Tilings in Lee metric

P. Horak

University of Washington, Tacoma, Tacoma, WA 98465, USA

ARTICLE INFO

Article history:
Received 26 February 2007
Accepted 8 April 2008
Available online 24 June 2008

ABSTRACT

Gravier et al. proved [S. Gravier, M. Mollard, Ch. Payan, On the existence of three-dimensional tiling in the Lee metric, European J. Combin. 19 (1998) 567–572] that there is no tiling of the three-dimensional space \( \mathbb{R}^3 \) with Lee spheres of radius at least 2. In particular, this verifies the Golomb–Welch conjecture for \( n = 3 \). Špacapan, [S. Špacapan, Non-existence of face-to-face four-dimensional tiling in the Lee metric, European J. Combin. 28 (2007) 127–133], using a computer-based proof, showed that the statement is true for \( \mathbb{R}^4 \) as well. In this paper we introduce a new method that will allow us not only to provide a short proof for the four-dimensional case but also to extend the result to \( \mathbb{R}^5 \). In addition, we provide a new proof for the three-dimensional case, just to show the power of our method, although the original one is more elegant. The main ingredient of our proof is the non-existence of the perfect Lee 2-error correcting code over \( \mathbb{Z} \) of block size \( n = 3, 4, 5 \).

1. Introduction

Let \((C, d)\) be a metric space. Then a code is any subset \( M \) of \( C \), \(|M| \geq 2\). The elements of \( C \) will be called \textit{words}, while elements of \( M \) will be referred to as \textit{codewords}. The most common metric in coding theory is the Hamming metric. In this paper we deal with another frequently used metric, the so-called Lee metric ( = the zig–zag metric, the Manhattan metric). The Lee metric \( d_L \) in \( \mathbb{R}^n \) is given by

\[
d_L(U, V) = \sum_{i=1}^{n} |u_i - v_i|,
\]

where \( U = (u_1, u_2, \ldots, u_n) \), \( V = (v_1, v_2, \ldots, v_n) \).

As usual \( \mathbb{Z} \) will stand for the set of integers. The perfect Lee \( t \)-error correcting code over \( \mathbb{Z} \) of block size \( n \), denoted \( PL(n, t) \), is a set \( M \subset \mathbb{Z}^n \) of codewords so that each word \( A \in \mathbb{Z}^n \) is at Lee distance at most \( t \) from exactly one codeword in \( M \). Since \( PL(n, t) \) code can be seen as a partition of \( \mathbb{Z}^n \) into spheres.
with radius $t$ centered at codewords, only a small step is needed to get a geometrical interpretation of $PL(n, t)$ codes. Consider the space $\mathbb{R}^n$. The $n$-cube centered at $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is the set: $C(X) = \{ Y = (y_1, \ldots, y_n), y_i = x_i + \alpha_i, \text{ where } -\frac{1}{2} \leq \alpha_i \leq \frac{1}{2} \}$. By a Lee sphere of radius $r$ in $\mathbb{R}^n$, $L(n, r)$, centered at $O$ we understand the union of $n$-cubes centered at $Y$, where $d_L(O, Y) \leq r$, and $Y - O$ has integer coordinates. Finally, a Lee sphere of radius $r$ in $\mathbb{R}^n$ centered at $X \in \mathbb{R}^n$ is a translation of $L(n, r)$ centered at $O$ along the coordinate axes so that $O$ is mapped on $X$. Clearly, a $PL(n, t)$ code exists if and only if there is a tiling of $\mathbb{R}^n$ by Lee spheres of radius $t$. The Lee spheres $L(2, 1)$, $L(2, 2)$, $L(3, 1)$, and $L(3, 2)$ are depicted in Fig. 1.

The most famous and intensively studied problem in the area of Lee codes is the Golomb–Welch conjecture. In [3] it is shown that $PL(n, 1)$ code exists for all $n \geq 1$, and $PL(2, t)$ code exists for all $t \geq 1$. In addition, it is proved there that there is no $P(3, 2)$ code, and that there are no $PL(3, a_n)$ codes, where $a_n \rightarrow \infty$ is not explicitly specified. The authors conjectured:

**Conjecture 1.** Golomb–Welch: There are no $PL(n, t)$ codes for $n > 2$ and $t > 1$.

There are many results supporting the conjecture. The strongest one was proved by Post [8]:

**Theorem 2.** $PL(n, t)$ codes do not exist for $n = 3$ and $t \geq 2$; for $4 \leq n \leq 5$ and $t \geq n - 2$; and for $n \geq 6$ and $t \geq \frac{\sqrt{2}}{2}n - \frac{1}{4}(3\sqrt{2} - 2)$.

In the final remark Post states that, by using a computer to evaluate coefficients of the Taylor series of a suitable function, it is possible to show that there are no perfect $t$-error correcting codes for $6 \leq n \leq 130$ and $t \geq \frac{1}{18}(9n - 15)$; and for $131 \leq n \leq 305$ and $t \geq \frac{1}{18}(9n - 14)$. The reader interested in the non-existence results for Lee codes over finite sets is referred to [1,2,7], and also to [9] for the size of the largest Lee codes over a finite set. It is speculated in [6] that the most difficult cases to prove in the Golomb–Welch conjecture are those for $t = 2$ because they are the threshold cases ($PL(n, 1)$ codes do exist). The Golomb–Welch conjecture has been verified there for the two smallest opened cases:

**Theorem 3.** There is no $PL(n, 2)$ code for $n = 5$ and 6.
Thus, the Golomb–Welch conjecture has been verified for all pairs \((n, t)\) where \(n \leq 6\).

In [4,5], the authors prove, for the three-dimensional case, a result even stronger than conjectured by Golomb and Welch. They formulate it in terms of tilings of \(\mathbb{R}^3\) by Lee spheres. It is shown in [4] that:

**Theorem 4.** There is no tiling of \(\mathbb{R}^3\) with Lee spheres of radii at least two, even with different radii.

Thus, as a special case, they showed that there is no \(\text{PL}(3, t)\) code for any \(t \geq 2\). The authors provide a very elegant “a picture says it all” proof. Yet, a stronger result is proved in [5], a sequel to [4], where it is shown that there is no tiling of \(\mathbb{R}^3\) with Lee spheres if radius of at least one sphere is greater than one. Recently, Špacapan [10] extended Theorem 4 to the four-dimensional case.

**Theorem 5.** There is no tiling of \(\mathbb{R}^4\) with Lee spheres of radii at least two, even with different radii.

The both proofs in [4] and in [10] have one feature in common; they are “from scratch”, they do not use any known result. On the other hand, the proof in [10] differs essentially from that one in [4]. It requires checking a large amount of cases and therefore it is computer-based.

In this paper we introduce a new method which provides a relatively short proof, not aided by a computer, for Theorem 5, but also for the five-dimensional case. We will give a new, short proof for Theorem 4 as well, although the original one given in [4] is more elegant, just to show the power of our method. The proof does not split into cases for \(n = 3\), and considers only two case for \(n = 4\), and three cases for \(n = 5\). Unlike the proofs in [4] and in [10], our method is based on a known result, namely on the non-existence of the perfect Lee 2–error correcting codes over \(\mathbb{Z}\) of block size \(n = 3, 4,\) and 5. Thus, “as a by-product”, our method provides some evidence that the most difficult cases in the Golomb–Welch conjecture are those for \(t = 2\), because they imply, as a special case, the non-existence of \(\text{PL}(n, t)\) for \(3 \leq n \leq 5\), and \(t \geq 3\). Our proof is “algebraic” in nature. Therefore we will first generalize the notion of the perfect Lee \(t\)-error correcting code. As usual, by a sphere \(S = (W, r_W)\), centered at \(W\) and of radius \(r_W\), we understand the set of all words \(V \in \mathbb{Z}^n\) so that \(d_l(W, V) \leq r_W\). For \(V \in S\), we will also say that \(S\) covers \(V\). The perfect Lee code over \(\mathbb{Z}\) of block size \(n\), denoted \(\text{PL}(n)\), is a set \(S\) of spheres \((W, r_W), W \in \mathbb{Z}^n, r_W \geq 2\), so that each word in \(\mathbb{Z}^n\) is covered by exactly one sphere in \(S\). The main theorem of the paper reads as follows:

**Theorem 6.** There is no \(\text{PL}(n)\) code for \(3 \leq n \leq 5\).

We believe that a further refinement of the method should provide a proof of the non-existence of \(\text{PL}(n)\) at least for \(n = 6\).

At the end of this introduction we mention a result which is related to the topic of this paper. A tiling of \(\mathbb{R}^n\) by Lee spheres is called regular if neighboring spheres meet along entire \((n - 1)\)-dimensional faces of the original cubes. It is shown in [4] and [5] that the results stated there hold even in the case if we admit non-regular tilings. At the first glance it seems obvious that there are no non-regular tilings of \(\mathbb{R}^n\) by Lee spheres. However, in [11] Szabo proved the following surprising result:

**Theorem 7.** There is a non-regular tiling of \(\mathbb{R}^n\) if and only if \(2n + 1\) is not a prime.

### 2. \(\text{PL}(n)\) codes for \(3 \leq n \leq 5\).

In this section we prove the main result of the paper.

Throughout the proof words in \(\mathbb{Z}^n\) will be denoted by upper case block letters, and their coordinates by the same lower case letter endowed with an index, e.g., a word \(W\) will have coordinates \((w_1, \ldots, w_n)\). Further, we drop subscript \(L\) when dealing with Lee metric, so the Lee distance will be denoted simply by \(d\). The statement will be proved by contradiction. Suppose that there is a \(\text{PL}(n)\) code \(\mathcal{P}\), where \(3 \leq n \leq 5\). By Theorems 2 and 3, there is no perfect Lee 2-error correcting code \(\text{PL}(n, 2)\) for \(3 \leq n \leq 5\) (note that Theorems 4 and 5 imply the statement for \(n = 3\) and \(n = 4\), respectively, as well). Thus, there is a sphere \(S_0 = (A, r_A) \in \mathcal{P}\) so that \(r_A \geq 3\). By a suitable translation of \(\mathcal{P}\) we
may assume that $A = (-r_A + 2, 0, \ldots, 0)$. Consider the set $V$ of words $V$ with $d(V, A) = r_A + 1$, and $v_1 \geq 0$. Clearly, $V \in \mathcal{V}$ iff

$$
\sum_{i=1}^{n} |v_i| = 3.
$$

Indeed, $d(V, A) = \sum_{i=1}^{n} |v_i - a_i| = |v_1 - (2 - r_A)| + \sum_{i=2}^{n} |v_i| = v_1 + r_A - 2 + \sum_{i=2}^{n} |v_i| = r_A + 1$, and (1) follows. Therefore, each word $V$ in $\mathcal{V}$ is either of type $[\pm 3]$, of type $[\pm 2, \pm 1]$, or $[\pm 1^3]$.

To prove the non-existence of $PL(n)$ code for $3 \leq n \leq 5$, we show that it is impossible to cover all words in $\mathcal{V}$, that is, we show that there is no set of pairwise disjoint spheres (and disjoint from $S_0$), each of radius at least 2, covering all words in $\mathcal{V}$. To this extent, let $\mathcal{S} \subset \mathcal{P}$ be the set of all spheres in $\mathcal{P}$ which cover at least one word in $\mathcal{V}$. The words $W$ so that $(W, r_W) \in \mathcal{P}$ will be called codewords, the words $W$ so that $(W, r_W) \in \mathcal{S}$ will be called codewords in $\mathcal{S}$. Moreover, if a word $V$ belongs to a sphere $S = (W, r_W) \in \mathcal{S}$, we will abuse slightly the language and sometimes instead of saying $S$ covers $V$ we will say that the codeword $W$ covers $V$.

Now we state a series of statements which are rather simple but will be applied over and over in this proof, although not always explicitly referred to. By definition of $PL(n)$ code we get:

Claim 8. If $W, Z$ are codewords then the spheres $(W, r_W)$ and $(Z, r_Z)$ are disjoint, that is, $d(W, Z) \geq r_W + r_Z + 1$.

For any two words $U, V$, their Lee distance $d(V, W)$ is invariant with respect to adding the same integer to a coordinate, multiplying a coordinate by $-1$, or swapping the order of coordinates. Therefore:

Claim 9. If $\mathcal{P}$ is a $PL(n)$ code, then (i) translating all codewords of $\mathcal{P}$, (ii) multiplying a coordinate of each codeword of $\mathcal{P}$ by $-1$, (iii) swapping the order of coordinates in all codewords of $\mathcal{P}$, results in a new $PL(n)$ code.

The following claim plays a crucial role in the description of words in $\mathcal{V}$ covered by a codeword in $\mathcal{S}$.

Claim 10. Let $W, Z$ be codewords (not necessarily in $\mathcal{S}$), $V$ be a word covered by $W$, and $d(Z, V) \leq r_Z + 2$. Then, for each coordinate $i$, it is either $z_i \leq v_i \leq w_i$, or $z_i \geq v_i \geq w_i$.

Proof. By Claim 9, we may assume that $Z = O = (0, 0, \ldots, 0)$. Then in fact we need to prove that $v_i w_i \geq 0$, and $|w_i| \geq |v_i|$ for all $i$. Since $W$ covers $V$ we have $r_W \geq d(W, V) = \sum_{i=1}^{n} |w_i - v_i|$. The spheres $(W, r_W)$ and $(O, r_O)$ are disjoint, therefore $d(W, O) = \sum_{i=1}^{n} |w_i| \geq r_W + r_O + 1 \geq \sum_{i=1}^{n} |w_i - v_i| + (d(O, V) - 2) + 1$ (by assumption $d(O, V) \leq r_O + 2$). After simple rearrangements we get $\sum_{i=1}^{n} |w_i - v_i| \geq \sum_{i=1}^{n} |w_i - v_i| - 1$. Trivially $|a - b| \leq |a - b|$ for all $a, b$, with equality if $ab \geq 0$, and $|a| \geq |b|$. To complete the proof it suffices to note that for $ab < 0$ it is $|a - b| < |a - b| - 1$. Thus $v_i w_i \geq 0$, and $|v_i| \geq |w_i|$ for all $i \leq n$. The claim follows.

As an immediate corollary of the above claim we get:

Claim 11. If $W$ is a codeword in $\mathcal{S}$ then $\sum_{i=1}^{n} |w_i| \geq 5$, and $w_1 \geq 0$.

Proof. By definition of $\mathcal{S}$ there is a word $V \in \mathcal{V}$ covered by $W$. For each word $V$ in $\mathcal{V}$ we have $d(A, V) = r_A + 1$. Since $v_1 \geq 0$ for all words in $\mathcal{V}$, by Claim 10, $w_1 \geq 0$ as well. In addition, $d(W, A) = (w_1 - (2 - r_A)) + \sum_{i=2}^{n} |w_i| \geq r_A + r_W + 1 \geq r_A + 2 + 1$, that is, $\sum_{i=1}^{n} |w_i| \geq 5$. The claim follows.

The following corollary is the most frequently used statement of all claims given here. It provides a simple but very useful description of all words $V$ in $\mathcal{V}$ covered by a codeword $W$ in $\mathcal{S}$. We point out that the description involves only coordinates of $W$ but not the radius $r_W$ of the sphere $S = (W, r_W) \in \mathcal{S}$.

Claim 12. Let $W$ be a codeword in $\mathcal{S}$. Then $W$ covers a word $V$ in $\mathcal{V}$ if and only if $v_i w_i \geq 0$, $|w_i| \geq |v_i|$ for all $i = 1, \ldots, n$. 


The statement will be proved by contradiction. We start with the case

Claim 10

Let \( Z \) be a codeword covering a word \( V \)

Theorems 4

Claim 12

and

Claim 12

Claim 10

Then the necessary part follows from Claim 14.

Example. Let \( W = (1, 7, -1, -2) \) be a codeword in \( \delta \). Then Claim 12 implies that \( W \) covers in \( \mathcal{V} \) only one word of type \([\pm 3]\), namely \( (0, 3, 0, 0) \), the following words of type \([\pm 2, \pm 1]\) : \((1, 2, 0, 0), (0, 2, -1, 0), (0, 2, 0, -1), (1, 0, 0, -2), (0, 1, 0, -2), (0, 0, -1, -2), \) and \((0, 0, -1, -2), \) and four words of type \([\pm 1^2]\) : \((1, 1, -1, 0), (1, 1, 0, -1), (1, 0, -1, -1), \) and \((0, 1, -1, -1) \).

The following three relations are essential for the proof of our theorem. Let \( C(t) \) be the set of codewords in \( \delta \) having \( t \) non-zero coordinates. Further, let \( C(t) \) be a subset of \( C(t) \) that contains codewords \( W \) so that \(| \{j : |w_j| > 1\} | = i \). Set \(| C(t) | = c(t) \) and \(| C(t) | = c(t) \).

There are \( 2n - 1 \) words of type \([\pm 3]\) in \( \mathcal{V} \) (note that for \( V \) in \( \mathcal{V} \) it is \( v_1 \geq 0) \). By Claim 11, for each codeword \( W \) in \( C(t) \) there is \( i, 1 \leq i \leq n \), so that \(|w_i| \geq 3\), and consequently, by Claim 12, \( W \) covers one of those words of type \([\pm 3]\); thus

\[
c_1(t) \leq 2n - 1. \tag{2}
\]

Further, there are \( 8 \binom{n}{2} - 4(n - 1) \) words of type \([\pm 2, \pm 1]\) in \( \mathcal{V} \). Hence

\[
4c(3) - 2c_1(t) + 3c(4) \leq 8 \binom{n}{2} - 4(n - 1) \tag{3}
\]
as, by Claim 12, each codeword in \( C(t) \) covers exactly two of those words, each codeword in \( C(3) - C_1(t) \) at least 4 of them, and each codeword in \( C(4) \) at least 3 of them (for each codeword \( W \) in \( C(4) \) there is at least one \( i \) so that \(|w_i| \geq 2\), see Claim 11). Finally, there are \( 8 \binom{n}{3} - 4 \binom{n - 1}{2} \) words of type \([\pm 1^2]\) in \( \mathcal{V} \), which leads to

\[
\sum_{i=3}^{n} \binom{i}{3} C(i) = 8 \binom{n}{3} - 4 \binom{n - 1}{2} \tag{4}
\]
because, by Claim 12, each codeword in \( C(i) \) covers exactly \( \binom{i}{3} \) words of type \([\pm 1^2]\).

Clearly, there are many solutions of Eqs. (2)–(4) in the non-negative integers. A solution, which corresponds to a perfect Lee code over \( \mathbb{Z} \), will be called a feasible solution. So our aim will be to show that there is no feasible solution of these equations.

2.1. \( n = 3, 4 \)

Here we provide alternative proofs to Theorems 4 and 5. We start with a statement, which plays an essential role in the closing argument of those proofs.

Theorem 13. For \( n = 3 \), there is no codeword \( W \in C_2(2) \) so that \( w_1 = 0 \). For \( n = 4 \), there is no codeword \( W \in C_2(2) \), so that \( w_1 = 2 \).

Proof. The statement will be proved by contradiction. We start with the case \( n = 4 \). Suppose that \( W \in C_2(2) \), where \( w_1 = 2 \). Then there is \( i, 2 \leq i \leq 4 \), so that \(|w_i| \geq 3\). (see Claim 11). By Claim 9 we assume, without loss of generality, that \( w_2 \geq 3 \), and \( w_i = 0 \), for \( 3 \leq i \leq 4 \). Let, for \( j = 1, 2, 3 \), \( \mathcal{M}_j \) be a set of words given by \( \mathcal{M}_1 = \{V, v_1 = 1, v_2 = 2, \sum_{i=3}^{4} |v_i| = 1\}, \mathcal{M}_2 = \{V, v_1 = 1, v_2 = 1, \sum_{i=3}^{4} |v_i| = 1\}, \mathcal{M}_3 = \{V, v_1 = 0, v_2 = 2, \sum_{i=3}^{4} |v_i| = 1\}. \) Clearly, for all \( j, |\mathcal{M}_j| = 4 \). The following claim is crucial for the proof of this theorem.

Claim 14. Let \( Z \) be a codeword covering a word \( V \in \mathcal{M}_j \), \( 1 \leq j \leq 3 \). Then \( z_1 = v_1 \), and \( z_2 = v_2 \).
**Proof of Claim.** We recall that $A = (2 - r_A, 0, 0, 0)$, $r_A \geq 3$, is a codeword. Let $V \in M_j$, $1 \leq j \leq 3$. Then $d(A, V) \leq r_A + 2$, and therefore, by Claim 10, $z_1 \geq v_1 \geq 0 > 2 - r_A$, and $z_2 \geq v_2 \geq 0$. On the other hand, $W$ is a codeword in $\mathcal{M}$, so, by Claim 12, $W$ covers both words $(2, 1, 0, 0)$ and $(1, 2, 0, 0)$. Hence $r_W = d(W, (2, 1, 0, 0)) = w_2 - 1$ (note that $r_W$ cannot be $w_2 - 1$ as then the spheres $(A, r_A)$ and $(W, r_W)$ would intersect). Therefore, $d(W, V) = \sum_{i=1}^{4} |w_i - v_i| = (w_1 - v_1) - (w_2 - v_2) + 1 \leq 2 + w_2 - 2 + 1 = w_2 + 1 = r_w + 2$ and, again by Claim 10, $z_1 \geq v_1 \leq w_1$, and $z_2 \geq v_2 \leq w_2$. The proof follows.

Now we will classify codewords $Z$ covering words in $\bigcup_{i=1}^{3} M_i$. Set, for $j = 1, 2, A_j = \{Z : Z$ is a codeword covering a word in $\bigcup_{i=1}^{3} M_j, \text{ and } |\{i, i \geq 3, z_i \neq 0\}| = j \}$. Put $a_i = |A_i|$. We prove two inequalities on $a_i$.

It is clear that each codeword from $A_i$ covers $i$ words in $\bigcup_{j=1}^{3} M_j$. Indeed, since $\bigcup_{i=2}^{3} M_j \subset V$, the statement in this case follows from Claim 12. For a codeword covering a word in $M_1$, the assumption that $U = (1, 2, d, 0)$, $d \neq 0$, covers a word $(1, 2, 0, c)$, $|c| = 1$, implies that spheres $(U, r_U)$ and $(W, r_W)$ intersect. The other part is obvious. As $|\bigcup_{j=1}^{3} M_j| = 12$, we get

$$2a_2 + a_1 = 12.$$  \tag{5}  

To finish the proof of the theorem we prove that

$$a_1 + a_2 \leq 4$$  \tag{6}

which contradicts (5).

First we point out that if $Z$ covers a word $V \in \bigcup_{j=1}^{3} M_j$ then

$$|z_3| + |z_4| \geq 3.$$  \tag{7}

Indeed, if $V$ is in $M_2 \cup M_3$, then the inequality follows from Claim 11: for a word $V$ in $M_1$ it can be routinely checked that $|z_3| + |z_4| < 3$ implies that the spheres $(Z, r_2)$ and $(W, r_W)$ intersect because $r_W \geq w_2 - 1$ and $d(Z, W) \leq w_2 + 1$ in this case. Let $Z, Z'$ be a codeword in $A_1 \cup A_2$ covering a word $V, V' \in \bigcup_{i=1}^{3} M_i$, respectively. To prove (6) it is sufficient to show, with respect to (7), that if $Z, Z' \in A_1 \cup A_2$, and $z_j z_j' > 0$, $j \geq 3$, then either $|z_j| = 1$, or $|z_j'| = 1$. Assume wlog $j = 3$. Suppose by contradiction that $|z_3| > 1$, and $|z_3'| > 1$. We have $z_3 z_3' > 0$, and $\min(\{|z_3|, |z_3'\}| \geq 2$, which yields

$$|z_3 - z_3'| \leq |z_3| + |z_3'| - 4.$$  \tag{8}

Further, the spheres $(Z, r_2)$ and $(Z', r_2')$ are disjoint, thus we get $d(Z, Z') = \sum_{i=1}^{4} |z_i - z_i'| \geq r_2 + r_2' + 1 \geq d(Z, W) + d(Z', W') + 1 = \sum_{i=1}^{4} |z_i - v_i| + \sum_{i=1}^{4} |z_i' - v_i'| + 1$. Applying Claim 14 into

$$\sum_{i=1}^{4} |z_i - z_i'| \geq \sum_{i=1}^{4} |z_i - v_i| + \sum_{i=1}^{4} |z_i' - v_i'| + 1 \text{ provides } |v_1 - v_1'| + |v_2 - v_2'| + \sum_{i=3}^{4} |z_i - z_i'| \geq \sum_{i=3}^{4} |z_i - v_i| + \sum_{i=3}^{4} |z_i' - v_i'| + 1 \text{ now using (8) and obvious } |z_4 - z_4'| \leq |z_4| + |z_4'| \text{ yields } |v_1 - v_1'| + |v_2 - v_2'| \geq 3.\text{ However, } |v_1 - v_1'| \leq 1, \text{ and } |v_2 - v_2'| \leq 1, \text{ a contradiction, and (6) follows.}$$

Finally, let $n = 3$. Assume wlog that $W = (0, a, b) \in C(2)$, where $a \geq 0, b \geq 0$. Consider the words $V_1 = (1, 1, 1)$, and $V_2 = (1, 1, 2)$. By the same argument as in the proof of Claim 14, we get that if a codeword $Z_i$ covers $V_i, i = 1, 2$, then $Z_1 = (c, 1, 1), Z_2 = (d, 1, 2)$, where, by Claim 11, $c \geq 3$, and $d \geq 2$ (otherwise the spheres $(A, r_A)$ and $(Z_2, r_2)$ would intersect). However, then the spheres $(Z_1, r_2)$ and $(Z_2, r_2)$ intersect. Indeed, $r_2 \geq c - 1$ while $r_2 \geq d - 1$, but $d(Z_1, Z_2) = |c - d| + 1 < (c - 1) + (d - 1) + 1 = r_2 + r_2 + 1$. The proof of the theorem is complete.

\[ \text{Theorem 15. There is no PL(3) code.} \]

\[ \text{Proof.} \] Putting $n = 3$ into Eq. (4), and taking into account that in this case $C(4) = C(5) = 0$ we get $c(3) = 4$.

First of all we show that there is a codeword $W \in C_1(3)$ so that $w_1 = 1$. Indeed, otherwise $c(3) - c_1(3) \geq 3$, and the codewords in $C(3) - C_1(3)$ would have to cover twice some word in $A = \{V,$

P. Horak / European Journal of Combinatorics 30 (2009) 480–489

485
Let \(z \in C_2(3)\) with \(z_1 = 0\), contradicting Theorem 13.

**Theorem 16.** There is no PL(4) code.

**Proof.** For the reader’s convenience we state that, for \(n = 4\), the equations (2)–(4) turn into

\[
\begin{align*}
c_1(3) & \leq 7 \\
4c(3) - 2c_1(3) + 3c(4) & \leq 36 \\
c(3) + 4c(4) & = 20.
\end{align*}
\]

First of all we show that if there were a feasible solution of Eqs. (2)–(4) then \(3 \leq c(4) \leq 4\). Let \(A = \{V, V\}\) be a word of type \([\pm 1]^3\), \(v_1 = 0\). Clearly, \(|A| = 8\). By Claim 12, each word in \(C(3) \cup C(4)\) covers at most one word in \(A\), thus \(c(3) + c(4) \geq 8\). Therefore, from (4), \(c(4) \leq 4\). On the other hand, it is easy to see that, for \(c(4) \leq m < 4\), \(m = \min(4c(3) - 2c_1(3) + 3c(4))\) is attained when \(c(3) = 2\) and \(c(4) = 4\) are maximum possible, i.e. \(c_1(3) = 2\) and \(c(4) = 2\). Thus, \(m = 48 - 14 + 6 = 40 > 36\), contradicting (3). We consider two cases.

I. \(c(4) = 4\). By Claim 9 we may assume that there is a codeword \(Z \in C(4)\) with \(z_1 > 0\) for all \(i = 1, \ldots, 4\). Then the coordinates of the four codewords in \(C(4)\) have the following signs: \((+, +, +, +), (+, +, -, -), (+, -, +, +),\) and \((+, -, -, +)\). Let \(B = \{V, V\}\) be a word of type \([\pm 1]^3\), \(v_1 = 2\). At most one codeword \(W \in C(4)\) has the property that \(w_1 > 1\), otherwise, by Claim 12, a word \(V\) of type \([\pm 1]^3\) with \(v_1 = 2\) and \(v_i = 1\), for some \(j, 2 \leq j \leq 4\), would be covered by two codewords from \(C(4)\). Thus, codewords in \(C(4)\) cover at most three words in \(B\). There are twelve words \(V\) of type \([\pm 1]^3\) with \(v_1 \neq 0\). Each codeword in \(C(4)\) covers three of them. As \(c(4) = 4\), all of those twelve words are covered by codewords in \(C(4)\). Hence, by Claim 12, for each codeword \(W \in C(3)\) we have \(w_1 = 0\). This in turn implies that no word in \(B\) is covered by a codeword in \(C(3)\). Hence, at least three words in \(B\) have to be covered by codewords \(Z \in C(2)\), with \(z_1 > 1\). By Claim 12, each codeword in \(C(2)\) covers at most one word from \(B\), so there are at least three codewords in \(C(2)\) covering a word in \(B\). Since at most one of them has its first coordinate \(\geq 3\) (otherwise the word \((3, 0, 0, 0)\) would be covered by more than one codeword), there is a codeword \(U \in C(2)\) with \(u_1 = 2\). However, this contradicts Theorem 13.

II. \(c(3) = 3\). Assume wlog that the coordinates of codewords in \(C(4)\) have the following signs: \((+, +, +, +), (+, +, -, -),\) and \((+, -, -, +)\). These three codewords in \(C(4)\) cover nine out of twelve codewords \(V\) of type \([\pm 1]^3\) with \(v_1 = 1\). To cover the three remaining words \(V\), there have to be, by Claim 12, codewords \(U_i\), \(i = 1, 2, 3\), in \(C(3)\) with coordinates \((+, 0, -, +), (+, -, 0, +),\) and \((+, -, -, 0)\), respectively. Moreover, there has to be a codeword \(U_i\) in \(C(3)\) with coordinates \((0, -, -, +)\) to cover the word \((0, -1, -1, -1)\). It is not difficult to check that, to avoid some word of type \([\pm 2, \pm 1]\) to be covered by two codewords \(U_i\), that \(U_i \in C(3)\) for \(i = 1, \ldots, 4\). Thus, there is \(i, 1 \leq i \leq 4\), so that the first coordinate of \(U_i\) is \(\geq 3\). Now we focus on the set of words \(B = \{V, V\}\) of type \([\pm 2, \pm 1]\), \(v_1 = 2\). As in the case \(c(4) = 4\), there is in \(C(4)\) at most one codeword \(W\) with \(w_1 \geq 2\). In addition, there is exactly one codeword \(U \in C(3)\) with \(u_1 \geq 2\) (in fact for this codeword its first coordinate \(\geq 3\), as mentioned above). Therefore, at most 5 words from \(B\) are covered by codewords in \(C(3) \cup C(4)\); hence there has to be a codeword \(Z \in C(2)\), covering a word in \(B\). To avoid covering the word \((3, 0, 0, 0)\) twice, it has to be \(z_1 = 2\), contradicting Theorem 13.

2.2. \(n = 5\)

In order to facilitate our discussion we introduce more notions and notation. Two words \(U\) and \(V\) are said to be sign equivalent in the \(i\)th coordinate if \(u_i v_i > 0\). We will deal very often with a set of words that are sign equivalent in some coordinate. For each coordinate we have two such sets. To simplify the language we will introduce the notion of the signed coordinate. For the rest of the paper
by the set of signed coordinates we will understand the set $I = \{+1, \ldots, +5, -2, \ldots, -5\}$ (we recall that by definition it is $v_1 \geq 0$ for each word $V$ in $V$; and by Claim 12 it is $w_1 \geq 0$ for each codeword $W$ in $\delta$). Let $A$ be a set of words, and $i, j, ||i|| \neq ||j|| \in I$. Then $A_i (A_j)$ is the set of all words in $A$ so that $i. w_{||i||} > 0 (i w_{||i||} > 0, j w_{||j||} > 0)$. In other words, $A_i$ is the set of words in $\mathbb{Z}^n$ that are pairwise sign equivalent in the $||i||$th coordinate, and their common sign in the $||i||$th coordinate coincide with the sign of $i$. We note that no two codewords in $\delta$ are sign equivalent in three coordinates because they would cover the same word of type $\{\pm1\}$, see Claim 12.

The Eqs. (2)–(4) describe “global” properties of parameters $c(t)$. The following three theorems describe their “local” properties.

**Theorem 17.** It is $\sum_{t=3}^{5} \left(\begin{array}{c}t-1 \\ 2\end{array}\right) |C(t+1)| = 24$, and for each $i \in I, i \neq +1$, we have $\sum_{t=3}^{5} \left(\begin{array}{c}t-1 \\ 2\end{array}\right) |C(t)_i| = 18$. Consequently, $|C(3)_i| \equiv 0 \pmod{3}$ for all $i \in I$.

**Proof.** Let $T$ be the set of all words in $V$ of type $[\pm1^3]$. Clearly, $|T| = 4 \left(\begin{array}{c}4 \\ 2\end{array}\right) = 24$, and for $i \in I, i \neq +1$, it is $|T_i| = 4 \left(\begin{array}{c}4 \\ 2\end{array}\right) - 6 = 18$. Further, each word in $T_i$ is covered by exactly one word from $\bigcup_{t=3}^{5} C(t)_i$, and, by Claim 12, each codeword from $C(t)_i$ covers $\left(\begin{array}{c}t-1 \\ 2\end{array}\right)$ of them. We are done with the first part of the statement. The second part follows trivially from the fact that $3| \left(\begin{array}{c}t-1 \\ 2\end{array}\right)$ for $t = 4, 5$. \hfill $\Box$

**Theorem 18.** For each $i, j \in I, +1 \in \{i, j\}, ||i|| \neq ||j||, \sum_{t=3}^{5} (t - 2) C(t)_{ij} = 6$; and for each $i, j \in I, +1 \notin \{i, j\}, \sum_{t=3}^{5} (t - 2) C(t)_{ij} = 5$. Consequently, $|C(3)_i|$ and $|C(5)_i|$ have the same parity for $+1 \in \{i, j\}$, and the opposite parity for $+1 \notin \{i, j\}$.

**Proof.** As in the previous proof, let $T$ be the set of all words of type $[\pm1^3]$. Then, for $||i|| \neq ||j||, |T_{ij}| = 2(5 - 2)$ for $+1 \in \{i, j\}$, otherwise $|T_{ij}| = 5$. Each word in $T_{ij}$ is covered by exactly one codeword from $\bigcup_{t=3}^{5} C(t)_i$. As, by Claim 12, each codeword from $C(t)_{ij}$ covers $t - 2$ words in $T_{ij}$, the main part of the statement follows. The second part is obvious. \hfill $\Box$

We state one more theorem that will significantly decrease the number of feasible solutions of (2)–(4).

**Theorem 19.** It is $|C(3) + 1| \leq 9$. For $i \in I$, $i \neq +1$, $|C(3)_i| \leq 3$ when $|C(5)_i| = 0$, and $|C(3)_i| \leq 6$ when $|C(5)_i| = 1$.

**Proof.** Let $i \in I$. Further, set $A_i = \{V, v_{||i||} = \pm1\}$, and $B_i = \{V, v_{||i||} = \pm2\}$. Clearly, $|A_i| = |B_i| = 7$ for each $i \in I, i \neq +1$, and $|A_{i+1}| = |B_{i+1}| = 8$. Let $a$ be the number of words in $A_i$ covered by codewords in $C(3)_i$. It is easy to see that if $W \in C(3)_i$ does not cover any word in $A_i$, then $W \in C(1)_i$ and $|w_{||i||}| \geq 3$. However, there is at most one such codeword in $C(3)_i$; thus $a \geq |C(3)_i| - 1$. This in turn implies $|C(3)_i| \leq 9$ as $|A_{i+1}| = 8$.

Let $i \in I, i \neq +1$. Put $D = \{W, W \in \bigcup_{t=3}^{5} C(t)_i, W$ covers a word in $B_i\}$. Now we state some bounds on $D$. Let $U \in C(t)_i, 3 \leq t \leq 5, |u_{||i||}| > 1$ (i.e., $U \in C$). Then $U$ covers $t - 1$ words in $B_i$. Thus, $|B_i| = 7$ implies $|D| \leq 3$, and consequently $|D| = 3$ implies $\bigcup_{t=3}^{5} C(t)_i \cap D \leq 1$. Further, if $\bigcup_{t=3}^{5} C(t)_i \cap D = 2$ then $C(3)_i \cap D = \emptyset$. Clearly, if $W \in C(4)_i$, but not in $B$ then $W$ covers at least one word in $A_i$ (by Claim 11 there is an index $s$ so that $w_s \geq 2$). Therefore, $|A_i| \geq a + |C(4)_i| - |C(4)_i \cap D| \geq |C(3)_i| + |C(4)_i| - 2$. Since $|A_i| = 7$, we get $|C(3)_i| + |C(4)_i| \leq 9$.

By Theorem 17, $|C(3)_i| + 3 |C(4)_i| = 18 - 6 |C(5)_i|$, and $|C(3)_i| \equiv 0 \pmod{3}$. Hence $|C(3)_i| \leq 3$ for $|C(5)_i| = 0$, and $|C(3)_i| \leq 6$ for $|C(5)_i| = 1$. The proof is complete. \hfill $\Box$
We are ready to prove the main result of the paper:

**Theorem 20.** There is no PL(5) code.

**Proof.** As in the previous cases, for the reader’s convenience we state that, for \( n = 5 \), the Eqs. (2)–(4) have the form

\[
c_1(3) \leq 9 \\
4c(3) - 2c_1(3) + 3c(4) \leq 64 \\
c(4) + 4c(4) + 10c(5) = 56.
\]

We point out that \( c(5) \leq 2 \). Indeed, by Claim 11, for each \( W \in C(5) \) it is \( w_1 \geq 0 \), which in turn implies \( |C(5)|_{+1} \leq 2 \), i.e., \( c(5) \leq 2 \), otherwise some word of type \([\pm 1]^3\) would be covered by two codewords from \( C(5) \). We will consider three cases.

I. \( c(5) = 2 \). Let \( w, z \in C(5) \). By Claim 9 we assume wlog \( w_1 \geq 0 \) for all \( i \). We will consider two subcases with respect to the number of coordinates in which \( W \) and \( Z \) are sign equivalent.

1a. Assume first that \( z_i < 0 \) for \( 2 \leq i \leq 5 \). Then \( |C(4)_{+1}| = 0 \), otherwise there would be a word of type \([\pm 1]^3\) covered by two codewords. Thus, by Theorem 17, \( |C(3)_{+1}| = 12 \), which contradicts Claim 11.

1b. There is \( i \), \( 2 \leq i \leq 5 \), so that \( z_i > 0 \). By Claim 9, we assume \( z_2 > 0 \), and \( z_1 < 0 \) for \( 3 \leq i \leq 5 \). It is easy to see that for each \( U \in C(4) \) it is \( u_2 < 0 \). Set \( A = \{ W, W \in C(4), w_1 \neq 0 \} \), and \( B = \{ W, W \in C(4), w_1 = 0 \} \). By Theorem 19, \( |C(3)_{-2}| \leq 3 \), which in turn implies \( |C(4)_{-2}| \leq 5 \). However, \( c(4) = |C(4)_{-2}| \), thus \( |A| + |B| \geq 5 \). Further, for \( |i| \geq 3 \), \( |j| \geq 3 \), \( |C(4)_{ij}| < 1 \), otherwise some word of type \([\pm 1]^3\) would be covered by two codewords. There are 6 pairs of indices \( i, j \), with \( |i| < 3 \), \( |j| < 3 \), and \( i \neq j \). Each codeword \( U \in A \) belongs to \( C(4)_{ij} \) for one of those six pairs of signed coordinates, and each codeword in \( B \) to two of those six pairs; thus \( |A| + 2 |B| \leq 6 \). Finally, for \( |i| \geq 3 \), it is \( |A| \leq 1 \), otherwise some word of type \([\pm 1]^3\) would be covered by two codewords in \( A \); hence \( |A| \leq 3 \). However, this contradicts \( |A| + 2 |B| \geq 5 \) because \( |A| + 2 |B| \leq 6 \).

II. \( c(5) = 1 \). Let \( w \in C(5) \). As in case I, we assume \( w_1 > 0 \) for all \( i \). By Theorem 18, \( |C(3)_{+1}| \) is odd for every \( i \) \( \in \{ \pm 2, \pm 3, \pm 4, \pm 5 \} \), hence \( |C(3)_{+1}| \geq 6 \) as \( |C(3)| \equiv 0 \pmod{3} \), see Theorem 19. Further, again by Theorem 18, \( |C(3)_{ij}| \) is odd if \( i \neq j \) and at least one of \( i \) and \( j \) is in \( \{ -2, \ldots, -5 \} \). Therefore, \( |C(3)| \geq 3 \) for all signed indices \( i \in I \). Moreover, for \( i \in \{ -2, \ldots, -5 \} \) if \( |C(3)_{+1,i}| > 0 \) then \( |C(3)| \geq 6 \) as then there are at least seven indices \( j \in I \) for which \( |C(3)_{ij}| > 0 \). No codeword in \( C(3)_{+1} \) has all its coordinates non-negative. Hence, \( |C(3)_{+1}| \geq 6 \) implies that there are at least three indices \( i \) in \( \{ -2, \ldots, -5 \} \) for which \( |C(3)| \geq 6 \), contradicting Theorem 19 because \( c(5) = 0 \).

III. \( c(5) = 0 \). By Theorem 18, \( |C(3)_{ij}| \) is odd for each \( i, j \in I \), \( |i| \neq |j| \), \( \pm 1 \neq \{ i, j \} \). Thus \( |C(3)| \geq 3 \), and by Theorem 18, \( |C(3)| = 3 \). By (4), we get \( c(3) \equiv 0 \pmod{4} \). Hence \( |C(3)_{ij}| \equiv 0 \pmod{4} \). As \( |C(3)_{+1}| = 12 \) contradicts Theorem 19, we get \( |C(3)_{+1}| = 0 \). Thus \( c(3) = 8 \), and by (4), \( c(4) = 12 \). Moreover, \( |C(3)_{+1}| = 0 \) implies \( |C(4)_{+1}| = 8 \). Therefore there are four codewords \( U \in C(4) \) with \( u_1 = 0 \). It is easy to see that there have to be two codewords \( U_1, U_2 \) among those four codewords which are sign equivalent in two coordinates, say \( i, j \). (In fact this would be true even if there were only three codewords \( U \in C(4) \) with \( u_1 = 0 \).) As \( |C(3)_{+1}| = 0 \), there has to be a codeword \( W \in C(4)_{+1} \) that is sign equivalent in the \( i \)-th and \( j \)-th coordinate with both \( U_1 \) and \( U_2 \). Let \( k, 2 \leq k \leq 5, k \notin \{ i, j \} \) be a coordinate so that \( w_k \neq 0 \). Clearly, \( U_1 \) and \( U_2 \) are not sign equivalent in \( k \), as they would be sign equivalent in three coordinates, thus one of them is sign equivalent with \( W \) in \( k \) as well, which again implies that there are two codewords sign equivalent in three coordinates, a contradiction. \( \blacksquare \)

References