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Remarks on the One-Phase Stefan Problem for the Heat Equation with the Flux Prescribed on the Fixed Boundary*

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1. INTRODUCTION

This paper is concerned with the following one-phase Stefan problem:

$$Lu \equiv u_{xx} - u_t = 0, \quad \text{in} \quad 0 < x < s(t),\ 0 < t \leqslant T, \tag{1.1}$$

$$u_x(0, t) = H(t), \quad 0 < t \leqslant T; \tag{1.2}$$

$$u(x, 0) = \Phi(x), \quad 0 \leqslant x \leqslant b, \tag{1.3}$$

$$u(s(t), t) = 0, \quad 0 \leqslant t \leqslant T,\ s(0) = b \geqslant 0, \tag{1.4}$$

and

$$\dot{s}(t) = - u_x(s(t), t), \quad 0 < t \leqslant T, \tag{1.5}$$

where T is an arbitrarily fixed positive number.

As is well known, the problem (1.1)-(1.5) is a mathematical description for the unidimensional heat conduction in a plane infinite slab of homogeneous thermally isotropic material with a phase occurring at one limiting plane and the thermal flux prescribed on the other.

For sake of simplicity, in writing down (1.1)-(1.5) we choose a system of variables such that the thermal coefficients (conductivity, heat capacity, density, latent heat) disappear.

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Problems of this type have been considered by various authors [1, 6, 7, 9, 10, 11, 12, 13]. Our discussion of this problem applies the maximum principle as the major tool in a constructive existence proof via the method of retarding the argument in the free boundary condition (1.5). Hence, existence is obtained under minimal smoothness assumptions upon the data. In the following sections we shall discuss existence (global), uniqueness, stability, monotone dependence, and the asymptotic behavior of the solution of (1.1)-(1.5). The techniques used and the results obtained are similar to those in [2, 3, 4, 5].

The assumptions we shall require on the Stefan data are as follows:

(A) $H(t)$ *is a bounded piecewise continuous nonpositive function;*

(B) $\Phi(x)$ *is a piecewise continuous function such that:*

$$0 \leqslant \Phi(x) \leqslant L(b - x).$$

Obviously, the assumption (B) on the Lipschitz continuity of $\Phi(x)$ near $x = b$ has a significance only in the case $b > 0$.

The assumption on the sign of $H(t)$ means that heat is entering the region, so that, for each t there is only one phase, say the liquid one (of course, the same reasoning holds if $\Phi(x) \leqslant 0$ and $H(t) \geqslant 0$).

We make the following definitions:

DEFINITION 1. We say that a real-valued function $u(x, t)$ is a *solution of the auxiliary problem* (1.1)-(1.4) for a *given real-valued function* $s(t)$ ($s(t) > 0$), if:

(a) u_{xx} and $u_t \in C$, $u_{xx} = u_t$, $0 < x < s(t)$, $0 < t \leqslant T$;

(b) $u \in C$ in $0 \leqslant x \leqslant s(t)$, $0 \leqslant t \leqslant T$ except at points of discontinuity of $\Phi(x)$;

(c) $u(x, 0) = \Phi(x)$ at points of continuity of $\Phi(x)$ and

$$0 \leqslant \varliminf_{t \to 0} u(x, t) \leqslant \varlimsup_{t \to 0} u(x, t) < \infty$$

at points of discontinuity;

(d) $\lim_{x \to 0} u_x(x, t) = H(t)$ at points of continuity of H and

$$0 \leqslant \varliminf_{x \to 0} | u_x(x, t)| \leqslant \varlimsup_{x \to 0} | u_x(x, t)| < \infty$$

at points of discontinuity.

DEFINITION 2. By a *solution of the given Stefan Problem* (1.1)-(1.5) we mean a pair of real-valued functions $(s(t), u(x, t))$, such that $s(0) = b$, and for $0 < t \leqslant T$:

(i) $s(t) \in C^1$, $s(t) > 0$; $t > 0$;

(ii) $u(x, t)$ is the solution of the corresponding auxiliary problem;

(iii) $u_x(s(t), t)$ exists and is continuous;

(iv) (1.5) is satisfied.

It is well known that (see [8, 2]), if $s(t)$ is Holder continuous with Holder coefficient $> \frac{1}{2}$, if (A) and (B) are satisfied and if $b \geqslant 0$, the auxiliary problem has a unique solution.

We state a useful result concerning the reformulation of the free boundary condition (1.5).

LEMMA 1. *Under assumptions* (A) *and* (B), *if* $s(t)$ *is a Lipschitz continuous function for* $0 \leqslant t \leqslant T$, *then condition* (1.5) *is equivalent to:*

$$s(t) = b + \int_0^b \Phi(x)\, dx - \int_0^t H(\tau)\, d\tau - \int_0^{s(t)} u(x, t)\, dx. \tag{1.6}$$

Proof. Suppose (u, s) is a solution of (1.1)-(1.5) and integrate (1.1) over its domain of validity; using (1.2)-(1.5), (1.6) follows directly.

Suppose conversely that (u, s) satisfies (1.1)-(1.4) and (1.6); by differentiating (1.6), (1.5) follows directly if $u_x(s(t), t)$ exists and is continuous for $0 < t \leqslant T$. But this last assumption is guaranteed by Lemma 1 of [2]; consequently Lemma 1 is proved.

2. EXISTENCE ($b > 0$)

First, we need:

LEMMA 2. *Under the assumptions* (A) *and* (B), *let* $s(t)$ *be a Lipschitz continuous, monotonic nondecreasing function, and define*

$$A = \max\{\max_{t \in [0,T]} |H(t)|, L\}. \tag{2.1}$$

Then the solution $u(x, t)$ *of the corresponding auxiliary problem is such that*

$$-A \leqslant u_x(s(t), t) \leqslant 0. \tag{2.2}$$

Proof. Let us prove (2.2) for each fixed instant t_0. First note that, from the maximum principle, it follows that $u(x, t)$ is non-negative and since $u(s(t), t) = 0$ the second inequality in (2.2) is proved. For each $t_0 \in [0, T]$ consider the function

$$v(x, t) = A(s(t_0) - x) - u(x, t)$$

for $0 < t \leqslant t_0$, $0 < x < s(t)$. Clearly,

$$\begin{aligned} Lv &\equiv v_{xx} - v_t = 0, & \\ v_x(0, t) &= -A - H(t) \leqslant 0 & \text{by (2.1) and (A)}, \\ v(s(t), t) &= A(s(t_0) - s(t)) \geqslant 0 & \text{by the monotonicity of } s(t), \\ v(x, 0) &= A(s(t_0) - x) - \Phi(x) \geqslant 0 & \text{by (2.1) and (B)}. \end{aligned}$$

Hence, by the maximum principle, $v(x, t) \geqslant 0$ in its domain of definition. Since $v(s(t_0)\, t_0) = 0$, it follows that $v_x(s(t_0)\, t_0) \leqslant 0$ which implies (2.2) for each t_0 and completes the proof of the Lemma.

Next, we will apply the retarded argument technique to construct the solution of the given problem. For each $\theta \in (0, b)$, let us define

$$\Phi^\theta(x) \begin{cases} = \Phi(x), & \text{for } 0 \leqslant x \leqslant b - \theta \\ = 0, & \text{for } b - \theta < x \leqslant b, \end{cases}$$

and find the solution $u^\theta(x, t)$ of the auxiliary problem (1.1)-(1.4), where $\Phi(x)$ is replaced by $\Phi^\theta(x)$, $s(t)$ is replaced by $s^\theta(t) \equiv b$, and T by θ. In this case it is easy to show that $u_x{}^\theta(b, t) \equiv u_x(s^\theta(t), t)$ exists and is continuous in $[0, \theta]$ and that by Lemma 2 we have $-A \leqslant u_x{}^\theta(s^\theta(t), t) \leqslant 0$. In the second time-interval $\theta \leqslant t \leqslant 2\theta$ let us define:

$$s^\theta(t) = b - \int_\theta^t u_x{}^\theta(s^\theta(\tau - \theta), \tau - \theta)\, d\tau \tag{2.3}$$

and solve the auxiliary problem for this choice of $s^\theta(t)$; these are the first steps of an inductive process that we can perform for each θ, $0 < \theta < b$. We prove the following result.

LEMMA 3. *For each $\theta \in (0, b)$, there exists a solution (s^θ, u^θ) of* (1.1)-(1.4) *where $\Phi(x) = \Phi^\theta(x)$. The function $s^\theta(t)$ is equal to b in $t \in [0, \theta]$, satisfies* (2.3) *in $t \in [\theta, T]$ and is C^1 in $[0, T]$. Moreover,*

$$0 \leqslant \dot{s}^\theta(t) \leqslant A. \tag{2.4}$$

Proof. Suppose that by the method given above we have constructed a pair (s^θ, u^θ) for $0 \leqslant t \leqslant n\theta$. Assume that $s^\theta(t) \in C^1$ and $\dot{s}^\theta \geqslant 0$. By Lemma 1 of [2], $u_x^\theta(s^\theta(t), t)$ exists, is continuous and satisfies (2.2). Suppose finally that (s^θ, u^θ) satisfy (2.3) for $0 \leqslant t \leqslant n\theta$. In the next step $n\theta \leqslant t \leqslant (n+1)\theta$, let us define s^θ by (2.3) and solve the auxiliary problem until $t = (n+1)\theta$. By the hypothesis on $u_x(s^\theta(t), t)$ in $[0, n\theta]$, $s^\theta(t)$ is C^1 in $[n\theta, (n+1)\theta]$ and satisfies (2.4). Hence, $u_x^\theta(s^\theta(t), t)$ is continuous in the same time interval and, by Lemma 2, satisfies (2.2).

We can now prove the main result of this section.

THEOREM 1. *Under the assumptions* (A) *and* (B), *there exists a solution* (s, u) *for the Stefan problem* (1.1)-(1.5) *when* $b > 0$. *The free boundary is* C^1 *in* $(0, T]$, *is monotonically nondecreasing and satisfies:*

$$0 \leqslant \dot{s}(t) \leqslant A \qquad \text{for } 0 \leqslant t \leqslant T, \tag{2.5}$$

where A *is defined by* (2.1).

Proof. By (2.4), the functions $s^\theta(t)$ form an equicontinuous, uniformly bounded family. Hence Ascoli-Arzela's theorem holds and we can select a subsequence $s^\theta(t)$ that converges uniformly to a monotonic Lipschitz continuous function $s(t)$ as θ tends to zero. Let $u(x, t)$ be the unique solution of the auxiliary problem with that choice of s. It is easy to show that, given any $\epsilon > 0$, it is possible to find a θ^* such that, for all $\theta < \theta^*$ $| u^\theta(x, t) - u(x, t)| \leqslant \epsilon$. Indeed consider the difference $u^\theta - u = w$ in the region

$$0 \leqslant x \leqslant \max(s^\theta(t), s(t)), \qquad 0 \leqslant t \leqslant T,$$

(where the two functions are extended by setting them identically zero outside their domain of definition). Since $w_x(0, t) = 0$, we can reflect the domain of definition about the line $x = 0$. Then, the maximum principle gives:

$$| u^\theta(x, t) - u(x, t)| \leqslant \max\{\sup | \Phi^\theta - \Phi | , \| u^\theta(s(\tau), \tau)\|_t , \| u(s^\theta(\tau), \tau)\|_t\},$$

where for any function $f = f(t)$

$$\| f \|_t = \sup_{0 \leqslant \tau \leqslant t} | f(\tau)|.$$

By the continuity of u^θ and u the right side of the inequality can be made less than ϵ, provided θ^* is chosen such that $| s^\theta(t) - s(t)|$ is sufficiently small. So we have shown that the subsequence u^θ, corresponding to the s^θ tending to s, converges uniformly to u. In order to prove that (s, u) is a solution of the Stefan problem, we must prove (i), (iii), (iv) of the Definition 2; (iii) follows directly by the Lipschitz continuity of $s(t)$. By Lemma 1 in order to prove (iv) it suffices to prove that (s, u) satisfies (1.6). Integrating (1.1) over its domain of definition, it is easy to see

$$s^\theta(t+\theta) = b + \int_0^b \Phi^\theta(x)\,dx - \int_0^t H(\tau)\,d\tau - \int_0^{s^\theta(t)} u^\theta(x,t)\,dx.$$

Taking the limit as θ tends to zero it follows from the uniform convergence of s^θ to s, of Φ^θ to Φ and of u^θ to u that (s, u) satisfies (1.6). Consequently (iv) is demonstrated and (i) follows directly by (1.5).

3. Stability and Uniqueness ($b \geqslant 0$)

Theorem 2. *If* (s_i, u_i) *is a solution of* (1.1)-(1.5) *for data* $H_i(t)$, $\Phi_i(x)$, b_i $(i = 1, 2)$ *satisfying assumptions* (A) *and* (B) $b_2 \geqslant b_1 \geqslant 0$, *then there exists a constant* $C = C(A, T)$ *such that:*

$$| s_1(t) - s_2(t)| \leqslant C \left\{ | b_1 - b_2 | + \int_0^{b_1} | \Phi_1(x) - \Phi_2(x)|\,dx + \int_{b_1}^{b_2} | \Phi_2(x)|\,dx \right.$$
$$\left. + \int_0^t | H_1(\tau) - H_2(\tau)|\,d\tau \right\}, \qquad t \in [0, T]. \tag{3.1}$$

Proof. The proof is given in [1] and will not be repeated here.

Theorem 3. *Under the assumptions* (A) *and* (B), *the solution to the Stefan problem* (1.1)-(1.5) *with* $b \geqslant 0$ *is unique.*

Proof. Theorem 3 is an immediate corollary of Theorem 2.

4. Existence ($b = 0$)

Theorem 4. *Under assumptions* (A) *and* (B), *there exists a solution* (s, u) *to the Stefan problem* (1.1)-(1.5) *when* $b = 0$. *The free boundary is* C^1 *in* $(0, T]$, *is monotonically nondecreasing and satisfies* (2.5).

Proof. For each $0 < b < b_0$ let (s^b, u^b) be the unique solution of the Stefan Problem (1.1)-(1.5) with $\Phi(x) \equiv 0$. Note that the proof of Lemma 2 and the constant A are independent on b. Hence, we have: $0 \leqslant \dot{s}^b(t) \leqslant A$ for $0 \leqslant t \leqslant T$ and $b \in (0, b_0)$. Consequently, the functions $s^b(t)$ form an equicontinuous, uniformly bounded family. Choose a sequence of b's tending to zero and apply the Ascoli-Arzela Theorem to obtain an subsequence s^b converging uniformly to s. Let $u(x, t)$ be the unique solution of the auxiliary problem (1.1)-(1.4) for such a choice of s. With the same argument used in Theorem 1, one can show that $u^b(x, t)$ converges uniformly to $u(x, t)$. Moreover, from (1.6) for each b we have

$$s^b(t) = b - \int_0^{s^b(t)} u^b(x, t)\, dx - \int_0^t H(\tau)\, d\tau,$$

and from the uniform convergence of $\{s^b(t)\}$ and $\{u^b(x, t)\}$, we obtain that (s, u) satisfies (1.6). Since $s(t)$ is a Lipschitz continuous function it follows that $u_x(s(t), t)$ exists and is continuous for $t > 0$. Consequently condition (iii) of Definition 2 is fulfilled. Since Lemma 1 applies, (iv) and (i) are satisfied. Since (ii) follows from the definition of $u(x, t)$, the existence of a solution is proved.

5. Monotone Dependence ($b \geqslant 0$)

Consider two sets $\{H_i(t), \Phi_i(x), b_i\}$ $i = 1, 2$ of Stefan data satisfying assumptions (A) and (B). Theorem 1, 3 and 4 state the existence of an unique solution (s_i, u_i) to each one of the two problems. We shall prove the following result.

Theorem 5. *Under the above assumptions, if*

$$0 \leqslant b_1 \leqslant b_2, \qquad \Phi_1 \leqslant \Phi_2, \qquad H_1 \geqslant H_2$$

then

$$s_1(t) \leqslant s_2(t). \tag{5.1}$$

Proof. Consider first the case $0 \leqslant b_1 < b_2$. We shall show that $s_1(t) < s_2(t)$. If not, then there exists a first time t_0 such that

$$s_1(t_0) = s_2(t_0) \qquad \text{and} \qquad \dot{s}_1(t_0) \geqslant \dot{s}_2(t_0). \tag{5.2}$$

Consider the difference $u_1(x, t) - u_2(x, t)$ in $0 \leqslant x \leqslant s_1(t)$, $0 \leqslant t \leqslant t_0$. By the maximum principle we have $u_1(x, t) - u_2(x, t) < 0$. Since

$$u_1(s_1(t_0), t_0) - u_2(s_1(t_0), t_0) = 0,$$
$$u_{1x}(s_1(t_0), t_0) - u_{2x}(s_1(t_0), t_0) > 0,$$

or $\dot{s}_1(t_0) < \dot{s}_2(t_0)$. For the case $b_1 = b_2 \geqslant 0$, we define $b_2^\delta = b_2 + \delta = b_1 + \delta$ and construct the solution (s_2^δ, u_2^δ) to the Stefan problem with data H_2, b_2^δ and Φ_2^δ where:

$$\Phi_2^\delta = \begin{cases} \Phi_2, & 0 \leqslant x \leqslant b_1 = b_2; \\ 0, & b_2 \leqslant x \leqslant b_2 + \delta. \end{cases}$$

By the previous argument, $s_2^\delta > s_1$, but, the stability theorem implies that $s_2^\delta(t)$ converges uniformly to $s_2(t)$ as δ tends to zero. Hence, (5.1) is proved.

6. Asymptotic Behavior

Throughout this section we shall be concerned with the asymptotic behavior of the free boundary $x = s(t)$ of the Stefan problem (1.1)-(1.5) as $t \to +\infty$. Under the assumptions (A) and (B), we have the existence and uniqueness of the solution in either case $b > 0$ or $b = 0$.

Theorem 6. *If*

$$\lim_{t\to\infty} \int_0^t H(\tau)\, d\tau = -\infty,$$

then

$$\lim_{t\to\infty} s(t) = +\infty.$$

If

$$\lim_{t\to\infty} \int_0^t H(\tau)\, d\tau = -F, \qquad 0 \leqslant F < +\infty,$$

then

$$\lim_{t\to\infty} s(t) = b + \int_0^b \Phi(x)\, dx + F \equiv \ell_0 \tag{6.1}$$

Proof. Consider first the case of $H(t)$ with compact support, i.e. suppose $H(t) = 0$ for each $t \geqslant \Sigma$.

From (1.6) we have

$$s(t) = \ell_0 - \int_0^{s(t)} u(x, t)\, dx \tag{6.2}$$

for $t \geqslant \Sigma$.

By the maximum principle, $u(x, t)$ is dominated by $y_1(x, t) + y_2(x, t)$, where y_1 and y_2 solve the heat equation in the half-space $x > 0$ with the following conditions:

$$y_1(x, 0) = \begin{cases} \Phi(x), & 0 \leqslant x \leqslant b, \\ 0, & b \leqslant x < \infty, \end{cases} \qquad y_{1x}(0, t) = 0; \tag{6.3}$$

and

$$y_2(x, 0) = 0, \qquad y_{2x}(0, t) = H(t);$$

respectively. But for $t \geqslant \Sigma$,

$$y_1(x, t) = c \int_0^\infty \frac{\Phi(\xi)}{\sqrt{t}} \exp\left[-\frac{(x-\xi)^2}{4t}\right] d\xi \leqslant cbt^{-1/2} \max \Phi \tag{6.3$'$}$$

and

$$y_2(x, t) = c' \int_0^t \frac{H(\tau)}{\sqrt{t-\tau}} \exp\left[-\frac{x^2}{4(t-\tau)}\right] d\tau \leqslant c'' \| H \|_\Sigma \{\sqrt{t} - \sqrt{t-\Sigma}\}. \tag{6.4$'$}$$

From (6.4′) and (6.3′) we find that $\lim_{t\to\infty} u(x, t) = 0$ and (6.1) is proved in the case of $H(t)$ with compact support.

For general $H(t)$, set

$$H^{(n)}(t) = \begin{cases} H(t), & 0 \leqslant t \leqslant n \\ 0, & n < t \end{cases}$$

and define the corresponding $\ell_0^{(n)}$ and $s^{(n)}$. Now $\lim_{t\to\infty} s^{(n)}(t) = \ell_0^{(n)}$. Since $H_n \geqslant H$, it follows from Theorem 5 that $s^{(n)}(t) \leqslant s(t)$. But, from (1.6), $s(t) \leqslant \ell_0$. Hence,

$$\ell_0^{(n)} = \lim_{t\to\infty} s^{(n)}(t) \leqslant \varliminf_{t\to\infty} s(t) \leqslant \varlimsup_{t\to\infty} s(t) \leqslant \ell_0 .$$

Now, let $n \to \infty$. Since $\ell_0^{(n)} \to \ell_0$, the proof of Theorem 6 is complete.

Next, we shall perform a deeper analysis of the behavior of $s(t)$ in the case $\lim_{t\to\infty} s(t) = \infty$. First we prove the following result.

THEOREM 7. *Assume* $\lim_{t\to\infty} \int_0^t H(\tau)\, d\tau = -\infty$ *and consider the solution* (s, u) *of the given problem and the solution* (σ, v) *of the Stefan problem*

$$\begin{aligned} Lv &= 0, && 0 < x < \sigma(t), \quad t_0 < t < \infty, \\ v_x(0, t) &= H(t), && \sigma(t_0) = 0, \quad t_0 < t \\ v(\sigma(t), t) &= 0, && \\ \dot\sigma(t) &= -v_x(\sigma(t), t), && t_0 < t. \end{aligned} \tag{6.5}$$

Then, as $t \to \infty$,

$$\frac{s(t)}{\sigma(t)} = 1 + 0\left(\frac{1}{\sigma(t)}\right), \tag{6.6}$$

which implies in particular that $s(t) \sim \sigma(t)$.

Proof. From the monotone dependence it follows $\sigma(t) \leqslant s(t)$; and from the maximum principle we have $v(x, t) \leqslant u(x, t)$. Now,

$$\sigma(t) \leqslant s(t) = s(t_0) + \int_0^{s(t_0)} u(x, t_0)\, dx - \int_{t_0}^{t} H(\tau)\, d\tau - \int_0^{s(t)} u(x, t)\, dx$$

$$\leqslant \sigma(t) + s(t_0) + \int_0^{s(t_0)} u(x, t_0)\, dx.$$

Hence,

$$1 \leqslant \frac{s(t)}{\sigma(t)} \leqslant 1 + (\sigma(t))^{-1} \left\{ s(t_0) + \int_0^{s(t_0)} u(x, t_0)\, dx \right\}$$

or, i.e.,

$$\frac{s(t)}{\sigma(t)} = 1 + 0\left(\frac{1}{\sigma(t)}\right).$$

Next we demonstrate the following result.

THEOREM 8. *If (s, u) is the solution of the given Stefan problem and if*

$$\text{(C)} \qquad \lim_{t\to\infty} - \int_0^t H(\tau)\, d\tau = \infty$$

and

$$\text{(D)} \qquad \lim_{t\to\infty} \int_0^t \frac{|H(\tau)|}{\sqrt{t-\tau}}\, d\tau = 0$$

then

$$s(t) \sim -\int_0^t H(\tau)\, d\tau \tag{6.7}$$

as $t \to \infty$.

Proof. Consider $\sigma(t)$ as defined by (6.5) with $t_0 = 0$. We have

$$-\int_0^t H(\tau)\, d\tau - \int_0^{\sigma(t)} y_2(x, t)\, dx \leqslant \sigma(t) \leqslant -\int_0^t H(\tau)\, d\tau, \tag{6.8}$$

where $y_2(x, t)$ is defined by (6.4) and (6.4'). From the first inequality in (6.8) we get

$$-\int_0^t H(\tau)\, d\tau \leqslant \sigma(t) \left\{ 1 + c \int_0^t \frac{|H(\tau)|\, d\tau}{\sqrt{t-\tau}}\, d\tau \right\}, \tag{6.9}$$

where c is a positive constant. Hence, the result follows immediately from (C) and (D).

Remark. The condition (C) and (D) are not contradictory. They express the delicate area where the total energy input (C) is infinite and yet the boundary temperature tends to zero as $t \to \infty$. As an example, consider

$$H(t) = \begin{cases} -1, & 0 \leqslant t \leqslant 1, \\ \dfrac{-1}{t}, & 1 \leqslant t \leqslant \infty. \end{cases}$$

An elementary quadrature yields

$$\int_0^t \frac{|H(\tau)|\, d\tau}{\sqrt{t-\tau}} = -\frac{1}{\sqrt{t}} \log \left| \frac{1-\sqrt{1-1/t}}{1+\sqrt{1+1/t}} \right| + 0\left(\frac{1}{\sqrt{t}}\right).$$

Other examples are

$$H(t) = \begin{cases} -1, & 0 \leqslant t \leqslant 1, \\ -\dfrac{1}{t^\gamma}, & 1 \leqslant t < \infty, \quad \dfrac{1}{2} < \gamma < 1. \end{cases}$$

The details are left to the reader.

Next, we study some particular cases in which

$$\lim_{t\to\infty} \int_0^t \frac{H(\tau)}{\sqrt{t-\tau}}\, d\tau \neq 0.$$

THEOREM 9. *Let (s, u) be the solution of the given Stefan problem. If*

$$H(t) \sim -c \exp[c^2 t], \qquad c > 0, \tag{6.10}$$

then

$$s(t) \sim ct. \tag{6.11}$$

And if,

$$H(t) \sim -t^{-\alpha} \sum_{n=1}^{\infty} \frac{c^{2n-1}\Gamma(2n\alpha - \alpha + 1)\, t^{2n\alpha - n}}{\Gamma(2n)\, \Gamma(2n\alpha - \alpha - n + 1)}, \tag{6.12}$$

then

$$s(t) \sim ct^\alpha. \qquad \alpha \geqslant \tfrac{1}{2}. \tag{6.13}$$

Proof. It is known that the formula [4, p. 434]

$$v(x, t) = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \frac{\partial^n}{\partial t^n} [x - s(t)]^{2n} \tag{6.14}$$

provides, whenever it makes sense, an explicit solution to the inverse Stefan problem. Consequently the asymptotic behavior of $H(t)$ corresponding to (6.11) can easily be found:

$$v_x(0, t) = -\sum_{n=1}^{\infty} \frac{c^{2n-1}}{(2n-1)!} (2n-1)(2n-2) \cdots (n) \cdot t^{n-1} = c \exp[c^2 t]$$

Conversely, it is easy to prove that an asymptotic behavior like (6.10) generates a solution whose boundary satisfies (6.11). The application of the same method provides the proof of the second statement of the Theorem. Details are omitted, as well as special cases of (6.12), as for example $\alpha = \frac{1}{2}$ in which

$$\lim_{t \to \infty} \int_0^t \frac{H(\tau)}{\sqrt{t - \tau}} > -\infty,$$

i.e. the boundary temperature is asymptotically finite (but not equal to zero) and the $s(t)$ goes to infinity approaching a parabola.

7. Regularity of the Boundary

For the case that $b > 0$ and $\Phi(x) \equiv 0$, we can state the following result.

THEOREM 10. *The free boundary s is infinitely differentiable if $b > 0$ and* $\Phi(x) = 0$.

Proof. The techniques of [5] can easily be applied to yield the results of the theorem.

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