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Note

Equality in a Result of Kleitman

DANIEL MCQUILLAN AND R. BRUCE RICHTER

Department of Mathematics, Carleton University, Ottawa, Ontario K1S 5B6, Canada

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An *upset* is a set $\mathscr{U}$ of subset of a finite set. S such that if $U \subseteq V$ and $U \in \mathscr{U}$, then $V \in \mathscr{U}$. A *downset* $\mathscr{D}$ is defined analogously. In 1966, Kleitman (*J. Combin. Theory* **1** (1966), 153–155) proved that if $\mathscr{U}$ and $\mathscr{D}$ are arbitrary up- and downsets, respectively, then $|\mathscr{U}|\,|\mathscr{D}| \geqslant 2^{|S|}\,|\mathscr{U} \cap \mathscr{D}|$. In this note, we show that a necessary and sufficient condition for equality to hold is: for every minimal element U of $\mathscr{U}$ and every maximal element D of $\mathscr{D}$, $U \subseteq D$. This result is extended to some related inequalities.

Let S be a finite set. We denote the set of all subsets of S by $\mathscr{P}(S)$. An *upset* is a subset $\mathscr{U}$ of $\mathscr{P}(S)$ such that if $U \subseteq V$ and $U \in \mathscr{U}$, then $V \in \mathscr{U}$. We state the following Theorem:

THEOREM 1. *Let $\mathscr{U}_1$ and $\mathscr{U}_2$ be upsets in $\mathscr{P}(S)$. Then $|\mathscr{U}_1|\,|\mathscr{U}_2| \leqslant 2^{|S|}\,|\mathscr{U}_1 \cap \mathscr{U}_2|$, with equality if and only if every minimal element of $\mathscr{U}_1$ is disjoint from every minimal element of $\mathscr{U}_2$.*

It is the main purpose of this note to prove Theorem 1 and to derive the equality conditions in some related inequalities.

If we define a *downset* $\mathscr{D}$ equivalently to an upset, then it is routine to verify Theorem 1's equivalence to the following statement: $|\mathscr{U}|\,|\mathscr{D}| \geqslant 2^{|S|}\,|\mathscr{U} \cap \mathscr{D}|$, with equality holding if and only if, for every minimal element U of $\mathscr{U}$ and every maximal element D of $\mathscr{D}$, $U \subseteq D$. We set $\mathscr{U} = \mathscr{U}_1$, and $\mathscr{D} = \mathscr{P}(S) \backslash \mathscr{U}_2$ to show this.

For an upset $\mathscr{U}$, define $m(\mathscr{U})$ to be the set of minimal elements of $\mathscr{U}$ and $\mathrm{sp}(\mathscr{U}) = \bigcup_{u \in m(\mathscr{U})} U$. The equality condition in Theorem 1 may be restated as $\mathrm{sp}(\mathscr{U}_1)$ and $\mathrm{sp}(\mathscr{U}_2)$ are disjoint.

Proof of Theorem 1. Our proof is basically the same as Kleitman's original proof [3], with a little more care to obtain the necessity of the equality condition.

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For each $a \in S$, and for $i=1,2$, let $\mathcal{U}_{ia}=\{U \in \mathcal{U}_i \mid a \in U\}$, $\mathcal{U}_{i\bar{a}}=\mathcal{U}_i \backslash \mathcal{U}_{ia}$, and $\mathcal{U}'_{ia}=\{U\backslash\{a\} \mid U \in \mathcal{U}_{ia}\}$. The proof proceeds by induction on $|S|$, so it is important to observe that $\mathcal{U}_{i\bar{a}}$ and $\mathcal{U}'_{ia}$ are upsets in $\mathscr{P}(S\backslash\{a\})$. Note that $\mathcal{U}_{i\bar{a}} \subseteq \mathcal{U}'_{ia}$.

Trivially, $|\mathcal{U}_i|=|\mathcal{U}'_{ia}|+|\mathcal{U}_{i\bar{a}}|$, so that

$$\begin{aligned} |\mathcal{U}_1|\,|\mathcal{U}_2| &= (|\mathcal{U}'_{1a}|+|\mathcal{U}_{1\bar{a}}|)(|\mathcal{U}'_{2a}|+|\mathcal{U}_{2\bar{a}}|) \\ &= |\mathcal{U}'_{1a}|\,|\mathcal{U}'_{2a}|+|\mathcal{U}_{1\bar{a}}|\,|\mathcal{U}_{2\bar{a}}|+|\mathcal{U}'_{1a}|\,|\mathcal{U}_{2\bar{a}}|+|\mathcal{U}_{1\bar{a}}|\,|\mathcal{U}'_{2a}| \\ &= 2(|\mathcal{U}'_{1a}|\,|\mathcal{U}'_{2a}|+|\mathcal{U}_{1\bar{a}}|\,|\mathcal{U}_{2\bar{a}}|)-(|\mathcal{U}'_{1a}|-|\mathcal{U}_{1\bar{a}}|)(|\mathcal{U}'_{2a}|-|\mathcal{U}_{2\bar{a}}|), \end{aligned}$$

the last equality following from $x_1y_2+y_1x_2=x_1x_2+y_1y_2-(x_1-y_1)\times(x_2-y_2)$.

Inductively, we have $|\mathcal{U}'_{1a}|\,|\mathcal{U}'_{2a}| \leqslant 2^{|S|-1}\,|\mathcal{U}'_{1a} \cap \mathcal{U}'_{2a}|$ and $|\mathcal{U}_{1\bar{a}}|\,|\mathcal{U}_{2\bar{a}}| \leqslant 2^{|S|-1}\,|\mathcal{U}_{1\bar{a}} \cap \mathcal{U}_{2\bar{a}}|$. Since it is clear that $|\mathcal{U}_1 \cap \mathcal{U}_2| = |\mathcal{U}'_{1a} \cap \mathcal{U}'_{2a}| + |\mathcal{U}_{1\bar{a}} \cap \mathcal{U}_{2\bar{a}}|$, we get

$$|\mathcal{U}_1|\,|\mathcal{U}_2| \leqslant 2^{|S|}\,|\mathcal{U}_1 \cap \mathcal{U}_2|-(|\mathcal{U}'_{1a}|-|\mathcal{U}_{1\bar{a}}|)(|\mathcal{U}'_{2a}|-|\mathcal{U}_{2\bar{a}}|), \tag{1}$$

from which the inequality of the theorem follows, since $\mathcal{U}_{i\bar{a}} \subseteq \mathcal{U}'_{ia}$.

To see the necessity of the equality condition, suppose $a \in \operatorname{sp}(\mathcal{U}_1) \cap \operatorname{sp}(\mathcal{U}_2)$. Then, for $i=1,2$, there is a $U_i \in m(\mathcal{U}_i)$ such that $a \in U_i$. Obviously, $U_i\backslash\{a\} \in \mathcal{U}'_{ia}$ and $U_i\backslash\{a\} \notin \mathcal{U}_{i\bar{a}}$. Therefore, both terms in the negative factor of (1) are positive and $|\mathcal{U}_1|\,|\mathcal{U}_2| < 2^{|S|}\,|\mathcal{U}_1 \cap \mathcal{U}_2|$.

On the other hand, if $\operatorname{sp}(\mathcal{U}_1) \cap \operatorname{sp}(\mathcal{U}_2)=\varnothing$, then, for $i=1,2$, let $S_i=\operatorname{sp}(\mathcal{U}_i)$ and let $\mathcal{U}_i^*=\mathcal{U}_i \cap \mathscr{P}(S_i)$. We have $|\mathcal{U}_i|=2^{|S|-|S_i|}\,|\mathcal{U}_i^*|$ and $|\mathcal{U}_1 \cap \mathcal{U}_2|=|\mathcal{U}_1^*|\,|\mathcal{U}_2^*|\,2^{|S|-|S_1|-|S_2|}$, the latter being verified by the bijection $U \leftrightarrow (U \cap S_1, U \cap S_2, U\backslash(S_1 \cup S_2))$. The equality is immediate. ∎

We now move on to two corollaries of Theorem 1; the inequality in the second is Kleitman's Theorem on the size of the union of intersecting families [3].

COROLLARY 1.1. *Let* $\mathcal{U}_1, \ldots, \mathcal{U}_k$ *be upsets in* $\mathscr{P}(S)$. *Then* $\prod_{i=1}^k |\mathcal{U}_i| \leqslant 2^{(k-1)|S|}\,|\bigcap_{i=1}^k \mathcal{U}_i|$, *with equality if and only if the* $\operatorname{sp}(\mathcal{U}_i)$ *are pairwise disjoint.*

Proof. The proof is by induction on k, the case $k=1$ being trivial and the case $k=2$ being Theorem 1. Therefore, assume $k>2$ and the result holds for $k-1$.

We note that the intersection of upsets is again an upset. Therefore,

$$\left|\bigcap_{i=1}^k \mathcal{U}_i\right| = \left|\mathcal{U}_1 \cap \left(\bigcap_{i=2}^k \mathcal{U}_i\right)\right| \geqslant \frac{|\mathcal{U}_1|\,|\bigcap_{i=2}^k \mathcal{U}_i|}{2^{|S|}} \tag{2}$$

$$\geqslant \frac{|\mathcal{U}_1|}{2^{|S|}}\frac{\prod_{i=2}^k |\mathcal{U}_i|}{2^{(k-2)|S|}} = \frac{\prod_{i=1}^k |\mathcal{U}_i|}{2^{(k-1)|S|}}, \tag{3}$$

where the inequality in (2) is from Theorem 1 and the inequality in (3) is from the inductive assumption.

For the equality condition, if $\mathrm{sp}(\mathscr{U}_i)$ and $\mathrm{sp}(\mathscr{U}_j)$ are not disjoint, then we can assume $2 \leqslant i < j$ and, inductively, the inequality in (3) is strict. On the other hand, if the $\mathrm{sp}(\mathscr{U}_i)$ are pairwise disjoint, then the inequality in (3) holds as an equality. Thus, it suffices to show that the inequality in (2) holds as an equality. That is, we must show that $\mathrm{sp}(\mathscr{U}_1) \cap \mathrm{sp}(\bigcap_{i=2}^k \mathscr{U}_i) = \varnothing$. But this follows from the simple observation that $\mathrm{sp}(\bigcap_{i=2}^k \mathscr{U}_i) \subseteq \bigcup_{i=2}^k \mathrm{sp}(\mathscr{U}_i)$. ∎

A subset $\mathscr{A}$ of $\mathscr{P}(S)$ is an *interesting family* if, for all A, $A' \in \mathscr{A}$, $A \cap A' \neq \varnothing$. Note that a maximal intersecting family is an upset.

COROLLARY 1.2 [3]. *Let $\mathscr{A}_1, ..., \mathscr{A}_k$ be intersecting families in $\mathscr{P}(S)$. Then*

$$\left| \bigcup_{i=1}^k \mathscr{A}_i \right| \leqslant 2^{|S|} - 2^{|S|-k}.$$

If the $\mathscr{A}_i$ are maximal intersecting families, then equality holds if and only if the $\mathrm{sp}(\mathscr{A}_i)$ are pairwise disjoint.

This generalizes the well-known fact that if $\mathscr{A}$ is a maximal intersecting family, then $|\mathscr{A}| = 2^{|S|-1}$.

Proof. We can assume that the $\mathscr{A}_i$ are maximal intersecting families. Let $\mathscr{A}^{\mathrm{C}} = \{S \backslash A \mid A \in \mathscr{A}\}$. Then

$$\begin{aligned}
\left| \bigcup_{i=1}^k \mathscr{A}_i \right| &= |\mathscr{P}(S)| - \left| \bigcap_{i=1}^k \mathscr{A}^{\mathrm{C}} \right| \\
&= |\mathscr{P}(S)| - \left| \bigcap_{i=1}^k \mathscr{A} \right| \qquad (4)\\
&\leqslant 2^{|S|} - \frac{\prod_{i=1}^k |\mathscr{A}_i|}{2^{(k-1)|S|}} \qquad (5)\\
&= 2^{|S|} - \frac{2^{k(|S|-1)}}{2^{(k-1)|S|}} \qquad (6)\\
&= 2^{|S|} - 2^{|S|-k},
\end{aligned}$$

where (4) follows from the bijection $A \leftrightarrow S \backslash A$, (5) from Corollary 1.1, and (6) from $|\mathscr{A}_i| = 2^{|S|-1}$. ∎

Also note that if equality holds in $|\bigcup_{i=1}^{k} \mathscr{A}_i| \leqslant 2^{|S|} - 2^{|S|-k}$, then every extension of the $\mathscr{A}_i$ to maximal intersecting families $\mathscr{A}_i^*$ must have the $\mathrm{sp}(\mathscr{A}_i^*)$ disjoint.

We conclude this article with two remarks. First, and foremost, it is not clear whether the cases of equality in several generalizations of Kleitman's Lemma can be so neatly characterized. In particular, we are curious about the following two generalizations, where $\mathscr{A} \vee \mathscr{B} = \{A \cup B \mid A \in \mathscr{A},\ B \in \mathscr{B}\}$ and $\mathscr{A} \wedge \mathscr{B} = \{A \cap B \mid A \in \mathscr{A}, B \in \mathscr{B}\}$.

THEOREM 2 [2]. *If $\mathscr{A}$ and $\mathscr{B}$ are subsets of $\mathscr{P}(S)$, then*

$$|\mathscr{A}|\ |\mathscr{B}| \leqslant |\mathscr{A} \wedge \mathscr{B}|\ |\mathscr{A} \vee \mathscr{B}|.$$

THEOREM 3 [1]. *Suppose α, β, γ, and δ are functions from $\mathscr{P}(S)$ to the non-negative reals. If, for any $A, B \in \mathscr{P}(S)$, $\alpha(A)\, \beta(B) \leqslant \gamma(A \cup B)\, \delta(A \cap B)$, then for any subsets $\mathscr{A}, \mathscr{B}$ of $\mathscr{P}(S)$,*

$$\left(\sum_{A \in \mathscr{A}} \alpha(A)\right)\left(\sum_{B \in \mathscr{B}} \beta(B)\right) \leqslant \left(\sum_{E \in \mathscr{A} \vee \mathscr{B}} \gamma(E)\right)\left(\sum_{F \in \mathscr{A} \wedge \mathscr{B}} \delta(F)\right).$$

The other comment we make is that the proof of Theorem 1 yields an injection $f\colon \mathscr{U}_1 \times \mathscr{U}_2 \to (\mathscr{U}_1 \cap \mathscr{U}_2) \times \mathscr{P}(S)$. Our proof that f is injective is very simple, but the description of f is complicated (it is an algorithm). It would be interesting to have a simply described injection, even if the proof of injectivity is somewhat complicated.

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