A numerical approach to a multi-objective optimal inventory control problem for deteriorating multi-items under fuzzy inflation and discounting

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Abstract

The optimal production and advertising policies for an inventory control system of deteriorating multi-items under a single management are formulated with resource constraints under inflation and discounting in fuzzy environment. Here, the deterioration of the items and depreciation of sales are at a constant rate. Deteriorated items are salvaged and the effect of inflation and time value of money are taken into consideration. The inflation and discount rates are assumed to be imprecise and represented by fuzzy numbers. These imprecise quantities are first transformed to corresponding intervals and then following interval mathematics, the related objective function is changed to respective multi-objective functions. Using Utility Function Method (UFM), the multi-objective problem is changed to a single objective problem. Here, the production and advertisement rates are unknown and considered as control(decision) variables. The production, advertisement and demand rates are functions of time $t$. The total profit which consists of the sales proceeds, production cost, inventory holding cost and advertisement cost is formulated as an optimal control problem and evaluated numerically using UFM and generalized reduced gradient (GRG) technique. Finally numerical experiment, sensitivity analysis and graphical representation are provided to illustrate the system. For the present model, expressions and graphical results are presented when the rates of advertisement are constant.

Keywords: Multi-item inventory; Optimal production; Deteriorating item; Advertising policies; Fuzzy inflation and discounting

1. Introduction

From financial standpoint, an inventory represents a capital investment and must compete with other assets within the firm’s limited capital funds. Most of the classical inventory models did not take into account the effects of inflation and time value of money. This has happened mostly because of the belief that inflation and time value of money will not influence the cost and price components (i.e., the inventory policy) to any significant degree. But, during the last few decades, due to high inflation and consequent sharp decline in the purchasing power of money in the developing
countries like Brazil, Argentina, India, Bangladesh, etc., the financial situation has been changed and so it is not possible to ignore the effect of inflation and time value of money any further. Following Buzacott [1], Misra [2] has extended the approach to different inventory models with finite replenishment, shortages, etc. by considering the time value of money, different inflation rates for the costs. Also Lo et al. [3] developed an integrated production-inventory model with a varying rate of deterioration under imperfect production process, partial backordering and inflation.

In the recent decades, multi-item classical inventory problems were approached by formulating proper mathematical models that considered the factors in real world situations, such as the deterioration of inventory items, depreciation of sales, advertising policies, effects of inflation and time value of money, etc. Deterioration is applicable to many inventory items in practice, like vegetables, rice, medicine, fruits, etc. Recently, Chang and Dye [4], Papachristos and Skouri [5], Maity and Maiti [6] and others formulated EOQ models of deteriorating items with time-varying demand. Also Goyal and Giri [7] have presented a review article on the recent trends in modeling with deteriorating items listing all important publications in this area up to 2001.

Again, some researchers (Cho [8] and others) have assumed depreciation rate of sales as a function of time, $t$. This assumption is supported by a general fact that, as time goes on, a firm usually faces more competition (thus it may lose its sales at an increasing rate). Again, to boost up the sale, the management goes for advertisement and thus advertisement policy plays an important role in increasing the demand. Also, a promotional cost (cf. Datta et al. [9]) is introduced to provide the advertisement that increase the demand of an item.

In the case of multi-item inventory models, it is possible to study each item separately as long as there is no interactions between the items. However, in general, interaction exist between the items, such as, limited warehouse space, available capital for investment, etc.

The production period of the seasonable products such as winter garments, etc. is normally finite. Moreover, in a production firm, production is discontinued once the level of the stock in godown is such that it is sufficient to fulfill the demand up to the end of the time period.

In a manufacturing system, the physical output (i.e., product) of a firm depends upon the combination of several product factors. These factors are (a) raw material (b) technical knowledge (c) production procedure (d) firm size (e) nature of the organization (f) quality of the product etc. Due to the changes of these factors, production rate and unit production cost are changed too. In the classical production lot size models, both production rate and unit production cost are assumed to be constant and dependent on each other. Several OR scientists developed inventory models for a single product or multiple products taking constant or variable production rate (as a function of demand and/or on hand inventory). In this connection, one may refer to the works of Misra [2], Mandal and Maiti [10]. In their models, the production cost is taken as constant. However, manufacturing flexibility has become much more important to firms and less expensive to acquire. Different types of flexibility in the manufacturing system have been identified in the literature among which volume flexibility is the most important one. Volume flexibility of a manufacturing system is defined as its ability to be operated profitably at different overall output levels. Khouja [11] developed an economic production lot size model under volume flexibility where unit production cost depends upon the raw material used, labour force engaged and tool wear-out cost incurred. Here, unit production cost is a function of production rate. If the production is more, the production related to some constant expenditures are spread over the number of produced units and hence the unit production cost decreases with the increase of produced units. Moreover, some expenditures do not increase linearly with the produced quantity. Bhandari and Sharma [12] extended the work of Khouja [11] including the marketing cost and taking a generalized unit cost function.

However, because of the dynamic nature of the manufacturing environment, the static models may not be adequate in analyzing the behavior of such systems. Dynamic models of production-inventory systems are available in many references (cf. Hu and Loulou [13], Worell and Hall [14], Chandra and Bahner [15], Misra [2], Maity and Maiti [16] and others).

In 1965, the first publication in fuzzy set theory by Zadeh [17] showed the intention to accommodate uncertainty in the non-stochastic sense. After that Bellman and Zadeh [18] defined a fuzzy decision making problem as the confluence of fuzzy objectives and constraints operated by max–min operators. Zimmermann [19] developed a tolerance approach to transform a fuzzy decision making problem to a regular crisp optimization problem and showed that it can be solved to obtain a unique exact optimal solution with highest membership degree using classical optimization algorithms. Recently, fuzzy set theoretic has been applied to several fields like project network, reliability, production planning, inventory problems, etc. Roy and Maiti [20], Mahapatra and Maiti [21] and others have solved the classical EOQ models in fuzzy environment. Now-a-days, some inventory problems have been developed by
considering fuzzy demand, fuzzy production quantity and/or fuzzy deterioration by several researchers (cf. Yao and Wu [22], Dey et al. [23] and others).

Though multi-objective decision making (MODM) problems have been formulated and solved in many other areas like air pollution, structural analysis, till now very few papers on MODM have been published in the field of optimal inventory control. Padmanabhan and Vrat [24] formulated an inventory problem of deteriorating items with two objectives — minimization of total average cost and wastage cost in crisp environment and solved by nonlinear goal programming method. Roy and Maiti [25] formulated an inventory problem of deteriorating items with two objectives, namely, maximizing total average profit and minimizing total waste cost in fuzzy environment.

In his paper, advertising and production policies are developed for a deteriorating multi-item inventory control problem. The system is under the control of fuzzy inflation and discounting. Deterioration and sales depreciation rates are assumed to be constant. The salvage value of deteriorated items is included. The warehouse to store the items is of limited capacity and the investment is also limited. The relevant inventory costs like production, holding, advertisement and promotional cost are considered. The profit out of the total proceeds is evaluated and maximized.

This maximization problem is formulated as an optimal control problem and solved numerically using UFM and GRG programming method. Roy and Maiti [25] formulated an inventory problem of deteriorating items with two objectives, specifically, maximizing total average profit and minimizing total waste cost in fuzzy environment.

In this paper, advertising and production policies are developed for a deteriorating multi-item inventory control problem. The system is under the control of fuzzy inflation and discounting. Deterioration and sales depreciation rates are assumed to be constant. The salvage value of deteriorated items is included. The warehouse to store the items is of limited capacity and the investment is also limited. The relevant inventory costs like production, holding, advertisement and promotional cost are considered. The profit out of the total proceeds is evaluated and maximized.

2. Interval arithmetic

Let $A = [A_L, A_R]$ and $B = [B_L, B_R]$ be two fuzzy numbers, $+, -$ be ordinary addition and subtraction on real numbers. Then according to Moore [27] the following binary operations on intervals can be defined.

$$A + B = [a_L, a_R] + [b_L, b_R] = [a_L + b_L, a_R + b_R]$$

$$A - B = [a_L, a_R] - [b_L, b_R] = [a_L - b_R, a_R - b_L].$$

Lemma 1. $e^{[a, b]} = [e^{a}, e^{b}]$.

Proof.

$$e^x = \lim_{k \to \infty} \left( 1 + \frac{x}{k} \right)^k \quad \text{[because, } e = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = \lim_{k \to \infty} \left( 1 + \frac{x}{k} \right)^k \text{]}$$

$$e^{[a, b]} = \lim_{k \to \infty} \left( 1 + \frac{[a, b]}{k} \right)^k$$

$$= \lim_{k \to \infty} \left[ \left( 1 + \frac{a}{k} \right)^k, \left( 1 + \frac{b}{k} \right)^k \right]$$

$$= \left[ \lim_{k \to \infty} \left( 1 + \frac{a}{k} \right)^k, \lim_{k \to \infty} \left( 1 + \frac{b}{k} \right)^k \right]$$

$$= [e^{a}, e^{b}].$$

Hence the Lemma 1. □

Lemma 2. If $f$ is a continuous interval-valued function of a real variable $x$ in $[a, b]$, then there is a pair of continuous real-valued functions $f_1, f_2$ such that $f(x) = [f_1(x), f_2(x)]$ and the integral of $f$ is equivalent to

$$\int_{[a,x]} f(x')dx' = \left[ \int_{[a,x]} f_1(x')dx', \int_{[a,x]} f_2(x')dx' \right].$$


3. The nearest interval approximation of a fuzzy number

If $\tilde{A}$ is a fuzzy number with $\eta$-cut $[A_L(\eta), A_R(\eta)]$ then according to Grzegorzewski [29], the nearest interval approximation of $\tilde{A}$ is $\left[ \int_0^1 A_L(\eta)d\eta, \int_0^1 A_R(\eta)d\eta \right].$
Therefore, interval number considering \( \tilde{A} = (a_1, a_2, a_3) \) as a triangle fuzzy number is \([(a_1 + a_2)/2, (a_2 + a_3)/2]\).

4. Optimal control framework

Assumption and notation:
For a deteriorating multi-item inventory control model, following assumptions and notations are used.

4.1. Assumptions

For \( i \th \) \((i = 1, 2, \ldots, n)\) item, the following assumptions are made.

(i) Deterioration and depreciation rates are known and constant.
(ii) Advertisements in different media are made to boost the demand.
(iii) As demand is artificially increased following the assumption (ii), allowing shortages will have adverse effect on the sale. Moreover, shortages always bring loss of goodwill to the firm. Hence, shortages are not allowed i.e. mathematically stock level is always greater than or equal to zero.
(iv) The promotional cost is introduced to provide the advertisement that increase the demand of an item and it is taken as an exponential function with zero promotional which implies zero advertisement,
(v) The inflation rate \( \tilde{k}_i \) and the discount rate \( \tilde{r}_i \) representing the time value of money are fuzzy in nature, therefore, the net discount rate of inflation, \( \tilde{R}_i \), is also a fuzzy number; \( \tilde{R}_i = \tilde{r}_i - \tilde{k}_i \). The present worth of \( p_t \) is \( p_t e^{-\tilde{R}_i t}, t \geq 0 \) where \( p_t \) is the value of \( p \) at time \( t \).
(vi) Deterioration of units occurs only when the item is effectively in stock and there is no repair or replacement of deteriorated units over the period \([0, T]\).
(vii) Unit production cost is produced-quantity-dependent. This means that as some constant expenditures in production are spread over the number of production units, unit production cost is inversely proportional to the produced quantity.
(viii) The inventory level, demand and production are continuous variables with appropriate units.
(ix) There are \( n \) items in the system.
(x) The maximum space and investment are limited.
(xi) This is a single period inventory model with finite time horizon.
(xii) The deteriorating units are salvaged.

4.2. Notations

\( n \) = number of items.
\( M \) = maximum space available for storage.
\( Z \) = maximum investment costs.
\( T \) = time length of the cycle.
For the \( i \th \) \((i = 1, \ldots, n)\) item, 
\( D_i(t) \) = sales rate at time \( t \).
\( U_i(t) = u_{i0} + u_{i1} t \) = production rate at time \( t \) where \( u_{i0} \) and \( u_{i1} \) are the control(decision) variables.
\( X_i(t) \) = the inventory level at time \( t \).
\( \alpha_i \) = rate of deterioration.
\( \beta_i \) = rate of depreciation.
\( C_{ui}(U_i(t)) = C_{ui0} + \frac{C_{ui1}}{U_i(t)} \) = production-dependent unit production cost where \( \gamma_i \) is the known constant. Here \( C_{ui0} \) is the constant cost due to raw materials and \( C_{ui1} \) is the mainly labour cost. This cost is spread over if the labourers produce more. \( \gamma_i \) is called production elasticity.
\( h_i \) = holding costs per unit item per unit time.
\( a_i \) = storage area per unit item.
\( V_i(t) = v_i t \) = the rate of demand created by the advertisement at time \( t \) where \( v_i \) is the control(decision) variable.
\( s_i \) = selling price per unit item.
\( c_i (e^{V_i(t)} - 1) \) = promotional costs.
\( b_i \) = salvage value per unit item.
5.1. Proposed production-inventory model in fuzzy environment

A production-inventory system for \( n \) deteriorating items is considered with warehouse capacity and investment constraints. Here, the items are produced at a variable rate \( U_i(t) \) and deteriorate at a constant rate, \( \alpha_i \). Demand of the items is time-dependent, it decreases due to the depreciation of sale and increases due to the advertising policy. So the promotional cost (cf. Datta et al. [9]) is introduced to provide the advertisement that increase the demand of an item and it is taken as an exponential function with zero promotional which implies zero advertisement. The stock level promotional cost (cf. Datta et al. [9]) decreases due to deterioration and sale. Shortages are not allowed. The effect of inflation and time value of money are taken into consideration as fuzzy numbers. These numbers are first transformed to corresponding intervals and then following interval mathematics, the objective function is changed to multi-objective functions. Using UFM, the multi-objectives are changed to a single objective function.

The differential equations for \( i \)th item representing the above system during a fixed time-horizon, \( T \) is

\[
\dot{X}_i(t) = U_i(t) - D_i(t) - \alpha_i X_i(t), \quad X_i(0) = 0, \quad X_i(T) = 0. \tag{1}
\]

The demand rate is created by the advertisement and it is destroyed due to the depreciation of the competition market, so the differential equation of demand for \( i \)th item during the fixed time-horizon, \( T \) is

\[
\dot{D}_i(t) = V_i(t) - \beta_i D_i(t). \tag{2}
\]

Assuming the produced-quantity-dependent unit production cost, the warehouse of finite capacity and investment constraint, maximization of total profit consisting of sales proceeds, holding, promotional and production costs lead to

\[
\text{Maximize } J(u) = \sum_{i=1}^{n} \int_{0}^{T} e^{-\tilde{R}_i t} (s_i D_i(t) + b_i \alpha_i X_i(t) - h_i X_i(t) - C_{ui0} U_i(t) - C_{ui1} U_i^{1-\gamma_i}(t) - c_i (e^{V_i(t)} - 1)) dt,
\]

subject to the constraints (1) and (2),

\[
\sum_{i=1}^{n} X_i(t) a_i \leq M \quad \text{(space constraint)} \tag{4}
\]

and

\[
\sum_{i=1}^{n} \int_{0}^{T} (C_{ui0} U_i(t) + C_{ui1} U_i^{1-\gamma_i}(t) + c_i (e^{V_i(t)} - 1)) dt \leq Z \quad \text{(investment constraint)}. \tag{5}
\]

5.1.1. Model with linear time-dependent advertisement cost

In this case, we take the advertisement rate \( V_i(t) = v_i t \) and solving Eq. (2), we get,

\[
D_i(t) = D_i(0) e^{-\beta_i t} + v_i \left( \frac{t}{\beta_i} - \frac{1 - e^{-\beta_i t}}{\beta_i^2} \right) \quad \text{in } (0, T)
\]

\[
i.e \quad D_i(t) = \frac{v_i}{\beta_i} \left( t - \frac{1}{\beta_i} \right) + \left( D_i(0) + \frac{v_i}{\beta_i^2} \right) e^{-\beta_i t} \quad \text{in } (0, T).
\]

\( \hat{k}_i \) = rate of inflation which is a fuzzy number and is converted to an appropriate interval number \( \hat{k}_i = [k_{i L}, k_{i R}] \).

\( \hat{r}_i \) = discount rate which is a fuzzy number and is converted to an approximate interval number \( \hat{r}_i = [r_{i L}, r_{i R}] \).

\( \tilde{R}_i \) = net discount rate of inflation i.e \( \tilde{R}_i = \hat{r}_i - \hat{k}_i \), where \( \tilde{R}_i = [R_{i L}, R_{i R}] \) and by interval arithmetic, \( R_{i L} = r_{i L} - k_{i R}, \) \( R_{i R} = r_{i R} - k_{i L} \).
Also we assume that as initially there is no stock, the production starts from initial time and after certain time (i.e. \( t_i, 0 < t_i < T \) time), the production is stopped such that the stock is exhausted after meeting the demand at the end of the fixed time-horizon, \( T \).

So \( U_i(t) = u_{i10} + u_{i1} t \) in \((0, t_i)\)
\[ \quad = 0 \quad \text{in} \quad (t_i, T), \quad i = 1, 2, \ldots n. \] (7)

Using the Eqs. (6)–(8), we have from (1),
\[ X_i(t) = u_{i10} \left( \frac{1 - e^{-a_i t}}{a_i} \right) + u_{i11} \left( \frac{t}{\alpha_i} - 1 - \frac{1 - e^{-a_i t}}{\alpha_i^2} \right) - D_{i11} \left( \frac{e^{-\beta_i t} - e^{-a_i t}}{\alpha_i - \beta_i} \right), \quad (0, t_i) \] (9)
\[ = -v_{i1} \left( \frac{e^{a_i (T-t) - 1}}{\alpha_i^2} \right) + \frac{v_i}{\beta_i} \left( T e^{a_i (T-t) - t} - 1 \right) + D_{i11} \left( \frac{e^{a_i (T-t) - \beta_i T} - e^{-\beta_i t}}{\alpha_i - \beta_i} \right), \quad (t_i, T) \] (10)
where \( u_{i10} = (u_{i10} + \frac{v_i}{\beta_i}), \ u_{i11} = u_{i1} - \frac{v_i}{\beta_i}, \ D_{i11} = D_i(0) + \frac{v_i}{\beta_i} \) and \( v_{i1} = \frac{v_i}{\beta_i} + \frac{v_i}{\beta_i} \) and \( t_i \) satisfy the continuity condition of \( X_i(t) \) of Eqs. (9) and (10) at \( t_i \).

5.2. Equivalent deterministic representation of the proposed model

As \( \tilde{R}_i [R_{iL}, R_{iR}] \) with \( R_{iL} = r_{iL} - k_{iL} \) and \( R_{iR} = r_{iR} - k_{iL} \), the problem is then reduced to the form

Maximize \( J(u) = \sum_{i=1}^{n} \int_{0}^{T} e^{-[R_{iL} R_{iR}] t} (s_i D_i(t) + b_i \alpha_i X_i(t) - h_i X_i(t)) \)
\[ - C_{ui0} U_i(t) - C_{ui1} U_i^{1-\gamma_i}(t) - c_i (e^{V_i(t)} - 1)) dt \] (11)
subject to the constraints (1), (2), (4) and (5).

Using Lemmas 1 and 2, the expression (11) is now expressed as

Maximize \( [J_L, J_R] \) (12)

where
\[ J_L(u) = \sum_{i=1}^{n} \int_{0}^{T} e^{-R_{iL} t} (s_i D_i(t) + b_i \alpha_i X_i(t) - h_i X_i(t)) - C_{ui0} U_i(t) - C_{ui1} U_i^{1-\gamma_i}(t) - c_i (e^{V_i(t)} - 1)) dt \]
\[ = \sum_{i=1}^{n} (ss_i \tilde{L} - H_{iL} - uv_{iL} - uu_{iL}) \quad (\text{cf. Appendix A}) \] (13)
and
\[ J_R(u) = \sum_{i=1}^{n} \int_{0}^{T} e^{-R_{iR} t} (s_i D_i(t) + b_i \alpha_i X_i(t) - h_i X_i(t)) - C_{ui0} U_i(t) - C_{ui1} U_i^{1-\gamma_i}(t) - c_i (e^{V_i(t)} - 1)) dt \]
\[ = \sum_{i=1}^{n} (ss_i \tilde{R} - H_{iR} - uv_{iR} - uu_{iR}) \quad (\text{cf. Appendix A}) \] (14)
subject to (4) and
\[ \sum_{i=1}^{n} \left( C_{ui0} \left( u_{i10} t_i + u_{i1} \frac{t_i^2}{2} \right) + C_{ui1} t_i + c_i \left( \frac{e^{V_i t_i} - 1}{v_i} - t_i \right) \right), \quad (\gamma_i = 1). \]

It is a multi-objective problem which is converted to a single objective problem by using the following utility function method (UFM).
6. Solution methodology

6.1. Utility function method (UFM)

In the utility function method, a utility function $Y_i(J_i)$ is defined for each objective depending on the importance of $J_i$ compared to the other objective functions. Then a total or overall utility function $Y$ is defined, for example, as

$$Y(u) = \sum_{i=L, R} Y_i(J_i(u)).$$

(15)

The solution vector $u^*$ is then found by maximizing the total utility $Y(u)$ subject to the some constraints. We may take a suitable form of the Eq. (15) for maximization formulation as

$$Y(u) = \sum_{i=L, R} w_i J_i(u)$$

subject to $\sum_{i=L, R} w_i = 1$ and $0 < w_R, w_L < 1$.

(In the literature, this is known as weighted sum method.)

Here $w_1$ and $w_2$ are the weights of the objective functions. Since the maximum of the above problem does not change if all the weights are multiplied by a constant, it is the usual practice to choose weights such that their sum is one. Here two theorems are presented concerning the weighted sum method.

**Theorem 1.** The solution of the weighted sum problem (16) is weakly Pareto optimal.

**Theorem 2.** The solution of the weighted sum problem (16) is Pareto optimal if the weighting coefficients are positive, that is $w_i > 0$ for all $i = 1, 2$.

**Proof.** Miettinen [30] proved the Theorems 1 and 2. □

In this case, we assume that $w_i = \frac{J_i^*}{\sum_{i=L, R} J_i^*}, i = L, R$ which obviously lies in $(0, 1)$ and also satisfy the condition $\sum_{i=L, R} w_i = 1$, $J_i^*$ being the optimum (here maximum) value of the individual function $J_i(u)$. Here it is obviously true that $w_1, i = R, L$ takes a single value $\frac{J_i^*}{\sum_{i=L, R} J_i^*}$ which satisfy the Theorems 1 and 2 respectively and as $w_R > w_L$, $w_R, w_L$ are the weights objective functions of $J_R(u), J_L(u)$. Hence the solution is pareto optimum.

Then $Y(u) = \sum_{i=L, R} \frac{J_i^*}{\sum_{i=L, R} J_i^*} (J_i(u))$

(17)

subject to (4) and

$$\sum_{i=1}^n \left( C_{ui0} \left( u_{i0} t_i + u_{i1} t_i^2 \right) + C_{ui1} t_i + c_i \left( \frac{e^{v_i t_i}}{v_i} - t_i \right) \right).$$

(18)

The objective value $Y(u)$ is optimized by using gradient based nonlinear optimization method i.e. GRG (cf. Gabriel and Ragsdoll [26]) technique and obtained the objective value and the corresponding production, demand and stock values.

7. Numerical experiment

7.1. Input data

We take two items i.e. $n = 2$, total investment amount, $Z = 350$, total space, $M = 75$ sq.m., $\gamma_1 = 1; \gamma_2 = 1$; unit area for each item, $a_1 = 2.2$ sq.m.; $a_2 = 2.5$ sq.m.; initial demand, $D_1(0) = 25.0$ units; $D_2(0) = 20.0$ units; initial stock, $X_1(0) = 0; X_2(0) = 0$; deterioration rate, $\alpha_1 = 0.04; \alpha_2 = 0.05$; depreciation rate, $\beta_1 = 0.03; \beta_2 = 0.02$; holding cost, $h_1 = 0.65; h_2 = 0.45$; sale revenue, $s_1 = 10.05; s_2 = 11.05$; salvage value, $b_1 = 1.5; b_2 = 1.2$; the coefficients production cost, $C_{u10} = 2.15; C_{u20} = 2.25; C_{u11} = 10; C_{u22} = 12.5$; promotional cost, $C_1 = 0.65; C_2 = 0.55$; time length of the system, $T = 5$ units. Imprecise discount rate is taken as triangular fuzzy numbers i.e.
7.2. Optimal results

The individual optimum values of the objective functions are $J^*_L = 980.3\$ and $J^*_R = 985.5\$ and using (4) and (5), from (17), we get the optimal revenue, rates of advertisement and production functions as $J = 983.5\$, $V_1 = 0.4t$, $V_2 = 0.3t$, $U_1(t) = 27.4 + 1.3t$, $0 \leq t \leq 3.5$ and $U_2(t) = 22.5 + 0.5t$, $0 \leq t \leq 3.6$. Using (6)–(10), we obtain the optimum values of $X_i(t), U_i(t), V_i(t)$ and $D_i(t)$ ($i = 1, 2$) and presented in Table 1 and depicted in Fig. 1 for the first item. The curves for $X_2(t), U_2(t), V_2(t)$ and $D_2(t)$ are also of similar nature.

7.3. Particular case: When $R$ is not fuzzy

If inflation and time value of many are crisp, i.e. $k_{iL} = k_1$ and $r_{iL} = r_1$ ($i = 1, 2$) and hence net discount rate of inflation is also crisp then $R_{iL} = R_1$ ($i = 1, 2$). For $k_{iL} = k_1 = 0.0785\$, $k_{2L} = k_2 = 0.0885\$ and $r_{iL} = r_1 = 0.0151\$, $r_{2L} = r_2 = 0.171\$, net discount rate of inflation $R_{iL} = R_1 = 0.0725\$, $R_{2L} = R_2 = 0.0825\$ and optimum revenue $J = 982.3\$. This does not differ much from the value of $J (=983.5\$) when $R$ is fuzzy because in this case, middle values of $(k_{iL}, k_1)$ and $(r_{iL}, r_1)$, $i = 1, 2$ have been assumed as the corresponding crisp values of $k$ and $r$ with $R = r - k$.

8. Model with constant advertisement cost

The case when rate of advertisement is constant, then the advertisement cost is stationary i.e. $V_i(t) = \text{constant} = v_i (=0.93)$(say), the corresponding expressions of the costs and the profit in the system are derived exactly in the same

### Table 1

<table>
<thead>
<tr>
<th>$t$</th>
<th>Item</th>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
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<td>10.8</td>
<td>12.4</td>
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<td>5</td>
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<tr>
<td>$U_i(t)$</td>
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<td>28.7</td>
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<td>$D_i(t)$</td>
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<td>19.80</td>
<td>20.15</td>
<td>20.80</td>
<td>21.73</td>
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</table>

Fig. 1. Optimal production, stock and demand for variable advertisement rates.

$[r_{11} = 0.149\$, $r_{12} = 0.151\$, $r_{13} = 0.153\$]; $[r_{21} = 0.169\$, $r_{22} = 0.171\$, $r_{23} = 0.173\$] which are transformed to two interval numbers $[0.15\$, $0.152\$] and $[0.17\$, $0.172\$], i.e. $r_{1L} = 0.15\$, $r_{1R} = 0.152\$ and $r_{2L} = 0.17\$, $r_{2R} = 0.172\$. Imprecise inflation rate is taken as triangular fuzzy numbers i.e. $R_{1L} = 0.075\$, $R_{12} = 0.079\$, $R_{13} = 0.081\$; $R_{11} = 0.086\$, $R_{12} = 0.088\$, $R_{13} = 0.092\$ which are transformed to two interval numbers $[0.077\$, $0.08\$] and $[0.087\$, $0.099\$], i.e. $R_{1L} = 0.077\$, $R_{12} = 0.08\$ and $R_{13} = 0.09\$. Hence net discount rate of inflation is then two interval numbers $[0.07\$, $0.075\$] and $[0.08\$, $0.085\$], i.e. $R_{1L} = 0.07\$, $R_{1R} = 0.075\$ and $R_{2L} = 0.08\$, $R_{2R} = 0.085\$.
way as done in the previous sections. These are

\[ D_i(t) = D_i(0)e^{-\beta_i t} + v_i \left( \frac{1 - e^{-\beta_i t}}{\beta_i} \right) \quad \text{in } (0, T) \] (19)

\[ = \frac{v_i}{\beta_i} + \left( D_i(0) - \frac{v_i}{\beta_i} \right) e^{-\beta_i t} \quad \text{in } (0, T). \] (20)

Using (18), (19), (7) and (8), from (1), we get

\[ X_i(t) = u_{i20} \left( \frac{1 - e^{-\alpha_i t}}{\alpha_i} \right) + u_{i1} \left( \frac{t}{\alpha_i} - \frac{1 - e^{-\alpha_i t}}{\alpha_i^2} \right) - D_{i2} \left( \frac{e^{-\beta_i t} - e^{-\alpha_i t}}{\alpha_i - \beta_i} \right), \quad (0, t_i) \] (21)

\[ = -v_{i2} \left( \frac{e^{\alpha_i (T-t)} - 1}{\alpha_i^2} \right) + D_{i2} \left( \frac{e^{\alpha_i (T-t)} - \beta_i T - e^{-\beta_i t}}{\alpha_i - \beta_i} \right), \quad (t_i, T) \] (22)

where \( u_{i20} = (u_{i0} - \frac{v_i}{\beta_i}) \), \( D_{i2} = D_i(0) - \frac{v_i}{\beta_i} \) and \( v_{i2} = \frac{v_i}{\beta_i} \) and \( t_i \) satisfy the continuity condition of \( X_i(t) \) of Eqs. (20) and (21) at \( t_i \).

In this case, we have,

\[ J_L(u) = \sum_{i=1}^{n} (s_{i1L} - H_{i1L} - v_{i1L} - uu_{i1L}) \quad \text{(cf. Appendix B)} \]

and

\[ J_R(u) = \sum_{i=1}^{n} (s_{i1R} - H_{i1R} - v_{i1R} - uu_{i1R}) \quad \text{(cf. Appendix B)} \]

subject to (4) and

\[ \sum_{i=1}^{n} \left( C_{ui0} \left( u_{i0} t_i + u_{i1} \frac{t_i^2}{2} \right) \right) + C_{ui1} t_i + c_i (e^{v_i} - 1) t_i, \quad (y_i = 1). \]

Using (7), (8) and (18)–(21) and with the input data in Section 7.1, the optimum expressions for production and demand functions and optimum levels for stocks are evaluated and presented in the table and graphically for the first item in Table 2 and Fig. 2 respectively.

9. Sensitivity analysis

Sensitivity analysis is made for linear time-dependent advertisement cost to study the effect of changing the holding cost, selling price and net discount rate of inflation, \( h_i, s_i \) and \( \tilde{R}_i, i = 1, 2 \) on the objective value. Table 3 shows that the values of the objective function \( J \) for different values of \( h_i, s_i \) and \( \tilde{R}_i, i = 1, 2 \). Percentage change of these values is shown with respect to the values used in the original problem (cf. input data in Section 7.1) which are \( h_1 = 0.6 \$, \ h_2 = 0.4 \$, \ s_1 = 10.0 \$, \ s_2 = 11.0 \$ and \( [R_{1L} R_{1R}] = [0.07$ $0.075$], \( [R_{2L} R_{2R}] = [0.08$ $0.085$].
10. Discussion

Figs. 1 and 2 pictorially represent the optimal results for production, demand, advertisement and stock. Here, for a period, the advertisement generates the demand and in the process, to meet the demand, production is accordingly adjusted so that whatever stock is built-up up to a certain period of time, it is completely exhausted at the end of the period. Such an advertisement is optimally evaluated so that profit is maximum. From Fig. 2, it is observed that in spite of continuous advertisement (constant), demand goes down gradually with time as the assumed demand function is a decreasing function. But, in Fig. 1, for increasing advertisement cost with time, demand decreases initially but after some time, it increases, even surpasses its initial value. Obviously profit is more in the case of increasing demand.

11. Practical implications

For any production-inventory model, it is the common practice that production is continued till sufficient stock is built-up in the godown. After that, demand is met from the stock and again the production is commenced once the stock is exhausted. Deterioration of the units and inflation and time value of money are taken into account to calculate the maximum profit. Such a real-life production-inventory model has been considered here. This model is applicable in the case of production of rice in a rice mill, production of fruit products such as Jelly, Jam, Fruit juice, etc. This model can be extended for the cases of production-inventory models with the defective items. All expressions will remain the same in this case also.
12. Conclusion

The present paper deals with the optimum production and advertising policy for a multi-item production-inventory system with deteriorating units, depreciation rate of sales, salvage value of deteriorated items, space capacity constraint, investment constraint and dynamic demand under the imprecise inflation and time discounting environment. Also some ideas such as (i) optimal control production problem for deteriorating multi-items, (ii) advertisement-dependent demand, (iii) dynamic production function, (iv) production-quantity-dependent unit cost and (v) imprecise inflation and imprecise depreciation in many value have been introduced for the first time.

In the solution approach, the new ideas are: (i) using interval mathematics for fuzzy numbers for their crisp value. In this connection, the Lemma 1 with an interval power of exponential has been reduced to an interval for the first time. As already mentioned earlier, for the first time, a new type of utility function with individual optimum values has been defined and used to convert multi-objective problem to a single objective one in an inventory control system. The formulation and analysis presented here can be extended to other production-inventory problems with different types of demand, advertisement, deterioration, defect, price discount, etc.

Appendix A

For simplicity, we take $\gamma_1 = 1 = \gamma_2$.

\[ H_{11L} = u_{i10} \left[ \frac{1 - e^{-R_i R_L}}{R_i R_L} - \frac{1 - e^{-\beta_i R_i R_L}}{\alpha_i (R_i R_L + \alpha_i)} \right] + \left[ \frac{1}{r_i} - \frac{1}{\beta_i} \right] \left( \frac{1 - e^{-\beta_i R_i R_L}}{R_i R_L} \right) + \frac{1 - e^{-\beta_i R_i R_L}}{R_i R_L} \left( \frac{1}{r_i} - \frac{1}{\beta_i} \right) \left( \frac{1}{r_i} - \frac{1}{\beta_i} \right), \]

\[ H_{11R} = u_{i10} \left[ \frac{1 - e^{-\beta_i R_i R_L}}{R_i R_L} - \frac{1 - e^{-\beta_i R_i R_L}}{\alpha_i (R_i R_L + \alpha_i)} \right] + \left[ \frac{1}{r_i} - \frac{1}{\beta_i} \right] \left( \frac{1 - e^{-\beta_i R_i R_L}}{R_i R_L} \right) + \frac{1 - e^{-\beta_i R_i R_L}}{R_i R_L} \left( \frac{1}{r_i} - \frac{1}{\beta_i} \right) \left( \frac{1}{r_i} - \frac{1}{\beta_i} \right), \]

\[ H_{12L} = u_{i10} \left[ \frac{1}{r_i} - \frac{1}{\beta_i} \right] \left( \frac{1 - e^{-\beta_i R_i R_L}}{R_i R_L} \right) + \left[ \frac{1}{r_i} - \frac{1}{\beta_i} \right] \left( \frac{1 - e^{-\beta_i R_i R_L}}{R_i R_L} \right) \left( \frac{1}{r_i} - \frac{1}{\beta_i} \right) \left( \frac{1}{r_i} - \frac{1}{\beta_i} \right), \]

\[ H_{12R} = u_{i10} \left[ \frac{1}{r_i} - \frac{1}{\beta_i} \right] \left( \frac{1 - e^{-\beta_i R_i R_L}}{R_i R_L} \right) + \left[ \frac{1}{r_i} - \frac{1}{\beta_i} \right] \left( \frac{1 - e^{-\beta_i R_i R_L}}{R_i R_L} \right) \left( \frac{1}{r_i} - \frac{1}{\beta_i} \right) \left( \frac{1}{r_i} - \frac{1}{\beta_i} \right), \]
\[ ss_{iL} = s_i \frac{v_i}{\beta_i} \left( \frac{1 - e^{-R_i R T}}{R_i^2} - \frac{1 - e^{-R_i R T}}{R_i \beta_i} \right) - s_i \frac{v_i}{\beta_i} \left( \frac{1 - e^{-R_i R T}}{\beta_i^2 R_i} \right) + D_{i1}(0) \left( \frac{1 - e^{(\beta_i + R_i R)T}}{R_i R + \beta_i} \right), \]

\[ ss_{iR} = s_i \frac{v_i}{\beta_i} \left( \frac{1 - e^{-R_i L T}}{R_i^2} - \frac{1 - e^{-R_i L T}}{R_i \beta_i} \right) - s_i \frac{v_i}{\beta_i} \left( \frac{1 - e^{-R_i L T}}{\beta_i^2 R_i} \right) + D_{i1}(0) \left( \frac{1 - e^{(\beta_i + R_i L)T}}{R_i L + \beta_i} \right). \]

\[ H_{iL} = (h_i - b_i \alpha_i)(H_{i1L} + H_{i2L}), \]

\[ H_{iR} = (h_i - b_i \alpha_i)(H_{i1R} + H_{i2R}), \]

\[ vv_{iL} = c_i \left( \frac{e^{(u_i - R_i R)T} - 1}{v_i - R_i R} - \frac{1 - e^{-R_i R T}}{R_i} \right), \]

\[ vv_{iR} = c_i \left( \frac{e^{(u_i - R_i L)T} - 1}{v_i - R_i L} - \frac{1 - e^{-R_i L T}}{R_i} \right). \]

\[ uu_{iL} = (C_{ui0 u_i0} + C_{ui1}) \left( \frac{1 - e^{-R_i R t_i}}{R_i} \right) + C_{ui0 u_i1} \left( \frac{1 - (1+t_i R_i R) e^{-R_i R t_i}}{R_i^2} \right), \]

and

\[ uu_{iR} = (C_{ui0 u_i0} + C_{ui1}) \left( \frac{1 - e^{-R_i L t_i}}{R_i} \right) + C_{ui0 u_i1} \left( \frac{1 - (1+t_i R_i L) e^{-R_i L t_i}}{R_i^2} \right) \]

where \( D_{i1}(0) = s_i (D_i(0) + \frac{v_i}{\beta_i}) \).

**Appendix B**

For simplicity, we take \( \gamma_1 = 1 = \gamma_2 \).

\[ H_{i11L} = u_{i20} \left( \frac{1 - e^{-R_i R t_i}}{R_i \alpha_i} - \frac{1 - e^{-(R_i R + \alpha_i) t_i}}{\alpha_i (R_i R + \alpha_i)} \right) + u_{i1} \left( \frac{-t_i e^{-R_i R t_i}}{R_i R \beta_i} + \frac{1 - e^{-R_i R t_i}}{R_i^2 \beta_i} - \frac{1 - e^{-R_i R t_i}}{\alpha_i^2 R_i} \right), \]

\[ H_{i11R} = u_{i20} \left( \frac{1 - e^{-R_i L t_i}}{R_i \alpha_i} - \frac{1 - e^{-(R_i L + \alpha_i) t_i}}{\alpha_i (R_i L + \alpha_i)} \right) + u_{i1} \left( \frac{-t_i e^{-R_i L t_i}}{R_i L \beta_i} + \frac{1 - e^{-R_i L t_i}}{R_i^2 \beta_i} - \frac{1 - e^{-R_i L t_i}}{\alpha_i^2 R_i} \right), \]

\[ H_{i21L} = u_{i21} \left( \frac{e^{(\alpha_i - \beta_i) (R_i R) t_i} - e^{-R_i R T}}{\alpha_i (\alpha_i + R_i R)} + \frac{e^{(\alpha R_i - \beta R_i) (R_i R + \alpha R_i) t_i} - e^{-(R_i R + \beta R_i) T}}{\alpha_i (\alpha R_i + R_i R)} \right), \]

\[ H_{i21R} = u_{i21} \left( \frac{e^{(\alpha_i - \beta_i) (R_i L) t_i} - e^{-R_i L T}}{\alpha_i (\alpha_i + R_i L)} + \frac{e^{(\alpha R_i - \beta R_i) (R_i L + \alpha R_i) t_i} - e^{-(R_i L + \beta R_i) T}}{\alpha_i (\alpha R_i + R_i L)} \right), \]
\[ s_{i1L} = s_i \left( v_i \left( 1 - e^{R_iR_i T} \right) + \left( D_i(0) - v_i \right) \left( 1 - e^{(\beta_i + R_i R_i)T} \right) \right), \]  \( i = 1, 2, \ldots \)

\[ s_{i1R} = s_i \left( v_i \left( 1 - e^{R_iL_i T} \right) + \left( D_i(0) - v_i \right) \left( 1 - e^{(\beta_i + R_i L_i)T} \right) \right), \]  \( i = 1, 2, \ldots \)

\[ H_{i1L} = (h_i - b_i \alpha_i)(H_{i11L} + H_{i21L}), \]  \( i = 1, 2, \ldots \)

\[ H_{i1R} = (h_i - b_i \alpha_i)(H_{i11R} + H_{i21R}), \]  \( i = 1, 2, \ldots \)

\[ v_{L_i1L} = c_i \left( e^{v_i} - 1 \right) \left( 1 - e^{-R_iR_i T} \right), \]  \( i = 1, 2, \ldots \)

\[ v_{L_i1R} = c_i \left( e^{v_i} - 1 \right) \left( 1 - e^{-R_iL_i T} \right), \]  \( i = 1, 2, \ldots \)

\[ u_{i1L} = (C_{u1i0}u_{i0} + C_{u1i1}) \left( 1 - e^{-R_iR_i h_i} \right) + u_{i1} \left( 1 - (1 + t_i R_i^2) e^{-R_iR_i h_i} \right), \]  \( i = 1, 2, \ldots \)

and

\[ u_{i1R} = (C_{u1i0}u_{i0} + C_{u1i1}) \left( 1 - e^{-R_iL_i h_i} \right) + C_{u1i0}u_{i1} \left( 1 - (1 + t_i R_i^2) e^{-R_iL_i h_i} \right). \]  \( i = 1, 2, \ldots \)

References


