Similarity and analytical solutions of free convective flow of dilatant nanofluid in a Darcian porous medium with multiple convective boundary conditions

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Abstract This paper deals with an analytical solution of free convective flow of dilatant nanofluid past a vertical cone/plate. A two-phase mixture model is used for nanofluid in which the Brownian motion and thermophoretic diffusivities are the important slip mechanisms between solid and liquid phases. The governing transport equations along with physically realistic thermal and mass convective boundary conditions are reduced to similarity equations using relevant similarity transformations before being solved by homotopy analysis method. The effects of the governing parameters (Brownian motion, thermophoresis, convection-conduction, convection-diffusion, Lewis number, buoyancy ratio, and power-law) on the dimensionless velocity, temperature and nanoparticle volume fraction, friction and heat transfer rates are plotted and discussed. It is found that friction factor decreases with the increase in $Le$ and $Nr$ for both vertical plate and cone. The local Nusselt number decreases with the increase in the thermophoresis and Brownian motion parameters for both the plate and cone. The local Sherwood number increases with the Brownian motion parameter and decreases for thermophoresis parameter. The results have been compared with the published ones and an excellent agreement has been noticed.

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1. Introduction

Including a dispersion of nanoparticles, nanofluid is a liquid in which the nanoparticles have been suspended in it without settlement and this can be pointed as the difference between nanoparticles and conventional particles [1]. Some researchers considered the free convective boundary-layer flow of nanofluids subject to different boundary conditions. Chamkha et al. [2] studied free convection past an isothermal sphere in a Darcy porous medium with a nanofluid and then presented numerical results for mass transfer rate, friction factor and surface heat transfer rate. Considering the effects of Brownian motion and thermophoresis, Nield and Kuznetsov [3,4] investigated the Cheng–Minkowycz problem and thermal instability in a porous medium (Darcy and Brinkman models) analytically. Xuan and Li [5] studied the behavior of nanofluid in turbulent flow through the tubes experimentally. According to their published results, the Reynolds number and volume fraction of nanoparticles can affect the convective heat transfer coefficient and Nusselt number of nanofluids. Hady et al. [6] studied the effect of radiation parameter over a nonlinear stretching sheet in a viscous flow of a nanofluid. An implicit finite-difference method was employed by Khan and Pop to study the steady nanofluid flow past a stretching surface [7]. Beg et al. [8] presented a comparative numerical solution for single-phase and two-phase models for Bio-nanofluid transport phenomena. Noghrehabadi et al. [9] considered the partial slip condition of nanofluids past a stretching sheet with the constant wall temperature. Bachok et al. [10] presented the results for a uniform free stream of a steady nanofluid flow over a semi-infinite flat plate. Abu-Nada et al. [11] took into consideration the effects of variable properties in a natural convective nanofluid flow. Rashidi and Erfani [12] used modified differential transform method to study the nano boundary-layer flow over the stretching surfaces with Navier boundary condition. Nadeem and Lee [13] studied the steady flow of a nanofluid over an exponential stretching surface. Stagnation-point nanofluid flow over a surface was studied by Rashidi et al. [14] via DTM-Pade´ . Khan et al. [15] considered the effect of momentum slip on Double-Diffusive natural convection of a nanofluid over a vertical plate.

Melts of polymers, biological solutions and paint which are non-Newtonian fluids play an important role in many industrial applications such as alternative energy technologies, microfluidic devices and biomedical devices [16]. The boundary-layer flow of non-Newtonian power-law nanofluids past a linearly stretching sheet was studied by Uddin et al. [17] with a linear hydrodynamic slip boundary condition numerically using Runge–Kutta–Fehlberg fourth-fifth order. Sheu [18] investigated the thermal instability in a porous medium horizontal layer saturated with a viscoelastic nanofluid by employing Oldroyd-B viscoelastic model. Akbar and Nadeem [19] used Eyring–Prandtl fluid model to study flow of a nanofluid in a diverging tube. To study heat transfer of Casson non-Newtonian fluid flow past a horizontal circular cylinder, Prasad et al. [20] employed the Keller box finite difference...
method. Niu et al. [21] studied the slip-flow and heat transfer of a non-Newtonian nanofluid in a micro tube. Lie group analysis for non-Newtonian nanofluid was employed by Uddin et al. [22] in the presence of internal heat generation. Many authors studied transport phenomena associated with nanofluid past various geometries. Saleem et al. [23] studied buoyancy and metallic particle effects on an unsteady water-based fluid flow past a rotating cone. Noor et al. [24] examined mixed convection stagnation slip flow of an unsteady water-based nanofluid past a stretching sheet. Magnetohydrodynamic (MHD) stagnation point flow of nanofluid past a stretching sheet with convective boundary condition is studied by Ibrahim and Haq [25]. Haq et al. [26] illustrated buoyancy and radiation effect on stagnation point flow of micropolar nanofluid along a convectively heated sheet. Haq et al. [27] studied heat transfer in MHD slip flow over a stretching surface in the presence of carbon nanotubes.

Homotopy analysis method (HAM) is one of the most applicable methods for highly-nonlinear problems. Liao was the first researcher who employed HAM and presented a general analytical method for nonlinear problems [28,9]. Rashidi et al. [30,31] used this method for Burger and Regularized Long Wave and Jaulent–Miodek equations. The stagnation-point flow of a nanofluid over a stretching sheet was investigated by Mustafa et al. [32] via HAM. Hayat et al. [33] considered MHD flow of an upper-convected (UCM) fluid over a stretching surface by means of HAM. This powerful method is being employed vastly by many researchers in different practical aspects of engineering and nonlinear problems [34–39].

The objective of this study was to investigate free convective flow of dilatant nanofluid along a vertical flat plate/cone located in a Darcy porous medium. The thermal and mass convective boundary conditions are taken into account. We have used the nanofluid model proposed by Buongiorno [40].

2. Formulation

Consider the steady two-dimensional free convective boundary-layer flow of a dilatant nanofluid past a vertical cone/flat plate embedded in a Darcy porous medium filled with nanofluid while the right side of the cone/plate is cooled by the convection from the cold fluid of temperature $T_\infty$ which provides a variable heat transfer coefficient $h_\infty(x)$. Thus a thermal convective boundary condition arises and the temperature in the left side of the cone is $T_f (> T_\infty > T_w)$. Further, the concentration in the left side of the cone/plate $C_f$ is higher than that of the plate concentration $C_w$ and free stream concentration $C_\infty$ which yields a variable mass transfer coefficient $h_m(x)$. Thus a mass convective boundary condition arises. Fig. 1 illustrates the system of coordinates and flow model in which the $x$-axis along the cone/plate in the upward direction and the $y$-axis in the normal direction to the cone/plate. There are three distinct boundary-layers (velocity, thermal, and nanoparticle volume fraction) of almost equal thickness formed near the cone/plate surface. However, only one of the boundary-layers is symbolically plotted as shown in Fig. 1. For simplicity it has been assumed that fluid properties are constant. The Oberbeck–Boussinesq approximation is adopted and the four field equations are the conservation of mass, momentum, thermal energy, and the nanoparticle volume fraction. These equations can be written in terms of dimensional forms, extending the formulations of Nield and Kuznetsov [3], Yih [41] and Chamkha and Rashad [43],

\[
\frac{\partial}{\partial x} (\rho u^m) + \frac{\partial}{\partial y} (\rho v^m) = 0, 
\]

\[
\frac{\partial (u^m)}{\partial y} = \frac{(1 - C_\infty)\gamma \rho c_p \cos \gamma K \beta}{K} \frac{\partial T}{\partial y} - \left( \rho_f - \rho_w \right) g K \cos \gamma \frac{\partial C}{\partial y}, 
\]

\[
\bar{u} \frac{\partial T}{\partial x} + \bar{v} \frac{\partial T}{\partial y} = 2 \frac{\partial^2 T}{\partial y^2} + \tau D_B \frac{\partial^2 C}{\partial y^2} + \left( \frac{D_T}{T_\infty} \right) \left( \frac{\partial T}{\partial y} \right)^2, 
\]

\[
\bar{u} \frac{\partial C}{\partial x} + \bar{v} \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \left( \frac{D_T}{T_\infty} \right) \frac{\partial^2 T}{\partial y^2}. 
\]

The relevant boundary conditions are as follows [42]:

\[
\bar{v} = 0, \quad -k \frac{\partial T}{\partial y} = h_f(\bar{T}_f - \bar{T}), \quad -D_B \frac{\partial C}{\partial y} = h_m(\bar{C}_f - \bar{C}) \quad \text{at} \quad \bar{y} = 0
\]

\[
\bar{u} = 0, \quad \bar{T} \rightarrow \bar{T}_\infty, \quad \bar{C} \rightarrow \bar{C}_\infty \quad \text{as} \quad \bar{y} \rightarrow \infty.
\]

The local radius to a point in the boundary-layer can be replaced by the radius of the cone $r$, i.e. $r = \bar{x} \sin \gamma$. It must be noted that $m = \gamma = 0$ corresponds to flow over vertical plate and $m = 1$ flow over vertical cone. Introducing the following transformations (Rashad et al. [43]), the ordinary form of the problem can be presented:

\[
\eta = \frac{\bar{y}}{x} \sqrt{\bar{r} \bar{a}_1}, \quad f(\eta) = \frac{\psi}{\bar{a}^m \bar{a}_1}, \quad \theta = \frac{\bar{T} - \bar{T}_\infty}{\Delta \bar{T}}, \quad \varphi = \frac{\bar{C} - \bar{C}_\infty}{\Delta \bar{C}}.
\]

where $\Delta \bar{T} = \bar{T}_f - \bar{T}_\infty$ and $\Delta \bar{C} = \bar{C}_f - \bar{C}_\infty$ are the characteristic temperature and concentration, respectively, $\bar{a}_1$ is the local Rayleigh number for a porous medium based on the position $\bar{x}$ and is defined by $\bar{a}_1 = \frac{\bar{g}}{\bar{\kappa}} \left( \frac{(1 - C_\infty)\gamma \rho c_p \cos \gamma K \beta \Delta \bar{T}}{K} \right)^{1/\nu}$, $\psi$ is the stream function which is defined as $\bar{r}^m \bar{u} = \frac{\partial \psi}{\partial \eta}$, and $\bar{r}^m \bar{v} = -\frac{\partial \psi}{\partial \eta}$. Substituting Eq. (6) into Eqs. (1)–(5):

\[
n(f')^{\nu-1}f'' - \theta' + N \varphi = 0,
\]
\[ \theta' + \left( m + \frac{1}{2} \right) \theta'' + \frac{\dot{\theta}}{\tau} \theta' + N_t \theta'' = 0, \]

\[ \phi'' + \left( m + \frac{1}{2} \right) Le f \phi' + \frac{N_t}{N_b} \phi'' = 0, \]

\[ f(\eta) = 0, \quad \theta'(\eta) = -N_m[1 - \theta(\eta)], \]

\[ \phi'(\eta) = -N_d[1 - \phi(\eta)], \quad \text{at} \quad \eta = 0 \]

\[ f'(\eta) = 0, \quad \theta'(\eta) = 0, \phi'(\eta) = 0 \quad \text{as} \quad \eta \to \infty. \]

where \( \prime \) denotes ordinary derivative with respect to \( \eta \), \( Le \) is the Lewis number, \( Nr \) is the buoyancy ratio parameter, \( Nb \) and \( Nt \) are the nanofluid parameters, and \( Nc \) and \( Nd \) are the convection-conduction and convection-diffusion parameters, respectively. The definitions of the parameter are (44.45) as follows:

\[ Le = \frac{x}{D_b}, \quad Nr = \frac{(\rho_p - \rho_{fl}) \Delta C}{\rho_{fl} (1 - C_m) \beta \Delta T}, \quad Nb = \frac{\tau D_b \Delta C}{x}, \]

\[ Nt = \frac{\tau D_b \Delta T}{2T_c}, \quad Nc = \frac{h_{nx}}{kR_{x}^{1/2}}, \quad Nd = \frac{h_{nx}}{D_b R_{x}^{1/2}}. \]

Note that for true similarity solutions \( h_{y} \) and \( h_{m} \) are proportional to \( x^{-1} \).

It is interesting to note that when we have Newtonian nanofluid \((n = 1)\) in the vertical cone \( m = 1 \), and isothermal surface temperature and concentration \((Nc, Nd \to \infty)\), the boundary value problem reduces to the problem considered by Rashad et al. [43]. These equations also reduce to the problem dealt by Nield and Kuznetsov [3] for Newtonian nanofluid with uniform surface temperature and concentration in vertical flat plate \( m = 0 \).

### 2.1. Quantities of engineering interest

Quantities of thermal engineering design of industrial equipment’s are the local Nusselt number, \( Nu_{x} \), and the local Sherwood number \( Sh_{x} \). Physically, \( Nu_{x} \) defines the heat transfer rates and \( Sh_{x} \) defines the mass transfer rates. They can be defined as

\[ Nu_{x} = \frac{\bar{q}_{x}}{k \Delta T}, \quad Sh_{x} = \frac{\bar{q}_{m}}{D_{b} \Delta C}, \]

where \( \bar{q}_{x} \) and \( \bar{q}_{m} \) are the wall heat and mass fluxes, respectively, and are defined as

\[ \bar{q}_{x} = -k (\frac{\partial T}{\partial y})_{y=0}, \quad \bar{q}_{m} = -D_{b} (\frac{\partial C}{\partial y})_{y=0}. \]

Using Eqs. \((6)\) and \((13)\), one can obtain from Eq. \((12)\) that

\[ Ra_{x}^{1/2} \cdot Nu_{a} = -\theta' \left( 0 \right), \quad Ra_{x}^{1/2} \cdot Sh_{a} = -\phi' \left( 0 \right). \]

### 3. Solution by homotopy analysis method

The initial approximations must be chosen in a way to satisfy the boundary conditions as follows in Eqs. \((7)\)–\((9)\):

\[ f_{0}(\eta) = 1 - e^{-\eta}, \]

\[ \theta_{0}(\eta) = \frac{N_{c}}{N_{c} + 1} e^{-\eta}, \]

\[ \phi_{0}(\eta) = \frac{N_{d}}{N_{d} + 1} e^{-\eta}. \]

The linear operators \( L_{f}(f), L_{\theta}(\theta) \) and \( L_{\phi}(\phi) \) are defined as

\[ L_{f}(f) = \frac{\partial \theta}{\partial \eta^{2}} - f, \]

\[ L_{\phi}(\phi) = \frac{\partial \theta}{\partial \eta^{2}} - \phi. \]

with the following properties:

\[ L_{f}(c_{1} e^{-\eta} + c_{2} e^{\eta}) = 0, \]

\[ L_{\phi}(c_{3} e^{-\eta} + c_{4} e^{\eta}) = 0, \]

where \( c_{1}, \ldots, c_{6} \), are arbitrary constants. It is clear that \( c_{2}, c_{4} \) and \( c_{6} \) are equal to zero.

The nonlinear operators are as follows:

\[ N_{f} \left[ f(\eta; q), \theta(\eta; q), \phi(\eta; q) \right] = 2 \frac{\partial f(\eta; q)}{\partial \eta} \frac{\partial^{2} f(\eta; q)}{\partial \eta^{2}} - \frac{\partial \theta(\eta; q)}{\partial \eta} \]

\[ + Nr \frac{\partial \theta(\eta; q)}{\partial \eta}, \quad \text{for} \quad n = 2, \]

\[ N_{f} \left[ f(\eta; q), \hat{\theta}(\eta; q), \phi(\eta; q) \right] = 3 \left( \frac{\partial f(\eta; q)}{\partial \eta} \right)^{2} \frac{\partial^{2} f(\eta; q)}{\partial \eta^{2}} - \frac{\partial \theta(\eta; q)}{\partial \eta} \]

\[ + Nr \frac{\partial \theta(\eta; q)}{\partial \eta}, \quad \text{for} \quad n = 3, \]

\[ N_{f} \left[ f(\eta; q), \hat{\theta}(\eta; q), \phi(\eta; q) \right] = \frac{\partial^{2} \theta(\eta; q)}{\partial \eta^{2}} + \frac{m + 1}{2} f(\eta; q) \frac{\partial \theta(\eta; q)}{\partial \eta} \]

\[ + Nb \frac{\partial \theta(\eta; q)}{\partial \eta} \frac{\partial \theta(\eta; q)}{\partial \eta} \]

\[ + Nr \left( \frac{\partial \theta(\eta; q)}{\partial \eta} \right)^{2}, \]

\[ N_{f} \left[ f(\eta; q), \theta(\eta; q), \phi(\eta; q) \right] = \frac{\partial^{2} \phi(\eta; q)}{\partial \eta^{2}} \frac{m + 1}{2} Le f(\eta; q) \frac{\partial \phi(\eta; q)}{\partial \eta} \]

\[ + Nr \frac{\partial \theta(\eta; q)}{\partial \eta} \frac{\partial \phi(\eta; q)}{\partial \eta} \]

Now, the \( i \)th order of deformation equations (Eqs. \((23)-(26)\)) may be solved by the symbolic software MATHEMATICA.

\[ L_{f} \left[ \varphi(\eta) \right] - \frac{z_{f_{-1}}(\eta)}{z_{f_{-1}}(\eta)} \right] = h_{x} R_{x}(\eta), \]

\[ L_{\theta} \left[ \varphi(\eta) \right] - \frac{z_{\theta_{-1}}(\eta)}{z_{\theta_{-1}}(\eta)} \right] = h_{x} R_{x}(\eta), \]

\[ L_{\phi} \left[ \varphi(\eta) \right] - \frac{z_{\phi_{-1}}(\eta)}{z_{\phi_{-1}}(\eta)} \right] = h_{x} R_{x}(\eta), \]

where \( h \) is the auxiliary nonzero parameter.

\[ R_{x}(\eta) = 2 \sum_{j=0}^{\infty} \frac{\partial f(\eta)}{\partial \eta} \frac{\partial f_{j-1}(\eta)}{\partial \eta} - \frac{\partial \theta_{j-1}}{\partial \eta} + Nr \frac{\partial \phi_{j-1}}{\partial \eta}, \quad \text{for} \quad n = 2, \]
R_{f_i}(\eta) = 3 \sum_{j=0}^{n-1} \left( \frac{\partial f_{i-1,j}(\eta)}{\partial \eta} + \sum_{j=0}^{n-1} \frac{\partial f_{i,j}(\eta)}{\partial \eta} \left( \sum_{j=0}^{n-1} \frac{\partial^2 f_{i-1,j}(\eta)}{\partial \eta^2} \right) \right) \frac{\partial \theta_{i-1}}{\partial \eta} \
+ N_r \frac{\partial \theta_{i-1}}{\partial \eta}, \quad \text{for } n = 3, \tag{31}

R_{\phi_i}(\eta) = \frac{\partial^2 \phi_{i-1}(\eta)}{\partial \eta^2} + \sum_{j=0}^{n-1} \left( \sum_{j=0}^{n-1} \frac{\partial f_{i,j}(\eta)}{\partial \eta} \sum_{j=0}^{n-1} \frac{\partial \phi_{i-1,j}(\eta)}{\partial \eta} \right) \frac{\partial \theta_{i-1}}{\partial \eta} \tag{32}

R_{w_i}(\eta) = \frac{\partial^2 \phi_{i-1}(\eta)}{\partial \eta^2} + \sum_{j=0}^{n-1} \left( \sum_{j=0}^{n-1} \frac{\partial f_{i,j}(\eta)}{\partial \eta} \sum_{j=0}^{n-1} \frac{\partial \phi_{i-1,j}(\eta)}{\partial \eta} \right) \frac{\partial \theta_{i-1}}{\partial \eta}, \tag{33}

\text{and}

\lambda_i = \begin{cases} 0, & i \leq 1, \\ 1, & i > 1. \end{cases} \tag{34}

For more information about the HAM solution see Refs. [28,29].

Figs. 2 and 3 represent h-curves for an especial case. A proper and optimal value for auxiliary parameter from the valid straight region must be selected to control the convergence of the approximation series in the so-called h-curve.

4. Optimal convergence control parameters

The series solutions (27)-(29) contain the nonzero auxiliary parameters \( h_i, h_0 \) and \( h_m \), which determine the convergence region. The average residual errors are as follows [46]:

\[ e_m^h = \frac{1}{k+1} \sum_{j=0}^{k} \left[ \mathcal{N}_f \left( \sum_{i=0}^{m} \phi_i(\eta), \sum_{i=0}^{m} \phi_i(\eta) \right) \right] \frac{1}{\eta} d\eta, \tag{35} \]

\[ e_m^\phi = \frac{1}{k+1} \sum_{j=0}^{k} \left[ \mathcal{N}_\phi \left( \sum_{i=0}^{m} \phi_i(\eta), \sum_{i=0}^{m} \phi_i(\eta) \right) \right] \frac{1}{\eta} d\eta, \tag{36} \]

\[ e_m^\theta = \frac{1}{k+1} \sum_{j=0}^{k} \left[ \mathcal{N}_\theta \left( \sum_{i=0}^{m} \phi_i(\eta), \sum_{i=0}^{m} \phi_i(\eta) \right) \right] \frac{1}{\eta} d\eta. \tag{37} \]

So

\begin{table}
\begin{tabular}{|c|c|c|c|}
\hline
Order of & & & \\
approximation & \( h_f \) & \( h_0 \) & \( h_m \) \\
\hline
2 & -0.461796 & -1.32768 & -1.25605 & 1.64 \times 10^{-3} \\
3 & -0.524027 & -1.20078 & -1.34049 & 2.64 \times 10^{-4} \\
4 & -0.490020 & -1.05973 & -1.47444 & 2.84 \times 10^{-4} \\
5 & -0.529268 & -0.896792 & -1.55199 & 6.01 \times 10^{-5} \\
6 & -0.492102 & -0.828646 & -1.55694 & 7.45 \times 10^{-5} \\
\hline
\end{tabular}
\caption{Optimal values of convergence control parameters versus different orders of approximation.}
\end{table}

\begin{table}
\begin{tabular}{|c|c|c|c|}
\hline
\( m \) & \( e_m^{\phi} \) & \( e_m^{\theta} \) & \( e_m^{\theta} \) \\
\hline
2 & 1.89 \times 10^{-3} & 1.89 \times 10^{-3} & 6.77 \times 10^{-3} \\
3 & 9.34 \times 10^{-4} & 1.30 \times 10^{-4} & 3.90 \times 10^{-4} \\
4 & 1.09 \times 10^{-4} & 1.64 \times 10^{-5} & 6.95 \times 10^{-6} \\
5 & 1.08 \times 10^{-5} & 1.48 \times 10^{-6} & 5.45 \times 10^{-7} \\
6 & 4.42 \times 10^{-6} & 3.15 \times 10^{-7} & 1.12 \times 10^{-7} \\
7 & 2.31 \times 10^{-7} & 2.38 \times 10^{-8} & 7.39 \times 10^{-9} \\
8 & 1.84 \times 10^{-7} & 2.65 \times 10^{-11} & 8.74 \times 10^{-12} \\
\hline
\end{tabular}
\caption{Averaged squared residual errors using optimal values of auxiliary parameters.}
\end{table}
where \( e_t \) is the total squared residual error. The total average squared residual error is minimized by employing MATHEMATICA package \textbf{BVPh2.0} \cite{47}. Employing the command \texttt{Minimize}, one can obtain the corresponding local optimal convergence control parameters. In an especial case when \( n = 2.0, \ N_r = 0.05, \ m = 1.0, \ N_b = 0.9, \ N_t = 0.1, \ L_e = 1.0, \ N_c = 0.3 \) and \( N_d = 0.3 \), Table 1 presents the optimal values of convergence control parameters as well as the minimum values of total averaged squared residual error for different orders of approximation. Using the optimal values from Table 1, the average squared residual error at different orders of approximations is presented in Table 2. In addition, Fig. 4 illustrates the maximum average squared residual error at different orders of approximation. It is obvious that the averaged squared residual errors and total averaged squared residual errors have decreasing trends.

To assess the accuracy of the solution, the results have been compared with the results obtained by Yih \cite{41}, Chen and Chen \cite{48}, and Cheng et al. \cite{49}. The comparison is displayed in Table 3.

Table 3 Comparison of \(-\theta'(0)\) for various values of \( n \) and \( m \) when \( N_r = N_t = N_b = 0.0 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( m )</th>
<th>Present results</th>
<th>Chen and Chen \cite{48}</th>
<th>Yih \cite{41}</th>
<th>Cheng et al. \cite{49}</th>
</tr>
</thead>
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<tr>
<td>1.0</td>
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<td>0.443853</td>
<td>0.4437</td>
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</tr>
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<td>0.7686</td>
<td>0.7686</td>
<td>0.7685</td>
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<td>–</td>
</tr>
<tr>
<td>1.0</td>
<td>0.855241</td>
<td>0.8552</td>
<td>0.8552</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Figure 4 Maximum average squared residual error at different orders of approximation.

Figure 5 The effect of \( N_r \) on the dimensionless velocity profile when \( N_t = N_b = 0.2, \ L_e = 1.0, \ N_c = N_d = 0.3 \) and \( n = 2.0 \).

\[
e'_m = e'_f + e'_a + e'_r, \tag{38}
\]

where \( e'_m \) is the total squared residual error. The total average squared residual error is minimized by employing MATHEMATICA package \textbf{BVPh2.0} \cite{47}. Employing the command \texttt{Minimize}, one can obtain the corresponding local optimal convergence control parameters. In an especial case when \( n = 2.0, \ N_r = 0.05, \ m = 1.0, \ N_b = 0.9, \ N_t = 0.1, \ L_e = 1.0, \ N_c = 0.3 \) and \( N_d = 0.3 \), Table 1 presents the optimal values of convergence control parameters as well as the minimum values of total averaged squared residual error for different orders of approximation. Using the optimal values from Table 1, the average squared residual error at different orders of approximations is presented in Table 2. In addition, Fig. 4 illustrates the maximum average squared residual error at different orders of approximation. It is obvious that the averaged squared residual errors and total averaged squared residual errors have decreasing trends.

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in Table 3. An excellent agreement is found between the present results and published ones.

5. Results and discussion

The influence of the buoyancy ratio parameter $Nr$ on the dimensionless velocity, temperature and nanoparticle volume fraction distributions is plotted in Figs. 5–10. This parameter arises only in the momentum boundary-layer Eq. (7). Physically, $Nr$ signifies the relative influence of the concentration buoyancy force to the thermal buoyancy force in the boundary-layer regime. In this study $Nr$ has been considered to be less than 1 which means that flow is driven by thermal buoyancy. In an examination of these figures, it is found that the dimensionless velocity reduces with an increase in the buoyancy ratio parameter for both flat pale ($m = 0$) and cone ($m = 1$) and observed that the enhancement in the value of $Nr$ has a tendency to decelerate the fluid flow in the vicinity of flat plate as well as cone surface. It is further observed that the dimensionless temperature and nanoparticle volume fraction increase with the increase in $Nr$ for both plate and cone. A similar trend is also reported by Khan et al. [50].

The effects of the nanofluid parameters $Nt$ and $Nb$ on the dimensionless temperature and nanoparticle volume fraction profiles are plotted in Figs. 11 and 12 for vertical plate. The Brownian motion of nanoparticle is supposed to increase the thermal conduction by one of the two mechanisms, either a
direct effect which transports heat or an indirect contribution due to micro-convection of fluid surrounding individual nanoparticle. The direct contribution of the Brownian motion is theoretically negligible. Nanoparticles are often in the form of agglomerates and/or aggregates. For larger diameter nanoparticles, $N_b$ is small and Brownian motion is weak; the converse is the case for small diameter nanoparticles ($N_b$ is high and Brownian motion vigorous). The temperature distributions show that Brownian motion effects exert a significant enhancing influence on the dimensionless temperature profiles. This is due to the fact that the diffusion of nanoparticles into the fluid enhances with the increase in $N_b$, and thereby, the temperature profiles are enhanced. Temperature is also enhanced with the increase in thermophoresis parameter. This is due to the fact that the thermophoresis force, which tends to move particles from the hot zone to the cold zone, increases with the increase in $N_t$. The effects of the same parameters on the temperature and nanoparticle volume fraction for cone are depicted in Figs. 13 and 14. The nanoparticle volume fraction (mass fraction i.e., concentration) is found to be diminished with the increase in the values of $N_b$. The nanofluid behavior is more like a fluid than the conventional solid-fluid mixtures. This two-phase fluid movement is in random in nature and the suspended nanoparticle increases energy exchange rates in the fluid. Brownian motion therefore augments temperatures in the boundary-layer but inhibits nanoparticle diffu-
An increase in the value of \( Nb \) will enhance concentration boundary-layer thickness (mass fraction).

The variation of the dimensionless temperature and nanoparticle volume fraction with the conduction–convection parameter \( (Nc) \) and conduction–diffusion parameter \( (Nd) \) for both vertical plate (Figs. 15 and 16) and cone (Figs. 17 and 18) is illustrated in Figs. 15–18. The dimensionless temperature and the nanoparticle volume fraction of the fluid enhance with an increase in the strength of \( Nc \) and \( Nd \). Increasingly stronger thermal convective boundary condition enhances the flow since the fluid on the right surface of the sheet is heated by the hot fluid on the left surface of the sheet, making it lighter and thus causing it to flow faster. Increasing convective boundary effect heats the boundary-layer, as expected, at the wall.

The slight increase in nanoparticle volume fraction is localized to a region at some distance from the wall. The thermal convective boundary condition \( \theta'(0) = -Nc[1 - \theta(0)] \) affects the temperature distribution and this in turn indirectly influences nanoparticle volume fraction via the coupling terms \( Nb\theta'\varphi' \) and \( (Nt/Nb)\theta' \). Similarly, the mass convective boundary condition \( \varphi'(0) = -Nd[1 - \varphi(0)] \) affects the concentration distribution and this in turn indirectly influences temperature via the coupling terms \( Nb\theta'\varphi' \). Note that one can recover thermal and mass slips boundary convections when \( Nc = \frac{1}{a} \), \( Nd = \frac{1}{b} \), \( a > 0 \), \( b > 0 \) (thermal and mass slip parameter). Note that no-slip assumptions are not true for fluid flows at the micro- and nanoscale. One can also recover isothermal

**Figure 15** The effect of \( Nc \), \( Nd \) on the dimensionless temperature profile when \( Nr = 0.2 \), \( Nt = Nb = 0.3 \), \( Le = 1.0 \), \( n = 2.0 \) and \( m = 0.0 \).

**Figure 16** The effect of \( Nc \), \( Nd \) on the dimensionless nanoparticle volume fraction profile when \( Nr = 0.2 \), \( Nb = Nt = 0.3 \), \( Le = 1.0 \), \( n = 2.0 \) and \( m = 0.0 \).

**Figure 17** The effect of \( Nc \), \( Nd \) on the dimensionless temperature profile when \( Nr = 0.2 \), \( Nt = Nb = 0.3 \), \( Le = 1.0 \), \( n = 2.0 \) and \( m = 1.0 \).

**Figure 18** The effect of \( Nc \), \( Nd \) on the dimensionless nanoparticle volume fraction profile when \( Nr = 0.2 \), \( Nb = Nt = 0.3 \), \( Le = 1.0 \), \( n = 2.0 \) and \( m = 1.0 \).
Figure 19 The effect of $Le$ on the dimensionless nanoparticle volume fraction profile when $Nr = 0.1$, $Nt = Nt = 0.1$, $Nc = Nd = 0.2$ and $n = 2.0$.

Figure 20 The effect of $Le$ on the dimensionless nanoparticle volume fraction profile when $Nr = 0.1$, $Nt = Nt = 0.1$, $Nc = Nd = 0.2$, and $n = 3.0$.

Figure 21 The effect of $Le$ and $Nr$ on the friction factor $-f'(0)$ when $Nt = Nb = 0.1$, $Nc = 0.1$, and $Nd = 0.1$.

Figure 22 The effect of $Nr$ and $Nb$ on the heat transfer rates $-\theta'(0)$ when $Nr = 0.05$, $Le = 1.0$, $Nc = 0.2$, and $Nd = 0.2$.

The effect of the Lewis number on the dimensionless nanoparticle volume fraction profile has been illustrated in Figs. 19 and 20. Physically, $Le$ is the ratio of the Schmidt number to the Prandtl number. Note that when $Le = 1$, thermal and nanoparticle volume fraction boundary-layer thicknesses are equal. For values of $Le > 1$, thermal diffusion is faster than species diffusion and species boundary-layer thickness is exceeded by thermal boundary-layer thickness. Thus, the nanoparticle volume fraction magnitudes are expected to decrease with increasing Lewis number. Furthermore, a greater value of the Lewis number implies a lower Brownian motion coefficient, $D_B$ for a base fluid with kinematic viscosity $v$. Hence, the higher Lewis number decreases concentration and its boundary-layer thickness. This decreasing behavior can be clearly seen for both cases of vertical plate and cone.

Now the focus is on the effects of the governing parameters on the dimensionless quantities of thermal engineering interest. In Figs. 21–23 the effect of different parameters on the skin friction factor, heat and nanoparticle volume fraction transfer rates is depicted. From Fig. 21 it is observed that friction factor proportional to $-f'(0)$ decreases with the increase in $Le$ and $Nr$ for both vertical plate and cone. From Fig. 23 it is
found that the local Nusselt number decreases with the increase in the thermophoresis and Brownian motion parameters for both plate and cone. The local Sherwood number increases with the Brownian motion parameter and decreases for the thermophoresis parameter.

6. Conclusion

In the present article, the free convective flow of dilatant nanofluid along a vertical flat plate/cone in a Darcian porous medium has been studied analytically by HAM. This analytical solution shows excellent agreement with the numerical data available in the literature. The dimensionless velocity profile shows a decreasing trend for vertical flat plate and cone with the increase in the buoyancy-ratio parameters for both cases of $n = 2$ and $n = 3$ but an opposite behavior occurs in the temperature and concentration distributions. With the increases in the value of $Nt$, both $\theta(\eta)$ and $\phi(\eta)$ increase. The impact of $Nt$ and $Nb$ is to enhance the thermal boundary-layer thickness but the opposite behavior can be seen for nanoparticle volume fraction boundary-layer thickness with increasing the value of $Nb$. With the increase in $Nc$ and $Nd$, one can see that the temperature and mass fraction will increase to the ultimate value of 1 in wall. Lewis number has a decreasing influence on the nanoparticle volume fraction distribution.

References


