LINEAR ALGEBRA AND ITS APPLICATIONS

ELSEVIER Linear Algebra and its Applications 290 (1999) 193–212

The symmetric inverse M-matrix completion problem

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Received 10 July 1997; accepted 26 October 1998

Submitted by W. Barrett

Abstract

Necessary and sufficient conditions are given on the data for completability of a partial symmetric inverse M-matrix, the graph of whose specified entries is a cycle, and these conditions coincide with those we identify to be necessary in the general (non-symmetric) case. Graphs for which all partial symmetric inverse M-matrices have symmetric inverse M-matrix completions are identified and these include those that arise in the general (positionally symmetric) case. However, the identification of all such graphs is more subtle than the general case. Finally, we show that our new cycle conditions are sufficient for completability of all partial symmetric inverse M-matrices, the graph of whose specified entries is a block graph.

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[1] The work of this author was supported, in part, by National Science Foundation grant DMS 92-00899 and by Office of Naval Research contract N00014-90-J1739.

[2] The work of this author was supported, in part, by a Summer Fellowship from the University of Chattanooga Foundation and, in part, by the Center of Excellence for Computer Applications at the University of Tennessee at Chattanooga and was completed while on subbatical at the College of William and Mary.

PII: S0024-3795(98)10230-6

1. Introduction

An M-matrix is an $n \times n$ matrix with nonpositive off-diagonal entries whose inverse exists and is entry-wise nonnegative. An $n \times n$ nonnegative matrix that occurs as the inverse of an M-matrix is called an inverse M-matrix, thus, an inverse M-matrix is simply a nonnegative matrix whose inverse has nonpositive off-diagonal entries. A good deal is known about the important classes of M-matrices and inverse M-matrices (e.g. [2,4,7,8]). We shall make use of the facts that the matrices in both classes necessarily have positive principal minors and hence positive diagonal entries and have LU factorizations with both factors in the class, that both classes are invariant under positive diagonal scaling, transposition, and permutation similarity, and that irreducible inverse M-matrices have only positive entries. Symmetric M-matrices are positive definite (and matrices with nonpositive off-diagonal entries that are symmetric positive definite are M-matrices). Symmetric M-matrices (sometimes called *Stieljtes matrices*) arise in a variety of applications. Our primary interest is in "inverse Stieltjes matrices". Since the Cholesky factors of Stieltjes matrices are M-matrices, inverse Stieltjes matrices are completely positive, i.e., factor as BB^{T} in which B is nonnegative. Henceforth, we abbreviate "inverse M" to IM and "symmetric IM" to SIM.

A *partial matrix* is a rectangular array in which some entries are specified, while the remaining (unspecified) entries are free to be chosen. A *completion* of a partial matrix is a particular choice of values for the unspecified entries, resulting in a conventional matrix. A *matrix completion problem* then asks for which partial matrices do there exist completions of a certain desired type. It is convenient to describe the positions of the specified entries in a partial matrix via a *graph*. For example, if the partial matrix is square, we may use the directed graph in which the edge (i,j) occurs if and only if the i, j entry is specified. Throughout, we assume that the diagonal entries of a square partial matrix are specified, but we omit loops from the graph. If the specified entries occur in symmetrically placed positions ("positionally symmetric"), the graph may be taken to be undirected, which will be the case throughout.

Here, we are primarily interested in the SIM completion problem, though the discussion is naturally relevant to the general IM completion problem. The problem is subject to investigation due to the intrinsic interest in these classes as well as its interesting relation to the well-studied positive definite completion problem. Since principal submatrices of IM matrices are IM (see e.g. Ref. [8]), every fully specified principal submatrix of a partial matrix that has an IM (SIM) completion would have to be IM (SIM). A partial matrix meeting this obvious necessary condition is called a *partial IM- (SIM-) matrix*, analogous to the notion of "partial positive definite" ([5]).

An undirected graph G is called *chordal* if no subgraph induced by a set of vertices is a cycle of length 4 or more. Chordal graphs are very important in completion problems and they may be viewed as follows.

A *clique* in a graph is a set of vertices that induce a complete subgraph, and a clique is called *maximal* if its vertices do not constitute a proper subset of a clique. The maximal cliques of a chordal graph may be viewed in a tree-like manner. A graph is chordal if and only if it may be sequentially built from complete graphs via the identification of a clique of the "next" complete graph to be added with a clique of the graph built so far. The complete building blocks are the maximal cliques of the resulting chordal graph while the cliques of identification (so-called "minimal vertex separators") are intersections of these maximal cliques. This brief introduction to the "clique-tree" structure of chordal graphs is sufficient for our needs here. See Ref. [3] for (many) more details.

It will be convenient, however, to categorize chordal graphs in terms of the number of vertices in the intersections of maximal cliques. We call a chordal graph *k-chordal* if no two distinct maximal cliques intersect in more than k vertices. In Ref. [5] it was shown that every partial positive definite matrix, the graph of whose specified entries is G, has a positive definite completion if and only if G is chordal. More recently in Ref. [12] it was shown that the answer to the analogous question for the positionally symmetric IM completion problem is the 1-chordal graphs (there called "block-clique"). In the symmetric case there is an additional subtlety, possibly involving 2-chordal graphs.

We will need some notation for submatrices. Let A be an $n \times n$ matrix. If α and β are two subsets of $\{1, 2, \ldots, n\}$, $A[\alpha, \beta]$ is the submatrix of A lying in rows α and columns β, while $A(\alpha, \beta)$ is the complementary submatrix of A. We abbreviate $A[\alpha, \alpha]$ to $A[\alpha]$ and $A(\alpha, \alpha)$ to $A(\alpha)$, and, if $\alpha = \{i\}$ and $\beta = \{j\}$, we write $A(i, j)$ for $A(\alpha, \beta)$.

In Section 2, we show that the 1-chordal graphs guarantee SIM completability (for partial SIM matrices) and give an example (3-chordal) to show that not all chordal graphs do. In Section 3 we derive new cycle conditions for the general IM completion problem, and in Section 4, we show that these conditions are sufficient for a cycle in the SIM completion problem. This allows us to exhibit a large class of graphs for which the cycle conditions are sufficient for the SIM completion problem in Section 5. Finally in Section 6, we further discuss graphs for which partial SIM matrices have SIM completions; here nonchordal graphs are again ruled out, but 2-chordal graphs play a role.

2. Sufficiency of 1-chordal graphs for the SIM completion problem

In Ref. [12] it was shown that the graphs (of the specified entries) for which every positionally symmetric partial IM matrix has an IM completion are exactly the 1-chordal graphs. Resolution of the analogous question for the SIM completion problem does not immediately follow because both the hypothesis and the requirements upon the completed matrix are strengthened by the requirement of symmetry. Indeed, as we shall see, there are interesting differences. However, as we shall exploit, 1-chordal graphs remain sufficient, and though the completion strategy is similar, we present a brief discussion here for completeness. As usual, the key is to first understand the case in which there are just two maximal cliques.

In the proof we shall have the need for the following definition, notation and fact from Ref. [13].

Definition 2.1. If A is an $n \times n$ matrix, then the *support* of A is the set $\{(i,j)\colon a_{ij} \neq 0\}$. If $\alpha \subset \{1,2,\dots,n\}$ and $A[\alpha]$ is invertible, let $\tilde{A}_\alpha$ denote the $n \times n$ matrix whose support is a subset of $\alpha \times \alpha$ and in which $\tilde{A}_\alpha[\alpha] = A[\alpha]^{-1}$. For example, $\tilde{A}_{\{1,2\}}$ is the $n \times n$ matrix which has $A[\{1,2\}]^{-1}$ in the upper left corner and 0's elsewhere.

Lemma 2.2 [13]. *Suppose α and β are two overlapping cliques whose union is the graph of a partial matrix A. If $A[\alpha], A[\beta]$, and $A[\alpha \cap \beta]$ are all invertible and B is the unique completion of A whose inverse has zeros in the positions that correspond to unspecified entries of A, then $B^{-1} = \tilde{A}_\alpha + \tilde{A}_\beta - \tilde{A}_{\alpha\cap\beta}$.*

Lemma 2.3. *Suppose that*

$$A = \begin{bmatrix} A_{11} & a_{12} & X \\ a_{12}^{\mathrm{T}} & a_{22} & a_{32}^{\mathrm{T}} \\ X^{\mathrm{T}} & a_{32} & A_{33} \end{bmatrix}$$

is an $n \times n$ partial SIM matrix in which a_{22} is 1×1 and X and X^{T} consist entirely of the only unspecified entries of A. Then

$$A_1 = \begin{bmatrix} A_{11} & a_{12} & \frac{a_{12}a_{32}^{\mathrm{T}}}{a_{22}} \\ a_{12}^{\mathrm{T}} & a_{22} & a_{32}^{\mathrm{T}} \\ \frac{a_{32}a_{12}^{\mathrm{T}}}{a_{22}} & a_{32} & A_{33} \end{bmatrix}$$

is an SIM completion of A. Moreover, A_1 is the unique completion of A whose inverse has 0's in the same positions that X and X^{T} occupy in A and is also the unique determinant maximizing completion among SIM completions of A.

Proof. Up to positive diagonal congruence Lemma 2.3 is a special case of [12], Theorem 1. □

By now standard techniques, first used in Ref. [5], the lemma may be extended to all 1-chordal graphs. Note that, implicitly, we use the fact that the SIM matrices are permutation similarity invariant.

Theorem 2.4. *Let G be a 1-chordal graph on n vertices. Then every $n \times n$ partial SIM matrix A, the graph of whose specified entries is G, has an SIM completion. Moreover, there is a unique SIM completion A_1 of A whose inverse entries are 0 in every unspecified position of A, and A_1 is the unique determinant maximizing SIM completion of A.*

Proof. The proof is by induction on the number of maximal cliques in G, beginning with the case of two cliques covered by Lemma 2.3. For brevity, we summarize and omit details. Suppose that G has k maximal cliques with index sets $C_1, \dots, C_k$, and suppose that G has been built by sequential addition of cliques in the indicated order, so that the cardinality of $C_k \cap (C_1 \cup \cdots \cup C_{k-1})$ is 1. By the induction hypothesis, we may complete $A[C_1 \cup \cdots \cup C_{k-1}]$ as indicated. Then, again apply Lemma 2.3 to C_k and the completion of $A[C_1 \cup \cdots \cup C_{k-1}]$, and Lemma 2.2 verifies the inverse zero pattern for the induction step. □

Note that any tree is 1-chordal, as the edges are the maximal cliques. Thus, the theorem applies to a partial SIM matrix whose specified entries lie in the tridiagonal part (G is a path), a fact we shall use later.

We shall discuss later (Section 4) whether there is completability for partial SIM matrices with more general graphs. But we note here that as in the positionally symmetric case [12] we do not have completability for nonchordal graphs (see Section 4). Moreover, also as in the prior case, there is no chordal theorem as shown by the following example whose graph is 3-chordal with two maximal cliques.

Example 2.5. Consider the partial SIM matrix

$$A = \begin{bmatrix} 1 & \frac{9}{40} & \frac{1}{5} & \frac{2}{5} & x \\ \frac{9}{40} & 1 & \frac{3}{10} & \frac{1}{2} & \frac{3}{8} \\ \frac{1}{5} & \frac{3}{10} & 1 & \frac{1}{5} & \frac{3}{4} \\ \frac{2}{5} & \frac{1}{2} & \frac{1}{5} & 1 & \frac{1}{4} \\ x & \frac{3}{8} & \frac{3}{4} & \frac{1}{4} & 1 \end{bmatrix}.$$

For there to be an x such that A is an SIM matrix, we must have det $A(1,2) = \det\ A[\{2,3,4,5\},\{1,3,4,5\}] \geqslant 0$ which, by Sylvester's identity for determinants [6], is equivalent to: $\det A[\{3,4,5\},\{1,3,4\}] \det A[\{2,3,4,\},\{3,4,5\}] \leqslant \det A[\{2,3,4\},\{1,3,4\}] \det A[\{3,4,5\}]$.

Since $\det A[\{2,3,4\},\{1,3,4\}] = 0$ and $\det A[\{2,3,4\},\{3,4,5\}] > 0$, for the principal submatrix $B = A[\{1,3,4,5\}]$, we must have $\det B(1,4) = \det A[\{3,4,5\},\{1,3,4\}] \leqslant 0$. But, since B must be IM, it follows from the cofactor form of the inverse that $\det B(1,4) \geqslant 0$. Thus, $\det B(1,4) = 0$ or $x = 3/16$. In this event, A^{-1} is positive in the 3, 4 and 4, 3 positions, so that A does not have an SIM completion.

3. Cycle conditions necessary for the IM completion problem

In Ref. [13] it was shown that the inverse M-matrices are contained among the "path-product" matrices (see Definition 5.5). This could be used to prove the necessity of the cycle conditions we are about to describe, but we give a simple, independent proof of their necessity.

Consider the partial IM matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & & & & a_{1n} \\ a_{21} & a_{22} & a_{23} & & ? & \\ & a_{32} & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & ? & & \cdot & \cdot & a_{n-1,n} \\ a_{n1} & & & & a_{n,n-1} & a_{nn} \end{bmatrix}.$$

If A has an IM completion $\hat{A} = (a_{ij})$, then each 3×3 principal submatrix

$$\hat{A}[\{i,j,k\}] = \begin{bmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{bmatrix}$$

would also be IM [8]. But, then the upper right 2×2 submatrix

$$\begin{bmatrix} a_{ij} & a_{ik} \\ a_{jj} & a_{jk} \end{bmatrix}$$

would have nonpositive determinant. Equivalently, we have

$$\frac{a_{ij}a_{jk}}{a_{jj}} \leqslant a_{ik}.$$

Concatenation of this inequality gives

$$\frac{a_{12}a_{23}\ldots a_{n-1,n}}{a_{22}a_{33}\ldots a_{n-1,n-1}} \leqslant a_{1n},$$

in which all entries are specified entries of A. By cyclic permutation and transposition, the above inequality implies a total of $2n$ inequalities. Notice that these inequalities preclude certain arrangements of 0's and nonzeros among the specified entries. (The diagonal entries are nonzero because of the prevailing assumption that A is partial IM.) For example, it cannot happen that exactly one, or only a symmetrically placed pair of specified off-diagonal entries be 0 if these $2n$ inequalities hold. (In these cases any completion would be irreducible, yet have 0's, which cannot happen for an IM matrix anyway.) On the other hand, if there are two (or more) pairs of symmetrically placed 0's, then the $2n$ conditions are trivially met, *and* there are completions as the problem reduces (via direct summation) to a collection of undirected paths (which are 1-chordal). Thus, in the case of symmetric data meeting the $2n$ conditions, we must have a completion if any specified entries are 0. So we assume henceforth that the specified off-diagonal entries are positive in any completion discussions for a cycle. This will be convenient for giving succinct statements of our cycle conditions, but we note that there are worthy questions of sufficiency for certain nonsymmetric arrangements of 0's. Multiplication of the cited $2n$ inequalities by appropriate factors then gives a symmetrized version of our "cycle conditions", which are equivalent to the $2n$ inequalities in case all data is nonzero.

Lemma 3.1. *If the partial IM matrix $A = (a_{ij})$, the graph of whose specified entries is the bidirectional cycle 1 to 2 to 3 to ... to n to 1, has an IM completion, then*

$$\max\left\{\frac{a_{12}a_{23}\ldots a_{n-1,n}a_{n1}}{a_{11}a_{22}\cdots a_{nn}}, \frac{a_{21}a_{32}\ldots a_{n,n-1}a_{1n}}{a_{11}a_{22}\cdots a_{nn}}\right\} \leqslant \min\left\{\frac{a_{12}a_{21}}{a_{11}a_{22}}, \frac{a_{23}a_{32}}{a_{22}a_{33}}, \ldots, \frac{a_{n1}a_{1n}}{a_{11}a_{nn}}\right\}.$$

By positive diagonal scaling, any partial IM matrix may be "normalized" to have 1's on the diagonal, without altering completability. Thus, without loss of generality we may assume

$$a_{11} = a_{22} = \cdots = a_{nn} = 1$$

in the partial matrix A and take A to be of the form

$$A = \begin{bmatrix} 1 & a_1 & & & & b_n \\ b_1 & 1 & a_2 & & ? & \\ & b_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & ? & & \cdot & \cdot & a_{n-1} \\ a_n & & & & b_{n-1} & 1 \end{bmatrix},$$

in which case the necessary cycle conditions assume the form

$$\max\{a_1a_2\ldots a_n, b_1b_2\ldots b_n\} \leqslant \min_{1\leqslant i\leqslant n}\{a_ib_i\}.$$

We note that the data are symmetrizable via positive diagonal similarity (which does not alter completability) if and only if $a_1a_2\ldots a_n = b_1b_2\ldots b_n$. In the event the data are symmetric: $b_i = a_i$, $i = 1,\ldots,n$, our cycle conditions become

$$a_1a_2\ldots a_n \leqslant \min_{1\leqslant i\leqslant n}\{a_i^2\}$$

or

$$\prod_{j\neq i} a_j \leqslant a_i, \quad i = 1,\ldots,n.$$

In Section 4 we show these cycle conditions are sufficient for the completability of a partial SIM matrix A to an SIM matrix. We conjecture that the general cycle conditions are also sufficient for completability of A in the nonsymmetric case.

4. Sufficiency of the cycle conditions for completability in the symmetric case

Just as in the positive definite completion problem, a natural step toward understanding the SIM completion problem is the understanding of completability when the graph of the specified entries is a single cycle (see Ref. [1]) and then the graphs for which these cycle conditions (along with partial SIM) are sufficient (see Ref. [1]). Cycles do require additional conditions (which are quite different from those for the positive definite case) and the next two sections are devoted to this program. In the 3×3 case, the (symmetric) cycle conditions are readily seen to be equivalent to a symmetric nonnegative matrix being SIM. This follow from an easily verified observation that we are about to use: if $0 \leqslant a_i < 1$, $i = 1,2$ and $0 \leqslant x < 1$ (which is necessary for the $\{1,3\}$ principal submatrix to be inverse M), then

$$A = \begin{bmatrix} 1 & a_1 & x \\ a_1 & 1 & a_2 \\ x & a_2 & 1 \end{bmatrix}$$

is invertible and the off-diagonal entries of A^{-1} nonpositive if and only if

(i)

$$a_1a_2 \leqslant x \leqslant \min\left\{\frac{a_1}{a_2}, \frac{a_2}{a_1}\right\}, \tag{4.1}$$

when $a_1a_2 > 0$,

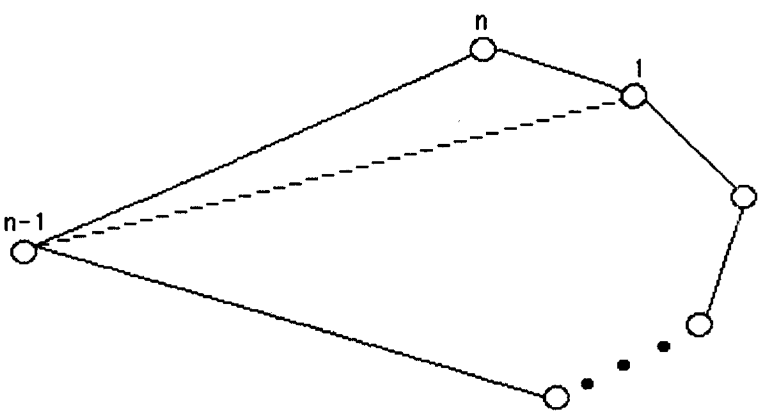
(ii) $x = 0$ when exactly one of $\{a_1, a_2\}$ is 0, or

(iii) $0 \leqslant x < 1$ when $a_1 = a_2 = 0$.

These conditions are equivalent to the cycle conditions.

In case $n > 3$, given the 1-chordal result of Section 2, one natural strategy for assessing completability of a given symmetric pattern is to determine conditions on the data for completability to a partial SIM matrix whose graph is some specific symmetric, 1-chordal super-pattern of the original pattern of specified entries. For the resulting partial SIM matrix with 1-chordal graph, completability is assured.

When the undirected graph of a partial SIM matrix A is an n-cycle, $n \geqslant 4$, we implement this strategy as follows. Data are assigned to the chord $\{n-1, 1\}$ so that (1) the resulting $(n-1)$-cycle $(1, 2, \ldots, n-1, 1)$ meets the cycle conditions and thus has a SIM completion by induction and (2) the original data on the edge $\{n, 1\}$ is consistent with a completion (by Theorem 2.4) of the 1-chordal data whose maximal cliques are $\{1, \ldots, n-1\}$ and $\{n-1, n\}$. Graphically, this is illustrated in the figure below.



Specifically, our result is stated as follows.

Theorem 4.1. *Let*

$$A = \begin{bmatrix} 1 & a_1 & & & & a_n \\ a_1 & 1 & a_2 & & ? & \\ & a_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & ? & & \cdot & \cdot & a_{n-1} \\ a_n & & & & a_{n-1} & 1 \end{bmatrix}$$

be an $n \times n$ partial SIM matrix, $n \geqslant 4$. Then A has an SIM completion if and only if the cycle conditions

$$\prod_{j \neq i} a_j \leqslant a_i, \quad i = 1, \ldots, n$$

are satisfied.

In the proof of the theorem we shall have need of the following easily verified lemma.

Lemma 4.2. *Let $A_n = (1-a)I_n + aJ_n$ in which $0 \leqslant a < 1$, I_n is the $n \times n$ identity matrix, and J_n is the $n \times n$ matrix consisting of all ones. Then,*

$$A_n^{-1} = \frac{1}{(1+(n-1)a)(1-a)}[(1+(n-1)a)I_n - aJ_n].$$

Notice that A_n, $n \geqslant 1$, is thus an SIM.

Proof of Theorem 4.1. In view of the remarks at the end of Section 3 it is enough to prove sufficiency. In case $a_i = a$ for all i, $1 \leqslant i \leqslant n$, we can, in light of the lemma, set each unspecified entry equal to a in order to obtain an SIM completion. If one of the a_i's is 0, the inequalities imply that at least two are. In this event, we have several SIM completion problems whose graphs are paths. Solution of these (guaranteed by Theorem 2.4), surrounded by 0's gives a completion of A.

Now assume no $a_i = 0$ and $a_i \neq a_j$ for some i and j, $1 \leqslant i, j \leqslant n$. By cyclic permutation we may also assume that a_n/a_1 or a_n/a_{n-1} equals

$$\min\left\{\frac{a_j}{a_{j+1}}, \frac{a_j}{a_{j-1}} : 1 \leqslant j \leqslant n\right\}$$

in which $a_0 := a_n$ and $a_{n+1} := a_1$.

First suppose a_n/a_{n-1} is the smallest "neighbor-to-neighbor" ratio among the data. If $n = 4$, then the partial matrix

$$A_1 = \begin{bmatrix} 1 & a_1 & \frac{a_4}{a_3} & a_4 \\ a_1 & 1 & a_2 & ? \\ \frac{a_4}{a_3} & a_2 & 1 & a_3 \\ a_4 & ? & a_3 & 1 \end{bmatrix}$$

is completable to an SIM matrix using Lemma 2.3. Specifically, $A_1[\{1,2,3\}]$ is SIM since $x = a_4/a_3$ (<1) satisfies the inequalities (4.1). Since $A_1[\{3,4\}]$ is SIM by assumption, if we apply Lemma 2.3, a completion of A (to an SIM) will result if the two remaining unspecified entries are replaced by a_2a_3 because $a_4 = a_3(a_4/a_3)$ already. Inductively, assume the theorem holds for all matrices of order $k, 4 \leqslant k < n$. Suppose that $a_i = \min_{1 \leqslant j \leqslant n-2} a_j$ and then define $a = \prod_{j=1, j\neq i}^{n-2} a_j$. Then, using the cycle conditions, we have

$$a_1a_2 \ldots a_{n-2} \leqslant \frac{a_n}{a_{n-1}} \leqslant \frac{a_i}{a_{i-1}}, \frac{a_i}{a_{i+1}}$$

since a_n/a_{n-1} is the smallest neighbor-to-neighbor ratio. Since at least one of $a_{i-1}, a_{i+1} \geqslant a$, this implies

$$a_1a_2 \ldots a_{n-2} \leqslant \frac{a_n}{a_{n-1}} \leqslant \frac{a_i}{a}. \tag{4.2}$$

Replace the unspecified $1, n-1$ and $n-1, 1$ entries of A by a_n/a_{n-1} to obtain the partial matrix

$$\begin{bmatrix} 1 & a_1 & & & \frac{a_n}{a_{n-1}} & a_n \\ a_1 & 1 & . & ? & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & . & . & a_{n-2} & \\ \frac{a_n}{a_{n-1}} & & ? & a_{n-2} & 1 & a_{n-1} \\ a_n & & & & a_{n-1} & 1 \end{bmatrix}.$$

The $(n-1) \times (n-1)$ partial SIM matrix $A_2[\{1,2,\ldots,n-1\}]$ satisfies the cycle conditions in Eq. (4.2) and is thus completable to an SIM matrix by the inductive assumption. Call this completion A_3. Then it is apparent that the last column of A_3 and a_{n-1} can be used to complete A itself (again using Lemma 2.3;

that is, for $k = 1, \dots, n-1$, we let $a_{kn} = a_{n-1}a_{k,n-1} = a_{n-1}a_{n-1,k} = a_{nk}$ to fill in the remaining unspecified entries where the double subscripts indicate entries from the completion of $A_2[\{1, \dots, n-1\}]$). The case in which a_n/a_1 is the minimum is analogous by placing a_n/a_1 in the 2, n entry. □

Remark. In the positive definite completion problem it follows from the chordal result ([5]) that the order of the data around the cycle is irrelevant to completability (without knowing the cycle conditions of Ref. [1]). In spite of the fact that there is no chordal result for the SIM completion problem, Theorem 4.1, interestingly, shows that again the order of the data around the cycle is irrelevant.

5. Graphs for which the cycle conditions are sufficient for completability

Definition 5.1. A *block graph* is a graph built from cliques and (simple) cycles as follows: starting with a clique or cycle, sequentially articulate the "next" clique or simple cycle at most one vertex of the current graph. Note that, if all cycles in a graph are completed to cliques, the resulting graph is 1-chordal.

A completability criterion for partial SIM matrices with block graphs is given in our next result.

Theorem 5.2. *Let A be an $n \times n$ partial SIM matrix, the graph of whose specified entries is a block graph G. Then A has an SIM completion if and only if all minimal cycles in G satisfy the cycle conditions.*

Proof. Suppose A is an $n \times n$ partial SIM matrix with block graph G such that all cycles in G satisfy the cycle conditions. Complete each simple cycle in G in the manner of Theorem 4.1, obtaining a new partial IM matrix, the graph of whose specified entries is 1-chordal. The resulting partial SIM matrix can be completed to an SIM matrix by Theorem 2.4. On the other hand, if G contains a minimal cycle that does not satisfy the cycle conditions, then the partial principal submatrix associated with this cycle does not have an SIM completion by Theorem 4.1. Thus, A itself cannot have an SIM completion, because the property of being IM is inherited by principal submatrices. □

In the next example, we complete a partial SIM matrix whose graph is a block graph in order to illustrate the algorithmic nature of the completion process.

Example 5.3. Consider the partial SIM matrix

$$A = \begin{bmatrix} 1 & 0.2 & ? & ? & 0.8 & 0.7 & 0.3 & ? & ? \\ 0.2 & 1 & 0.5 & ? & ? & ? & ? & ? & ? \\ ? & 0.5 & 1 & 0.4 & ? & ? & ? & ? & ? \\ ? & ? & 0.4 & 1 & 0.3 & ? & ? & ? & ? \\ 0.8 & ? & ? & 0.3 & 1 & ? & ? & ? & ? \\ 0.7 & ? & ? & ? & ? & 1 & 0.4 & ? & ? \\ 0.3 & ? & ? & ? & ? & 0.4 & 1 & 0.5 & 0.3 \\ ? & ? & ? & ? & ? & ? & 0.5 & 1 & 0.2 \\ ? & ? & ? & ? & ? & ? & 0.3 & 0.2 & 1 \end{bmatrix}.$$

The block graph G of A consists of the cycle $1 \to 2 \to 3 \to 4 \to 5 \to 1$ together with the cliques induced by each of $\{1, 6, 7\}$ and $\{7, 8, 9\}$. It is easily seen that the cycle satisfies the cycle conditions and thus we are ensured of a completion. We will complete A by first completing the cycle in the algorithmic fashion of the proof of Theorem 4.1, and then combining the resulting clique with the other two using the "1-chordal" method of Theorem 2.4.

The smallest neighbor-to-neighbor ratio in $A[\{1, 2, 3, 4, 5\}]$ is $0.2/0.8 = 0.25$. Thus, we form the partial completion

$$A_1 = \begin{bmatrix} 1 & 0.5 & ? & 0.25 & 0.2 \\ 0.5 & 1 & 0.4 & ? & ? \\ ? & 0.4 & 1 & 0.3 & ? \\ 0.25 & ? & 0.3 & 1 & 0.8 \\ 0.2 & ? & ? & 0.8 & 1 \end{bmatrix}$$

of $A[\{2, 3, 4, 5, 1\}]$. A "1-chordal" completion is possible using the cliques induced by each of $\{2, 3, 4, 5\}$ and $\{5, 1\}$.

The smallest neighbor-to-neighbor ratio in $A_1[\{2, 3, 4, 5\}]$ is $0.25/0.5 = 0.5$. So we form the partial completion

$$A_2 = \begin{bmatrix} 1 & 0.3 & 0.5 & 0.25 \\ 0.3 & 1 & 0.4 & ? \\ 0.5 & 0.4 & 1 & 0.5 \\ 0.25 & ? & 0.5 & 1 \end{bmatrix}$$

of $A_1[\{5, 4, 3, 2\}]$. We can then complete A_2 in 1-chordal fashion using the cliques induced by $\{5, 4, 3\}$ and $\{3, 2\}$. Similarly, we can complete $A[\{1, 2, 3, 4, 5\}]$ in 1-chordal fashion using the cliques induced by $\{2, 3, 4, 5\}$ and $\{5, 1\}$ to obtain the completion

$$A_3 = \begin{bmatrix} 1 & 0.2 & 0.4 & 0.24 & 0.8 \\ 0.2 & 1 & 0.5 & 0.2 & 0.25 \\ 0.4 & 0.5 & 1 & 0.4 & 0.5 \\ 0.24 & 0.2 & 0.4 & 1 & 0.3 \\ 0.8 & 0.25 & 0.5 & 0.3 & 1 \end{bmatrix}$$

of $A[\{1,2,3,4,5\}]$.

In 1-chordal fashion we can then combine the cliques induced by $\{1,2,3,4,5\}$ and $\{1,6,7\}$ to obtain the clique induced by $\{1,2,3,4,5,6,7\}$. This latter clique can then be combined with the clique induced by $\{7,8,9\}$ to obtain the completion

$$\begin{bmatrix} 1 & 0.2 & 0.4 & 0.24 & 0.8 & 0.7 & 0.3 & 0.15 & 0.09 \\ 0.2 & 1 & 0.5 & 0.2 & 0.25 & 0.14 & 0.06 & 0.03 & 0.018 \\ 0.4 & 0.5 & 1 & 0.4 & 0.5 & 0.28 & 0.12 & 0.06 & 0.036 \\ 0.24 & 0.2 & 0.4 & 1 & 0.3 & 0.168 & 0.072 & 0.036 & 0.0216 \\ 0.8 & 0.25 & 0.5 & 0.3 & 1 & 0.56 & 0.24 & 0.12 & 0.072 \\ 0.7 & 0.14 & 0.28 & 0.168 & 0.56 & 1 & 0.4 & 0.2 & 0.12 \\ 0.3 & 0.06 & 0.012 & 0.072 & 0.24 & 0.4 & 1 & 0.5 & 0.3 \\ 0.15 & 0.03 & 0.06 & 0.036 & 0.12 & 0.2 & 0.5 & 1 & 0.2 \\ 0.09 & 0.018 & 0.036 & 0.0216 & 0.072 & 0.12 & 0.3 & 0.2 & 1 \end{bmatrix}$$

of A. It may be verified that this completion is an SIM matrix by direct calculation.

We next examine the relationship between the inverse M-matrix and Euclidean distance matrix completion problems.

Definition 5.4. Let $\|\ \|$ denote Euclidean length on some R^k. For two points P and Q, we use $d(P,Q)$ for $\|P-Q\|$. The $n\times n$ matrix $D=(d_{ij})$ is a (*Euclidean*) *distance matrix* if there exist points $P_1,\ldots,P_n$ in R^k such that $d_{ij}=d(P_i,P_j)$.

Definition 5.5. We call a nonnegative $n\times n$ matrix A a path product (PP) matrix [13] if, for every path $i_1\to i_2\to\cdots\to i_{k-1}\to i_k$ in the complete graph K_n on n vertices, we have

$$\frac{a_{i_1i_2}a_{i_2i_3}\ldots a_{i_{k-1}i_k}}{a_{i_2i_2}a_{i_3i_3}\ldots a_{i_{k-1}i_{k-1}}} \leqslant a_{i_1i_k}. \tag{5.1}$$

If, whenever $i_1=i_k$ in Eq. (5.1), we further require that the inequality is strict, we call A a strict path product (SPP) matrix. Further, if $a_{ii}=1$, $i=1,\ldots,k$, we call a PP (respectively SPP) matrix a normalized (respectively strict normalized) path product (NPP, respectively SNPP) matrix. We call the inequalities (5.1) the path product inequalities and, in case $i_1 = i_k$ in Eq. (5.1), the cycle product inequalities.

We note by the Theorem 4.1 that any SIM matrix with diagonal entries equal to 1 must be NPP.

Observe that a matrix A is NPP if and only if $B = -\log A$ (interpreted entrywise) satisfies the polygonal inequalities

$$b_{i_1i_2} + b_{i_2i_3} + \cdots + b_{i_{k-1}i_k} \geqslant b_{i_1i_k}. \tag{5.2}$$

Thus, if B is a distance matrix, then e^{-B} (interpreted entrywise) is NPP and, similarly, since a normalized inverse M-matrix is NPP [13], if A is an normalized inverse M-matrix, then $B = -\log A$ (again interpreted entrywise) satisfies the polygonal inequalities.

It was shown in Ref. [9] that the polygonal inequalities are necessary and sufficient for cycle completability in the Euclidean distance matrix completion problem. However, it is not necessarily the case that e^{-B} is inverse M (and so, by itself, the cycle result for distance matrices does not imply our SIM cycle completion result), as seen by the following example.

Example 5.6. Consider the distance matrix

$$B = \begin{bmatrix} 0 & 0.80 & \sqrt{0.10} & \sqrt{0.29} \\ 0.80 & 0 & \sqrt{0.90} & \sqrt{0.13} \\ \sqrt{0.10} & \sqrt{0.90} & 0 & \sqrt{0.37} \\ \sqrt{0.29} & \sqrt{0.13} & \sqrt{0.37} & 0 \end{bmatrix},$$

which is the Euclidean distance matrix in R^2 for the points $P_1(0.9, 0.5)$, $P_2(0.1, 0.5)$, $P_3(1.0, 0.2)$, and $P_4(0.4, 0.3)$. It is easily verified that the 2, 3 and 3, 2 entries of the inverse of $A = e^{-B}$ are positive and thus A is not an SIM matrix.

6. Graphs for which partial SIM matrices have SIM completions

Now we turn to the question: if all partial SIM matrices with graph G are completable, what must the structure of G be? This is the question of a converse to Theorem 2.4.

In view of the results of Section 4, G cannot contain an induced cycle of length 4 or more, because, then, properly more conditions are needed for completability; thus, G must be chordal. Using simple embedding techniques, Example 2.5 shows that G cannot contain an induced subgraph that is the complete graph on 5 vertices less an edge. Thus, G must be 2-chordal. Must G, in fact, be 1-chordal as in the positionally symmetric case [12]? This is a very delicate question. In the positionally symmetric case, it was possible to give robust examples, for the complete graph on 4 vertices less an edge (K_4^-), in which a partial IM matrix has no completion even on the boundary of the IM matrices. In the symmetric case no such example is possible, although there is a thin set of examples for which there are completions only in the closure of the SIM matrices. Thus, if the notion of completability is expanded slightly, then K_4^- also insures completability. However, without such an expansion, the following discussion indicates a converse to Theorem 2.4 (i.e. 1-chordal is necessary).

By a "singular IM (SIM)" matrix we simply mean a singular matrix in the topological closure of the IM (SIM) matrices. For example, $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ is singular SIM as it is the limit of matrices of the form

$$\begin{bmatrix} 4+\varepsilon & 2 \\ 2 & 1 \end{bmatrix}$$

for $\varepsilon > 0$.

Theorem 6.1. *Let A be a 4×4 partial SIM matrix with a 2-chordal graph G. Then A has a (possibly singular) SIM completion.*

Proof. If the graph G of A is, in fact, 1-chordal, we are done by Theorem 2.4. So we may assume, without loss of generality, that

$$A = \begin{bmatrix} 1 & a_{12} & a_{13} & x \\ a_{21} & 1 & a_{23} & a_{24} \\ a_{31} & a_{32} & 1 & a_{34} \\ x & a_{42} & a_{43} & 1 \end{bmatrix}$$

in which $a_{ij} = a_{ji}$.

First, suppose that one of the specified 3×3 submatrices of A contains a symmetrically placed pair of 0 entries. Then, since an inverse M-matrix with 0 entries is reducible, this specified submatrix must contain a second pair of symmetrically placed 0 entries. It is then seen by considering cases that either $x = 0$ gives an SIM completion of A, or we can apply Lemma 2.3 to complete A to a SIM matrix. Hence, we will assume that A has no 0 entry for the remainder of the proof.

The completion that will have zeros in the inverse in the 1, 4 and 4, 1 positions is obtained by setting x equal to

$$\begin{aligned} x_0 &= A[\{1\},\{2,3\}](A[\{2,3\}])^{-1}A[\{2,3\},4] \\ &= \frac{a_{34}(a_{13} - a_{12}a_{23}) + a_{24}(a_{12} - a_{13}a_{32})}{1 - a_{23}^2}. \end{aligned}$$

Similarly, for a completion to have zeros in the 2, 3 and 3, 2 positions of the inverse, $\det A[\{1,3,4\},\{1,2,4\}] = 0$ or, equivalently, $ax^2 + bx + c = 0$ in which $a = a_{23}$, $b = -(a_{12}a_{34} + a_{13}a_{24})$, and $c = a_{12}a_{13} + a_{24}a_{34} - a_{23}$. Then

$$\begin{aligned} d^2 = b^2 - 4ac &= (a_{12}a_{34} + a_{13}a_{24})^2 + 4a_{23}(a_{23} - a_{12}a_{13} - a_{24}a_{34}) \\ &= (a_{12}a_{34} - a_{13}a_{24})^2 + 4(a_{23} - a_{21}a_{13})(a_{23} - a_{24}a_{43}) \geqslant 0 \end{aligned}$$

by the path product conditions.

Thus, there are two completions that have zeros in the 2, 3 and 3, 2 positions of the inverse, namely

$$x_1 = \frac{a_{12}a_{34} + a_{13}a_{24} - d}{2a_{23}} \quad \text{and} \quad x_2 = \frac{a_{12}a_{34} + a_{13}a_{24} + d}{2a_{23}}.$$

We will show that either x_0 or x_1 provides a (possibly singular) completion of A.

Claim 1. *$a_{12}a_{24},\ a_{13}a_{34} \leqslant x_0 \leqslant x_v$ in which*

$$x_v = \frac{x_1 + x_2}{2} = \frac{a_{12}a_{34} + a_{13}a_{24}}{2a_{23}}.$$

Proof of Claim 1. After some algebraic manipulations, one sees that $a_{12}a_{24} \leqslant x_0$ is equivalent to $(a_{13} - a_{12}a_{23})(a_{34} - a_{32}a_{24}) \geqslant 0$ which holds by the path product inequalities. The other left-hand inequality is similar.

Algebraic manipulations show that $x_0 \leqslant x_v$ if and only if $(a_{12} - a_{13}a_{32})(a_{34} - a_{32}a_{24}) + (a_{13} - a_{12}a_{23})(a_{24} - a_{23}a_{34}) \geqslant 0$ which again holds by the path product inequalities.

Claim 2.

$$x_1 \leqslant \min\left\{\frac{a_{12}}{a_{24}}, \frac{a_{24}}{a_{12}}, \frac{a_{13}}{a_{34}}, \frac{a_{34}}{a_{13}}\right\}.$$

Proof of Claim 2. Without loss of generality, assume $a_{12} \leqslant a_{24}$ and consider the inequality $x_1 \leqslant a_{12}/a_{24}$ which is equivalent to $a_{24}(a_{12}a_{34} + a_{13}a_{24}) - 2a_{12}a_{23} \leqslant a_{24}d$. Now, if the left-hand side of the latter inequality is <0, there is nothing to prove. So assume the contrary (so that both sides of the inequality are nonnegative). Squaring both sides of the inequality and simplifying, we see that the inequality is equivalent to $a_{23}(a_{24}^2 - a_{12}^2)(a_{23} - a_{24}a_{43}) \geqslant 0$. Since each of the factors on the left is nonnegative (the second by assumption), it follows that

$$x_1 \leqslant \frac{a_{12}}{a_{24}}, \frac{a_{24}}{a_{12}}.$$

In a similar fashion one can argue that

$$x_1 \leqslant \frac{a_{13}}{a_{34}}, \frac{a_{34}}{a_{13}}$$

and Claim 2 follows.

Since $x_1 \leqslant x_v \leqslant x_2$, we may just consider the following two cases.

Case 1: $x_1 \leqslant x_0 \leqslant x_v \leqslant x_2$: Let B be the nonnegative completion obtained by setting $x = x_0$. By Sylvester's identity for determinants,

$$\det\ B = \frac{\det\ B[\{1,2,3\}]\ \det\ B[\{2,3,4\}]}{1 - a_{23}^2} > 0.$$

Let $B^{-1} = (\beta_{ij})$. We have $B^{-1} = \tilde{B}_{\{1,2,3\}} + \tilde{B}_{\{2,3,4\}} - \tilde{B}_{\{2,3\}}$ by Lemma 2.2. Since $\det A(3,2) \geqslant 0$ for $x_1 \leqslant x \leqslant x_2$, $\beta_{23} \leqslant 0$. Thus, B^{-1} is in Z which implies B is an SIM completion of A.

Case 2: $x_0 < x_1 \leqslant x_v < x_2$: Let C be the completion obtained by setting $x = x_1$. By Claims 1 and 2,

$$a_{12}a_{24}, a_{13}a_{34} < x_1 \leqslant \frac{a_{12}}{a_{24}}, \frac{a_{24}}{a_{12}}, \frac{a_{13}}{a_{34}}, \frac{a_{34}}{a_{13}}.$$

This implies that $0 < x_1 \leqslant 1$. First suppose $0 < x_1 < 1$. Then, by Eq. (4.1), $C[\{1,2,4\}]$ and $C[\{1,3,4\}]$ are inverse M-matrices. Also,

$$\det\ C = \frac{\det\ C[\{1,2,4\}]\ \det\ C[\{1,3,4\}]}{1 - x_1^2} > 0$$

and $\det C(1,4) > 0$ since $x_0 < x_1$, i.e.,

$$\begin{aligned} \det\ C(1,4) &= x_1(1 - a_{23}^2) - a_{24}(a_{12} - a_{13}a_{32}) - a_{34}(a_{13} - a_{12}a_{23}) \\ &> x_0(1 - a_{23}^2) - a_{24}(a_{12} - a_{13}a_{32}) - a_{34}(a_{13} - a_{12}a_{23}) \\ &= \det\ B(1,4) = 0. \end{aligned}$$

Applying Lemma 2.2 again, $C^{-1} = \tilde{C}_{\{1,2,4\}} + \tilde{C}_{\{1,3,4\}} - \tilde{C}_{\{1,4\}}$ is in Z which implies C is an SIM completion of A. On the other hand, if $x_1 = 1$, then

$$\frac{a_{12}}{a_{24}} = 1 = \frac{a_{13}}{a_{34}}$$

or, equivalently, $a_{12} = a_{24}$ and $a_{13} = a_{34}$. Further, upon simplifying $x_1 = 1$, we see that $a_{23} = a_{12}a_{13} = a_{24}a_{34}$. So, if we let $r = a_{12}$ and $s = a_{13}$, then

$$C = \begin{bmatrix} 1 & r & s & 1 \\ r & 1 & rs & r \\ s & rs & 1 & s \\ 1 & r & s & 1 \end{bmatrix}.$$

It can be easily checked that C is a singular SIM matrix by showing that det $C = 0$ and, for $\varepsilon > 0$, det $(C + \varepsilon I) > 0$ and all 3×3 minors of $C + \varepsilon I$ have the appropriate sign.

Thus, either x_0 or x_1 provides a (possibly singular) completion of A which completes the proof. □

A necessary condition that A have an SIM completion is that det $A(2,3) \geqslant 0$, equivalently, $x_1 \leqslant x \leqslant x_2$. Thus, if $x_1 = 1$, there is a unique singular SIM completion.

Example 6.3. The only SIM completion of the partial SIM matrix

$$A = \begin{bmatrix} 1 & 0.4 & 0.5 & ? \\ 0.4 & 1 & 0.2 & 0.4 \\ 0.5 & 0.2 & 1 & 0.5 \\ ? & 0.4 & 0.5 & 1 \end{bmatrix}$$

is the singular completion which replaces the unspecified entries by 1's.

However, it is not true that every partial SIM matrix with a 2-chordal graph has a (possibly singular) SIM completion. This is illustrated by the following example.

Example 6.4. Consider the partial SIM matrix

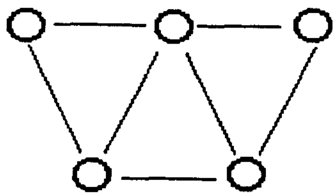
$$A = \begin{bmatrix} 1 & \frac{9}{40} & \frac{1}{5} & \frac{2}{5} & x \\ \frac{9}{40} & 1 & \frac{3}{10} & \frac{1}{2} & y \\ \frac{1}{5} & \frac{3}{10} & 1 & \frac{1}{5} & \frac{3}{4} \\ \frac{2}{5} & \frac{1}{2} & \frac{1}{5} & 1 & \frac{1}{4} \\ x & y & \frac{3}{4} & \frac{1}{4} & 1 \end{bmatrix}.$$

Then x (respectively y) is the only unspecified entry of $B = A[\{1,3,4,5\}]$ (respectively $C = A[\{2,3,4,5\}]$). Adopting the notation in the proof of Theorem 6.1 and applying it to the matrices B and C, we find that

$$x_0 = \frac{3}{16} < 0.875 - 2.5\sqrt{0.0685} = x_1 \quad \text{and}$$
$$y_0 = \frac{13}{48} < 1.125 - 2.5\sqrt{0.0925} = y_1.$$

Thus, if A has a (possibly singular) SIM completion, we must have $x \geqslant x_1 > x_0$ and $y \geqslant y_1 > y_0$. Also notice that if $D = A[\{1,2,3,4\}]$, then $\det D(1,2) = 0$. Let A_1 be the corresponding completion of A. Applying Sylvester's identity for determinants, $\det A_1(1,2) \geqslant 0$ if and only if $\det A[\{3,4,5\},\{1,3,4\}] \det A[\{2,3,4\}, \{3,4,5\}] \leqslant \det A[\{2,3,4\}, \{1,3,4\}] \det A\{3,4,5\}]$, or, equivalently, $\det B(1,4) \det C(4,1) \leqslant \det D(1,2) \det C[\{2,3,4\}] = 0$. Since $x > x_0$ implies $\det B(1.4) > 0$ and since $y > y_0$ implies $\det C(4,1) > 0$, we have a contradiction. Therefore, A has no SIM completion.

We have been unable to determine whether partial SIM matrices with graph



have at least a singular SIM completion.

References

[1] W. Barrett, C.R. Johnson, P. Tarazaga, The real positive definite completion problem for a simple cycle, Linear Algebra Appl. 192 (1993) 3–31.

[2] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Academic Press, New York, 1979.

[3] J.R.S. Blair, B. Peyton, An introduction to chordal graphs and clique trees, in: A. George, J.R. Gilbert, J.W.H. Liu (Eds.), Graph Theory and Sparse Matrix Computation, IMA, vol. 56, Springer, New York, 1993.

[4] M. Fielder, V. Pták, On matrices with nonpositive off-diagonal elements and positive principal minors, Czechoslovak Math. J. 42 (1962) 382–400.

[5] R. Grone, C.R. Johnson, E. Sá, H. Wolkowicz, Positive definite completions of partial Hermitian matrices, Linear Algebra Appl. 58 (1984) 109–124.

[6] R. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.

[7] R. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, New York, 1991.

[8] C.R. Johnson, Inverse M-matrices, Linear Algebra Appl. 47 (1982) 195–216.

[9] C.R. Johnson, C.A. Jones, B.K. Kroeschel, The Euclidean distance completion problem: Cycle completability, Linear and Multilinear Algebra 39 (1995) 195–207.

[10] C.R. Johnson, M. Lundquist, Matrices with chordal inverse zero-patterns, Linear and Multilinear Algebra 36 (1993) 1–17.

[11] C.R. Johnson, M. Lundquist, Local inversion of matrices with sparse inverses, Linear Algebra Appl. 277 (1998) 33–39.

[12] C.R. Johnson, R.L. Smith, The completion problem for M-matrices and inverse M-matrices, Linear Algebra Appl. 241–43 (1996) 655–667.

[13] C.R. Johnson, R.L. Smith, Path product matrices, Linear and Multilinear Algebra to appear.