varepsilon|^2\, dx \to \int_\Omega |\nabla w_n|^p\, dx \qquad \text{as } \varepsilon\to 0, \tag{4.16}$$

whence easily follow

$$\int_\Omega (|\nabla w_n^\varepsilon|^2+\varepsilon)^{p/2}\, dx \to \int_\Omega |\nabla w_n|^p\, dx \qquad \text{as } \varepsilon\to 0, \tag{4.17}$$

and

$$\int_\Omega |\nabla w_n^\varepsilon|^p\, dx \to \int_\Omega |\nabla w_n|^p\, dx \qquad \text{as } \varepsilon\to 0. \tag{4.18}$$

Then, since $W_0^{1,p}(\Omega)$ is uniformly convex, (4.14) and (4.18) assure that $w_n$ converges to $w_n^\varepsilon$ strongly in $W_0^{1,p}(\Omega)$. Q.E.D.

We here claim that $w_n$ satisfy the following inequalities.

LEMMA 4.2. *Let $w_n$ be solutions of* (E)$_n$. *Then $w_n$ satisfy*

$$\frac{N}{q}\int_\Omega |w_n|^q\, dx + \frac{N-p}{p}\int_\Omega |\nabla w_n|^p\, dx$$
$$+\sum_{i=1}^N \int_\Omega 2|u_n|^{q-2} u_n x_i \frac{\partial w_n}{\partial x_i}\, dx + R_n \leqq 0, \tag{4.19}$$

*for all $n\in\mathbb{N}$, where $R_n = \limsup_{\varepsilon\to 0} ((p'-1)/p)\int_{\partial\Omega} (|\nabla w_n^\varepsilon|^2+\varepsilon)^{p/2}$ $(x\cdot n(x))\, dS$, $p'=\min(p,2)$, and $n(x)=(n_1(x), n_2(x), ..., n_N(x))$ is the outward normal unit vector at $x\in\partial\Omega$.*

*Proof.* In conformity with [16], we are going to calculate $\sum_{i=1}^N \int_\Omega (4.3)\, x_i\, \partial w_n^\varepsilon(x)/\partial x_i\, dx$. First, it is easy to see that

$$\sum_{i=1}^N \int_\Omega |w^\varepsilon|^{q-2} w^\varepsilon(x)\, x_i \frac{\partial w^\varepsilon}{\partial x_i}(x)\, dx = -\frac{N}{q}\int_\Omega |w^\varepsilon(x)|^q\, dx. \tag{4.20}$$

Next we get

$$\sum_{i=1}^{N} \int_{\Omega} A_{\varepsilon} w^{\varepsilon}(x)\, x_i \frac{\partial w^{\varepsilon}}{\partial x_i}(x)\, dx$$

$$= \int_{\Omega} (|\nabla w^{\varepsilon}|^2 + \varepsilon)^{(p-1)/2} |\nabla w^{\varepsilon}(x)|^2\, dx + I_1 + I_2,$$

where

$$I_1 = \sum_{i,j=1}^{N} \int_{\Omega} (|\nabla w^{\varepsilon}|^2 + \varepsilon)^{(p-2)/2} x_i \frac{1}{2} \frac{\partial}{\partial x_i} \left( \frac{\partial w^{\varepsilon}}{\partial x_j} \right)^2 dx$$

and

$$I_2 = -\sum_{i,j=1}^{N} \int_{\partial\Omega} (|\nabla w^{\varepsilon}|^2 + \varepsilon)^{(p-2)/2} x_i \frac{\partial w^{\varepsilon}}{\partial x_i}(x) \frac{\partial w^{\varepsilon}}{\partial x_j}(x)\, n_j(x)\, dS.$$

Here

$$I_1 = \frac{1}{p} \sum_{i=1}^{N} \int_{\Omega} x_i \frac{\partial}{\partial x_i} (|\nabla w^{\varepsilon}|^2 + \varepsilon)^{p/2}\, dx$$

$$= -\frac{N}{p} \int_{\Omega} (|\nabla w^{\varepsilon}|^2 + \varepsilon)^{p/2}\, dx + \frac{1}{p} \int_{\partial\Omega} (|\nabla w^{\varepsilon}|^2 + \varepsilon)^{p/2} (x \cdot n(x))\, dS.$$

Since $w^{\varepsilon}(x)$ is constant on $\partial\Omega$, $\nabla w^{\varepsilon}(x) = |\nabla w(x)|\, n(x)$ or $-|\nabla w(x)|\, n(x)$ holds for all $x \in \partial\Omega$. Then we find

$$I_2 = -\int_{\partial\Omega} (|\nabla w^{\varepsilon}|^2 + \varepsilon)^{(p-2)/2} |\nabla w^{\varepsilon}|^2 (x \cdot n(x))\, dS.$$

Therefore we derive

$$\sum_{i=1}^{N} \int_{\Omega} A_{\varepsilon} w^{\varepsilon}(x)\, x_i \frac{\partial w^{\varepsilon}}{\partial x_i}(x)\, dx$$

$$= \int_{\Omega} \left\{ (|\nabla w^{\varepsilon}|^2 + \varepsilon)^{(p-2)/2} |\nabla w^{\varepsilon}|^2 - \frac{N}{p} (|\nabla w^{\varepsilon}|^2 + \varepsilon)^{p/2} \right\} dx$$

$$+ \frac{1-p}{p} \int_{\partial\Omega} (|\nabla w^{\varepsilon}|^2 + \varepsilon)^{p/2} (x \cdot n(x))\, dS$$

$$+ \int_{\partial\Omega} \varepsilon (|\nabla w^{\varepsilon}|^2 + \varepsilon)^{(p-2)/2} (x \cdot n(x))\, dS. \tag{4.21}$$

Here, since $(x \cdot n(x)) \geqq 0$ for all $x \in \partial\Omega$, we get

$$\int_{\partial\Omega} \varepsilon(|\nabla w^\varepsilon|^2 + \varepsilon)^{(p-2)/2} (x \cdot n(x))\, dS$$

$$\leqq \begin{cases} \displaystyle\int_{\partial\Omega} \varepsilon^{p/2}(x \cdot n(x))\, dS & \text{if } 1 < p \leqq 2, \\ \displaystyle\frac{p-2}{p} \int_{\partial\Omega} (|\nabla w^\varepsilon|^2 + \varepsilon)^{p/2} (x \cdot n(x))\, dS & \\ \displaystyle\qquad + \frac{2}{p} \int_{\partial\Omega} \varepsilon^{p/2}(x \cdot n(x))\, dS & \text{if } 2 < p. \end{cases} \tag{4.22}$$

Thus, letting $\varepsilon \to 0$ in (4.20)–(4.22) and recalling (4.2), (4.16), (4.17), and Lemma 4.1, we can deduce (4.19). Q.E.D.

Now we are ready to introduce a "*Pohozaev-type inequality*," which is valid for every weak solution $u$ in the class $P$.

LEMMA 4.3. *Let $u$ be a weak solution of* (E) *belonging to $P$. Then $w_n$ converges to $u$ strongly in $W_0^{1,p}(\Omega)$ and $L^q(\Omega)$. Moreover $u$ satisfies the Pohozaev-type inequality*

$$\left(\frac{N-p}{p} - \frac{N}{q}\right) \int_\Omega |u|^q\, dx + R \leqq 0, \tag{4.23}$$

*where* $R = \limsup_{n \to +\infty} \left(\limsup_{\varepsilon \to 0} ((p'-1)/p) \int_{\partial\Omega} (|\nabla w_n^\varepsilon|^2 + \varepsilon)^{p/2} (x \cdot n(x))\, dS\right)$.

*Proof.* Let us first note that

$$|u_n|^{q-2} u_n \to |u|^{q-2} u \qquad \text{strongly in } L^{q'}(\Omega),$$
$$q' = q/(q-1), \text{ as } n \to +\infty, \tag{4.24}$$

$$x_i |u_n|^{q-2} u_n \to x_i |u|^{q-2} u \qquad \text{strongly in } L^{p'}(\Omega),$$
$$p' = p/(p-1), \text{ as } n \to +\infty, \text{ for all } i. \tag{4.25}$$

Multiplying (4.5) by $w_n$, we obtain

$$\|w_n\|_{L^q}^q + \|\nabla w_n\|_{L^p}^p = 2 \int_\Omega |u_n|^{q-2} u_n w_n\, dx \leqq 2\|u_n\|_{L^q}^{q-1} \|w_n\|_{L^q}. \tag{4.26}$$

Hence, since $w_n$ is bounded in $L^q(\Omega)$ and $W_0^{1,p}(\Omega)$, we can extract a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\begin{aligned} w_{n_k} \to w \text{ weakly in } W_0^{1,p}(\Omega) \text{ and } L^q(\Omega),\\ \text{and strongly in } L^{q-\theta}(\Omega) \qquad (4.27)\\ \text{for all } \theta \in (0, q-1] \text{ as } k \to +\infty, \end{aligned}$$

$$|w_{n_k}|^{q-2} w_{n_k} \to |w|^{q-2} w \text{ weakly in } L^{q'}(\Omega) \text{ as } k \to +\infty, \qquad (4.28)$$

where we used the demiclosedness of the operator $w \mapsto |w|^{q-2} w$ from $L^1(\Omega)$ into $L^{q'}(\Omega)$. On the other hand, since $w_n$ is a solution of $(\mathrm{E})_n$, $w_n$ satisfies

$$\begin{aligned} &\int_\Omega |\nabla v|^p \, dx - \int_\Omega |\nabla w_n|^p \, dx \\ &\geqq \int_\Omega (-|w_n|^{q-2} w_n + 2|u_n|^{q-2} u_n)(v - w_n) \, dx \\ &= \int_\Omega |w_n|^q \, dx - \int_\Omega |w_n|^{q-2} w_n v \, dx \\ &\quad + \int_\Omega 2|u_n|^{q-2} u_n (v - w_n) \, dx \qquad \text{for all} \quad v \in C_0^\infty(\Omega). \qquad (4.29) \end{aligned}$$

Then, letting $n = n_k \to +\infty$ in (4.29) and considering (4.24), (4.27), and (4.28), we obtain

$$\begin{aligned} &\int_\Omega |\nabla v|^p \, dx - \int_\Omega |\nabla w|^p \, dx \\ &\geqq \int_\Omega (-|w|^{q-2} w + 2|u|^{q-2} u)(v - w) \, dx \qquad \text{for all} \quad v \in L^q(\Omega). \end{aligned} \qquad (4.30)$$

Then, by the standard argument (putting $v = w + tz$, $z \in C_0^\infty(\Omega)$, and letting $t \uparrow 0$ and $t \downarrow 0$), it is shown that $w$ satisfies

$$-\Delta_p w = -|w|^{q-2} w + 2|u|^{q-2} u \qquad (4.31)$$

in the sense of distribution. Since $-\Delta_p w$ and $|u|^{q-2}u$ belong to $V^* = (W_0^{1,p}(\Omega))^*$, $w$ satisfies (4.31) in $V^*$. Hence

$$(|w|^{q-2} w - \Delta_p w) - (|u|^{q-2} u - \Delta_p u) = 0 \qquad \text{in } V^*. \qquad (4.32)$$

Then, multiplying (4.32) by $w - u$, we easily find that $w = u$. The above argument does not depend on the choice of $\{n_k\}$, so (4.27) and (4.28) hold good with $n_k = n$ and $w = u$. Therefore it follows from (4.26) that

$$2\|u\|_{L^q}^q = \|u\|_{L^q}^q + \|\nabla u\|_{L^p}^p \leqq \liminf_{n\to+\infty} (\|w_n\|_{L^q}^q + \|\nabla w_n\|_{L^p}^p)$$
$$\leqq \limsup_{n\to+\infty} (\|w_n\|_{L^q}^q + \|\nabla w_n\|_{L^p}^p)$$
$$\leqq 2\|u\|_{L^q}^q,$$

whence follows $w_n \to u$ strongly in $W_0^{1,p}(\Omega)$ and $L^q(\Omega)$. Furthermore, (4.19) together with (4.25) and (4.27) gives

$$\int_\Omega \left(\frac{N}{q}|u|^q + \frac{N-p}{p}|\nabla u|^p + \sum_{i=1}^N 2|u|^{q-2} u x_i \frac{\partial u}{\partial x_i}\right) dx + R \leqq 0.$$

Then, to verify (4.23), it suffices to note that $\|\nabla u\|_{L^p}^p = \|u\|_{L^q}^q$ and

$$\sum_{i=1}^N \int_\Omega 2|u|^{q-2} u x_i \frac{\partial u}{\partial x_i} dx = \frac{-2N}{q}\|u\|_{L^q}^q. \qquad \text{Q.E.D.}$$

*Proof of Theorem* III. Here we are going to complete the proof of Theorem III. The first part of Theorem III follows from (4.23) at once, since $R \geqq 0$. Let $\Omega$ be strictly star shaped and $q = Np/(N-p)$, $p < N$, then (4.23) yields $R = 0$, whence follows

$$\lim_{n\to+\infty}\left(\limsup_{\varepsilon\to 0}\int_{\partial\Omega} (|\nabla w_n^\varepsilon|^2 + \varepsilon)^{p/2}\, dS\right) = 0. \tag{4.33}$$

On the other hand, multiplying (4.3) by $v(x) \equiv 1$, we obtain

$$\int_\Omega (|w_n^\varepsilon|^{q-2} w_n^\varepsilon - v_n^\varepsilon)\, dx = \int_{\partial\Omega} (|\nabla w_n^\varepsilon|^2 + \varepsilon)^{(p-2)/2} \nabla w_n^\varepsilon(x)\cdot n(x)\, dS. \tag{4.34}$$

Then, letting $\varepsilon \to 0$ and $n \to +\infty$ in (4.34), we get

$$\left|\int_\Omega |u|^{q-2} u\, dx\right| \leqq \limsup_{n\to+\infty}\left(\limsup_{\varepsilon\to 0}\int_{\partial\Omega} (|\nabla w_n^\varepsilon|^2 + \varepsilon)^{(p-1)/2}\, dS\right).$$

Hence, by (4.33), we deduce that $\int_\Omega |u|^{q-2} u\, dx = 0$. This completes the proof of Theorem III. Q.E.D.

*Remark* 4.4. If the underlying inequality (SP) is replaced by

$$\|u\|_{L^q} \leqq C\left(\sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p}^p\right)^{1/p} \qquad \text{for all} \quad u \in W_0^{1,p}(\Omega), \tag{SP)'$$

then we are led to another elliptic equation of the form

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}(x)\right) = |u|^{q-2} u(x), \qquad x \in \Omega,$$
$$u(x) = 0, \qquad x \in \partial\Omega, \tag{E)'$$

whose degeneracy is slightly more complicated than that of (E). However, the same results as in Theorems I–IV are still verified by the same arguments as before with obvious modifications, except replacing the operator $A_\varepsilon$ in the approximate equations $(E)_n^\varepsilon$ by

$$A'_\varepsilon : u \mapsto -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^2 + \varepsilon|\nabla u|^2 + \varepsilon\right)^{(p-2)/2} \frac{\partial u}{\partial x_i}.$$

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